

As usual, k is a perfect field, but not necessarily algebraically closed. Throughout this lecture C/k denotes a curve (smooth projective variety of dimension one) and F/k the corresponding function field. To simplify the notation, for any place P of F/k and divisor $D = \sum n_P P$, we define $\text{ord}_P(D) = n_P$.

21.1 Riemann-Roch spaces

We have seen that the degree of a divisor is a key numerical invariant that is preserved under linear equivalence; recall that two divisors are linearly equivalent if their difference is a principal divisor, equivalently, they correspond to the same element of the Picard group. We now want to introduce a second numerical invariant associated to each divisor. In order to do this we first partially order divisors by defining the relation \leq on $\text{Div}_k C$ by

$$A \leq B \iff \text{ord}_P(A) \leq \text{ord}_P(B) \text{ for all } P.$$

As usual, P ranges over all closed points of C/k , equivalently, all places of $k(C)$, but of course the inequality on the right is automatically satisfied for all but finitely many P . This partial ordering is compatible with divisor addition, since

$$A \leq B \implies A + C \leq B + C,$$

for any divisor C . We also note that

$$A \leq B \text{ and } C \leq D \implies A + C \leq B + D.$$

It is important to remember that \leq is not a total ordering on $\text{Div}_k C$; most pairs of divisors are incomparable.

Definition 21.1. A divisor $D \geq 0$ is said to be *effective*. As with principal divisors $\text{div } f = \text{div}_0 f - \text{div}_\infty f$, every divisor can be written uniquely as the difference of two effective divisors, as $D = D_0 - D_\infty$, where

$$D_0 := \sum_{\text{ord}_P(D) > 0} \text{ord}_P(D) P \quad \text{and} \quad D_\infty := \sum_{\text{ord}_P(D) < 0} -\text{ord}_P(D) P.$$

We now define the Riemann-Roch space of a divisor.

Definition 21.2. The *Riemann-Roch* space of a divisor D is the k -vector space

$$\mathcal{L}(D) := \{f \in k(C)^\times : \text{div } f \geq -D\} \cup \{0\}.$$

That $\mathcal{L}(D)$ is a vector space follows immediately from:

1. $\text{div } \lambda f = \text{div } f + \text{div } \lambda = \text{div } f$ for all $\lambda \in k^\times$;
2. $\text{ord}_P(f + g) \geq \min(\text{ord}_P(f), \text{ord}_P(g))$ for all $f, g \in F^\times$.

Example 21.3. If $D = 3P - 2Q$ then $\mathcal{L}(D)$ is the set of functions in $k(C)$ that have at most a triple pole at P , and at least a double zero at Q , and poles nowhere else (but they may have zeros of any order at points other than Q).

Example 21.4. If $D = -P$ then $\mathcal{L}(D)$ is the set of functions that have a zero at P and no poles at all. The only such function is the zero function (which lies in $\mathcal{L}(D)$ by definition). More generally, for any $D < 0$ we have $\mathcal{L}(D) = \{0\}$.

Example 21.5. If $D = 0$ then $\mathcal{L}(D)$ is the set of functions that have no poles at all. By Corollary 19.23, for $f \in \mathcal{L}(0)$ we have $\deg \operatorname{div}_\infty f = 0$ if and only if $f \in k^\times$, so $\mathcal{L}(0) = k$.

We now show that $\mathcal{L}(D)$ is preserved (up to isomorphism) by linear equivalence.

Lemma 21.6. *For any linearly equivalent divisors $A \sim B$ we have $\mathcal{L}(A) \simeq \mathcal{L}(B)$.*

Proof. We have $A - B = \operatorname{div} f$ for some $f \in k(C)^\times$, and we claim that the maps $g \mapsto fg$ and $g \mapsto g/f$ are inverse k -linear maps from $\mathcal{L}(A)$ to $\mathcal{L}(B)$ and from $\mathcal{L}(B)$ to $\mathcal{L}(A)$, respectively. Linearity is clear, and if $\operatorname{div} g \geq -A$ then

$$\operatorname{div} fg = \operatorname{div} f + \operatorname{div} g \geq \operatorname{div} f - A = -B.$$

Similarly, if $\operatorname{div} g \geq -B$ then

$$\operatorname{div} g/f = \operatorname{div} g - \operatorname{div} f \geq -B - \operatorname{div} f = -A.$$

Thus we have defined linear transformations from $\mathcal{L}(A)$ to $\mathcal{L}(B)$, hence $\mathcal{L}(A) \simeq \mathcal{L}(B)$. \square

The following lemma shows that non-trivial Riemann-Roch spaces arise only (and always) for divisors that are linearly equivalent to an effective divisor.

