Description

These problems are related to the material covered in Lectures 4-5. I have made every effort to proof-read them, but there are may still be errors that I have missed. The first person to spot each error will receive 1-5 points of extra credit on their problem set, depending on the severity of the error.

The problem set is due before the start of class (2:30 pm) on 09/24/2013 and should be submitted electronically as a pdf-file e-mailed to drew@math.mit.edu. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions (you can just replace the due date in the header with your name). Remember to identify any collaborators or sources that you consulted that are not listed in the syllabus.

Problem 1. The Chevalley-Warning theorem. (35 points)

A C_1 *field* is a field k in which every homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of nonzero degree $d < n$ has a nontrivial zero (equivalently, the projective variety $f(x_1, \ldots, x_n) = 0$ has a rational point). The Chevalley-Warning theorem states that finite fields are C_1 fields.

(a) For each nonnegative integer n and finite field $\mathbb{F}_q$, determine the value of $\sum_{x\in\mathbb{F}_q} x^n$. (Note that $0^0 = 1$).

(b) Prove that $\mathbb{F}_q$ is a C_1 field. (Hint: consider $\sum_{a_1,\ldots,a_n\in\mathbb{F}_q} \left(1 - f(a_1, \ldots, a_n)^{q-1}\right)$).

(c) Prove more generally that if $f_1, \ldots, f_m \in \mathbb{F}_q[x_1, \ldots, x_n]$ are homogeneous polynomials for which $\deg f_1 + \cdots \deg f_m < n$, then the system of equations

$$\begin{aligned} f_1(a_1, \ldots, a_n) &= 0 \\ f_2(a_1, \ldots, a_n) &= 0 \\ &\vdots \\ f_m(a_1, \ldots, a_n) &= 0 \end{aligned}$$

has a nonzero solution.

Problem 2. Nonarchimedean absolute values. (30 points)

Let $\|\cdot\|$ be a nonarchimedean absolute value on a field k. Define

$$\begin{aligned} R &= \{x \in k : \|x\| \le 1\}, \\ \mathfrak{m} &= \{x \in k : \|x\| < 1\}. \end{aligned}$$

Prove each of the following statements.

(a) R is a subring of k.

(b) $\mathfrak{m}$ is an ideal of R.

(c) $R - \mathfrak{m}$ is the unit group $R^\times$ of R.

(d) $\mathfrak{m}$ is a maximal ideal of R.

(e) $\mathfrak{m}$ is the only maximal ideal of R.

A commutative ring R with a unique maximal ideal $\mathfrak{m}$ is called a *local ring.*[1] The field $R/\mathfrak{m}$ is the *residue field* of R. Now let p be a prime and define

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : p \nmid b \right\}.$$

(f) Prove that $\mathbb{Z}_{(p)}$ is a local ring and describe its unique maximal ideal $\mathfrak{m}$.

(g) Determine the residue field of $\mathbb{Z}_{(p)}$.

Problem 3. The ring of p-adic integers $\mathbb{Z}_p$ (35 points)

(a) Prove that $\mathbb{Z}_p$ is a Euclidean domain.

(b) Prove that $\mathbb{Z}_p$ and $\mathbb{R}$ have the same cardinality.

(c) Prove that $\mathbb{Z}_p \bigcap \mathbb{Q} = \mathbb{Z}_{(p)}$ (take the intersection in $\mathbb{Q}_p$).

(d) Show that $\mathbb{Z}_{(p)}$ is the set of elements in $\mathbb{Z}_p$ whose p-adic expansion eventually repeats.

(e) Characterize the set of elements in $\mathbb{Z}_p$ whose p-adic expansion is eventually constant. Is it ring? If not, is it closed under either addition or multiplication?

Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), and the novelty of the material (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Novelty
9/17	p-adic integers			
9/19	p-adic numbers, Ostrowski's theorem			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.

[1] Discrete valuation rings are local rings that are also principal ideal domains and not fields. One can define non-commutative local rings, but we will not use these in this course.