Throughout this lecture C/k is a curve over a perfect but not necessarily algebraically closed field k and F/k denotes the corresponding function field.[1]

In the last lecture we defined $\ell(D)$ as the dimension of the Riemann-Roch space $\mathcal{L}(D)$ of the divisor D, and we proved that $\ell(D)$ is invariant under linear equivalence. It is immediate from the definitions that for any divisors $A \leq B$ we have $\ell(A) \leq \ell(B)$ and $\deg(A) \leq \deg(B)$. Less obvious is the fact that

$$\deg(A) - \ell(A) \leq \deg(B) - \ell(B),$$

but we proved that this holds for all $A \leq B$; see Lemma 21.8. As we are particularly interested in the quantities on the two sides of the above inequality, let us define

$$r(D) := \deg(D) - \ell(D).$$

Then $r(D)$ is preserved under linear equivalence and $A \leq B \implies r(A) \leq r(B)$. At the end of Lecture 21 we proved that for every curve C/k there is an integer $g \geq 0$ such that

$$r(D) \leq g - 1 \tag{1}$$

for all $D \in \mathrm{Div}_k\, C$. We also showed that for $C \simeq \mathbb{P}^1$ we can take $g = 0$, and that $r(D) = -1$ for all $D \geq 0$. We always have $r(0) = 0 - 1 = -1$, so $r(D) \geq -1$ for all $D \geq 0$.

22.1   The genus of a curve

We now define the genus of a curve.

Definition 22.1. The *genus* of the curve C/k is defined by

$$g := \max\{r(D) + 1 : D \in \mathrm{Div}_k(C)\}.$$

In other words, g is the least integer for which (1) holds.

Remark 22.2. This definition of the genus of a curve is sometimes called the *geometric genus* to distinguish it from other notions of genus that we won't consider in this course. For (smooth projective) curves the different definitions all agree.

We now give the complete statement of Riemann's Theorem, most of which was proved in Theorem 21.15.

Theorem 22.3 (Riemann's Theorem). *Let C/k be a curve of genus g. Then $r(D) \leq g - 1$ for all $D \in \mathrm{Div}_k\, C$, and equality holds for all divisors of sufficiently large degree.*

Proof. We have already proved the inequality. Let us pick a divisor A for which $r(A) = g-1$; some such A exists, by the definition of g. We will show that $r(D) = g - 1$ whenever $\deg D \geq \deg A + g = c$.

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[1]Recall that our curves are smooth projective varieties of dimension one, and that our varieties are geometrically irreducible.

So assume $\deg D \geq c$. We have $r(D - A) = \deg(D - A) - \ell(D - A) \leq g - 1$, so

$$\ell(D - A) \;\geq\; \deg(D - A) + 1 - g \;\geq\; c - \deg A + 1 - g \;=\; 1.$$

There is a nonzero $f \in \mathcal{L}(D - A)$, so let $D' = D + \operatorname{div} f \geq D + A - D = A$. Then

$$r(D) = r(D') \geq r(A) = g - 1,$$

and we already know that $r(D) \leq g - 1$, so $r(D) = g - 1$. $\qquad\square$

We now want to refine Riemann's Theorem to obtain a more precise statement that will tell us exactly what "sufficiently large" means and give us a measure of how far the inequality $r(D) \leq g - 1$ is from being an equality for any particular divisor D; this is the Riemann-Roch theorem.

Definition 22.4. Let C/k be a curve of genus g. For $D \in \operatorname{Div}_k C$, the non-negative integer

$$i(D) := g - 1 - r(D)$$

is the *index of speciality* of D. Divisors for which $i(D) > 0$ are said to be *special*.

We know from Riemann's Theorem that $i(D) = 0$ for all D of sufficiently large degree, and we also know that $i(0) = g$, since

$$i(0) = g - 1 - r(0) = g - 1 - \deg 0 + \ell(0) = g - 1 - 0 + 1 = g.$$

22.2   The ring of adeles

To compute the index of speciality we introduce the adele ring. Our presentation roughly follows that in [2, §1.5].

