16 Elliptic curves over $\mathbb{C}$ (part 2)

Last time we showed that every lattice $L \subseteq \mathbb{C}$ gives rise to an elliptic curve

$$E_L\colon y^2 = 4x^3 - g_2(L)x - g_3(L),$$

where

$$g_2(L) = 60G_4(L) := 60\sum_{L^*}\frac{1}{\omega^4}, \qquad g_3(L) = 140G_6(L) = 140\sum_{L^*}\frac{1}{\omega^6},$$

with $L^* := L - \{0\}$, and we defined a map

$$\begin{aligned}\Phi\colon \mathbb{C}/L &\to E_L(\mathbb{C})\\ z &\mapsto \begin{cases}(\wp(z),\wp'(z)) & z\notin L\\ 0 & z\in L\end{cases}\end{aligned}$$

where

$$\wp(z) = \wp(z;L) = \frac{1}{z^2} + \sum_{\omega\in L^*}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

is the Weierstrass $\wp$-function for the lattice L, and

$$\wp'(z) = -2\sum_{\omega\in L}\frac{1}{(z-\omega)^3}.$$

In this lecture is to prove two theorems. First we will prove that Φ is an isomorphism of additive groups; it is also an isomorphism of complex manifolds [3, Cor. 5.1.1], and of complex Lie groups, but we won't prove this now.[1] Second, we will prove that every elliptic curve $E/\mathbb{C}$ is isomorphic to E_L for some lattice L; this is also known as the Uniformization Theorem.

16.1 The isomorphism from a torus to the corresponding elliptic curve

Theorem 16.1. *Let $L \subseteq \mathbb{C}$ be a lattice and let $E_L\colon y^2 = 4x^3 - g_2(L)x - g_3(L)$ be the corresponding elliptic curve. The map $\Phi\colon \mathbb{C}/L \to E_L(\mathbb{C})$ is a group isomorphism.*

Proof. We first note that $\Phi(0) = 0$, so Φ preserves the identity, and for all $z \notin L$ we have

$$\Phi(-z) = (\wp(-z),\wp'(-z)) = (\wp(z), -\wp'(z)) = -\Phi(z),$$

since $\wp$ is even and $\wp'$ is odd, so Φ is compatible with taking inverses.

Let $L = [\omega_1,\omega_2]$. There are three points of order 2 in $\mathbb{C}/L$; if $L = [\omega_1,\omega_2]$ these are $\omega_1/2, \omega_2/2$, and $(\omega_1+\omega_2)/2$. By Lemma 15.31, $\wp'$ vanishes these points, hence Φ maps points of order 2 in $\mathbb{C}/L$ to points of order 2 in $E_L(\mathbb{C})$, since the latter are the points with y-coordinate zero. Moreover, Φ is injective on points of order 2, since $\wp(z)$ maps each point of order 2 in $\mathbb{C}/L$ to a distinct root of $4\wp(z)^3 - g_2(L)\wp(z) - g_3(L)$, as shown in the proof of Lemma 15.32. The restriction of Φ to $(\mathbb{C}/L)[2]$ defines a bijection of $(\mathbb{C}/L)[2] \xrightarrow{\sim} E_L[2] \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ with $\Phi(0)=0$, which must be a group isomorphism.

[1] This is not difficult to show, but it would distract us from our immediate goal. We will see an explicit isomorphism of complex manifolds in a few lectures when we study modular curves, and in that case we will take the time to define precisely what this means and to prove it.

To show that Φ is surjective, let $(x_0, y_0) \in E_L(\mathbb{C})$. The elliptic function $f(z) = \wp(z) - x_0$ has order 2, hence it has two zeros in the fundamental parallelogram $\mathcal{F}_0$, by Theorem 15.18. Neither of these zeros occurs at $z = 0$, since f has a pole at 0. So let $z_0 \neq 0$ be a zero of $f(z)$ in $\mathcal{F}_0$. Then $\wp(z_0) = x_0$, which implies $\Phi(z_0) = (x_0, \pm y_0)$ and therefore $(x_0, y_0) = \Phi(\pm z_0)$; thus Φ is surjective.

