18 The CM torsor

Over the course of the last three lectures we have established an equivalence of categories between complex tori $\mathbb{C}/L$ and elliptic curves $E/\mathbb{C}$:

$$\begin{aligned}\{\text{lattices } L \subseteq \mathbb{C}\}/_\sim &\xrightarrow{\sim} \{\text{elliptic curves } E/\mathbb{C}\}/_\simeq\\ L &\longmapsto E_L\colon y^2 = 4x^3 - g_2(L)x - g_3(L)\\ j(L) &= j(E_L)\end{aligned}$$

in which homothetic lattices correspond to isomorphic elliptic curves, and we have established ring isomorphisms

$$\mathrm{End}(\mathbb{C}/L) \simeq \mathcal{O}(L) \simeq \mathrm{End}(E_L)$$

where the ring

$$\mathcal{O}(L) := \{\alpha \in \mathbb{C} : \alpha L \subseteq L\}$$

is necessarily equal to $\mathbb{Z}$ or an order $\mathcal{O}$ in an imaginary quadratic field. In the latter case, which we will assume throughout this lecture, the elliptic curve E_L is said to have *complex multiplication* (CM) by $\mathcal{O}$, and the lattice L is necessarily homothetic to an $\mathcal{O}$-ideal.

If we fix the order $\mathcal{O}$, the $\mathcal{O}$-ideals L for which $\mathrm{End}(E_L) \simeq \mathcal{O}$ are precisely those for which $\mathcal{O}(L) = \mathcal{O}$; in the previous lecture we defined such $\mathcal{O}$-ideals to be *proper*. Note that $\mathcal{O} \subseteq \mathcal{O}(L)$ always holds, since L is an $\mathcal{O}$-ideal, but in general $\mathcal{O}(L)$ be be larger than $\mathcal{O}$.

The sets

$$\{L \subseteq \mathbb{C} : \mathcal{O}(L) = \mathcal{O}\}/_\sim \longleftrightarrow \{E/\mathbb{C} : \mathrm{End}(E) = \mathcal{O}\}/_\simeq$$

are both in bijection with the *ideal class group*

$$\mathrm{cl}(\mathcal{O}) := \{\text{proper } \mathcal{O}\text{-ideals } \mathfrak{a}\}/_\sim$$

where the equivalence relation on proper $\mathcal{O}$-ideals is defined by

$$\mathfrak{a} \sim \mathfrak{b} \qquad \Longleftrightarrow \qquad \alpha\mathfrak{a} = \beta\mathfrak{b} \text{ for some nonzero } \alpha, \beta \in \mathcal{O},$$

and the group operation is given by multiplying representative ideals. As noted in the previous lecture it is not immediately obvious that $\mathrm{cl}(\mathcal{O})$ is a group (associativity is clear but the existence of inverses is not); one of our first goals is to prove this.

Remark 18.1. Recall that that an order in a $\mathbb{Q}$-algebra K of dimension r is a subring of K that is also a free $\mathbb{Z}$-module of rank r; see Definition 13.22. When K is an imaginary quadratic field embedded in the complex numbers, every order $\mathcal{O}$ in K is automatically a lattice in $\mathbb{C}$, since in this case $r = \dim K = 2$ and K is not contained in $\mathbb{R}$. Not every lattice in $\mathbb{C}$ is an imaginary quadratic order, but every imaginary quadratic order $\mathcal{O}$ is a lattice in $\mathbb{C}$ (once we fix an embedding of its fraction field), as is every $\mathcal{O}$-ideal (as a free $\mathbb{Z}$-module an $\mathcal{O}$-ideal must have the same rank as $\mathcal{O}$ because it is closed under multiplication by $\mathcal{O}$). Notice that the equivalence relation we have defined on $\mathcal{O}$-ideals coincides with our notion of homethety for lattices.

Recalling that isomorphism classes of elliptic curves over an algebraically closed field are identified by their j-invariants, we now define the set

$$\mathrm{Ell}_{\mathcal{O}}(\mathbb{C}) = \{j(E) : E \text{ is defined over } \mathbb{C} \text{ and } \mathrm{End}(E) = \mathcal{O}\},$$

and we then have a bijection of sets

$$\begin{aligned} \mathrm{cl}(\mathcal{O}) &\xrightarrow{\sim} \mathrm{Ell}_{\mathcal{O}}(\mathbb{C}) \\ [\mathfrak{a}] &\longmapsto j(E_{\mathfrak{a}}) = j(\mathfrak{a}). \end{aligned}$$

As you will prove in Problem Set 9, the ideal class group $\mathrm{cl}(\mathcal{O})$ is finite, thus the set $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ is finite. Its cardinality is the *class number* $h(\mathcal{O}) = \#\,\mathrm{cl}(\mathcal{O})$. Remarkably, not only are the sets $\mathrm{cl}(\mathcal{O})$ and $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ in bijection, the set $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ admits a group action by $\mathrm{cl}(\mathcal{O})$. In order to define this action, and to gain a better understanding of what it means for an $\mathcal{O}$-ideal to be proper, we first introduce the notion of a fractional $\mathcal{O}$-ideal.

