18.782 Introduction to Arithmetic Geometry — **Spring 2023**
Problem Set #5 — **Due: 03/16/2023**

These problems are related to the material covered in Lectures 4-15 (unlike previous problem sets, you may need to refer to material covered in lectures and/or problem sets from earlier weeks). I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due at 11:59pm Eastern time on 03/16/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions, and remember to identify any collaborators or sources that you consulted that are not listed in the syllabus.

Problem 1. Hilbert symbol over $\mathbb{Q}$ is not bilinear (10 points)

(a) Now that we have proved the Hasse-Minkowski theorem, give an explicit description of the Hilbert symbol $(a,b)_{\mathbb{Q}}$ for $a,b \in \mathbb{Q}^\times$ in terms of local Hilbert symbols.

(b) Use part (a) to produce elements $a_1, a_2, b \in \mathbb{Q}^\times$ such that $(a_1a_2, b)_{\mathbb{Q}}$ is not equal to the product $(a_1,b)_{\mathbb{Q}}(a_2,b)_{\mathbb{Q}}$.

Problem 2. Extensions and Norm subgroups: A peek into local class field theory (10 points)

This problem gives another proof of bilinearity of the Hilbert symbol for $\mathbb{Q}_p$ without using the explicit formula.

Over any field k, we have $(a_1a_2, b)_k = 1 \implies (a_1,b)_k = (a_2,b)_k$. Let H_b denote the image of the norm map $\mathrm{Nm} : k(\sqrt{b})^\times \to k^\times$. To turn the one-way implication into an equivalence and thus get bilinearity, we need that the index of H_b as a subgroup of $k^\times$ is 1 or 2.

Let p be an odd prime. Use Problem 3(a) from Pset 4 to show that this is indeed the case for $k = \mathbb{Q}_p$.
(**Hint:** There are three distinct quadratic extensions of $\mathbb{Q}_p$. List them, and for each of them, show that the norm subgroup is strictly bigger than $(\mathbb{Q}_p^\times)^2$.)

In Problems 3-5 we will recapitulate much of the material covered over the past month, replacing the field $\mathbb{Q}$ with the field $\mathbb{F}_q(t)$, the field of rational functions in t with coefficients in a finite field $\mathbb{F}_q$. The fields $\mathbb{Q}$ and $\mathbb{F}_q(t)$ have much in common and are the two essential sources of global fields; by definition, every global field is a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_q(t)$.

Recall that $\mathbb{F}_q(t)$ is the fraction field of the polynomial ring $\mathbb{F}_q[t]$. The ring $\mathbb{F}_q[t]$ is a principal ideal domain, hence a unique factorization domain, and its unit group is $\mathbb{F}_q^\times$. In particular, every irreducible polynomial in $\mathbb{F}_q[t]$ is prime, and (up to scaling by units) the prime elements of $\mathbb{F}_q[t]$ are precisely the monic irreducible polynomials. We will use π to denote such a prime (and also ∞, see below).

Problem 3. Absolute values on $\mathbb{F}_q(t)$ (25 points)

For each prime $\pi \in \mathbb{F}_q[t]$ and any nonzero $f \in \mathbb{F}_q[t]$, let $v_\pi(f)$ be the largest integer for which $\pi^n | f$, equivalently, the largest n for which $f \in (\pi^n)$. For each $f/g \in \mathbb{F}_q(t)^\times$ in lowest terms (meaning f and g have no common factor in $\mathbb{F}_q[t]$), we define

$$v_\pi(f/g) = v_\pi(f) - v_\pi(g),$$

and we let $v_\pi(0) = \infty$ (with the usual conventions that $n + \infty = \infty$, $\max(n, \infty) = \infty$, and $n^{-\infty} = 0$ if $n \neq 0$). We also define

$$\deg(f/g) = \deg f - \deg g,$$

and $\deg 0 = -\infty$.

(a) For each prime $\pi \in \mathbb{F}_q[t]$, define

$$|r|_\pi = (q^{\deg \pi})^{-v_\pi(r)}$$

for all $r \in \mathbb{F}_q(t)$. Show that $|\ |_\pi$ is a nonarchimedean absolute value on $\mathbb{F}_q(t)$.

(b) Let

$$|r|_\infty = q^{\deg r},$$

for all $r \in \mathbb{F}_q(t)$. Prove that $|\ |_\infty$ is a nonarchimedean absolute value on $\mathbb{F}_q(t)$.[1]

Recall from Problem Set 3 (problem 4e) that in a field k with nonarchimedean absolute value $\|\ \|$,

$$R = \{x \in k : \|x\| \le 1\}$$

is a local ring with unique maximal ideal $\mathfrak{m} = \{x \in k : \|x\| < 1\}$. The field $R/\mathfrak{m}$ is called the *residue field* of R (and of k, with respect to $\|\ \|$).

(c) Determine the residue field of $\mathbb{F}_q(t)$ with respect to $|\ |_\pi$, and with respect to $|\ |_\infty$.

(d) Prove Ostrowski's theorem for $\mathbb{F}_q(t)$: every nontrivial absolute value on $\mathbb{F}_q(t)$ is equivalent to $|\ |_\infty$ or $|\ |_\pi$ for some prime $\pi \in \mathbb{F}_q[t]$.

More precisely, show that if $\|\ \|$ is a nontrivial absolute value on $\mathbb{F}_q(t)$, either $\|t\| > 1$ and $\|\ \| \sim |\ |_\infty$, or $\|t\| \le 1$ and $\|\ \| \sim |\ |_\pi$ for some prime $\pi \in \mathbb{F}_q[t]$.