We now show that Φ is injective. Let $z_1, z_2 \in \mathcal{F}_0$ and suppose that $\Phi(z_1) = \Phi(z_2)$. If $2z_1 \in L$ then z_1 is a 2-torsion element and we have already shown that Φ restricts to a bijection on $(\mathbb{C}/L)[2]$, so we must have $z_1 = z_2$. We now assume $2z_1 \notin L$, which implies $\wp'(z_1) \neq 0$. As argued above, the roots of $f(z) = \wp(z) - \wp(z_1)$ in $\mathcal{F}_0$ are $\pm z_1$, thus $z_2 \equiv \pm z_1 \bmod L$. We also have $\wp'(z_1) = \wp'(z_2)$, and this forces $z_2 \equiv z_1 \bmod L$, since $\wp'(-z_1) = -\wp'(z_1) \neq \wp'(z_1)$ because $\wp'(z_1) \neq 0$.

It remains only to show that $\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2)$. So let $z_1, z_2 \in \mathcal{F}_0$; we may assume that $z_1, z_2, z_1 + z_2 \notin L$ since the case where either z_1 or z_2 lies in L is immediate, and if $z_1 + z_2 \in L$ then z_1 and z_2 are inverses modulo L, a case treated above.

The points $P_1 = \Phi(z_1)$ and $P_2 = \Phi(z_2)$ are affine points in $E_L(\mathbb{C})$, and the line ℓ between them cannot be vertical because P_1 and P_2 are not inverses (since z_1 and z_2 are not). So let $y = mx + b$ be an equation for this line, and let P_3 be the third point where the line intersects the curve E_L. Then $P_1 + P_2 + P_3 = 0$, by the definition of the group law on $E_L(\mathbb{C})$.

Now consider the function $\ell(z) = -\wp'(z) + m\wp(z) + b$. It is an elliptic function of order 3 with a triple pole at 0, so it has three zeros in the fundamental region $\mathcal{F}_0$, two of which are z_1 and z_2. Let z_3 be the third zero in $\mathcal{F}_0$. The point $\Phi(z_3)$ lies on both the line ℓ and the elliptic curve $E_L(C)$, hence it must lie in $\{P_1, P_2, P_3\}$; moreover, we have a bijection from $\{z_1, z_2, z_3\}$ to $\{\Phi(z_1), \Phi(z_2), \Phi(z_3)\} = \{P_1, P_2, P_3\}$, and this bijection must send z_3 to P_3 if P_3 is distinct from P_1 and P_2. If P_3 coincides with exactly one of P_1 or P_2, say P_1, then $\ell(z)$ has a double zero at z_1 and we must have $z_3 = z_1$; and if $P_1 = P_2 = P_3$ then clearly $z_1 = z_2 = z_3$. Thus in every case we must have $P_3 = \Phi(z_3)$.

We have $P_1 + P_2 + P_3 = 0$, so it suffices to show $z_1 + z_2 + z_3 \in L$, since this implies

$$\Phi(z_1 + z_2) = \Phi(-z_3) = -\Phi(z_3) = -P_3 = P_1 + P_2 = \Phi(z_1) + \Phi(z_2).$$

Let $\mathcal{F}_\alpha$ be a fundamental region for L whose boundary does not contain any zeros or poles of $\ell(z)$ and replace z_1, z_2, z_3 by equivalent points in $\mathcal{F}_\alpha$ if necessary.

Applying Theorem 15.17 to $g(z) = z$ and $f(z) = \ell(z)$ yields

$$\frac{1}{2\pi i}\int_{\partial \mathcal{F}_\alpha} z\frac{\ell'(z)}{\ell(z)}dz = \sum_{w \in F_\alpha} \operatorname{ord}_w(\ell)w = z_1 + z_2 + z_3 - 3 \cdot 0 = z_1 + z_2 + z_3, \tag{1}$$

where the boundary $\partial\mathcal{F}_\alpha$ of $\mathcal{F}_\alpha$ is oriented counter-clockwise.