Definition 22.5. The *adele ring* of the function field F/k is the subring $\mathcal{A} = \mathcal{A}_F$ of the direct product $\prod_P F$ consisting of those elements $\alpha = (\alpha_P)$ for which $\alpha_P \in \mathcal{O}_P$ for all but finitely many P. The elements of $\mathcal{A}$ are called *adeles*.

Remark 22.6. The adele ring $\mathcal{A}$ is also called the *ring of repartitions*. It is often defined in terms of the P-adic completions of F, but we don't need to take completions to prove the Riemann-Roch theorem so we won't (some authors refer to $\mathcal{A}$ as the ring of *pre-adeles*).

The function field F/k is canonically embedded in $\mathcal{A}$ via the diagonal embedding

$$f \mapsto (f, f, f, \ldots).$$

Adeles of this form are called *principal adeles*, terminology that is consistent with our notion of a principal divisor; those divisors that correspond to elements of the function field. Like F, the adele ring $\mathcal{A}$ is a k-vector space. We extend each valuation ord_P of F/k to a valuation on $\mathcal{A}$ by defining $\operatorname{ord}_P(\alpha) = \operatorname{ord}_P(\alpha_P)$ for $\alpha_P \neq 0$ and setting $\operatorname{ord}_P(0) = \infty$, where ∞ is greater than any element of $\mathbb{Z}$.

Definition 22.7. For a divisor D the *adele space* of D is the k-vector space

$$\mathcal{A}(D) := \{\alpha \in \mathcal{A} : \operatorname{ord}_P(\alpha) \geq -\operatorname{ord}_P(D) \text{ for all places } P\}.$$

It contains the Riemann-Roch space $\mathcal{L}(D) = \mathcal{A}(D) \cap F$ as a subspace, and it is in turn a subspace of the adele ring $\mathcal{A}$.

The adele space of a divisor gives us additional information beyond what we get from the Riemann-Roch space that will allow us to characterize the index of speciality in a canonical way. We first prove three lemmas.

Lemma 22.8. *For any two divisors $A \leq B$ we have $\mathcal{A}(A) \subseteq \mathcal{A}(B)$ and*

$$\dim \mathcal{A}(B)/\mathcal{A}(A) = \deg B - \deg(A),$$

as k-vector spaces.

Proof. The inclusion $\mathcal{A}(A) \subseteq \mathcal{A}(B)$ is clear. As in the proof of Lemma 21.8, it suffices to consider the case $B = A + P$ for some place P, and the proof is exactly the same. We pick a uniformizer t for P and define the linear map $\phi \colon \mathcal{A}(B) \to k(P)$ by $\phi(f) = (t^n f)(P)$, where $n = \operatorname{ord}_P(B)$. The map ϕ is surjective and its kernel is $\mathcal{A}(A)$, hence

$$\dim(\mathcal{A}(B)/\mathcal{A}(A)) = \dim k(P) = \deg P = \deg B - \deg A. \qquad \square$$

Lemma 22.9. *For any two divisors $A \leq B$ we have $\mathcal{A}(A) + F \subseteq \mathcal{A}(B) + F$ and*

$$\dim \frac{\mathcal{A}(B) + F}{\mathcal{A}(A) + F} = r(B) - r(A),$$

as k-vector spaces, where F is embedded diagonally in $\mathcal{A}$.

Proof. The inclusion is clear, and the map

$$\mathcal{A}(B) \to \mathcal{A}(B) + F \to (\mathcal{A}(B) + F)/(\mathcal{A}(A) + F)$$

is surjective, with kernel $\mathcal{A}(B) \cap (\mathcal{A}(A) + F)$. We therefore have

$$\frac{\mathcal{A}(B) + F}{\mathcal{A}(A) + F} \simeq \frac{\mathcal{A}(B)}{\mathcal{A}(B) \cap (\mathcal{A}(A) + F)} = \frac{\mathcal{A}(B)}{\mathcal{A}(A) + \mathcal{L}(B)} \simeq \frac{\mathcal{A}(B)/\mathcal{A}(A)}{(\mathcal{A}(A) + \mathcal{L}(B))/\mathcal{A}(A)}.$$