Lemma 21.7. *We have $\mathcal{L}(D) \neq \{0\}$ if and only if $D \sim D'$ for some $D' \geq 0$.*

Proof. If $f \in \mathcal{L}(D)$ is nonzero, then $\operatorname{div} f \geq -D$, and $D \sim D' = D + \operatorname{div} f \geq 0$. Conversely, if $D \sim D' \geq 0$ then $-D \leq D' - D = \operatorname{div} f$ for some $f \in k(C)^\times$, hence $\mathcal{L}(D) \neq \{0\}$. \square

Lemma 21.8. *For any two divisors $A \leq B$ we have $\mathcal{L}(A) \subseteq \mathcal{L}(B)$ and*

$$\dim(\mathcal{L}(B)/\mathcal{L}(A)) \leq \deg B - \deg A.$$

Proof. It is clear that $\mathcal{L}(A) \subseteq \mathcal{L}(B)$, and that the inequality holds if $A = B$. We now prove the inequality in the case $B = A + P$, for some place P . Let t be a uniformizer at P , let $k(P) = \mathcal{O}_P/P$ be the residue field of P , and let $n = \operatorname{ord}_P(B)$. Now define the linear transformation $\phi: \mathcal{L}(B) \rightarrow k(P)$ by $\phi(f) = (t^n f)(P) = t^n f \bmod P$; we have

$$\operatorname{ord}_P(t^n f) = n + \operatorname{ord}_P(f) \geq 0$$

for $f \in \mathcal{L}(B)$, so $t^n f \in \mathcal{O}_P$ and ϕ is well-defined. The image of ϕ lies in $k(P) = k^{\deg P}$, and its kernel consists of subspace of functions $f \in \mathcal{L}(B)$ for which $\operatorname{ord}_P(t^n f) \geq 1$, equivalently, $\operatorname{ord}_P(f) \geq 1 - n = -\operatorname{ord}_P(A)$, which is precisely $\mathcal{L}(A)$. We have $\mathcal{L}(B)/\ker \phi \simeq \operatorname{im} \phi$, so

$$\dim(\mathcal{L}(B)/\mathcal{L}(A)) = \dim \operatorname{im} \phi \leq \dim k(P) = \deg P = \deg B - \deg A. \quad (1)$$

The general case follows from repeated application of the same result. If

$$A = B_0 < B_1 < B_2 < \cdots < B_m = B,$$

where $B = \sum n_P P$ and $m = \sum n_P$, then each difference $B_{i+1} - B_i$ is a single place P_i . Applying (1) gives $\dim(\mathcal{L}(B_{i+1})/\mathcal{L}(B_i)) = \deg B_{i+1} - \deg B_i = \deg P_i$. Summing yields the desired result $\dim(\mathcal{L}(B)/\mathcal{L}(A)) \leq \deg B - \deg A$. \square

We now prove that the dimension of a Riemann-Roch space is finite.

Theorem 21.9. *For any divisor D we have $\dim \mathcal{L}(D) \leq \deg D_0 + 1$.*

Proof. Applying Lemma 21.8 with $B = D$ and $A = 0$ yields

$$\dim(\mathcal{L}(D_0)/\mathcal{L}(0)) \leq \deg D_0 - \deg 0 = \deg D_0.$$

As noted in Example 21.5, we have $\mathcal{L}(0) = k$, and therefore

$$\dim \mathcal{L}(D_0) = \dim(\mathcal{L}(D_0)/\mathcal{L}(0)) + 1 \leq \deg D_0 + 1.$$

We also have $D \leq D_0$, so by Lemma 21.8, $\mathcal{L}(D) \subseteq \mathcal{L}(D_0)$, and we have

$$\dim \mathcal{L}(D) \leq \dim \mathcal{L}(D_0) \leq \deg D_0 + 1$$

as claimed. □

Definition 21.10. The *dimension* $\ell(D)$ of a divisor is the dimension of $\mathcal{L}(D)$.

The following corollary summarizes what we know about $\ell(D)$ so far.

Corollary 21.11. *The following hold:*

- (a) $\ell(0) = 1$.
- (b) If $A \sim B$ then $\ell(A) = \ell(B)$ and $\deg(A) = \deg(B)$.
- (c) For any $A \leq B$ we have $\ell(B) - \ell(A) \leq \deg B - \deg A$.
- (d) For all $D \geq 0$ we have $\ell(D) \leq \deg D + 1$.
- (e) If $\deg D < 0$ then $\ell(D) = 0$.