Let us now evaluate the integral in (1); to ease the notation, define $f(z) := \ell'(z)/\ell(z)$, which we note is an elliptic function (hence periodic with respect to L). We then have

$$\begin{aligned}
\int_{\partial F_\alpha} zf(z)\,dz &= \int_\alpha^{\alpha+\omega_1} zf(z)dz + \int_{\alpha+\omega_1}^{\alpha+\omega_1+\omega_2} zf(z)dz + \int_{\alpha+\omega_1+\omega_2}^{\alpha+\omega_2} zf(z)dz + \int_{\alpha+\omega_2}^{\alpha} zf(z)dz \\
&= \int_\alpha^{\alpha+\omega_1} zf(z)dz + \int_\alpha^{\alpha+\omega_2} (z+\omega_1)f(z)dz + \int_{\alpha+\omega_1}^{\alpha} (z+\omega_2)f(z)dz + \int_{\alpha+\omega_2}^{\alpha} zf(z)dz \\
&= \omega_1\int_\alpha^{\alpha+\omega_2} f(z)dz + \omega_2\int_{\alpha+\omega_1}^{\alpha} f(z)dz.
\end{aligned} \tag{2}$$

Note that we have used the periodicity of $f(z)$ to replace $f(z+\omega_i)$ by $f(z)$, and to cancel integrals in opposite directions along lines that are equivalent modulo L.

For any closed (not necessarily simple) curve C and a point $z_0 \notin C$, the quantity

$$\frac{1}{2\pi i}\int_C \frac{dz}{z-z_0}$$

is the *winding number* of C about z_0, and it is an integer (it counts the number of times the curve C "winds around" the point z_0); see [1, Lem. 4.2.1] or [4, Lem. B.1.3].

The function $\ell(\alpha+t\omega_2)$ parametrizes a closed curve C_1 from $\ell(\alpha)$ to $\ell(\alpha+\omega_2)=\ell(\alpha)$, as t ranges from 0 to 1. The winding number of C_1 about the point 0 is the integer

$$c_1 := \frac{1}{2\pi i}\int_{C_1}\frac{dz}{z-0} = \frac{1}{2\pi i}\int_0^1 \frac{\ell'(\alpha+t\omega_2)}{\ell(\alpha+t\omega_2)}\omega_2 dt = \frac{1}{2\pi i}\int_\alpha^{\alpha+w_2}\frac{\ell'(z)}{\ell(z)}dz = \frac{1}{2\pi i}\int_\alpha^{\alpha+\omega_2} f(z)dz. \quad (3)$$

Similarly, the function $\ell(\alpha+t\omega_1)$ parameterizes a closed curve C_2 from $\ell(\alpha)$ to $\ell(\alpha+\omega_1)$, and we obtain the integer

$$c_2 := \frac{1}{2\pi i}\int_{C_2}\frac{dz}{z-0} = \frac{1}{2\pi i}\int_0^1 \frac{\ell'(\alpha+t\omega_1)}{\ell(\alpha+t\omega_1)}\omega_1 dt = \frac{1}{2\pi i}\int_\alpha^{\alpha+\omega_1}\frac{\ell'(z)}{\ell(z)}dz = \frac{1}{2\pi i}\int_\alpha^{\alpha+\omega_1} f(z)\,dz. \quad (4)$$

Plugging (3), and (4) into (2), and applying (1), we see that

$$z_1+z_2+z_3 = c_1\omega_1 - c_2\omega_2 \in L,$$

as desired. □

16.2 The j-invariant of a lattice

Definition 16.2. The *j-invariant* of a lattice L is defined by

$$j(L) = 1728\frac{g_2(L)^3}{\Delta(L)} = 1728\frac{g_2(L)^3}{g_2(L)^3-27g_3(L)^2}.$$

Recall that $\Delta(L)\neq 0$, by Lemma 15.32, so $j(L)$ is always defined.

