18.782 Introduction to Arithmetic Geometry **Spring 2023**

Problem Set #10 **Due: 05/06/2023**

These problems are related to the material covered in Lectures 30-32. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due at 11:59pm Eastern time on 05/06/2023. It is to be submitted electronically as a pdf-file through gradescope. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions, and identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective (irreducible) variety of dimension one.

Problem 1. Differentials (30 points)

Let F/k be a function field of transcendence degree 1, with $\operatorname{char} k = 0$ and k algebraically closed in F. Define Δ_F to be the F-vector space generated by the set of formal symbols $\{dx : x \in F\}$, subject to the relations

$$(1)\ d(x+y) = dx + dy, \qquad (2)\ d(xy) = xdy + ydx, \qquad (3)\ da = 0 \text{ for } a \in k.$$

Note that x and y denote elements of F (functions), not free variables, so Δ_F reflects the structure of F and will be different for different function fields.

(a) Prove that $\dim_F \Delta_F = 1$, and that any dx with $x \notin k$ is a basis.

The set $\Delta = \Delta_F$ is often used as an alternative to the set of Weil differentials Ω. They are both one-dimensional F-vector spaces, hence isomorphic (as F-vector spaces). But in order to be useful, we need to associate divisors to differentials in Δ, as we did for Ω.

For any differential $\omega \in \Delta$ and any place P, we may pick a uniformizer t for P and write $\omega = wdt$ for some function $w \in F$ that depends on our choice of t; note that t is necessarily transcendental over k, since it is a uniformizer. We then define $\operatorname{ord}_P(\omega) := \operatorname{ord}_P(w)$, and the divisor of ω is then given by

$$\operatorname{div}\omega := \sum_P \operatorname{ord}_P(\omega)P.$$

The value $\operatorname{ord}_P(\omega)$ does not depend on the choice of the uniformizer t.

(b) Prove that $\operatorname{div} udv = \operatorname{div} u + \operatorname{div} dv$ for any $u, v \in F$ with v transcendental over k. Conclude that the set of nonzero differentials in Δ constitutes a linear equivalence class of divisors.

(c) Let $F = k(t)$ be the rational function field. Compute $\operatorname{div} dt$ and prove that it is a canonical divisor. Conclude that a divisor $D \in \operatorname{Div}_k \mathbb{P}^1$ is canonical if and only if $D = \operatorname{div} df$ for some transcendental $f \in F$.

Part (c) holds for arbitrary curves, but you are not asked to prove this. It follows that the space of differentials Δ plays the same role as the space of Weil differentials Ω, and it has the virtue of making explicit computations much easier.

(d) Prove that the curve $x^2 + y^2 + z^2$ over $\mathbb{Q}$ has genus 0 (even though it is not isomorphic to $\mathbb{P}^1$ because it has no rational points) by explicitly computing a canonical divisor.

Problem 2. A genus 1 curve with no rational points (20 points)

Consider the homogeneous polynomial

$$f(x, y, z) = x^3 + 2y^3 + 4z^3.$$

(a) Prove that the zero locus of f is a plane curve $C/\mathbb{Q}$. (That is, prove smoothness and deduce irreducibility as in Lemma 23.13.)

(b) Prove that C has genus one. (**Hint:** It will be helpful to look at Theorem 22.25, and recall Problem 4 of Pset 1.)

(c) Prove that C has no $\mathbb{Q}$-rational points (so it is not an elliptic curve over $\mathbb{Q}$).

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.