As usual, k is a perfect field and $\bar{k}$ is a fixed algebraic closure of k. Recall that an affine (resp. projective) variety is an irreducible alebraic set in $\mathbb{A}^n = \mathbb{A}^n(\bar{k})$ (resp. $\mathbb{P}^n = \mathbb{P}^n(\bar{k})$).

15.1 Rational maps of affine varieties

Before defining rational maps we want to nail down two points on which we we were intentional vague in the last lecture. We defined a morphism $\phi\colon X \to Y$ of varieties $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ as a "map defined by a tuple of polynomials $(\phi_1, \ldots, \phi_n)$." This definition is vague in two ways. First, is a morphism a map between two sets X and Y, or is it a tuple of polynomials? We shall adopt the second view; we still get a function by evaluating the polynomials at points in X.

Given that we regard ϕ as a tuple $(\phi_1, \ldots, \phi_n)$, the next question is in which ring do its components ϕ_i lie? Are they elements of $\bar{k}[x_1, \ldots, x_m]$ or $\bar{k}[X]$. The function they define is the same in either case, but we shall regard the ϕ_i as elements of $\bar{k}[X]$. This means that in order to evaluate ϕ_i at a point $P \in X$, we need to lift $\phi_i = \hat{\phi}_i + I(X)$ to a representative $\hat{\phi}_i \in \bar{k}[x_1, \ldots, x_m]$ and then compute $\hat{\phi}_i(P)$. Of course it does not matter which representative $\hat{\phi}_i$ we pick; we define $\phi_i(P)$ to be the value $\hat{\phi}_i(P)$ for any/every choice of $\hat{\phi}$, and thereby define $\phi(P)$ for $P \in X$ (but note that $\phi(P)$ is *not defined* for any $P \notin X$).

One advantage of this approach is that there is then a one-to-one correspondence between morphisms and the functions they define. If $\phi = (\phi_1, \ldots, \phi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$ define the same function from X to Y then each of the polynomials $\hat{\phi}_i - \hat{\psi}_i$ in $\bar{k}[x_1, \ldots, x_m]$ contains X in its zero locus and therefore lies in the ideal $I(X)$. This implies, by definition, that in $\bar{k}[X] = \bar{k}[x_i, \ldots, x_m]/I(X)$ we have $\phi_i = \psi_i$ for $1 \le i \le m$ and therefore $\phi = \psi$.

We now want to extend these ideas to the function field $\bar{k}(X)$. Elements of $\bar{k}(X)$ have the form $r = f/g$, with $f, g \in \bar{k}[X]$ and $g \neq 0$, and are called *rational functions* (or even just *functions*), on X, even though they are formally elements of the fraction field of $\bar{k}[X]$ and typically do *not* define a function from X to $\bar{k}$; indeed, this is precisely the issue we must now address.

It seems natural to say that for a point $P \in X$ we should define $r(P)$ to be $f(P)/g(P)$ whenever $g(P) \neq 0$ and call it undefined otherwise. But there is a problem with this approach: the representation $r = f/g$ is not necessarily unique.[1] We also have $r = p/q$ whenever $pg = fq$ holds in $\bar{k}[X]$ (recall that this is part of the definition of a fraction field, it is a set of equivalence classes of fractions). The values $f(P)/g(P)$ and $p(P)/q(P)$ are necessarily equal wherever both denominators are nonzero, but it may be that $q(P) \neq 0$ at points where $g(P) = 0$ (and vice versa).

Example 15.1. Consider the the zero locus X of $x_1x_2 - x_3x_4$ in $\mathbb{A}^4$ (which is in fact a variety) and the rational function $r = x_1/x_3 = x_4/x_2$. At the point $P = (0,1,0,0) \in X$ the value $x_1(P)/x_3(P)$ is not defined, but $x_4(P)/x_2(P) = 0$ is defined, and the reverse occurs for $P = (0,0,1,0) \in X$. But we can assign a meaningful value to $r(P)$ at both points; the only points in X where r is not defined are those with $x_2 = x_3 = 0$.

This motivates the following definition.