The elliptic curve $E_L\colon y^2=4x^3-g_2(L)x-g_3(L)$ is isomorphic to the elliptic curve $y^2=x^3+Ax+B$, where $g_2(L)=-4A$ and $g_3(L)=-4B$. Thus

$$j(L) = 1728\frac{g_2(L)^3}{g_2(L)^3-27g_3(L)^2} = 1728\frac{(-4A)^3}{(-4A)^3-27(-4B)^2} = 1728\frac{4A^3}{4A^3+27B^2} = j(E_L).$$

Thus the j-invariant of a lattice L is the same as the j-invariant of the corresponding elliptic curve E_L. We now define the discriminant of an elliptic curve so that it agrees with the discriminant of the corresponding lattice.

Definition 16.3. The *discriminant* of an elliptic curve $E\colon y^2=x^3+Ax+B$ is

$$\Delta(E) = -16(4A^3+27B^2).$$

This definition applies to any elliptic curve E/k defined by a short Weierstrass equation, whether $k=\mathbb{C}$ or not, but for the moment we continue to focus on elliptic curves over $\mathbb{C}$.

Recall from Theorem 14.14 that elliptic curves E/k and E'/k are isomorphic over $\bar{k}$ if and only if $j(E)=j(E')$. Thus over an algebraically closed field like $\mathbb{C}$, the j-invariant characterizes elliptic curves up to isomorphism. We now define an analogous notion of isomorphism for lattices.

Definition 16.4. Lattices L and L' are said to be *homothetic* if $L' = \lambda L$ for some $\lambda \in \mathbb{C}^\times$.

Theorem 16.5. *Two lattices L and L' are homothetic if and only if $j(L) = j(L')$.*

Proof. Suppose L and L' are homothetic, with $L' = \lambda L$. Then

$$g_2(L') \;=\; 60 \sum_{\omega \in L'^*} \frac{1}{w^4} \;=\; 60 \sum_{\omega \in L^*} \frac{1}{(\lambda\omega)^4} \;=\; \lambda^{-4} g_2(L).$$

Similarly, $g_3(L') = \lambda^{-6} g_3 L$, and we have

$$j(L') = 1728 \frac{(\lambda^{-4} g_2(L))^3}{(\lambda^{-4} g_2(L))^3 - 27(\lambda^{-6} g_3(L))^2} = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27 g_3(L)^2} = j(L).$$

To show the converse, let us now assume $j(L) = j(L')$. Let E_L and $E_{L'}$ be the corresponding elliptic curves. Then $j(E_L) = j(E_{L'})$. We may write

$$E_L \colon y^2 = x^3 + Ax + B,$$

with $-4A = g_2(L)$ and $-4B = g_3(L)$, and similarly for $E_{L'}$, with $-4A' = g_2(L')$ and $-4B' = g_3(L')$. By Theorem 14.13, there is a $\mu \in \mathbb{C}^\times$ such that $A' = \mu^4 A$ and $B' = \mu^6 B$, and if we let $\lambda = 1/\mu$, then $g_2(L') = \lambda^{-4} g_2(L) = g_2(\lambda L)$ and $g_3(L') = \lambda^{-6} g_3(L) = g_3(\lambda L)$, as above. We now show that this implies $L' = \lambda L$.

Recall from Theorem 15.29 that the Weierstrass $\wp$-function satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Differentiating both sides yields

$$\begin{aligned} 2\wp'(z)\wp''(z) &= 12\wp(z)^2\wp'(z) - g_2\wp'(z) \\ \wp''(z) &= 6\wp(z)^2 - \frac{g_2}{2}. \end{aligned} \tag{5}$$

By Theorem 15.28, the Laurent series for $\wp(z; L)$ at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n} = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^{2n},$$

where $a_1 = g_2/20$ and $a_2 = g_3/28$.

Comparing coefficients for the z^{2n} term in (5), we find that for $n \geq 2$ we have

$$(2n+2)(2n+1)a_{n+1} = 6\left(\sum_{k=1}^{n-1} a_k a_{n-k} + 2a_{n+1}\right),$$

and therefore

$$a_{n+1} = \frac{6}{(2n+2)(2n+1) - 12} \sum_{k=1}^{n-1} a_k a_{n-k}.$$

This allows us to compute a_{n+1} from $a_1, \ldots, a_{n-1}$, for all $n \geq 2$. It follows that $g_2(L)$ and $g_3(L)$ uniquely determine the function $\wp(z) = \wp(z; L)$ (and therefore the lattice L where $\wp(z)$ has poles), since $\wp(z)$ is uniquely determined by its Laurent series expansion about 0.