18.1 Fractional ideals

Definition 18.2. Let $\mathcal{O}$ be an integral domain with fraction field K. For any $\lambda \in K^\times$ and $\mathcal{O}$-ideal $\mathfrak{a}$, the $\mathcal{O}$-module $\mathfrak{b} = \lambda\mathfrak{a} := \{\lambda\alpha : \alpha \in \mathfrak{a}\}$ is called a *fractional $\mathcal{O}$-ideal.*[1] Multiplication of fractional ideals $\mathfrak{b} = \lambda\mathfrak{a}$ and $\mathfrak{b}' = \lambda\mathfrak{a}'$ is defined in the obvious way:

$$\mathfrak{b}\mathfrak{b}' := (\lambda\lambda')\mathfrak{a}\mathfrak{a}',$$

where $\mathfrak{a}\mathfrak{a}'$ is the product of the $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{a}'$.[2]

Without loss of generality we can assume $\lambda = 1/\beta$ for some $\beta \in \mathcal{O}$ (if $\lambda = \alpha/\beta$, replace $\mathfrak{a}$ with $\alpha\mathfrak{a}$), and in the case of interest to us, where K is a number field, we can assume $\lambda = 1/b$ for some positive integer b (if $f \in \mathbb{Z}[x]$ is the minimal polynomial of β then $f(\beta) - f(0)$ is divisible by β with $(f(\beta) - f(0))/\beta = -f(0)/\beta \in \mathcal{O}$, and we can take $b = \pm f(0) > 0$).

Fractional $\mathcal{O}$-ideals that lie in $\mathcal{O}$ are $\mathcal{O}$-ideals, and every $\mathcal{O}$-ideal is a fractional $\mathcal{O}$-ideal. Note that $\mathcal{O}$ is itself an $\mathcal{O}$-ideal, hence a fractional $\mathcal{O}$-ideal, and it acts as the multiplicative identity with respect to multiplication of fractional $\mathcal{O}$-ideals. Fractional $\mathcal{O}$-ideals $\mathfrak{b}$ for which there exists a fractional $\mathcal{O}$-ideal $\mathfrak{b}^{-1}$ such that $\mathfrak{b}\mathfrak{b}^{-1} = \mathcal{O}$ are said to be *invertible.* Not every fractional $\mathcal{O}$-ideal is invertible (the zero ideal never is, and in general there may be nonzero fractional $\mathcal{O}$-ideals that are not invertible). The set of invertible fractional $\mathcal{O}$-ideals form a group under multiplication (this is sometimes called the *ideal group of* $\mathcal{O}$, even though its elements are fractional $\mathcal{O}$-ideals many of which are not $\mathcal{O}$-ideals).

18.2 Norms

Let $\mathcal{O}$ be an order in an imaginary quadratic field K. We want to define the norm of fractional $\mathcal{O}$-ideal $\mathfrak{b} = \lambda\mathfrak{a}$, a rational number that is the product of the norms of λ and $\mathfrak{a}$. We first define the norm of a field element $\lambda \in K^\times$, and the norm of an $\mathcal{O}$-ideal $\mathfrak{a}$.

Definition 18.3. Let K/k be a field extension and let $\lambda \in K^\times$. The multiplication-by-λ map $K \to K$ is an invertible linear transformation $M_\lambda \in \mathrm{GL}(K)$ of K as a k-vector space. The (field) *norm* and *trace* of λ are defined by

$$\mathrm{N}_{K/k}\lambda := \det M_\lambda \in k^\times \qquad \text{and} \qquad \mathrm{T}_{K/k}\lambda := \mathrm{tr}\, M_\lambda \in k.$$

[1] Some authors define fractional $\mathcal{O}$-ideals to be finitely generated $\mathcal{O}$-submodules of K. Every finitely generated $\mathcal{O}$-module in K is a fractional ideal under our definition, and when $\mathcal{O}$ is noetherian (which applies to orders in number fields), the definitions are equivalent.

[2] One can also add fractional $\mathcal{O}$-ideals via $\mathfrak{b} + \mathfrak{b}' := \{b + b' : b \in \mathfrak{b}, b' \in \mathfrak{b}\}$, but we won't need this.

One typically computes the norm and trace by fixing a basis for K as a k vector space and writing M_λ as a matrix using this basis, but the norm and trace of M_λ do not depend on the choice of basis. When K is a number field and $k = \mathbb{Q}$ it is common to simply write $\mathrm{N} := \mathrm{N}_{K/\mathbb{Q}}$ and $\mathrm{T} := \mathrm{T}_{K/\mathbb{Q}}$ when the number field K is clear from context, but note that for $\lambda \in \mathbb{Q}$ we have $\mathrm{N}\lambda = \lambda^{[K:\mathbb{Q}]}$ and $\mathrm{T}\lambda = [K : \mathbb{Q}]\lambda$, which depend on K, not just λ.

When $K \simeq \mathrm{End}^0(E)$ is an imaginary quadratic field, Definition 18.3 coincides with our definition of the (reduced) norm and trace of an element of $\mathrm{End}^0(E)$ (see Definition 13.6). When K is an imaginary quadratic field embedded in $\mathbb{C}$ we have $\mathrm{N}\alpha = \alpha\bar{\alpha}$ and $\mathrm{T}\alpha = \alpha + \bar{\alpha}$, where $\bar{\alpha}$ denotes complex conjugation (equivalently, the action of the unique non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$). Thus in this setting the complex conjugate

$$\bar{\alpha} = \mathrm{T}\alpha - \alpha = \hat{\alpha}$$

is the dual of $\alpha \in \mathrm{End}^0(E) = K \hookrightarrow \mathbb{C}$.

Definition 18.4. Let $\mathcal{O}$ be an order in a number field K and let $\mathfrak{a}$ be a nonzero $\mathcal{O}$-ideal. The (absolute) *norm* of the ideal $\mathfrak{a}$ is

$$\mathrm{N}\mathfrak{a} := [\mathcal{O} : \mathfrak{a}] = \#\mathcal{O}/\mathfrak{a} \in \mathbb{Z}_{>0}.$$

We can also interpret $\mathrm{N}\mathfrak{a}$ as the ratio of the volumes of fundamental parallelepipeds for $\mathfrak{a}$ and $\mathcal{O}$, viewed as lattices in the $\mathbb{Q}$-vector space K.

We now show that our two definitions of norm agree on principal $\mathcal{O}$-ideals.

Lemma 18.5. *Let α be a nonzero element of an order $\mathcal{O}$ in a number field K. Then*

$$\mathrm{N}(\alpha) = |\mathrm{N}\alpha|,$$

where (α) denotes the principal $\mathcal{O}$-ideal generated by α.