[1] If $\bar{k}[X]$ is a UFD then we can put $r = f/g$ in lowest terms to get a unique representation. However, the coordinate ring $\bar{k}[X]$ is usually *not* a UFD, even though $\bar{k}[x_1, \ldots, x_m]$ is. A quotient of a UFD is typically not a UFD, even when it is an integral domain; consider $\mathbb{Z}[x]/(x^2 + 5)$, for example.

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Definition 15.2. A function $r \in \bar{k}(X)$ is said to be *regular* at a point $P \in X$ if $gr \in \bar{k}[X]$ for some $g \in \bar{k}[X] \subseteq \bar{k}(X)$[2] for which $g(P) \neq 0$ (we then have $r = f/g$ for some $f \in \bar{k}[X]$).

The set of points at which a function $r \in \bar{k}(X)$ is regular form a nonempty open (hence dense) subset $\operatorname{dom}(r)$ of the subspace X: the complement of $\operatorname{dom}(r)$ in X is the closed subset of X defined by the *denominator ideal* $\{g \in \bar{k}[X] : gr \in \bar{k}[X]\}$, which we note is not the zero ideal, since $r = f/g$ for some nonzero g.[3]

We now associate to r the function from $\operatorname{dom}(r)$ to $\bar{k}$ that maps P to

$$r(P) = (f/g)(P) = f(P)/g(P),$$

where g is chosen so that $g(P) \neq 0$ and $gr = f \in \bar{k}[X]$. Now if r is actually an element of $\bar{k}[X]$, then r is regular at every point in X and we have $\operatorname{dom}(r) = X$. The following lemma says that the converse holds.

Lemma 15.3. *A rational function $r \in \bar{k}(X)$ lies in $\bar{k}[X]$ if and only if* $\operatorname{dom}(r) = X$.

Proof. The forward implication is clear, and if $\operatorname{dom}(r) = X$ then the complement of $\operatorname{dom}(r)$ in X is the empty set and the denominator ideal is (1), which implies $r \in \bar{k}[X]$. □

Definition 15.4. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. We say that a tuple $(\phi_1, \ldots, \phi_n)$ with $\phi_i \in \bar{k}(X)$ is *regular* at $P \in X$ if the ϕ_i are all regular at P. A *rational map* $\phi\colon X \to Y$ is a tuple $(\phi_1, \ldots, \phi_n)$ with $\phi_i \in \bar{k}(X)$ such that $\phi(P) := (\phi_1(P), \ldots, \phi_n(P)) \in Y$ for all points $P \in X$ where ϕ is regular. If ϕ is regular at every point $P \in X$ then we say that ϕ is *regular*.

The set of points where ϕ is regular form an open subset $\operatorname{dom}(\phi) = \cap_i \operatorname{dom}(\phi_i)$ of X. Thus ϕ defines a function from $\operatorname{dom}(\phi)$ to Y, which we may also regard as a partial function from X to Y. We get a complete function from X to Y precisely when $\operatorname{dom}(\phi) = X$, that is, when ϕ is regular. This occurs precisely when ϕ is a morphism.

Theorem 15.5. *A rational map of affine varieties is a morphism if and only if it is regular.*

Proof. A morphism is clearly a regular rational map. For the converse, apply Lemma 15.3 to each component of $\phi = (\phi_1, \ldots, \phi_n)$. □

We now want to generalize the categorical equivalence between affine varieties and their function fields, analogous to what we proved in the last lecture for affine varieties and their coordinate rings (affine algebras), but with morphisms of varieties replaces by the more general notion of a rational map. But there is a problem with this. In order to even define a category of varieties with rational maps, we need to be able to compose rational maps. But this is not always possible!

Example 15.6. Let $X = Y = Z = \mathbb{A}^2$, and let $\phi\colon X \to Y$ be the rational map $(1/x_1, 0)$ and let $\psi\colon Y \to Z$ be the rational map $(0, 1/x_2)$. Then the image of ϕ is disjoint from $\operatorname{dom}(\psi)$. There is no rational function that corresponds to the composition of ϕ and ψ (or ψ with ϕ). Even formally, the fractions $1/0$ that we get by naively composing ϕ with ψ are not elements of $\bar{k}(X)$, and the function defined by the composition of the functions defined by ϕ and ψ has the empty set as its domain, which is not true of any $r \in \bar{k}(X)$.