Applying Lemma 22.8 and taking dimensions gives

$$\dim \frac{\mathcal{A}(B) + F}{\mathcal{A}(A) + F} = \deg B - \deg A - \dim \frac{\mathcal{A}(A) + \mathcal{L}(B)}{\mathcal{A}(A)}.$$

Finally we note that

$$\dim \frac{\mathcal{A}(A) + \mathcal{L}(B)}{\mathcal{A}(A)} = \dim \frac{\mathcal{L}(B)}{\mathcal{A}(A) \cap \mathcal{L}(B)} = \dim \frac{\mathcal{L}(B)}{\mathcal{L}(A)} = \ell(B) - \ell(A),$$

thus

$$\dim \frac{\mathcal{A}(B) + F}{\mathcal{A}(A) + F} = \deg B - \deg A - (\ell(B) - \ell(A)) = r(B) - r(A). \qquad \square$$

Lemma 22.10. *For any divisor D for which $r(D) = g - 1$ we have*

$$\mathcal{A} = \mathcal{A}(D) + F.$$

Proof. Let $\alpha \in \mathcal{A}$. We will show $\alpha \in \mathcal{A}(D) + F$. Let us pick a divisor $D' \geq D$ such that $\alpha \in \mathcal{A}(D') + F$; this is clearly possible. We have $g - 1 = r(D) \leq r(D') \leq g - 1$, by Riemann's Theorem, so $r(D') = g - 1$. By Lemma 22.9 we have

$$\dim \frac{\mathcal{A}(D') + F}{\mathcal{A}(D) + F} = r(D') - r(D) = (g - 1) - (g - 1) = 0,$$

so $\mathcal{A}(D') + F = \mathcal{A}(D) + F$ and therefore $\alpha \in \mathcal{A}(D) + F$. $\qquad \square$

We can now determine the index of speciality of a divisor in terms of its adele space.

Theorem 22.11. *Let F/k be a function field. For any divisor $D \in \mathrm{Div}_k F$ we have*

$$i(D) = \dim \mathcal{A}/(\mathcal{A}(D) + F).$$

Proof. By Riemann's Theorem there exists a divisor $D' \geq D$ for which $r(D') = g - 1$; we just need to make the degree of $D' \geq D$ large enough, and this is clearly possible; if $D \neq 0$ we can take a multiple of $D_0 + D_\infty$. By Lemma 22.10 we then have $\mathcal{A} = \mathcal{A}(D') + F$, thus

$$\dim \frac{\mathcal{A}}{\mathcal{A}(D) + F} = \dim \frac{\mathcal{A}(D') + F}{\mathcal{A}(D) + F} = r(D') - r(D) = g - 1 - r(D) = i(D),$$

where we use Lemma 22.9 to get the second equality. □

We now have an equality that holds for all divisors

$$g = r(D) + 1 + i(D),$$

and we know that $i(D) = 0$ for all divisors of sufficiently large degree. But we would like to characterize $i(D)$ in a canonical way that does not involve the adele ring; this will yield the Riemann-Roch theorem.

22.3 Canonical divisors

Definition 22.12. Let F/k be a function field and let $\mathcal{A}$ be its adele ring. For a divisor $D \in \mathrm{Div}_k F$ the space of *Weil differentials* $\Omega(D)$ is the orthogonal complement of $\mathcal{A}(D) + F$ (its annihilator in the dual space $\mathcal{A}^\vee$). Explicitly, this is the set of all linear functionals $\omega\colon \mathcal{A} \to k$ that vanish on $\mathcal{A}(D) + F$. The k-vector space

$$\Omega = \Omega_F := \bigcup_{D \in \mathrm{Div}_k F} \Omega(D)$$

is the space of *Weil differentials* for F/k.

It is clear that Ω is a k-vector space: if $\omega_1 \in \Omega(D_1)$ and $\omega_2 \in \Omega(D_2)$ then $\omega_1 + \omega_2$ lies in $\Omega(D)$, where $D = D_1 \wedge D_2$ is defined by $\mathrm{ord}_P(D) = \min(\mathrm{ord}_P(D_1), \mathrm{ord}_P(D_2))$.