Proof. The lemma follows from the fact that the determinant of $M_\alpha \in \mathrm{GL}(K) \simeq \mathrm{GL}_n(\mathbb{Q})$ can be interpreted as the signed volume of the fundamental parallelepiped of the lattice (α) in the $\mathbb{Q}$-vector space $K \simeq \mathbb{Q}^n$, where $n = [K : \mathbb{Q}]$ is the degree of K. Notice that $\mathrm{N}(\alpha) = [\mathcal{O} : (\alpha)] = [\mathcal{O} : \alpha\mathcal{O}] = [\mathcal{O}_K : \alpha\mathcal{O}_K]$ depends only on α and K, not the order $\mathcal{O}$ (N.B. this holds for principal ideals but not in general). □

Warning 18.6. Given that the field norm is multiplicative and that we can view the ideal norm as the absolute value of a determinant, it would be reasonable to expect the ideal norm to be multiplicative. **This is not always true**. As an example, consider the ideal $\mathfrak{a} = [2, 2i]$ in the order $\mathcal{O} = [1, 2i]$, which has norm $\mathrm{N}\mathfrak{a} = [\mathcal{O} : \mathfrak{a}] = 2$. Then $\mathfrak{a}^2 = [4, 4i]$ and

$$\mathrm{N}\mathfrak{a}^2 = 8 \neq 2^2 = (\mathrm{N}\mathfrak{a})^2.$$

However, as we shall see, the ideal norm is multiplicative when $\mathfrak{a}$ and $\mathfrak{b}$ are both proper $\mathcal{O}$-ideals, and when either $\mathfrak{a}$ or $\mathfrak{b}$ is a principal ideal.

Corollary 18.7. *Let $\mathcal{O}$ be an order in a number field, let $\alpha \in \mathcal{O}$ be nonzero, and let $\mathfrak{a}$ be a nonzero $\mathcal{O}$-ideal. Then*

$$\mathrm{N}(\alpha\mathfrak{a}) = \mathrm{N}(\alpha)\mathrm{N}\mathfrak{a}.$$

Proof. $\mathrm{N}(\alpha\mathfrak{a}) = [\mathcal{O} : \alpha\mathfrak{a}] = [\mathcal{O} : \mathfrak{a}][\mathfrak{a} : \alpha\mathfrak{a}] = [\mathcal{O} : \mathfrak{a}][\mathcal{O} : \alpha\mathcal{O}] = \mathrm{N}\mathfrak{a}\mathrm{N}(\alpha) = \mathrm{N}(\alpha)\mathrm{N}\mathfrak{a}$. □

This allows us to make the following definition.

Definition 18.8. Let $\mathfrak{b} = \frac{1}{b}\mathfrak{a}$ be a nonzero fractional ideal in an order $\mathcal{O}$ of a number field, with $b \in \mathbb{Z}_{>0}$ (as above, we can always write $\mathfrak{b}$ this way). The (absolute) *norm* of $\mathfrak{b}$ is

$$\mathrm{N}\mathfrak{b} := \frac{\mathrm{N}\mathfrak{a}}{\mathrm{N}b} \in \mathbb{Q}_{>0}^{\times}.$$

Corollary 18.7 ensures that this does not depend on the choice of b and $\mathfrak{a}$.

When $\mathfrak{b} \subseteq \mathcal{O}$ we can take $b = 1$, in which case this agrees with Definition 18.4.

18.3 Proper and invertible fractional ideals

We now return to our original setting, where $\mathcal{O}$ is an order in an imaginary quadratic field. Extending our terminology for $\mathcal{O}$-ideals, for any fractional $\mathcal{O}$-ideal $\mathfrak{b}$ we define

$$\mathcal{O}(\mathfrak{b}) := \{\alpha : \alpha\mathfrak{b} \subseteq \mathfrak{b}\},$$

and say that $\mathfrak{b}$ is *proper* if $\mathcal{O}(\mathfrak{b}) = \mathcal{O}$. In this section we will show that $\mathfrak{b}$ is proper if and only if it $\mathfrak{b}$ is invertible (there is a fractional $\mathcal{O}$-ideal $\mathfrak{b}^{-1}$ for which $\mathfrak{b}\mathfrak{b}^{-1} = \mathcal{O}$). Let us first note that for $\mathfrak{b} = \lambda\mathfrak{a}$, whether $\mathfrak{b}$ is proper or invertible depends only on the $\mathcal{O}$-ideal $\mathfrak{a}$.

Lemma 18.9. *Let $\mathcal{O}$ be an order in an imaginary quadratic field, let $\mathfrak{a}$ be a nonzero $\mathcal{O}$-ideal, and let $\mathfrak{b} = \lambda\mathfrak{a}$ be a fractional $\mathcal{O}$-ideal. Then $\mathfrak{a}$ is proper if and only if $\mathfrak{b}$ is proper, and $\mathfrak{a}$ is invertible if and only if $\mathfrak{b}$ is invertible.*

Proof. For the first statement, note that $\{\alpha : \alpha\mathfrak{b} \subseteq \mathfrak{b}\} = \{\alpha : \alpha\lambda\mathfrak{a} \subseteq \lambda\mathfrak{a}\} = \{\alpha : \alpha\mathfrak{a} \subseteq \mathfrak{a}\}$. For the second, if $\mathfrak{a}$ is invertible then $\mathfrak{b}^{-1} = \lambda^{-1}\mathfrak{a}^{-1}$, and if $\mathfrak{b}$ is invertible then $\mathfrak{a}^{-1} = \lambda\mathfrak{b}^{-1}$, since $\mathfrak{a}\mathfrak{a}^{-1} = \mathfrak{a}\lambda\mathfrak{b}^{-1} = \mathfrak{b}\mathfrak{b}^{-1} = \mathcal{O}$. □

We now prove that the invertible $\mathcal{O}$-ideals are precisely the proper $\mathcal{O}$-ideals and give an explicit formula for the inverse when it exists. Our proof follows the presentation in [1, §7].