In view of (d) we now regard ∞ as a prime of $\mathbb{F}_q(t)$ and let π range over both monic irreducible polynomials in $\mathbb{F}_q[t]$ and ∞.

(e) Prove the product formula for absolute values on $\mathbb{F}_q(t)$: for every $r \in \mathbb{F}_q(t)^\times$ we have

$$\prod_\pi |r|_\pi = 1.$$

[1] You may be surprised that $|\ |_\infty$ is nonarchimedean, but recall from Corollary 5.4 of Lecture 5 that this is necessarily the case in a field of positive characteristic.

Problem 4. Completions of $\mathbb{F}_q(t)$ (30 points)

Let $\mathbb{F}_q(t)_\pi$ denote the completion of $\mathbb{F}_q(t)$ with respect to the absolute value $|\ |_\pi$. Recall that for any field k, ring of formal power series $\sum_{n\geq 0} a_n T^n$ with $a_n \in k$ is denoted $k[[T]]$, and its fraction field $k((T))$ is the field of formal Laurent series $\sum_{n\geq n_0} a_n T^n$, where $n_0 \in \mathbb{Z}$.

(a) Prove that every π-adic field $\mathbb{F}_q(t)_\pi$ is isomorphic to $\mathbb{F}_\pi((T))$, where $\mathbb{F}_\pi$ denotes the residue field of $\mathbb{F}_q(t)$ with respect to $|\ |_\pi$. For each π determine the field $\mathbb{F}_\pi$ and describe this isomorphism by identifying the element of $\mathbb{F}_q(t)_\pi$ that corresponds to T (this element is called a uniformizing parameter, or simply a *uniformizer*).

It follows from (a) that if we fix a uniformizer (i.e., choose an isomorphism from $\mathbb{F}_q(t)_\pi$ to $\mathbb{F}_\pi((T))$), then every element a of $\mathbb{F}_q(t)_\pi$ has a unique π-adic expansion of the form $\sum_{n\geq n_0} a_n T^n$, where $n_0 = v_\pi(a)$ is the π-adic valuation of a (extended from $\mathbb{F}_q(t)$ to its completion $\mathbb{F}_q(t)_\pi$). We can also write a as $T^{v_\pi(a)}u$, where $u \in \mathbb{F}_\pi[[T]]^\times$ is a π-adic unit.

(b) Determine for which π we have $\mathbb{F}_\pi = \mathbb{F}_q$. Conclude that, unlike the p-adic fields $\mathbb{Q}_p$, distinct π-adic fields $\mathbb{F}_q(t)_\pi$ may be isomorphic as fields, even though they have inequivalent absolute values.

(c) Determine the group of square classes $\mathbb{F}_q(t)_\pi^\times/\mathbb{F}_q(t)_\pi^{\times 2}$ (your answer may depend on q and π, be sure to consider all possibilities).

Problem 5. Hilbert symbols over $\mathbb{F}_q(t)$ (25 points)

We now assume q is odd. For $a, b \in \mathbb{F}_q(t)_\pi^\times$, we define the Hilbert symbol

$$(a,b)_\pi = \begin{cases} 1 & \text{if } ax^2 + by^2 = 1 \text{ has a solution in } \mathbb{F}_q(t)_\pi, \\ -1 & \text{otherwise.} \end{cases}$$

As noted in class, the proofs of Lemma 10.2 and Corollary 10.3 work over any field, so you can assume they apply to $(a, b)_\pi$.

(a) For $a, b \in \mathbb{F}_q(t)_\pi^\times \simeq \mathbb{F}_\pi((T))^\times$, write $a = T^\alpha u$ and $b = T^\beta v$, with $u, v \in \mathbb{F}_\pi[[T]]^\times$. Prove that

$$(a,b)_\pi = (-1)^{\alpha\beta\frac{\#\mathbb{F}_\pi - 1}{2}} \left(\frac{u}{\pi}\right)^\beta \left(\frac{v}{\pi}\right)^\alpha,$$

where $\left(\frac{u}{\pi}\right)$ is the Legendre symbol of the residue field $\mathbb{F}_\pi$ applied to the image of u in $\mathbb{F}_\pi$ (so $\left(\frac{u}{\pi}\right) = 1$ if and only if the first π-adic digit of u is a square in $\mathbb{F}_\pi^\times$).

(b) Prove that the Hilbert symbol $(\cdot,\cdot)_\pi$ is bilinear and nondegenerate.

(c) Prove that for all $a, b \in \mathbb{F}_q(t)^\times$ we have $(a, b)_\pi = 1$ for all but finitely many π. Then prove the Hilbert reciprocity law:

$$\prod_\pi (a,b)_\pi = 1.$$

You may assume the quadratic reciprocity law for $\mathbb{F}_q[t]$, which states that for any monic irreducible polynomials $\pi_1, \pi_2 \in \mathbb{F}_q[t]$ we have

$$\left(\frac{\pi_1}{\pi_2}\right) = (-1)^{\deg(\pi_1)\deg(\pi_2)\frac{q-1}{2}} \left(\frac{\pi_2}{\pi_1}\right),$$

where $\left(\frac{\pi_1}{\pi_2}\right)$ is defined to be 1 if π_1 is a square modulo π_2 and -1 otherwise.

Problem 6. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.