**18.782 Introduction to Arithmetic Geometry** **Spring 2023**
**Problem Set #7** **Due: 04/13/2023**

---

These problems are related to the material covered in Lectures 19-23. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due at 11:59pm Eastern time on 04/13/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions, and identify collaborators and any sources not listed in the syllabus.

## Problem 1. Short answer questions (20 points)

Please justify each of the following assertions in 25 words or less. They all have short easy proofs and are meant to check your understanding, not to challenge you.

1. Every affine variety is connected.
2. The closure of the image of a morphism of affine varieties is a variety.
3. A morphism of affine varieties is dominant if and only if the corresponding morphism of coordinate rings is injective.
4. $\mathbb{A}^n$ and $\mathbb{P}^n$ are birationally equivalent (prove this explicitly by writing down the rational maps; indicate in each case whether your rational map is a morphism or not).

## Problem 2. Products of varieties (30 points)

1. Define the map $\phi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ by
$$\phi((x_0 : x_1), (y_0 : y_1)) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$
Prove that $\phi$ is a morphism whose image $V$ is a variety (specify $I(V)$ explicitly). Then prove that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq V$ by giving an inverse morphism.

2. Let $x_0, \ldots, x_m$ and $y_0, \ldots, y_n$ be homogeneous coordinates for $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively, and for $0 \le i \le m$ and $0 \le j \le n$ let $z_{ij}$ be homogeneous coordinates for $\mathbb{P}^N$, where $N = (m + 1)(n + 1) - 1$. Prove that the *Segre morphism* from $\varphi\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ defined by $z_{ij} = x_iy_j$ is an isomorphism to a projective variety $V \subseteq \mathbb{P}^N$.

3. The field $\mathbb{C}$ is an affine $\mathbb{R}$-algebra (an integral domain finitely generated over $\mathbb{R}$). Show that the tensor product of $\mathbb{R}$-algebras $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain, hence not an affine $\mathbb{R}$-algebra. Thus Lemma 16.4, which states that a tensor product of affine algebras is again an affine algebra, depends on the assumption that $k$ is algebraically closed. Explain exactly where in the proof this assumption is used.

## Problem 3. Valuation rings (50 points)

An *ordered abelian group* is an abelian group $\Gamma$ with a total order $\leq$ that is compatible with the group operation. This means that for all $a, b, c \in \Gamma$ the following hold:

$$\begin{array}{rcll} a \leq b \leq a & \Longrightarrow & a = b & \text{(antisymmetry)} \\ a \leq b \leq c & \Longrightarrow & a \leq c & \text{(transitivity)} \\ a \not\leq b & \Longrightarrow & b \leq a & \text{(totality)} \\ a \leq b & \Longrightarrow & a + c \leq b + c & \text{(compatibility)} \end{array}$$

Note that totality implies reflexivity $(a \leq a)$ . Given an ordered abelian group $\Gamma$, we define the relations $\geq, <, >$ and the sets $\Gamma_{\leq 0}, \Gamma_{\geq 0}, \Gamma_{<0}$, and $\Gamma_{>0}$ in the obvious way.

A *valuation* $v$ on a field $K$ is a surjective homomorphism $v\colon K^\times \to \Gamma$ to an ordered abelian group $\Gamma$ that satisfies $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in K^\times$. The group $\Gamma$ is called the *value group* of $v$, and when $\Gamma = \{0\}$ we say that $v$ is the *trivial valuation*.

1. Let $R$ be a valuation ring with fraction field $F$, and let $v\colon F^\times \to F^\times/R^\times = \Gamma$ be the quotient map. Show that the relation $\leq$ on $\Gamma$ defined by

$$v(x) \leq v(y) \Longleftrightarrow y/x \in R,$$

   makes $\Gamma$ an ordered abelian group and that $v$ is a valuation on $F$.

2. Let $F$ be a field and let $v\colon F^\times \to \Gamma$ be a non-trivial valuation. Prove that the set

$$R_v := v^{-1}(\Gamma_{\geq 0}) \cup \{0\}$$

   is a valuation ring of $F$ and that $v(x) \leq v(y) \Longleftrightarrow y/x \in R_v$ (note that this includes proving that $R_v$ is actually a ring).

3. Let $v\colon F^\times \to \Gamma_v$ and $w\colon F^\times \to \Gamma_w$ be two valuations on a field $F$, and let $R_v$ and $R_w$ be the corresponding valuation rings. Prove that $R_v = R_w$ if and only if there is an order preserving isomorphism $\rho\colon \Gamma_v \to \Gamma_w$ for which $\rho \circ v = w$ (in which case we say that $v$ and $w$ are equivalent). Thus there is a one-to-one correspondence between valuation rings and equivalence classes of valuations.

4. Let $D$ be an integral domain that is properly contained in its fraction field $F$. Let $\mathcal{R}$ be the set of local rings that contain $D$ and are properly contained in $F$. Give the following partial ordering on $\mathcal{R}$: $R_1 \leq R_2$ if $R_1 \subseteq R_2$ and the maximal ideal of $R_1$ is contained in the maximal ideal of $R_2$.[1] Zorn's lemma states that any partially ordered set in which every chain has an upper bound contains at least one maximal element. Use this to prove that $\mathcal{R}$ contains a maximal element $R$, and then show that any such $R$ is a valuation ring (note: the hypothesis of Zorn's lemma is understood to include the empty chain, so you must prove that $\mathcal{R}$ is nonempty).
   (**Hint:** To show that $R$ is a valuation ring, use proof by contradiction. If it were not a valuation ring, show that you can contradict its maximality.)

5. Prove that every valuation ring $R$ is *integrally closed*. This means that every element of the fraction field of $R$ that is the root of a monic polynomial in $R[x]$ lies in $R$.
   (**Hint:** Argue that you can always reduce the degree of the monic polynomial further.)

[1] This is known as the *dominance* ordering.

## Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

| | Interest | Difficulty | Time Spent |
|---|---|---|---|
| Problem 1 | | | |
| Problem 2 | | | |
| Problem 3 | | | |

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.