Theorem 18.10. *Let $\mathcal{O}$ be an order in an imaginary quadratic field and let $\mathfrak{a} = [\alpha, \beta]$ be an $\mathcal{O}$-ideal. Then $\mathfrak{a}$ is proper if and only if $\mathfrak{a}$ is invertible. Whenever $\mathfrak{a}$ is invertible we have $\mathfrak{a}\bar{\mathfrak{a}} = (\mathrm{N}\mathfrak{a})$, where $\bar{\mathfrak{a}} = [\bar{\alpha}, \bar{\beta}]$ and $(\mathrm{N}\mathfrak{a})$ is the principal $\mathcal{O}$-ideal generated by the integer $\mathrm{N}\mathfrak{a}$; the inverse of $\mathfrak{a}$ is then the fractional $\mathcal{O}$-ideal $\mathfrak{a}^{-1} = \frac{1}{\mathrm{N}\mathfrak{a}}\bar{\mathfrak{a}}$.*

Proof. If $\mathfrak{a}$ is invertible, then for any $\gamma \in \mathbb{C}$ we have

$$\gamma\mathfrak{a} \subseteq \mathfrak{a} \implies \gamma\mathfrak{a}\mathfrak{a}^{-1} \subseteq \mathfrak{a}\mathfrak{a}^{-1} \implies \gamma\mathcal{O} \subseteq \mathcal{O} \implies \gamma \in \mathcal{O},$$

so $\mathcal{O}(\mathfrak{a}) \subseteq \mathcal{O}$, and therefore $\mathfrak{a}$ is a proper $\mathcal{O}$-ideal, since we always have $\mathcal{O} \subseteq \mathcal{O}(\mathfrak{a})$.

We now assume that $\mathfrak{a} = [\alpha, \beta]$ is a proper $\mathcal{O}$-ideal and show that $\mathfrak{a}\bar{\mathfrak{a}} = (\mathrm{N}\mathfrak{a})$, which implies $\mathfrak{a}^{-1} = \frac{1}{\mathrm{N}\mathfrak{a}}\bar{\mathfrak{a}}$. Let $\tau = \beta/\alpha$, so that $\mathfrak{a} = \alpha[1, \tau]$, and let $ax^2 + bx + c \in \mathbb{Z}[x]$ be the minimal polynomial of τ made integral by clearing denominators, with $a > 0$ minimal. The fractional ideal $[1, \tau]$ is homothetic to $\mathfrak{a}$, so $\mathcal{O}([1, \tau]) = \mathcal{O}(\mathfrak{a}) = \mathcal{O}$, since $\mathfrak{a}$ is proper.

Let $\mathcal{O} = [1, \omega]$. Then $\omega \in [1, \tau]$ and $\omega = m + n\tau$ for some $m, n \in \mathbb{Z}$; after replacing ω with $\omega - m$, we may assume $\omega = n\tau$. We also have $\omega\tau \in [1, \tau]$, since $[1, \tau]$ is an $\mathcal{O}$-module, so $n\tau^2 \in [1, \tau]$, which implies that $a|n$, by the minimality of a (Gauss's lemma implies that we must have $\{f \in \mathbb{Z}[x] : f(\tau) = 0\} = (ax^2 + bx + c)$). We also have $a\tau[1, \tau] \subseteq [1, \tau]$ (since

$a\tau$ and $a\tau^2 = -b\tau - c$ lie in $[1,\tau]$), so $a\tau \in \mathcal{O}([1,\tau]) = \mathcal{O} = [1, n\tau]$, and we must have $n = a$ and $\mathcal{O} = [1, a\tau]$. Thus

$$\mathrm{N}(\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}] = [[1, a\tau] : \alpha[1,\tau]] = \frac{1}{a}[[1, a\tau] : \alpha[1, a\tau]] = \frac{1}{a}[\mathcal{O} : \alpha\mathcal{O}] = \frac{\mathrm{N}(\alpha)}{a}.$$

We also have

$$\mathfrak{a}\bar{\mathfrak{a}} = \alpha[1,\tau]\bar{\alpha}[1,\bar{\tau}] = \mathrm{N}(\alpha)[1,\tau,\bar{\tau},\tau\bar{\tau}].$$

Using $a\tau^2 + b\tau + c = 0$, we see that $\tau + \bar{\tau} = -b/a$, and $\tau\bar{\tau} = c/a$. We then have

$$\mathfrak{a}\bar{\mathfrak{a}} = \mathrm{N}(\alpha)[1,\tau,\bar{\tau},\tau\bar{\tau}] = \frac{\mathrm{N}(\alpha)}{a}[a, a\tau, -b, c] = \mathrm{N}\mathfrak{a}[1, a\tau] = (\mathrm{N}\mathfrak{a})\mathcal{O} = (\mathrm{N}\mathfrak{a})$$

as claimed, where we have used $\gcd(a,b,c) = 1$ to get $[a, a\tau, -b, c] = [1, a\tau]$, and it follows that $\mathfrak{a}^{-1} = \frac{1}{\mathrm{N}\mathfrak{a}}\bar{\mathfrak{a}}$. □

Corollary 18.11. *The ideal class group* $\mathrm{cl}(\mathcal{O})$ *is the group of invertible fractional* $\mathcal{O}$*-ideals modulo its subgroup of principal fractional* $\mathcal{O}$*-ideals (in particular* $\mathrm{cl}(\mathcal{O})$ *is a group).*

Proof. Recall that $\mathrm{cl}(\mathcal{O}) = \{\text{proper } \mathcal{O}\text{-ideals}\}/_\sim$, where $\sim$ denotes homethety. Let G be the group of invertible fractional $\mathcal{O}$-ideals and H its subgroup of principal fractional $\mathcal{O}$-ideals.