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[2] Recall that an integral domain can always be embedded in its fraction field by identifying g with the equivalence class of $g/1$, so we assume $\bar{k}[X] \subseteq \bar{k}(X)$ henceforth.

[3] When restricting our attention to a variety X in $\mathbb{A}^n$ it is simpler to work with ideals in $\bar{k}[X] = \bar{k}[x_1, \ldots, x_m]/I(X)$ rather than $\bar{k}[x_1, \ldots, x_m]$. The one-to-one corrsepondence between radical ideals and closed sets still holds, as does the correspondence between prime ideals and (sub-) varieties.

To fix this problem we want to restrict our attention to rational maps whose image is dense in its codomain.

Definition 15.7. A rational map $\phi\colon X \to Y$ is *dominant* if $\overline{\phi(\mathrm{dom}(\phi))} = Y$.

If $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ are dominant rational maps then the intersection of $\phi(\mathrm{dom}(\phi))$ and $\mathrm{dom}(\psi)$ must be nonempty; the complement of $\mathrm{dom}(\psi)$ in Y is a proper closed subset of Y and therefore contains no sets that are dense in Y, including $\phi(\mathrm{dom}(\phi))$. It follows that we can always compose dominant rational maps, and since the identity map is also dominant rational map, we can now speak of the category of affine varieties and rational maps. Not every morphism is a dominant rational map, but affine varieties and dominant morphisms form a subcategory of affine varieties and dominant rational maps. As you will show on the problem set, the closure of the image of a morphism of varieties is a variety, so one can always make a morphism dominant be restricting its codomain.

We now prove the analog of Theorem 14.8, replacing morphisms with dominant rational maps, and coordinate rings with function fields. We now use the term function field to refer to any finitely generated extension of $\bar{k}$, and we require morphisms of function fields to fix $\bar{k}$ (we could also call them $\bar{k}$-algebra homomorphisms). Field homomorphisms are always injective, so a morphism of function fields is just a field embedding that fixes $\bar{k}$. Note that the all the interesting function fields $F/\bar{k}$ are transcendental. If $F/\bar{k}$ is algebraic then $F = \bar{k}$; this corresponds to the function field of a zero-dimensional variety (a point).

Given a function field F generated by elements $\alpha_1, \ldots, \alpha_n$ over $\bar{k}$, let R be the $\bar{k}$-algebra generated by $\alpha_1, \ldots, \alpha_n$ in F; this means that R is equal to the set of all polynomial expressions in $\alpha_1, \ldots, \alpha_n$, but there may be algebraic relations between the α_i that make many of these expressions equivalent. In any case, R is isomorphic to the quotient of the polynomial ring $\bar{k}[x_1, \ldots, x_n]$ by an ideal I corresponding to all the algebraic relations that exist among the α_i. The ring R is an integral domain (since it is contained in a field), therefore I is a prime ideal that defines a variety X whose coordinate ring is isomorphic to R and whose function field is isomorphic to F, the fraction field of R.

Theorem 15.8. *The following hold:*

(i) Every dominant rational map $\phi\colon X \to Y$ of affine varieties, induces a morphism $\phi^*\colon \bar{k}(Y) \to \bar{k}(X)$ of function fields such that $\phi^*(r) = r \circ \phi$.

(ii) Every morphism $\theta\colon K \to L$ of function fields induces a dominant rational map of affine varieties $\theta^*\colon X \to Y$, with $K \simeq \bar{k}(Y)$ and $L \simeq \bar{k}(X)$, such that the image of $\theta(r)$ in $\bar{k}(X)$ is $r \circ \theta^*$.

