18.782 Introduction to Arithmetic Geometry
Spring 2023

Problem Set #2 **Due: 02/23/2023**

Description

These problems are related to the material covered in Lectures 4-6. I have made every effort to proof-read them, but there are may still be errors that I have missed. The first person to spot each error will receive 1-5 points of extra credit on their problem set, depending on the severity of the error.

The problem set is due at 11:59pm Eastern time on 02/23/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions (you can just replace the due date in the header with your name). Remember to identify any collaborators or sources that you consulted that are not listed in the syllabus.

Problem 1. The Chevalley-Warning theorem. (35 points)

A C_1 *field* is a field k in which every homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of nonzero degree $d < n$ has a nontrivial zero (equivalently, the projective variety $f(x_1, \ldots, x_n) = 0$ has a rational point). The Chevalley-Warning theorem states that finite fields are C_1 fields.

- **(a)** For each nonnegative integer n and finite field $\mathbb{F}_q$, determine the value of $\sum_{x \in \mathbb{F}_q} x^n$. (Note that $0^0 = 1$).
- **(b)** Prove that $\mathbb{F}_q$ is a C_1 field. (Hint: consider $\sum_{a_1, \ldots, a_n \in \mathbb{F}_q} \left(1 - f(a_1, \ldots, a_n)^{q-1}\right)$).
- **(c)** Prove more generally that if $f_1, \ldots, f_m \in \mathbb{F}_q[x_1, \ldots, x_n]$ are nonconstant homogeneous polynomials for which $\deg f_1 + \cdots \deg f_m < n$, then the system of equations

$$\begin{aligned} f_1(a_1, \ldots, a_n) &= 0 \\ f_2(a_1, \ldots, a_n) &= 0 \\ &\vdots \\ f_m(a_1, \ldots, a_n) &= 0 \end{aligned}$$

has a nonzero solution.

Problem 2. The ring of p-adic integers $\mathbb{Z}_p$ (35 points)

- **(a)** Prove that $\mathbb{Z}_p$ is a Euclidean domain.
- **(b)** Prove that $\mathbb{Z}_p$ and $\mathbb{R}$ have the same cardinality.
- **(c)** Define

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : p \nmid b \right\}.$$

Prove that $\mathbb{Z}_p \bigcap \mathbb{Q} = \mathbb{Z}_{(p)}$ (take the intersection in $\mathbb{Q}_p$).

(d) Show that $\mathbb{Z}_{(p)}$ is the set of elements in $\mathbb{Z}_p$ whose p-adic expansion eventually repeats.

(e) Characterize the set of elements in $\mathbb{Z}_p$ whose p-adic expansion is eventually constant. Is it ring? If not, is it closed under either addition or multiplication?

Problem 3. The ring of profinite integers. (30 points)

A *partially ordered set* (poset) is a set together with a relation $\preceq$ that is reflexive, transitive and antisymmetric. Here are two examples:

(i) The set $A = \mathbb{Z}_{\geq 1}$ of positive integers with the standard order relation $\leq$.

(ii) The set $B = \mathbb{Z}_{\geq 1}$ of positive integers with relation $n \preceq m$ if and only if $n \mid m$.

Given a poset $(I, \preceq)$, an *inverse system* is a collection of objects $\{R_i : i \in I\}$ with maps $f_{ij} : R_j \to R_i$ for $i \preceq j$ such that

$$f_{ik} = f_{ij} \circ f_{jk} \text{ whenever } i \preceq j \preceq k.$$

The p-adic integers $\mathbb{Z}_p$ are formed from the inverse system defined with respect to A where R_i is the ring $\mathbb{Z}/p^i\mathbb{Z}$.

(a) Show that $\{\mathbb{Z}/n\mathbb{Z}\}_{n\geq 1}$ together with the projection maps $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ for $n \mid m$ forms an inverse system of rings with respect to the partially ordered set $(\mathbb{Z}_{\geq 1}, \cdot \mid \cdot)$.

(b) Define the inverse limit $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ of this system. Show that the natural map $\mathbb{Z} \to \hat{\mathbb{Z}}$ is injective.

(c) Construct an isomorphism

$$\hat{\mathbb{Z}} \to \prod_{p \text{ prime}} \mathbb{Z}_p.$$

(Hint: use the Chinese remainder theorem)

Remark. *One can define the topology on an inverse limit* $R = \varprojlim R_i$ *as the coarsest topology making all of the projection maps* $R \to R_i$ *continuous.*

Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.