Every invertible fractional $\mathcal{O}$-ideal $\mathfrak{b} = \frac{1}{b}\mathfrak{a}$ is the product of an invertible principal fractional $\mathcal{O}$-ideal $(\frac{1}{b})$ and an invertible $\mathcal{O}$-ideal $\mathfrak{a}$, by Lemma 18.9. It follows that G/H consists of all cosets $\mathfrak{a}H$, where $\mathfrak{a}$ is any invertible, equivalently, proper $\mathcal{O}$-ideal (by Theorem 18.10). Every nonzero principal fractional $\mathcal{O}$-ideal is invertible, since $(\alpha)^{-1} = (\alpha^{-1})$, so H contains every nonzero principal fractional $\mathcal{O}$-ideal and for any two proper/invertible $\mathcal{O}$-ideals $\mathfrak{a}, \mathfrak{b}$ we have $\mathfrak{a} \sim \mathfrak{b}$ if and only if $\mathfrak{a}H = \mathfrak{b}H$. It follows that $\mathrm{cl}(\mathcal{O}) = G/H$. □

Corollary 18.12. *Let* $\mathcal{O}$ *be an order in an imaginary quadratic field and let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be invertible (equivalently, proper) fractional* $\mathcal{O}$*-ideals. Then* $\mathrm{N}(\mathfrak{a}\mathfrak{b}) = \mathrm{N}\mathfrak{a}\mathrm{N}\mathfrak{b}$*.*

Proof. If $\mathfrak{a} = \frac{1}{a}\mathfrak{a}'$ and $\mathfrak{b} = \frac{1}{b}\mathfrak{b}'$ with $a, b \in \mathbb{Z}_{>0}$ and $\mathfrak{a}', \mathfrak{b}' \subseteq \mathcal{O}$ then $\mathrm{N}(\mathfrak{a}\mathfrak{b}) = \frac{\mathrm{N}(\mathfrak{a}'\mathfrak{b}')}{\mathrm{N}a\mathrm{N}b}$, so it is enough to consider the case where $\mathfrak{a}$ and $\mathfrak{b}$ are invertible $\mathcal{O}$-ideals. We have

$$(\mathrm{N}(\mathfrak{a}\mathfrak{b})) = \mathfrak{a}\mathfrak{b}\overline{\mathfrak{a}\mathfrak{b}} = \mathfrak{a}\mathfrak{b}\bar{\mathfrak{a}}\bar{\mathfrak{b}} = \mathfrak{a}\bar{\mathfrak{a}}\mathfrak{b}\bar{\mathfrak{b}} = (\mathrm{N}\mathfrak{a})(\mathrm{N}\mathfrak{b}),$$

and it follows that $\mathrm{N}(\mathfrak{a}\mathfrak{b}) = \mathrm{N}\mathfrak{a}\mathrm{N}\mathfrak{b}$, since $\mathrm{N}\mathfrak{a}, \mathrm{N}\mathfrak{b}, \mathrm{N}(\mathfrak{a}\mathfrak{b}) \in \mathbb{Z}_{>0}$. □

18.4 The action of the ideal class group on CM elliptic curves

Let $\mathcal{O}$ be an order in an imaginary quadratic field. We are ready to define the action of $\mathrm{cl}(\mathcal{O})$ on $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C}) = \{j(E) : E/\mathbb{C} \text{ with } \mathrm{End}(E) = \mathcal{O}\}$, which we will do by defining an action of proper $\mathcal{O}$-ideals on elliptic curves $E/\mathbb{C}$ with CM by $\mathcal{O}$ (up to isomorphism).

Every $E/\mathbb{C}$ with $\mathrm{End}(E) = \mathcal{O}$ is isomorphic to $E_{\mathfrak{b}}$, for some proper $\mathcal{O}$-ideal $\mathfrak{b}$. For any proper $\mathcal{O}$-ideal $\mathfrak{a}$ we define the action of $\mathfrak{a}$ on $E_{\mathfrak{b}}$ via

$$\mathfrak{a}E_{\mathfrak{b}} = E_{\mathfrak{a}^{-1}\mathfrak{b}} \tag{1}$$

(we $E_{\mathfrak{a}^{-1}\mathfrak{b}}$ rather than $E_{\mathfrak{a}\mathfrak{b}}$ because $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}$ but $\mathfrak{b} \subseteq \mathfrak{a}^{-1}\mathfrak{b}$). The action of the equivalence class $[\mathfrak{a}]$ on the isomorphism class $j(E_{\mathfrak{b}})$, is then defined by

$$[\mathfrak{a}]j(E_{\mathfrak{b}}) = j(E_{\mathfrak{a}^{-1}\mathfrak{b}}), \tag{2}$$

which we can also write as
$$[\mathfrak{a}]j(\mathfrak{b}) = j(\mathfrak{a}^{-1}\mathfrak{b}),$$
which does not depend on the choice of $\mathfrak{a}$ and $\mathfrak{b}$.

If $\mathfrak{a}$ is a nonzero principal $\mathcal{O}$-ideal, then the lattices $\mathfrak{b}$ and $\mathfrak{a}^{-1}\mathfrak{b}$ are homothetic, and we have $\mathfrak{a}E_{\mathfrak{b}} \simeq E_{\mathfrak{b}}$. Thus the identity element of $\mathrm{cl}(\mathcal{O})$ acts trivially on $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$. For any proper $\mathcal{O}$-ideals $\mathfrak{a}$,$\mathfrak{b}$, and $\mathfrak{c}$ we have

$$\mathfrak{a}(\mathfrak{b}E_{\mathfrak{c}}) = \mathfrak{a}E_{\mathfrak{b}^{-1}\mathfrak{c}} = E_{\mathfrak{a}^{-1}\mathfrak{b}^{-1}\mathfrak{c}} = E_{(\mathfrak{b}\mathfrak{a})^{-1}\mathfrak{c}} = (\mathfrak{b}\mathfrak{a})E_{\mathfrak{c}} = (\mathfrak{a}\mathfrak{b})E_{\mathfrak{c}}.$$

Thus we have a group action of $\mathrm{cl}(\mathcal{O})$ on $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$.