Now consider L' and λL, where we have $g_2(L') = g_2(\lambda L)$ and $g_3(L') = g_3(\lambda L)$. It follows that $\wp(z; L') = \wp(z; \lambda L)$ and $L' = \lambda L$, as desired. $\square$

Corollary 16.6. *Two lattices L and L' are homothetic if and only if the corresponding elliptic curves E_L and $E_{L'}$ are isomorphic.*

Thus homethety classes of lattices correspond to isomorphism classes of elliptic curves over $\mathbb{C}$, and both are classified by the j-invariant. Recall from Theorem 14.12 that every complex number is the j-invariant of an elliptic curve $E/\mathbb{C}$. To prove the Uniformization Theorem we just need to show that the same is true of lattices.

16.3 The j-function

Every lattice $[\omega_1, \omega_2]$ is homothetic to a lattice of the form $[1, \tau]$, with τ in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{im} z > 0\}$; we may take $\tau = \pm\omega_2/\omega_1$ with the sign chosen so that $\operatorname{im}\tau > 0$. This leads to the following definition of the j-function.

Definition 16.7. The *j-function* $j \colon \mathbb{H} \to \mathbb{C}$ is defined by $j(\tau) = j([1, \tau])$. We similarly define $g_2(\tau) = g_2([1, \tau])$, $g_3(\tau) = g_3([1, \tau])$, and $\Delta(\tau) = \Delta([1, \tau])$.

Note that for any $\tau \in \mathbb{H}$, both $-1/\tau$ and $\tau + 1$ lie in $\mathbb{H}$ (the maps $\tau \mapsto 1/\tau$ and $\tau \mapsto -\tau$ both swap the upper and lower half-planes; their composition preserves them).

Theorem 16.8. *The j-function is holomorphic on $\mathbb{H}$, and satisfies $j(-1/\tau) = j(\tau)$ and $j(\tau + 1) = j(\tau)$.*

Proof. From the definition of $j(\tau) = j([1, \tau])$ we have

$$j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)} = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

The series defining

$$g_2(\tau) = 60 \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0}} \frac{1}{(m + n\tau)^4} \qquad \text{and} \qquad g_3(\tau) = 140 \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m + n\tau)^6}$$

converge absolutely for any fixed $\tau \in \mathbb{H}$, by Lemma 15.22, and they converge uniformly over τ in any compact subset of $\mathbb{H}$. The proof of this last fact is straight-forward but slightly technical; see [2, Thm. 1.15] for the details. It follows that $g_2(\tau)$ and $g_3(\tau)$ are holomorphic on $\mathbb{H}$, and therefore $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$ is also holomorphic on $\mathbb{H}$. Since $\Delta(\tau)$ is nonzero for all $\tau \in \mathbb{H}$, by Lemma 15.32, the j-function $j(\tau)$ is holomorphic on $\mathbb{H}$ as well.

The lattices $[1, \tau]$ and $[1, -1/\tau] = -{}^1\!/\!_\tau[1, \tau]$ are homothetic, and the lattices $[1, \tau + 1]$ and $[1, \tau]$ are equal; thus $j(-1/\tau) = j(\tau)$ and $j(\tau + 1) = j(\tau)$, by Theorem 16.5. □

16.4 The modular group

We now consider the *modular group*

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

As proved in Problem Set 8, the group Γ acts on $\mathbb{H}$ via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d},$$

and it is generated by the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. This implies that the j-function is invariant under the action of the modular group; in fact, more is true.

