These problems are related to the material covered in Lectures 28-29. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due at 11:59pm Eastern time on 04/27/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions, and identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective variety of dimension one.

Problem 1. Weil reciprocity (50 points)

Let C/k be a curve over an algebraically closed field k. For $f \in k(C)^\times$ and $D \in \operatorname{Div}_k C$ such that $\operatorname{div} f$ and D have disjoint support, define $f(D)$ as in part (d) of Problem 3 of pset8. Your goal in this problem is to prove the *Weil reciprocity law*, which states that for all $f, g \in k(C)^\times$ such that $\operatorname{div} f$ and $\operatorname{div} g$ have disjoint support,

$$f(\operatorname{div} g) = g(\operatorname{div} f).$$

(a) Prove that for $C = \mathbb{P}^1$ we have

$$\prod_P (-1)^{\operatorname{ord}_P(g)\operatorname{ord}_P(f)} \left(\frac{f^{\operatorname{ord}_P(g)}}{g^{\operatorname{ord}_P(f)}}\right)(P) = 1,$$

for all $f, g \in k(C)^\times$, whether or not $\operatorname{div} f$ and $\operatorname{div} g$ have disjoint support.

(b) Use (a) to prove the Weil reciprocity law in the case $C = \mathbb{P}^1$.

Associated to any morphism of curves $\phi\colon C_1 \to C_2$ we have defined three maps

$$\begin{array}{ll} \phi^*\colon k(C_2) \to k(C_1) & \phi^*\colon \operatorname{Div}_k C_2 \to \operatorname{Div}_k C_1 \\ ? & \phi_*\colon \operatorname{Div}_k C_1 \to \operatorname{Div}_k C_2 \end{array}$$

It is natural to ask whether there is a fourth map $\phi_*\colon k(C_1) \to k(C_2)$ that completes the table above, and indeed there is. It is defined by

$$f \mapsto (\phi^*)^{-1} N(f),$$

where $N\colon k(C_1) \to \phi^*(k(C_2))$ is the norm map associated to the extension $k(C_1)/\phi^*(k(C_2))$.[1] It is obviously not a morphism of fields (unless ϕ is an isomorphism), but it is a morphism of their multiplicative groups.

We then have identities analogous to those proved in (c) and (d) of Problem 3 of pset8:

$$\phi_*(\operatorname{div} f) = \operatorname{div}(\phi_* f), \tag{1}$$

[1] Recall from Definition 7.1 that the norm map of a finite extension L/K sends an element of L with characteristic polynomial $F \in K[x]$ to $(-1)^{[L:K]}F(0)$ which is an element of K.

for all $f \in k(C_1)^\times$, and

$$f(\phi^* D) = (\phi_* f)(D), \tag{2}$$

for all $f \in k(C_1)^\times$ and $D \in \operatorname{Div}_k C_2$ where both sides are defined.

We now turn to the general case of the Weil reciprocity law, where f and g are elements of $k(C)$ with disjoint support, where C is now any curve. Your objective is to prove Weil's reciprocity law in the general case using Weil reciprocity for $\mathbb{P}^1$, which you proved in (b).

As noted in lecture, any function in $k(C)$ defines a morphism to $\mathbb{P}^1$, so we may view both f and g as morphisms, and we then have associated pullback maps f^*, g^* and pushforward maps f_*, g_* that can be applied both to the divisor groups and function fields of C and $\mathbb{P}^1$. We also need the identity morphism $i \colon \mathbb{P}^1 \to \mathbb{P}^1$.

(c) Prove that $\operatorname{div} g = g^* \operatorname{div} i$ for any $g \in k(C)^\times$.

(d) Use (b) and (c), parts (c) and (d) of Problem 3 of pset8, and identities (1) and (2) to prove the Weil reciprocity law in the general case.

Problem 2. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.