For any proper $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$, we have $[\mathfrak{a}]j(\mathfrak{b}) = j(\mathfrak{a}^{-1}\mathfrak{b}) = j(\mathfrak{b})$ if and only if $\mathfrak{b}$ is homothetic to $\mathfrak{a}^{-1}\mathfrak{b}$, by Theorem 16.5, and in this case we have $\mathfrak{a}\mathfrak{b} = \lambda\mathfrak{b}$ for $\lambda \in K^{\times}$, and then $\mathfrak{a} = \lambda\mathcal{O}$ is principal. This implies that the action of $\mathrm{cl}(\mathcal{O})$ is not only faithful (only the identity fixes every element), it is *free* (every stabilizer is trivial).

The fact that the sets $\mathrm{cl}(\mathcal{O})$ and $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ have the same cardinality implies that the action must also be transitive: if we fix any $j_0 \in \mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ the images $[\mathfrak{a}]j_0$ of j_0 under the action of each $[\mathfrak{a}] \in \mathrm{cl}(\mathcal{O})$ must all be distinct, otherwise the action would not be free; there are only $\#\mathrm{Ell}_{\mathcal{O}}(\mathbb{C}) = \#\,\mathrm{cl}(\mathcal{O})$ possibilities, so the $\mathrm{cl}(\mathcal{O})$-orbit of j_0 is all of $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$.

A group action that is both free and transitive is said to be *regular*. Equivalently, the action of a group G on a set X is regular if and only if for all $x, y \in X$ there is a unique $g \in G$ for which $gx = y$. In this situation the set X is said to be a G-*torsor* (or *principal homogeneous space*) for G. We have thus shown that the set $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ is a $\mathrm{cl}(\mathcal{O})$-torsor.

If we fix a particular element x of a G-torsor X, we can then view X as a group that is isomorphic to G under the map that sends $y \in X$ to the unique element $g \in G$ for which $gx = y$. Note that this involves an arbitrary choice of the identity element x; rather than thinking of elements of X as group elements, it is more appropriate to think of the "differences" or "ratios" of elements of X as group elements. In the case of the $\mathrm{cl}(\mathcal{O})$-torsor $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ there is an obvious choice for the identity element: the isomorphism class $j(E_{\mathcal{O}})$. But when we reduce to a finite field $\mathbb{F}_q$ and work with the $\mathrm{cl}(\mathcal{O})$-torsor $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_q)$, as we shall soon do, we cannot readily distinguish the element of $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_q)$ that corresponds to $j(E_{\mathcal{O}})$, and make an arbitrary choice.

18.5 The CM action via isogenies

To better understand the $\mathrm{cl}(\mathcal{O})$-action on $\mathrm{Ell}_{\mathcal{O}}(\mathbb{C})$ we now want to look at isogenies between elliptic curves with CM by $\mathcal{O}$; but first let us consider the situation more generally.

Let $\phi\colon E_1 \to E_2$ be an isogeny of elliptic curves over $\mathbb{C}$, and let L_1 and L_2 be corresponding lattices, so that $E_1 = E_{L_1}$ and $E_2 = E_{L_2}$. By Theorem 17.4, there is a unique $\alpha = \alpha_\phi$ with $\alpha L_1 \subseteq L_2$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}/L_1 & \xrightarrow{\;\alpha\;} & \mathbb{C}/L_2 \\ \big\downarrow{\scriptstyle \Phi_1} & & \big\downarrow{\scriptstyle \Phi_2} \\ E_1(\mathbb{C}) & \xrightarrow{\;\phi\;} & E_2(\mathbb{C})\,. \end{array}$$

As we are only interested in lattices up to homethety and elliptic curves up to isomorphism, we can replace L_1 with the homothetic lattice αL_1 and E_1 by an isomorphic elliptic curve so that $\alpha = 1$ and the isogeny ϕ is induced by the inclusion $L_1 \subseteq L_2$; note that this amounts to composing ϕ with an isomorphism and does not change its degree. Up to an isomorphism of elliptic curves and a homethety of lattices, every isogeny $\phi\colon E_1 \to E_2$ arises from an inclusion of lattices $L_1 \subseteq L_2$. In this situation it is clear what the kernel of ϕ is. By commutativity, since $\alpha = 1$, the kernel of ϕ consists of the images $\Phi_1(z)$ of points $z \in \mathbb{C}$ for which $\Phi_2(z) = 0$; these are precisely the $z \in L_2$ (which includes $L_1 \subseteq L_2$ but may also include $z \in L_2 - L_1$, since L_2 is a finer lattice). We have $\Phi_1(z) = 0$ if and only if $z \in L_1$, and it follows that

$$\#\ker\phi = [L_2 : L_1].$$

We are in characteristic zero, so ϕ is automatically separable and $\deg\phi = \#\ker\phi = [L_2 : L_1]$.

The discussion above applies to any isogeny of elliptic curves over $\mathbb{C}$; up to isomorphism they all arise from lattice inclusions; in particular, the inclusion $nL \subseteq L$ induces the multiplication-by-n endomorphism of E_L.