(iii) If $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ are dominant rational maps of affine varieties then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

As in the analogous Theorem 14.8, to compute $r \circ \phi$ one needs to lift/compose/reduce, that is, pick representatives for $\phi_1, \ldots, \phi_n$ that are rational functions in $\bar{k}(x_1, \ldots, x_m)$, pick a representative of r in $\bar{k}(y_1, \ldots, y_n)$, then compose and reduce the numerator and denominator modulo $I(X)$. The fact that ϕ is dominant ensures that the denominator of the composition does not lie in $I(X)$, so $r \circ \phi$ is an element of $\bar{k}(X)$.

Proof. The proofs of parts (i) and (iii) follow the proof of Theorem 14.8 verbatim, with coordinate rings replaced by function fields, only part (ii) merits further discussion. As discussed above, we can write $K \simeq \bar{k}(Y)$ and $L \simeq \bar{k}(X)$ for some varieties X and Y, and any morphism $K \to L$ induces a morphism $\bar{k}(Y) \to \bar{k}(Y)$ that is compatible with these isomorphisms, so let us assume $\theta\colon \bar{k}(Y) \to \bar{k}(X)$.

As in the proof of Theorem 14.8 we define $\theta^*\colon X \to Y$ by $\theta^* = (\theta(y_1), \ldots, \theta(y_n))$, where we now regard the coordinate functions y_i as elements of $\bar{k}(Y)$. For any $r \in \bar{k}(Y)$ we have

$$r \circ \theta^* = \hat{r}(\theta(y_1), \ldots, \theta(y_n)) = \theta(r(y_1, \ldots, y_n)) = \theta(r).$$

The fact that $r \in \bar{k}(Y)$ ensures that the denominator of $\hat{r} \in \bar{k}[y_1, \ldots, y_n]$ is not in $I(Y)$, so this composition is well defined. The proof that the image of θ^* actually lies in Y is the same as in Theorem 14.8: for any $f \in I(Y)$ we have $f \circ \theta^* = \theta(f) = 0$, so certainly $f(\theta^*(P)) = 0$ for all $P \in \operatorname{dom}(\theta^*)$. Thus θ^* is a rational map from X to Y.

But we also need to check that θ^* is dominant (this is the only new part of the proof). This is equivalent to showing that the only element of $\bar{k}[Y]$ that vanishes on the image of θ^* is the zero element, which is in turn equivalent to showing that the only element of $\bar{k}(Y)$ that vanishes on the intersection of its domain and the image of θ^* is the zero element. This is in turn equivalent to showing that if $r \circ \theta^* = \theta(r)$ vanishes at every point in its domain then $r = 0$. But the only element of $\bar{k}(X)$ that vanishes at every point in its domain is the zero element, and θ is injective, so we are done. □

Corollary 15.9. *The category of affine varieties with dominant rational maps and the category of function fields are contravariantly equivalent.*

Proof. As in the proof of Corollary 14.9, the only thing left to show is that $(\phi^*)^* = \phi$ and $(\theta^*)^* = \theta$, up to isomorphism, but both follow from Theorem 15.8 and its proof. □

Definition 15.10. Two affine varieties X and Y are said to be *birationally equivalent* if there exist dominant rational maps $\phi\colon X \to Y$ and $\psi\colon Y \to X$ such that $(\phi \circ \psi)(P) = P$ for all $P \in \operatorname{dom}(\phi \circ \psi)$ and $(\psi \circ \phi)(P) = P$ for all $P \in \operatorname{dom}(\psi \circ \phi)$.

Corollary 15.11. *Two affine varieties are birationally equivalent if and only if their function fields are isomorphic.*

As with morphisms, if $\phi\colon X \to Y$ is a rational map of varieties that are defined over k, we say that $\phi = (\phi_1, \ldots, \phi_n)$ is defined over k if the ϕ_i all lie in $k(X)$.

Corollary 15.12. *Let X and Y be affine varieties defined over k. If $\phi\colon X \to Y$ is a dominant rational map defined over k then $\phi^*\colon \bar{k}(Y) \to \bar{k}(X)$ restricts to a morphism $k(Y) \to k(X)$.*

15.2 Morphisms and rational maps of projective varieties

We now want to generalize everything we have done for maps between affine varieties to maps between projective varieties. This is completely straight-forward, we just need to account for the equivalence relation on $\mathbb{P}^n$.