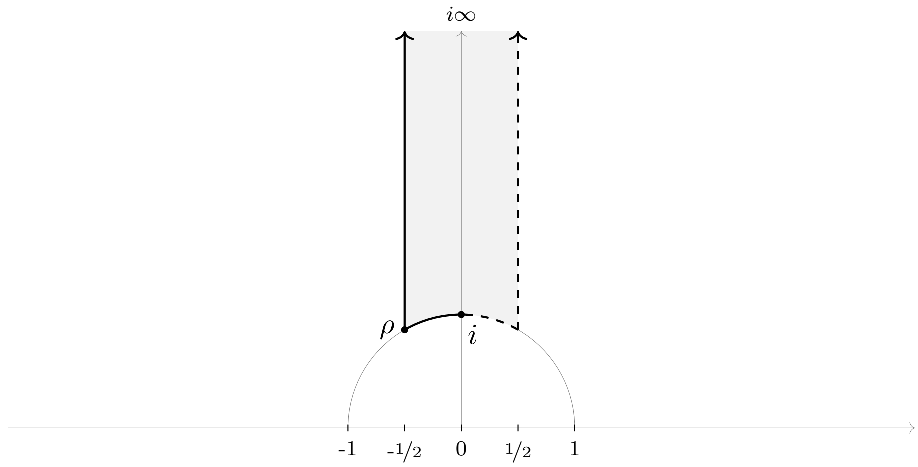

Figure 1: Fundamental domain $\mathcal{F}$ for $\mathbb{H}/\Gamma$, with $i = e^{\pi/2}$ and $\rho = e^{2\pi i/3}$.

Lemma 16.9. *We have $j(\tau) = j(\tau')$ if and only if $\tau' = \gamma\tau$ for some $\gamma \in \Gamma$.*

Proof. We have $j(S\tau) = j(-1/\tau) = j(\tau)$ and $j(T\tau) = j(\tau+1) = j(\tau)$, by Theorem 16.8, It follows that if $\tau' = \gamma\tau$ then $j(\tau') = j(\tau)$, since S and T generate Γ.

To prove the converse, let us suppose that $j(\tau) = j(\tau')$. Then by Theorem 16.5, the lattices $[1,\tau]$ and $[1,\tau']$ are homothetic So $[1,\tau'] = \lambda[1,\tau]$, for some $\lambda \in \mathbb{C}^\times$. There thus exist integers a, b, c, and d such that

$$\begin{aligned}\tau' &= a\lambda\tau + b\lambda\\ 1 &= c\lambda\tau + d\lambda\end{aligned}$$

From the second equation, we see that $\lambda = \frac{1}{c\tau+d}$. Substituting this into the first, we have

$$\tau' = \frac{a\tau+b}{c\tau+d} = \gamma\tau, \qquad \text{where } \gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathbb{Z}^{2\times 2}.$$

Similarly, using $[1,\tau] = \lambda^{-1}[1,\tau']$, we can write $\tau = \gamma'\tau'$ for some integer matrix γ'. The fact that $\tau' = \gamma\gamma'\tau'$ implies that $\det\gamma = \pm 1$ (since γ and γ' are integer matrices). But τ and τ' both lie in $\mathbb{H}$, so we must have $\det\gamma = 1$; therefore $\gamma \in \Gamma$ as desired. $\square$

Lemma 16.9 implies that when studying the j-function it suffices to study its behavior on Γ-equivalence classes of $\mathbb{H}$, that is, the orbits of $\mathbb{H}$ under the action of Γ. We thus consider the quotient of $\mathbb{H}$ modulo Γ-equivalence, which we denote by $\mathbb{H}/\Gamma$.[2] The actions of γ and $-\gamma$ are identical, so taking the quotient by $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ yields the same result, but for the sake of clarity we will stick with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

We now wish to determine a fundamental domain for $\mathbb{H}/\Gamma$, a set of unique representatives in $\mathbb{H}$ for each Γ-equivalence class. For this purpose we will use the set

$$\mathcal{F} = \{\tau \in \mathbb{H} : \mathrm{re}(\tau) \in [-1/2, 1/2) \text{ and } |\tau| \ge 1, \text{ such that } |\tau| > 1 \text{ if } \mathrm{re}(\tau) > 0\}.$$

Lemma 16.10. *The set $\mathcal{F}$ is a fundamental domain for $\mathbb{H}/\Gamma$.*

[2]Some authors write this quotient as $\Gamma\backslash\mathbb{H}$ to indicate that the action is on the left.