Let us now specialize to the case where $E_1/\mathbb{C}$ has CM by $\mathcal{O}$. Then L_1 is homothetic to a proper (hence invertible) $\mathcal{O}$-ideal $\mathfrak{b}$, so let us put $L_1 = \mathfrak{b}$ and $E_1 = E_{\mathfrak{b}}$. If $\mathfrak{a}$ is any invertible $\mathcal{O}$-ideal, the inclusion of lattices $\mathfrak{b} \subseteq \mathfrak{a}^{-1}\mathfrak{b}$ (given by $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}$) induces an isogeny

$$\phi_{\mathfrak{a}}\colon E_{\mathfrak{b}} \to E_{\mathfrak{a}^{-1}\mathfrak{b}} = \mathfrak{a}E_{\mathfrak{b}}$$

that corresponds to the action of $\mathfrak{a}$ on $E_{\mathfrak{b}}$ defined in (1). Moreover, if $E_2 = E_{L_2}$ has CM by $\mathcal{O}$, then L_2 is homothetic to an invertible $\mathcal{O}$-ideal $\mathfrak{c}$, and if we replace $\mathfrak{b}$ by the homothetic $\mathcal{O}$-ideal $(\mathrm{N}\mathfrak{c})\mathfrak{b}$, then $\mathfrak{c}$ divides (hence contains) $\mathfrak{b}$, because $\mathrm{N}\mathfrak{c} = \mathfrak{c}\bar{\mathfrak{c}}$, by Theorem 18.10. If we now put $\mathfrak{a} = \mathfrak{b}\mathfrak{c}^{-1}$, then the isogeny $\phi_{\mathfrak{a}}\colon E_{\mathfrak{b}} \to E_{\mathfrak{c}} = \mathfrak{a}E_{\mathfrak{b}}$ induced by the inclusion $\mathfrak{b} \subseteq \mathfrak{c}$ corresponds to the action of $\mathfrak{a}$ on $E_{\mathfrak{b}}$. After rescaling $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ by integer multiples if necessary, we can assume $\mathfrak{a}$ is an invertible $\mathcal{O}$-ideal.

Thus all elliptic curves over $\mathbb{C}$ with CM by $\mathcal{O}$ are isogenous, and up to isomorphism, every isogeny between elliptic curves over $\mathbb{C}$ with CM by $\mathcal{O}$ is of the form $E_{\mathfrak{b}} \to \mathfrak{a}E_{\mathfrak{b}}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are invertible $\mathcal{O}$-ideals.

Definition 18.13. Let E/k be any elliptic curve with CM by an imaginary quadratic order $\mathcal{O}$, and let $\mathfrak{a}$ be an $\mathcal{O}$-ideal. The $\mathfrak{a}$-*torsion subgroup* of E is defined by

$$E[\mathfrak{a}] := \{P \in E(\bar{k}) : \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\},$$

where we are viewing each $\alpha \in \mathfrak{a} \subseteq \mathcal{O} \simeq \mathrm{End}(E)$ as an endomorphism.

Theorem 18.14. *Let $\mathcal{O}$ be an imaginary quadratic order, let $E/\mathbb{C}$ be an elliptic curve with endomorphism ring $\mathcal{O}$, let $\mathfrak{a}$ be an invertible $\mathcal{O}$-ideal, and let $\phi_{\mathfrak{a}}$ be the corresponding isogeny from E to $\mathfrak{a}E$. The following hold:*

(i) $\ker\phi_{\mathfrak{a}} = E[\mathfrak{a}]$*;*

(ii) $\deg\phi_{\mathfrak{a}} = \mathrm{N}\mathfrak{a}$.

Proof. By composing $\phi_{\mathfrak{a}}$ with an isomorphism if necessary, we assume without loss of generality that $E = E_{\mathfrak{b}}$ for some invertible $\mathcal{O}$-ideal $\mathfrak{b}$. Let Φ be the isomorphism from $\mathbb{C}/\mathfrak{b} \to E_{\mathfrak{b}}$

that sends z to $(\wp(z), \wp'(z))$. We have

$$\begin{aligned}
\Phi^{-1}(E[\mathfrak{a}]) &= \{z \in \mathbb{C}/\mathfrak{b} : \alpha z = 0 \text{ for all } \alpha \in \mathfrak{a}\}\\
&= \{z \in \mathbb{C} : \alpha z \in \mathfrak{b} \text{ for all } \alpha \in \mathfrak{a}\}/\mathfrak{b}\\
&= \{z \in \mathbb{C} : z\mathfrak{a} \subseteq \mathfrak{b}\}/\mathfrak{b}\\
&= \{z \in \mathbb{C} : z\mathcal{O} \subseteq \mathfrak{a}^{-1}\mathfrak{b}\}/\mathfrak{b}\\
&= (\mathfrak{a}^{-1}\mathfrak{b})/\mathfrak{b}\\
&= \ker\left(\mathbb{C}/\mathfrak{b} \xrightarrow{z \to z} \mathbb{C}/\mathfrak{a}^{-1}\mathfrak{b}\right)\\
&= \Phi^{-1}(\ker \phi_{\mathfrak{a}}),
\end{aligned}$$

which proves (i). We then note that

$$\#E[\mathfrak{a}] = [\mathfrak{a}^{-1}\mathfrak{b} : \mathfrak{b}] = [\mathfrak{b} : \mathfrak{a}\mathfrak{b}] = [\mathcal{O} : \mathfrak{a}\mathcal{O}] = [\mathcal{O} : \mathfrak{a}] = \mathrm{N}\mathfrak{a},$$

which proves (ii). □

Corollary 18.15. *Let $\mathcal{O}$ be an imaginary quadratic order and let $\mathfrak{a}$ be an invertible $\mathcal{O}$-ideal. For every elliptic curve $E/\mathbb{C}$ with CM by $\mathcal{O}$ the elliptic curves E and $\mathfrak{a}E$ are related by an isogeny $\phi_{\mathfrak{a}} \colon E \to \mathfrak{a}E$ of degree* $\mathrm{N}\mathfrak{a}$.

