18.782 Introduction to Arithmetic Geometry **Fall 2013**
Problem Set #9 **Due: 11/19/2013**

These problems are related to the material covered in Lectures 19-20. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by the start of class on 11/19/2013 and should be submitted electronically as a pdf-file e-mailed to drew@math.mit.edu. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective variety of dimension one.

Problem 1. Divisor identities (50 points)

Let k be a perfect (but not necessarily algebraically closed) field. Let $\phi\colon C_1 \to C_2$ be a morphism of curves defined over k, and let $\phi^*\colon k(C_2) \to k(C_1)$ be the corresponding embedding of function fields. We also use ϕ^* to denote the pullback map $\operatorname{Div}_k C_2 \to \operatorname{Div}_k C_1$, and ϕ_* denotes the pushforward map $\operatorname{Div}_k C_1 \to \operatorname{Div}_k C_2$, as defined in Lecture 19.

(a) Prove that $\phi_* \circ \phi^*$ acts as multiplication by $\deg\phi$ on $\operatorname{Div}_k C_2$.

(b) Show that $\phi^* \circ \phi_*$ is not multiplication by $\deg\phi$ on $\operatorname{Div}_k C_1$, in general, but that we do have

$$\deg(\phi^*\phi_* D) = (\deg\phi)(\deg D)$$

for all $D \in \operatorname{Div}_k C_1$.

(c) Prove that $\operatorname{ord}_P(\phi^* f) = e_\phi(P)\operatorname{ord}_{\phi(P)}(f)$ for all closed points P of C_1/k and functions f in $k(C_2)^\times$. Then use this to show $\phi^*(\operatorname{div} f) = \operatorname{div}(\phi^* f)$ for all $f \in k(C_2)^\times$.

(d) For any curve C/k and $f \in k(C)^\times$ and $D \in \operatorname{Div}_k(C)$ such that $\operatorname{div} f$ and $D = \sum n_P P$ have disjoint supports, define

$$f(D) = \prod f(P)^{n_P \deg P},$$

where P ranges over the closed points of C/k. Prove that $f(\phi_* D) = (\phi^* f)(D)$ for all $f \in k(C_2)^\times$ and $D \in \operatorname{Div}_k(C_1)$ where both sides are defined.

(e) Prove that the sequence

$$1 \to k^\times \to k(\mathbb{P}^1)^\times \xrightarrow{\operatorname{div}} \operatorname{Div}_k \mathbb{P}^1 \xrightarrow{\deg} \mathbb{Z} \to 0$$

is exact.

Problem 2. Weil reciprocity (50 points)

Let C/k be a curve over an algebraically closed field k. For $f \in k(C)^\times$ and $D \in \operatorname{Div}_k C$ such that $\operatorname{div} f$ and D have disjoint support, define $f(D)$ as in part (d) of Problem 1. Your goal in this problem is to prove the *Weil reciprocity law*, which states that for all $f, g \in k(C)^\times$ such that $\operatorname{div} f$ and $\operatorname{div} g$ have disjoint support,

$$f(\operatorname{div} g) = g(\operatorname{div} f).$$

(a) Prove that for $C = \mathbb{P}^1$ we have

$$\prod_P (-1)^{\mathrm{ord}_P(g)\,\mathrm{ord}_P(f)} \left(\frac{f^{\mathrm{ord}_P(g)}}{g^{\mathrm{ord}_P(f)}}\right)(P) = 1,$$

for all $f, g \in k(C)^\times$, whether or not $\operatorname{div} f$ and $\operatorname{div} g$ have disjoint support.

(b) Use (a) to prove the Weil reciprocity law in the case $C = \mathbb{P}^1$.

Associated to any morphism of curves $\phi\colon C_1 \to C_2$ we have defined three maps

$$\begin{array}{ll} \phi^*\colon k(C_2) \to k(C_1) & \phi^*\colon \operatorname{Div}_k C_2 \to \operatorname{Div}_k C_1 \\ ? & \phi_*\colon \operatorname{Div}_k C_1 \to \operatorname{Div}_k C_2 \end{array}$$

It is natural to ask whether there is a fourth map $\phi_*\colon k(C_1) \to k(C_2)$ that completes the table above, and indeed there is. It is defined by

$$f \mapsto (\phi^*)^{-1} N(f),$$

where $N\colon k(C_1) \to \phi^*(k(C_2))$ is the norm map associated to the extension $k(C_1)/\phi^*(k(C_2))$.[1] It is obviously not a morphism of fields (unless ϕ is an isomorphism), but it is a morphism of their multiplicative groups.

We then have an identities analogous to those proved in (c) and (d) of Problem 1:

$$\phi_*(\operatorname{div} f) = \operatorname{div}(\phi_* f), \tag{1}$$

for all $f \in k(C_1)^\times$, and

$$f(\phi^* D) = (\phi_* f)(D), \tag{2}$$

for all $f \in k(C_1)^\times$ and $D \in \operatorname{Div}_k C_2$ where both sides are defined.

We now turn to the general case of the Weil reciprocity law, where f and g are elements of $k(C)$ with disjoint support, where C is now any curve. Your objective is to prove Weil's reciprocity law in the general case using Weil reciprocity for $\mathbb{P}^1$, which you proved in (b).

As noted in lecture, any function in $k(C)$ defines a morphism to $\mathbb{P}^1$, so we may view both f and g as morphisms, and we then have associated pullback maps f^*, g^* and pushforward maps f_*, g_* that we can apply both to the divisor groups and function fields of C and $\mathbb{P}^1$. We also need the identity morphism $i\colon \mathbb{P}^1 \to \mathbb{P}^1$.

(c) Prove that $\operatorname{div} g = g^* \operatorname{div} i$ for any $g \in k(C)^\times$.

(d) Use (b) and (c), parts (c) and (d) of Problem 1, and identities (1) and (2) to prove the Weil reciprocity law in the general case.

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

[1] Recall from Lecture 7 that the norm map of a finite extension L/K sends an element of L with minimal polynomial $F \in K[x]$ to $(-1)^{[L:K]}F(0)$ which is an element of K.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast"), and the novelty of the material (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
11/7	Smooth projective curves				
11/12	Divisors, closed points				
11/14	Ramification, Picard group				

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.