Proof. We need to show that for every $\tau \in \mathbb{H}$, there is a unique $\tau' \in \mathcal{F}$ such that $\tau' = \gamma\tau$, for some $\gamma \in \Gamma$. We first prove existence. Let us fix $\tau \in \mathbb{H}$. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\operatorname{im}(\gamma\tau) = \operatorname{im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\operatorname{im}((a\tau+b)(c\bar{\tau}+d))}{|c\tau+d|^2} = \frac{(ad-bc)\operatorname{im}\tau}{|c\tau+d|^2} = \frac{\operatorname{im}\tau}{|c\tau+d|^2} \tag{6}$$

Let $c\tau + d$ be a shortest vector in the lattice $[1, \tau]$. Then c and d must be relatively prime, and we can pick integers a and b so that $ad - bc = 1$. The matrix $\gamma_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then maximizes the value of $\operatorname{im}(\gamma\tau)$ over $\gamma \in \Gamma$. Let us now choose $\gamma = T^k\gamma_0$, where k is chosen so that $\operatorname{re}(\gamma\tau) \in [1/2, 1/2)$, and note that $\operatorname{im}(\gamma\tau) = \operatorname{im}(\gamma_0\tau)$ remains maximal. We must have $|\gamma\tau| \geq 1$, since otherwise $\operatorname{im}(S\gamma\tau) > \operatorname{im}(\gamma\tau)$, contradicting the maximality of $\operatorname{im}(\gamma\tau)$. Finally, if $\tau' = \gamma\tau \notin \mathcal{F}$, then we must have $|\gamma\tau| = 1$ and $\operatorname{re}(\gamma\tau) > 0$, in which case we replace γ by $S\gamma$ so that $\tau' = \gamma\tau \in \mathcal{F}$.

It remains to show that τ' is unique. This is equivalent to showing that any two Γ-equivalent points in $\mathcal{F}$ must coincide. So let τ_1 and $\tau_2 = \gamma_1\tau_1$ be two elements of $\mathcal{F}$, with $\gamma_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and assume $\operatorname{im}\tau_1 \leq \operatorname{im}\tau_2$. By (6), we must have $|c\tau_1 + d|^2 \leq 1$, thus

$$1 \geq |c\tau_1 + d|^2 = (c\tau_1 + d)(c\bar{\tau}_1 + d) = c^2|\tau_1|^2 + d^2 + 2cd\operatorname{re}\tau_1 \geq c^2|\tau_1|^2 + d^2 - |cd| \geq 1,$$

where the last inequality follows from $|\tau_1| \geq 1$ and the fact that c and d cannot both be zero (since $\det\gamma = 1$). Thus $|c\tau_1 + d| = 1$, which implies $\operatorname{im}\tau_2 = \operatorname{im}\tau_1$. We also have $|c|, |d| \leq 1$, and by replacing γ_1 by $-\gamma_1$ if necessary, we may assume that $c \geq 0$. This leaves 3 cases:

1. $c = 0$: then $|d| = 1$ and $a = d$. So $\tau_2 = \tau_1 \pm b$, but $|\operatorname{re}\tau_2 - \operatorname{re}\tau_1| < 1$, so $\tau_2 = \tau_1$.
2. $c = 1, d = 0$: then $b = -1$ and $|\tau_1| = 1$. So τ_1 is on the unit circle and $\tau_2 = a - 1/\tau_1$. Either $a = 0$ and $\tau_2 = \tau_1 = i$, or $a = -1$ and $\tau_2 = \tau_1 = \rho$.
3. $c = 1, |d| = 1$: then $|\tau_1 + d| = 1$, so $\tau_1 = \rho$, and $\operatorname{im}\tau_2 = \operatorname{im}\tau_1 = \sqrt{3}/2$ implies $\tau_2 = \rho$.