Proof. This follows immediately from the theorem and discussion above. □

18.6 Discriminants

To streamline our work with imaginary quadratic orders, we define the *discriminant* of $\mathcal{O}$, a negative integer that uniquely determines $\mathcal{O}$. Since $\mathcal{O}$ is a subring of an imaginary quadratic field that has rank 2 as a $\mathbb{Z}$-module, we can always write $\mathcal{O}$ as $[1, \tau]$, where τ is an algebraic integer that does not lie in $\mathbb{Z}$; its minimal polynomial is necessarily of the form $x^2 + bx + c$ with discriminant $b^2 - 4c \in \mathbb{Z}_{<0}$.

Definition 18.16. Let $\mathcal{O} = [1, \tau]$ be an imaginary quadratic order. The *discriminant* of $\mathcal{O}$ is the discriminant of the minimal polynomial of τ, which we can compute as

$$\operatorname{disc}(\mathcal{O}) = (\tau + \bar{\tau})^2 - 4\tau\bar{\tau} = (\tau - \bar{\tau})^2 = \det\begin{pmatrix} 1 & \tau \\ 1 & \bar{\tau} \end{pmatrix}^2.$$

If A is the area of a fundamental parallelogram of $\mathcal{O}$ then

$$\operatorname{disc}(\mathcal{O}) = (\tau - \bar{\tau})^2 = -4|\operatorname{im} \tau|^2 = -4A^2,$$

thus the discriminant does not depend on our choice of τ, it is intrinsic to the lattice $\mathcal{O}$.

Since the discriminant $\operatorname{disc}(\mathcal{O})$ is a negative integer of the form $\mathfrak{b}^2 - 4c$ with $b, c \in \mathbb{Z}$, it is necessarily a square modulo 4 (hence congruent ot 0 or 1 mod 4).

Definition 18.17. A negative integer D that is a square modulo 4 is an (imaginary quadratic) *discriminant*. Discriminants not of the form u^2D' for some integer $u > 1$ and discriminant D' are said to be *fundamental*. Every discriminant can be written uniquely as the product of a square and a fundamental discriminant.

There is a one-to-one relationship between imaginary quadratic discriminants and orders in imaginary quadratic fields; fundamental discriminants correspond to maximal orders.

Theorem 18.18. *Let D be an imaginary quadratic discriminant. There is a unique imaginary quadratic order $\mathcal{O}$ with $\operatorname{disc}(\mathcal{O}) = D = u^2 D_K$, where D_K is the fundamental discriminant of the maximal order $\mathcal{O}_K$ in $K = \mathbb{Q}(\sqrt{D_K})$, and $u = [\mathcal{O}_K : \mathcal{O}]$.*

Proof. Write $D = \operatorname{disc}(\mathcal{O})$ as $D = u^2 D_K$, with $u \in \mathbb{Z}_{>0}$ and D_K a fundamental discriminant. Let $K = \mathbb{Q}(\sqrt{D_K})$, and let $\mathcal{O}_K$ be its ring of integers, the maximal order of K, by Theorem 13.26. Now define

$$\tau := \begin{cases} \frac{\sqrt{D_K}}{2} & \text{if } D_K \equiv 0 \bmod 4; \\ \frac{1+\sqrt{D_K}}{2} & \text{if } D_K \equiv 1 \bmod 4. \end{cases}$$

Then $\operatorname{disc}([1,\tau]) = (\tau - \bar{\tau})^2 = D_K$, and $\tau + \bar{\tau}$ and $\tau\bar{\tau}$ are integers, so $\tau \in O_K$ and $[1,\tau]$ is a suborder of $\mathcal{O}_K$. But $\mathcal{O}_K$ is the maximal order of K, so $O_K = [1,\tau]$ and $\operatorname{disc}(\mathcal{O}_K) = D_K$. The order $\mathcal{O} = [1, u\tau]$ then has discriminant $(u\tau - \overline{u\tau})^2 = u^2 D_K = D$.

Conversely, if $\mathcal{O} = [1,\omega]$ is any imaginary quadratic order of discriminant D, than ω is the root of a quadratic equation of discriminant D and therefore an algebraic integer in the field $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{D_K}) = K$. We must have $\mathcal{O} \subseteq \mathcal{O}_K$, since $\mathcal{O}_K$ is the unique maximal order. The ratio of the squares of the areas of the fundamental parallelograms of $\mathcal{O}_K$ and $\mathcal{O}$ must be $D/D_K = u^2$, which implies $[\mathcal{O}_K : \mathcal{O}] = u$. Let $\mathcal{O}_K = [1,\tau]$ with τ defined as above. By Lemma 18.19 below, $u\mathcal{O}_k \subseteq \mathcal{O}$, so $u\tau \in \mathcal{O}$, and the lattice $[1, u\tau] \subseteq \mathcal{O}$ has index u in $\mathcal{O}_K$ and is therefore equal to $\mathcal{O}$. It follows that $[1, u\tau]$ is the unique imaginary quadratic order of discriminant D. $\square$

The index $u = [\mathcal{O}_K : \mathcal{O}]$ is also called the *conductor* of the order $\mathcal{O}$.

Lemma 18.19. *If L' is an index n sublattice of L then nL is an index n sublattice of L'.*

Proof. Without loss of generality, $L = [1,\tau]$ and $L' = [a, b + c\tau]$ (let a be the least positive integer in L'). Comparing areas of fundamental parallelograms yields

$$\begin{aligned} n|\operatorname{im}\tau| &= |a \operatorname{im} c\tau| = |ac||\operatorname{im}\tau| \\ n &= |ac|, \end{aligned}$$

Thus $a|n$, so $n \in L'$, and $a(b+c\tau) - ba = ac\tau = \pm n\tau$, so $n\tau \in L'$; therefore $nL = [n, n\tau] \subseteq L'$. We have $[L : L'] = n$ and $[L : L'][L' : nL] = [nL : L] = n^2$, so $[L' : nL] = n$. $\square$

References

[1] David A. Cox, *Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication*, second edition, Wiley, 2013.