Lemma 22.13. *For any divisor $D \in \mathrm{Div}_k F$ we have $\dim \Omega(D) = i(D)$.*

Proof. The quotient space $\mathcal{A}/(\mathcal{A}(D) + F)$ has finite dimension $i(A)$, by Theorem 22.11, thus it has the same dimension as its dual, which is canonically isomorphic to the orthogonal complement of $\mathcal{A}(D) + F$, which is precisely $\Omega_F(D)$.[2] Thus $\dim \Omega(D) = i(D)$. □

We have seen that the space of Weil differentials Ω is a k-vector space; we now make Ω an F-vector space by defining $f\omega \in \Omega$ for $f \in F$ and $\omega \in \Omega$ as the linear functional $\mathcal{A} \to k$ that sends α to $\omega(f\alpha)$, in other words

$$(f\omega)(\alpha) = \omega(f\alpha),$$

for all $\alpha \in \mathcal{A}$.

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[2]If V/W is any quotient, the map $\Phi\colon (V/W)^\vee \to W^\perp$ defined by $\Phi(\lambda)(v) = \lambda(v + W)$ is an isomorphism.

Theorem 22.14. *Let F/k be a function field and let Ω be its space of Weil differentials. Then $\dim_F \Omega = 1$.*

Proof. Clearly $\Omega \neq 0$, so let $\omega_1, \omega_2 \in \Omega$ be nonzero. We will show that $\omega_1/\omega_2 \in F$.

For $i = 1, 2$, let D_i be such that $\omega_i \in \Omega(D_i)$ and define the k-linear map

$$\phi_i \colon \mathcal{L}(D_i + D) \to \Omega(-D),$$
$$f \mapsto f\omega_i,$$

where D is a fixed divisor to be determined. For any $\alpha + g$ in $\mathcal{A}(-D) + F$ we have

$$(f\omega_i)(\alpha + g) = \omega_i(f\alpha) + \omega_i(fg) = 0 + 0 = 0,$$

since ω_i vanishes on $fg \in F$ and

$$\operatorname{ord}_P(f\alpha) = \operatorname{ord}_P(f) + \operatorname{ord}_P(\alpha) \geq \operatorname{ord}_P(-D_i - D) + \operatorname{ord}_P(D) = \operatorname{ord}_P(-D_i)$$

for all P, so ω_i vanishes on $f\alpha$. Thus ϕ_i is well defined, and it is clearly injective.

We claim that for an appropriate choice of D we have

$$\phi_1(\mathcal{L}(D_1 + D)) \cap \phi_2(\mathcal{L}(D_2 + D)) \neq \{0\}. \tag{2}$$

Assuming the claim, we may pick nonzero $f_1 \in \mathcal{L}(D_1 + D)$ and $f_2 \in \mathcal{L}(D_2 + D)$ such that $\phi_1(f_1) = \phi_2(f_2)$. Then $f_1\omega_1 = f_2\omega_2$ and $\omega_1/\omega_2 = f_2/f_1 \in F$ as desired.

We now prove that there is a divisor D for which (2) holds. By Riemann's Theorem, we can pick $D > 0$ of sufficiently large degree so that $r(D_i + D) = g - 1$ for $i = 1, 2$. Let U_i be the image of $\mathcal{L}(D_i + D)$ in $\Omega(-D)$ under ϕ_i. We want to show $\dim(U_1 \cap U_2) > 0$. We have

$$\dim \Omega(-D) = i(-D) = g - 1 - r(-D) = g - 1 - \deg(-D) + \ell(-D) = g - 1 + \deg(D),$$

since $\ell(-D) = 0$ for $D > 0$. We have $U_1 + U_2 \subseteq \Omega(-D)$, and therefore

$$\dim \Omega(-D) \geq \dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2),$$

where all the dimensions are as k-vector spaces. Thus

$$\begin{aligned}
\dim(U_1 \cap U_2) &\geq \dim U_1 + \dim U_2 - \dim \Omega(-D) \\
&= \ell(D_1 + D) + \ell(D_2 + D) - g + 1 - \deg D \\
&= \deg(D_1 + D) - r(D_1 + D) + \deg(D_2 + D) - r(D_2 + D) - g + 1 - \deg D \\
&= \deg(D_1 + D) - g + 1 + \deg(D_2 + D) - g + 1 - g + 1 - \deg D \\
&= \deg D + \deg D_1 + \deg D_2 - 3g + 3.
\end{aligned}$$

