18.782 Introduction to Arithmetic Geometry **Spring 2023**
Problem Set #8 **Due: 04/20/2023**

These problems are related to the material covered in Lectures 24-27. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due at 11:59pm Eastern time on 04/20/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions, and identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective variety of dimension one.

Problem 1. Singularities (20 points)

Let X be the projective closure of the affine curve $y^2 = x^5$ over an algebraically closed field of characteristic 0.

(a) Find the singularities of X.

(b) Find a smooth projective curve Y that is birational to X.

Problem 2. Smooth projective model of hyperelliptic curve (30 points)

Let k be an algebraically closed field and $f(x) = \sum a_i x^i$ be a squarefree polynomial of degree 5 or 6 in $k[x]$. Let C be the affine curve defined by the equation $y^2 = f(x)$, with function field $F = k(x, \sqrt{f(x)})$. In this exercise, we want to show that $X \subseteq \mathbb{P}^4$ defined by

$$\begin{aligned}
&Y^2 = a_0X_0^2 + a_1X_0X_1 + a_2X_1^2 + a_3X_1X_2 + a_4X_2^2 + a_5X_2X_3 + a_6X_3^2\\
&X_0X_2 - X_1^2 = 0\\
&X_1X_3 - X_2^2 = 0\\
&X_0X_3 - X_1X_2 = 0
\end{aligned}$$

is a smooth projective curve with function field isomorphic to F.

(a) Show that the affine curve C is isomorphic to the affine part of X in the affine chart given by $X_0 = 1$, thus getting the assertion about function field of X.

(b) Show all points of C are smooth, and deduce the same about points of X in this affine chart.

(c) Find all points of X not in this affine chart. By choosing another suitable affine chart, and working out the Jacobian matrix, show that these points are smooth as well.

Problem 3. Divisor identities (50 points)

Let k be a perfect (but not necessarily algebraically closed) field. Let $\phi\colon C_1 \to C_2$ be a morphism of curves defined over k, and let $\phi^*\colon k(C_2) \to k(C_1)$ be the corresponding embedding of function fields. We also use ϕ^* to denote the pullback map $\operatorname{Div}_k C_2 \to \operatorname{Div}_k C_1$, and ϕ_* denotes the pushforward map $\operatorname{Div}_k C_1 \to \operatorname{Div}_k C_2$, as defined in Definition 19.19.

(a) Prove that $\phi_* \circ \phi^*$ acts as multiplication by $\deg \phi$ on $\operatorname{Div}_k C_2$.

(b) Show that $\phi^* \circ \phi_*$ is not multiplication by $\deg \phi$ on $\operatorname{Div}_k C_1$, in general, but that we do have

$$\deg(\phi^* \phi_* D) = (\deg \phi)(\deg D)$$

for all $D \in \operatorname{Div}_k C_1$.

(c) Prove that $\operatorname{ord}_P(\phi^* f) = e_\phi(P) \operatorname{ord}_{\phi(P)}(f)$ for all closed points P of C_1/k and functions f in $k(C_2)^\times$. Then use this to show $\phi^*(\operatorname{div} f) = \operatorname{div}(\phi^* f)$ for all $f \in k(C_2)^\times$.

(d) For any curve C/k and $f \in k(C)^\times$ and $D \in \operatorname{Div}_k(C)$ such that $\operatorname{div} f$ and $D = \sum n_P P$ have disjoint supports, define

$$f(D) = \prod f(P)^{n_P \deg P},$$

where P ranges over the closed points of C/k. Prove that $f(\phi_* D) = (\phi^* f)(D)$ for all $f \in k(C_2)^\times$ and $D \in \operatorname{Div}_k(C_1)$ where both sides are defined.

(e) Prove that the sequence

$$1 \to k^\times \to k(\mathbb{P}^1)^\times \xrightarrow{\operatorname{div}} \operatorname{Div}_k \mathbb{P}^1 \xrightarrow{\deg} \mathbb{Z} \to 0$$

is exact.

Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.