Recall from Lecture 13 that although we defined the function field $\bar{k}(X)$ of a projective variety $X \subseteq \mathbb{P}^n$ as the function field of any of its non-empty affine parts, we can always represent r by an element $\hat{r} \in \bar{k}(x_0, \ldots, x_n)$ whose numerator and denominator are homogeneous polynomials of the same degree, and for any point $P \in X$ where $\hat{r}(P)$ is defined (has nonzero denominator), we can unambiguously define $r(P) = \hat{r}(P)$.

Definition 15.13. Let X be a projective variety. We say that $r \in \bar{k}(X)$ is *regular* at a point $P \in X$ if it has a representation $\hat{r}$ that is defined at P. The set of points $P \in X$ at which r is regular form an open subset of X that we denote $\operatorname{dom}(r)$.[4] For any point $P \in \operatorname{dom}(r)$ we define $r(P) = \hat{r}(P)$, where $\hat{r}$ is chosen so that $\hat{r}$ is defined at P.

Definition 15.14. Let $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ be projective varieties. A *rational map* $\phi\colon X \to Y$ is an equivalence class of tuples $\phi = (\phi_0 : \ldots : \phi_n)$ with $\phi_i \in \bar{k}(X)$ not all zero such that at any point $P \in X$ where all the ϕ_i are regular and at least one is nonzero, the point $(\phi_0(P) : \ldots : \phi_n(P))$ lies in Y. The equivalence relation is given by

$$(\phi_0 : \ldots : \phi_n) = (\lambda\phi_0 : \ldots : \lambda\phi_n)$$

for any $\lambda \in \bar{k}(X)^\times$. We say that ϕ is *regular* at P if there is a tuple $(\lambda\phi_0 : \ldots : \lambda\phi_n)$ in its class with each component regular at P and at least one nonzero at P. The open subset of X at which ϕ is regular is denoted $\operatorname{dom}(\phi)$.

Remark 15.15. We can alternatively represent the rational map $\phi\colon X \to Y$ as a tuple of homogeneous polynomials in $\bar{k}[x_0, \ldots, x_m]$ that all have the same degree and not all of which lie in $I(X)$. To ensure that the image lies in Y one requires that for all $f \in I(Y)$ we have $f(\phi_0, \ldots, \phi_n) \in I(X)$. The equivalence relation is then $(\phi_0 : \ldots : \phi_n) = (\psi_0 : \ldots : \psi_n)$ if and only if $\phi_i\psi_j - \phi_j\psi_i \in I(X)$ for all $i.j$.

Remark 15.16. One can also define rational maps $X \to Y$ where one of X, Y is an affine variety and the other is a projective variety. When Y is projective the definition is exactly the same as in the case that both are projective (but we don't use homegenized functions to represent elements of $\bar{k}(X)$ when X is affine). When X is projective and Y is affine, a rational map is no longer an equivalence class of tuples, it is a particular tuple of rational functions on X. Of course there is still a choice of representation for each rational function (and the choice may vary with P), but note that in this case Remark 15.15 *no longer applies.*

Now that we have defined rational maps for projective varieties we can define morphisms and dominant rational maps; the definitions are exactly the same as in the affine case, so we can now state them generically.

Definition 15.17. A *morphism* is a regular rational map. A rational map is *dominant* if its image is dense in its codomain.

The analogs of Theorem 15.8 and Corollary 15.9 both apply to dominant rational maps between projective varieties. The proofs are exactly the same, modulo the equivalence relations for projective points and rational maps between projective varieties. Alternatively, one can simply note that any dominant rational map $X \to Y$ of projective varieties restricts to a dominant rational map between any pair of the nonempty affine parts of X and Y (the nonempty affine parts of X (resp. Y) are all dense in X (resp. Y), and they are all isomorphic as affine varieties; see Corollary 14.10). Conversely, any dominant rational map of affine varieties can be extended to a dominant rational map of their projective closures (but this is *not* true of morphisms; see Example 15.20 below).