In every case we have $\tau_1 = \tau_2$ as desired. □

Theorem 16.11. *The restriction of the j-function to $\mathcal{F}$ defines a bijection from $\mathcal{F}$ to $\mathbb{C}$.*

Proof. Injectivity follows immediately from Lemmas 16.9 and 16.10. It remains to prove surjectivity. We have

$$g_2(\tau) \;=\; 60 \sum_{\substack{n,m\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^4} \;=\; 60\left(2\sum_{m=1}^{\infty}\frac{1}{m^4} + \sum_{\substack{n,m\in\mathbb{Z} \\ n\neq 0}} \frac{1}{(m+n\tau)^4}\right).$$

The second sum tends to 0 as $\operatorname{im}\tau \to \infty$. Thus we have

$$\lim_{\operatorname{im}\tau\to\infty} g_2(\tau) = 120\sum_{m=1}^{\infty} m^{-4} = 120\,\zeta(4) = 120\,\frac{\pi^4}{90} = \frac{4\pi^4}{3},$$

where $\zeta(s)$ is the Riemann zeta function. Similarly,

$$\lim_{\operatorname{im}\tau\to\infty} g_3(\tau) = 280\,\zeta(6) = 280\,\frac{\pi^6}{945} = \frac{8\pi^6}{27}.$$

Thus

$$\lim_{\operatorname{im}\tau\to\infty} \Delta(\tau) = \left(\frac{4}{3}\pi^4\right)^3 - 27\left(\frac{8}{27}\pi^6\right)^2 = 0.$$

(this explains the coefficients 60 and 140 in the definitions of g_2 and g_3; they are the smallest pair of integers that ensure this limit is 0). Since $\Delta(\tau)$ is the denominator of $j(\tau)$, the quantity $j(\tau) = g_2(\tau)^3/\Delta(\tau)$ is unbounded as $\operatorname{im}\tau \to \infty$.

In particular, the j-function is non-constant, and by Theorem 16.8 it is holomorphic on $\mathbb{H}$. The open mapping theorem implies that $j(\mathbb{H})$ is an open subset of $\mathbb{C}$; see [4, Thm. 3.4.4].

We claim that $j(\mathbb{H})$ is also a closed subset of $\mathbb{C}$. Let $j(\tau_1), j(\tau_2), \ldots$ be an arbitrary convergent sequence in $j(\mathbb{H})$, converging to $w \in \mathbb{C}$. The j-function is Γ-invariant, by Lemma 16.9, so we may assume the τ_n all lie in $\mathcal{F}$. The sequence $\operatorname{im}\tau_1, \operatorname{im}\tau_2, \ldots$ must be bounded, say be B, since $j(\tau) \to \infty$ as $\operatorname{im}\tau \to \infty$, but the sequence $j(\tau_)), j(\tau_2), \ldots$ converges; it follows that the τ_n all lie in the compact set

$$\Omega = \{\tau : \operatorname{re}\tau \in [-1/2, 1/2], \operatorname{im}\tau \in [1/2, B]\}.$$

There is thus a subsequence of the τ_n that converges to some $\tau \in \Omega \subset \mathbb{H}$. The j-function is holomorphic, hence continuous, so $j(\tau) = w$. It follows that the open set $j(\mathbb{H})$ contains all its limit points and is therefore closed.

The fact that the non-empty set $j(\mathbb{H}) \subseteq \mathbb{C}$ is both open and closed implies that $j(\mathbb{H}) = \mathbb{C}$, since $\mathbb{C}$ is connected. It follows that $j(\mathcal{F}) = \mathbb{C}$, since every element of $\mathbb{H}$ is Γ-equivalent to an element of $\mathcal{F}$ (Lemma 16.10) and the j-function is Γ-invariant (Lemma 16.9). $\square$

Corollary 16.12 (Uniformization Theorem)**.** *For every elliptic curve $E/\mathbb{C}$ there exists a lattice L such that $E = E_L$.*

Proof. Given $E/\mathbb{C}$, pick $\tau \in \mathbb{H}$ so that $j(\tau) = j(E)$ and let $L' = [1, \tau]$. We have

$$j(E) = j(\tau) = j(L') = j(E_{L'}),$$

so E is isomorphic to $E_{L'}$, by Theorem 14.13, where the isomorphism is given by the map $(x, y) \mapsto (\mu^2 x, \mu^3 y)$ for some $\mu \in \mathbb{C}^\times$. If now let $L = \frac{1}{\mu}L'$, then $E = E_L$. $\square$

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