18.782 Introduction to Arithmetic Geometry **Spring 2023**
Problem Set #6 **Due: 03/23/2023**

These problems are related to the material covered in Lectures 16-18. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due at 11:59pm Eastern time on 03/23/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions, and identify collaborators and any sources not listed in the syllabus.

Problem 1. Homogeneous ideals and projective algebraic sets (50 points)

Let k be a perfect field and fix an algebraic closure $\overline{k}$. Fix a positive integer n, let R be the polynomial ring $\overline{k}[x_0, \ldots, x_n]$, and let R^h be the set of all homogeneous polynomials in R. Recall from class that (1) a homogenous ideal in R is an ideal generated by elements of R^h, (2) for any subset $S \subset R$ the algebraic set Z_S is the zero locus in $\mathbb{P}^n$ of $S \cap R^h$, and (3) for any subset $Z \subset \mathbb{P}^n$ the ideal $I(Z)$ is the homogeneous ideal generated by the polynomials in R^h that vanish at every point in Z.

(a) Prove the "homogeneous *Nullstellensatz*," which says that if $I \subseteq R$ is a homogeneous ideal and $f \in R^h$ is a nonconstant polynomial that vanishes at every point in Z_I, then $f^r \in I$ for some $r > 0$.
(**Hint:** Re-interpret the problem in $\mathbb{A}^{n+1}$ and use the usual *Nullstellensatz*).

(b) Let R_+ denote the set of all polynomials in R with no constant term. Prove that for any homogeneous ideal $I \subseteq R$ the following are equivalent:

- (i) Z_I is the empty set;
- (ii) Either $\sqrt{I} = R$ or $\sqrt{I} = R_+$.
- (iii) I contains every polynomial in R^h of degree d, for some $d > 0$.

(c) Prove the following:

- (i) If $S \subseteq T$ are subsets of R^h then $Z_T \subseteq Z_S$.
- (ii) If $Y \subseteq Z$ are subsets of $\mathbb{P}^n$ then $I(Z) \subseteq I(Y)$.
- (iii) For any two subsets Y, Z in $\mathbb{P}^n$ we have $I(Y \cup Z) = I(Y) \cap I(Z)$.
- (iv) For any algebraic set $Z \subseteq \mathbb{P}^n$ we have $Z_{I(Z)} = Z$.
- (v) For any homogeneous radical ideal I for which $Z_I \neq \emptyset$ we have $I(Z_I) = I$.

(d) Conclude from (a), (b), and (c) that there is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals in R that are not equal to R_+, given by the maps $Z \to I(Z)$ and $I \to Z_I$.[1]

(e) Prove that an algebraic set $Z \in \mathbb{P}^n$ is irreducible if and only if $I(Z)$ is a prime ideal.

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[1] The ideal R_+ that does not appear in this correspondence is called the *irrelevant* ideal.

Problem 2. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.