By choosing D of sufficiently large degree, we can make the RHS positive. $\qquad\square$

Lemma 22.15. *For any nonzero $\omega \in \Omega$ there is a unique divisor D_ω such that $D \leq D_\omega$ for all divisors D for which $\omega \in \Omega(D)$.*

Proof. By Lemma 22.13, we have $\dim \Omega(D) = i(D)$, so $i(D) > 0$ for all divisors D such that $\omega \in \Omega(D)$. At least one such D exists, since $\omega \in \Omega$, so let us choose D_ω maximal subject to the constraint $\omega \in \Omega(D_\omega)$; a maximal D_ω exists because $i(D) = 0$ for all D of sufficiently large degree, by Riemann's Theorem. We now prove that D_ω is unique.

Suppose not. Then there are two distinct divisors D_1 and D_2 that are maximal subject to the constraints $\omega \in \Omega(D_1)$ and $\omega \in \Omega(D_2)$. Since D_1 and D_2 are incomparable, there exist distinct places P_1 and P_2 such that

$$\mathrm{ord}_{P_1}(D_1) > \mathrm{ord}_{P_1}(D_2) \quad \text{and} \quad \mathrm{ord}_{P_2}(D_2) > \mathrm{ord}_{P_2}(D_1).$$

We claim that $\omega \in \Omega(D_1 + P_2)$, contradicting the maximality of D_1. Write $\alpha \in \mathcal{A}(D_1 + P_2)$ as $\alpha = \alpha_1 + \alpha_2$, where α_1 is zero at P_2 and equal to α otherwise, while α_2 is equal to α at P_2 and zero otherwise. Then $\alpha_1 \in \mathcal{A}(D_1)$ and $\alpha_2 \in \mathcal{A}(D_2)$ and $\omega(\alpha) = \omega(\alpha_1) + \omega(\alpha_2) = 0$, since $\omega \in \Omega(D_1)$ and $\omega \in \Omega(D_2)$. But then $\omega \in \Omega(D_1 + P_2)$ as claimed. $\qquad\square$

Definition 22.16. For a nonzero Weil differential $\omega \in \Omega$ we define the *divisor of* ω to be the unique divisor

$$\mathrm{div}\,\omega := D_\omega$$

given by Lemma 22.15. A divisor D is said to be *canonical* if $D = \mathrm{div}\,\omega$ for some $\omega \in \Omega$. We also define $\mathrm{ord}_P(\omega) := \mathrm{ord}_P(\mathrm{div}\,\omega)$.

Lemma 22.17. *For any nonzero $f \in F$ and nonzero $\omega \in \Omega$ we have*

$$\mathrm{div}(f\omega) = \mathrm{div}\,f + \mathrm{div}\,\omega.$$

Proof. We have $f\omega \in \Omega(\mathrm{div}\,f + \mathrm{div}\,\omega)$, since for any $g + \alpha$ in $\mathcal{A}(\mathrm{div}\,f + \mathrm{div}\,\omega) + F$:

$$(f\omega)(g + \alpha) = \omega(fg + f\alpha) = \omega(fg) + \omega(f\alpha) = 0 + 0 = 0,$$

because ω vanishes on F and

$$\mathrm{ord}_P(f\alpha) = \mathrm{ord}_P(f) + \mathrm{ord}_P(\alpha) \geq \mathrm{ord}_P(\mathrm{div}\,f) + \mathrm{ord}_P(-D - \mathrm{div}\,f) = \mathrm{ord}_P(-D)$$

for all places P of F, so $\omega(f\alpha) = 0$. It follows that $\mathrm{div}\,f\omega \geq \mathrm{div}\,f + \mathrm{div}\,\omega$.