Theorem 15.18. *Theorem 15.8 and Corollary 15.9 hold for projective varieties as well as affine varieties.*

[4] It is clear that $\operatorname{dom}(r)$ is open in X; its intersection with each affine patch is an open subset of X.

Let us now look at a couple of examples.

Example 15.19. Let $X \subseteq \mathbb{A}^2$ be the affine variety defined by $x^2 + y^2 = 1$, and let P be the point $(-1, 0) \in X$. The rational map $\phi\colon X \to \mathbb{A}^1$ defined by

$$\phi(x, y) = \left(\frac{y}{x+1}\right) = \left(\frac{1-x}{y}\right)$$

sends each point $Q = (x, y) \in X$ different from P to the slope of the line $\overline{PQ}$. The map ϕ is not regular (hence not a morphism), because it is not regular at P, but it is dominant (even surjective). The rational map $\phi^{-1}\colon \mathbb{A}^1 \to X$ defined by

$$\phi^{-1}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

is an inverse to ϕ. Note that ϕ^{-1} is also not regular (it is not defined at $\sqrt{-1}$), but it is dominant (but not surjective). Thus X is birationally equivalent to $\mathbb{A}^1$, but not isomorphic to $\mathbb{A}^1$, as expected. The function fields of X and $\mathbb{A}^1$ are both isomorphic to $\bar{k}(t)$.

Now let us consider the corresponding map of projective varieties. The projective closure $\overline{X}$ of X in $\mathbb{P}^2$ is defined by $x^2 + y^2 = z^2$. We now define the rational map $\varphi\colon \overline{X} \to \mathbb{P}^1$ by

$$\varphi(x : y : z) = \left(\frac{y}{x+z} : 1\right) = \left(\frac{z-x}{y} : 1\right) = \left(1 : \frac{y}{z-x}\right)$$

Per Remark 15.15, we could also write φ as

$$\varphi(x : y : z) = (y : x + z) = (z - x : y)$$

The first RHS is defined everywhere except $(1 : 0 : -1)$ and the second RHS is defined everywhere except $(1 : 0 : 1)$, thus φ is regular everywhere, hence a morphism.

We also have the rational map $\varphi\colon \mathbb{P}^1 \to \overline{X}$ defined by

$$\varphi^{-1}(s : t) = \left(\frac{s^2 - t^2}{s^2 + t^2} : \frac{2st}{s^2 + t^2} : 1\right) = \left(1 : \frac{2st}{s^2 - t^2} : \frac{s^2 + t^2}{s^2 - t^2}\right)$$

which can also be written as

$$\varphi^{-1}(s : t) = (s^2 - t^2 : 2st : s^2 + t^2).$$

The map φ^{-1} is regular everywhere, hence a morphism, and the compositions $\varphi \circ \varphi^{-1}$ and $\varphi^{-1} \circ \varphi$ are both the identity maps, thus $\overline{X}$ and $\mathbb{P}^1$ are isomorphic.

Example 15.20. Recall the morphism $\phi\colon \mathbb{A}^2 \to \mathbb{A}^2$ defined by $\phi(x, y) = (x, xy)$ from Lecture 14, where we noted that the image of ϕ is not closed (but it is dense in $\mathbb{A}^2$, so ϕ is dominant). Let us now consider the corresponding rational map $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2$ defined by

$$\varphi(x : y : z) = \left(\frac{x}{z} : \frac{xy}{z^2} : 1\right) = (xz : xy : z^2).$$

We might expect φ to be a morphism, but this is not the case! It is not regular at $(0 : 1 : 0)$.

This is not an accident. As we will see in the next lecture, morphisms of projective varieties are *proper*, and in particular this means that they are closed maps (so unlike the affine case, the image of a morphism of projective varieties *is* a variety). But there is clearly no way to extend the morphism $\phi\colon \mathbb{A}^2 \to \mathbb{A}^2$ to a proper morphism $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2$ (the image of φ in the affine patch $z \neq 0$ must be dense but not surjective), and this means that φ cannot be a morphism.