The same argument shows that $\mathrm{div}\,\omega = \mathrm{div}\,f^{-1}f\omega \geq \mathrm{div}\,f^{-1} + \mathrm{div}\,f\omega = \mathrm{div}\,f\omega - \mathrm{div}\,f$, and therefore $\mathrm{div}\,f\omega \leq \mathrm{div}\,f + \mathrm{div}\,\omega$, so the claimed equality holds. $\qquad\square$

Corollary 22.18. *The canonical divisors form a single linear equivalence class.*

Proof. Let $D_1 = \mathrm{div}\,\omega_1$ and $D_2 = \mathrm{div}\,\omega_2$ be two canonical divisors for F/k. Then ω_1 and ω_2 are both nonzero, and by Theorem 22.14, we have $\omega_2 = f\omega_1$ for some $f \in F^\times$. But then $D_2 = \mathrm{div}\,\omega_2 = \mathrm{div}\,f\omega_1 = \mathrm{div}\,f + \mathrm{div}\,\omega_1 = \mathrm{div}\,f + D_1$, so $D_1 \sim D_2$.

Now suppose $D_1 = \mathrm{div}\,\omega_1$ is a canonical divisor and $D_2 = D_1 + \mathrm{div}\,f$ for some $f \in F^\times$. Then $D_2 = \mathrm{div}\,\omega_1 + \mathrm{div}\,f = \mathrm{div}\,f\omega_1$ is canonical. $\qquad\square$

Thus their is a unique element of the Picard group $\mathrm{Pic}_k\,C$ corresponding to the class of canonical divisors. This is a truly remarkable fact; given the rather abstract definition of the Picard group, there is no *a priori* reason to expect that it should have a uniquely distinguished element other than zero. As we shall see in the next lecture, the canonical divisor class is typically not the zero divisor, and the case where it is is actually quite interesting.

We now show that, like elements of the function field, Weil differentials are determined up to a scalar factor in $k^\times$ by their divisors.

Corollary 22.19. *Two nonzero Weil differentials $\omega_1, \omega_2 \in \Omega$ have the same divisor if and only if one $\omega_2 = c\omega_1$ for some $c \in k^\times$.*

Proof. Since $\omega_1 \neq 0$ and $\dim_F \Omega = 1$, we can write $\omega_2 = f\omega_1$ for some $f \in F^\times$. Then $\mathrm{div}\,\omega_2 = \mathrm{div}\,f\omega_1 = \mathrm{div}\,f + \mathrm{div}\,\omega_1$, so if $\mathrm{div}\,\omega_1 = \mathrm{div}\,\omega_2$ then $\mathrm{div}\,f = 0$ and $f \in k^\times$. Conversely, $\mathrm{div}\,\omega_2 = \mathrm{div}\,c\omega_1 = \mathrm{div}\,c + \mathrm{div}\,\omega_1 = \mathrm{div}\,\omega_1$, for any $c \in k^\times$. $\qquad\square$

22.4 The Riemann-Roch Theorem

We now have almost everything we need to prove the Riemann-Roch Theorem. The last ingredient is the Duality Theorem, which gives us an isomorphism between Riemann-Roch spaces and spaces of Weil differentials.

Theorem 22.20 (Duality). *For any divisor D and canonical divisor $W = \operatorname{div} \omega$, the linear map $\phi \colon \mathcal{L}(W - D) \to \Omega(D)$ defined by $\phi(f) = f\omega$ is an isomorphism of k-vector spaces. In particular, we have $i(D) = \ell(W - D)$ for all divisors D.*

Proof. For any nonzero $f \in \mathcal{L}(W - D)$ and $\omega \in \Omega(D)$ we have

$$\operatorname{div} f\omega = \operatorname{div} f + \operatorname{div} \omega \geq -(W - D) + W = D,$$

thus $f\omega \in \Omega(D)$, and $\operatorname{im} \phi \subseteq \Omega(D)$. It is clear that ϕ is linear, and its kernel is obviously trivial, so it is injective. To show that ϕ is surjective, let ω' be any nonzero element of $\Omega(D)$. By Theorem 22.14 we can write $\omega' = f\omega$ for some $f \in F^\times$, and since

$$\operatorname{div} f + W = \operatorname{div} f + \operatorname{div} \omega = \operatorname{div} f\omega = \operatorname{div} \omega_1 \geq D,$$

we have $\operatorname{div} f \geq -(W - D)$ and therefore $f \in \mathcal{L}(W - D)$, so $\omega' = \phi(f)$. Thus ϕ is surjective, hence an isomorphism, and $i(D) = \dim \Omega(D) = \ell(W - D)$, by Lemma 22.13. $\qquad\square$

Theorem 22.21 (Riemann-Roch Theorem). *Let W be a canonical divisor of the genus g curve C/k. For every divisor D we have*

$$\ell(D) = \deg(D) + 1 - g + \ell(W - D).$$

Proof. Immediate from Definition 22.4 and Theorem 22.20. $\qquad\square$

Corollary 22.22. *For any canonical divisor W of a genus g curve we have*

$$\ell(W) = g, \qquad \deg W = 2g - 2, \qquad i(W) = 1.$$

Proof. We apply the Riemann-Roch Theorem twice, first with $D = 0$, which gives

$$\ell(0) = \deg 0 + 1 - g + \ell(W),$$

and since $\deg 0 = 0$ and $\ell(0) = 1$, we have $\ell(W) = g$. Taking $D = W$ gives

$$\ell(W) = \deg W + 1 - g + \ell(0),$$

which implies $\deg W = 2g - 2$ and $i(W) = \ell(W - W) = 1$. $\qquad\square$

We can now give an exact value for the constant c in Riemann's Theorem.

Corollary 22.23. *For all divisors D of a genus g curve C/k with $\deg D > 2g - 2$ we have*

$$\ell(D) = \deg D + 1 - g,$$

equivalently, $i(D) = 0$.

Proof. By the Riemann-Roch Theorem,

$$\ell(D) = \deg(D) + 1 - g + \ell(W - D)$$

where W is a canonical divisor. We have

$$\deg(W - D) = \deg W - \deg D < 2g - 2 - (2g - 2) = 0,$$

so $\ell(W - D) = 0$ and the corollary follows. $\qquad\square$

We can also give some more down-to-earth characterization of a canonical divisor.

Corollary 22.24. *For a divisor D of a genus g curve, the following are equivalent:*

(a) *D is a canonical divisor.*

(b) *$\ell(D) = g$ and $\deg D = 2g - 2$.*

(c) *$i(D) = 1$ and $\deg D$ is maximal among divisors with $i(D) = 1$.*

Proof. That (a) implies (b) is immediate from Corollary 22.22, and the implications (b)$\Rightarrow$(c) and (c)$\Rightarrow$(a) both follow from the combination of Corollaries 22.22 and 22.23. $\qquad\square$

Finally we note a very useful fact.

Theorem 22.25. *The genus of a curve C/k over a perfect field k is preserved under base extension.*[3]

Proof. Let k'/k be an extension of the perfect field k (hence a separable extension). It suffices to show that if $D \in \mathrm{Div}_k C$ is a canonical divisor for C/k then it is also a canonical divisor for C/k'. Clearly $\deg D$ is not changed under base extension (some closed points may split, but the total degree does not change), so it suffices to show that the dimension $\ell(D)$ of the Riemann-Roch space $\mathcal{L}(D) \subseteq k(C)$ does not change under base extension. The key point here is that any finite-dimensional k'-vector subspace of $k'(C)$ has a basis that lies in $k(C)$; this follows from a general algebraic result that we will not prove here; see Proposition 1 in §3 of the appendix to [1]. Thus $\ell(D)$ does not change under base extension. $\qquad\square$

References

[1] I. R. Shafarevich, *Basic algebraic geometry*, 2nd edition, Springer-Verlag, 1994.

[2] H. Stichtenoth, *Algebraic function fields and codes*, Springer, 2009.

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[3]This theorem is not necessarily true when k is not perfect. It is possible for the genus to decrease under an inseparable base extension.