Proof. (a) follows from Example 21.5, (b) is Lemma 21.6 and Corollary 19.23, (c) is Lemma 21.6, (d) is Theorem 21.9, and (e) follows from Lemma 21.7. □

An equivalent form of (c) that we will often use is

$$A \leq B \implies \deg A - \ell(A) \leq \deg B - \ell(B).$$

Lemma 21.12. *If $\deg D = 0$ then $\ell(D) = 1$ if D is principal and $\ell(D) = 0$ otherwise.*

Proof. If $D = \operatorname{div} f$ is principal, then $f \in \mathcal{L}(D)$, so $\ell(D) \geq 1$ and by Lemma 21.7 we must have $D \sim D' \geq 0$. But $\deg D' = \deg D = 0$, so $D' = 0$ and $\ell(D) = \ell(0) = 1$. Now suppose $\ell(D) \geq 1$. As just argued, we must have $\ell(D) = 1$, so there is a nonzero $f \in \mathcal{L}(D)$, and since $\operatorname{div} f \geq -D$, we have $D + \operatorname{div} f \geq 0$. But $\deg(D + \operatorname{div} f) = 0$, so $D + \operatorname{div} f = 0$ and therefore $D = -\operatorname{div} f = \operatorname{div} 1/f$ is principal. Taking the contrapositive, if D is not principal then we must have $\ell(D) = 0$. □

It follows from Lemma 21.12 that the inequality in Theorem 21.9 is not tight for curves for which $\operatorname{Pic}_k^0 C$ is not trivial, since this implies the existence of non-principal divisors of degree 0. On the other hand, for $C = \mathbb{P}^1$, the inequality is tight for all effective divisors (as noted at the end of Lecture 20, there is a gap between these two cases, one can have $\operatorname{Pic}_k^0 C = \{0\}$ and $C \not\cong \mathbb{P}^1$; we will address this gap in the next lecture).

Lemma 21.13. *If C is isomorphic to \mathbb{P}^1 then $\ell(D) = \deg D + 1$ for all $D \geq 0$.*

Proof. If $C \simeq \mathbb{P}^1$ then $k(C)$ is the field of all rational functions over k . We claim that given any effective divisor $A = \sum n_P P$ we can construct a function $f_A \in k(C)$ with $\text{div}_\infty f = A$.

Proof of claim: We just need to show that we can construct $\text{div}_0 f$ with support disjoint from $\text{div}_\infty f$. If k is infinite this is easy: pick a degree one place $P \notin \text{Supp}(A)$ and let $\text{div}_0 f = (\deg A)P$. If k is finite, then, as noted in Lecture 3, there exist monic irreducible polynomials of every degree in $k[t]$, and each corresponds to a place of $k(C)$. If A consists of more than a single place, no place of degree $\deg A$ can lie in the support of A , so pick one such place P and let $\text{div}_0 f = P$. Otherwise A consists of a single place and we can pick a degree one place P not in the support of A and let $\text{div}_0 f = (\deg A)P$ as above.

Now let $0 = A_0 < A_1 < \dots < A_m = D$ be a maximal chain of divisors, let $P_i = A_i - A_{i-1}$ for $1 \leq i \leq m$, and let $f_1, \dots, f_m \in k(C)$ satisfy $\text{div}_\infty f_i = A_i$ (note that the list P_1, \dots, P_m may contain repetitions). These functions are linearly independent over k , since for any nonempty subset the f_i with maximal index i has a pole at P_i of order greater than that of any f_j with $j < i$, and the triangle equality then precludes any non-trivial relations. Finally, for each point P_i there is a subspace $V_i \subseteq \mathcal{L}(D)$ corresponding to functions f for which $\text{div} f = \text{div} f_i$, and $\mathcal{L}(D)$ contains the direct sum of these subspaces, since no pair intersects non-trivially. If we consider the linear transformation $\phi: V_i \rightarrow k(P_i)$ defined by $f \mapsto (t_i^{n_i} f)(P_i)$, where t_i is a uniformizer for P_i and $n_i = -\text{ord}_{P_i}(f)$, it is clear that $\ker \phi$ is trivial, and ϕ is surjective because $k(C)$ is the rational function field. So $\dim V_i = \deg P_i$.

We then have

$$\ell(D) = \dim \mathcal{L}(D) \geq \dim \mathcal{L}(0) + \sum \dim V_i = 1 + \sum \deg P_i = 1 + \deg D,$$

as claimed. □

Remark 21.14. If you think the proof of Lemma 21.13 is a lot of effort to prove something that should be obvious, you are right. Once we prove the Riemann-Roch theorem it will follow trivially (as will many other results). Our purpose in proving it now is to help motivate the definition of genus.

We know that the inequality in Theorem 21.9 is tight when C is *rational* (isomorphic to \mathbb{P}^1), but not in general. As we will show, for suitable divisors D (which will turn out to be almost all of them), the quantity $\deg D + 1 - \ell(D)$ tells us something intrinsic to the function field $k(C)$; roughly speaking, it measures how far C is from being rational.¹ One way to think about this metric is as a measure of the functions that are “missing” from $k(C)$.

We now show that the $\deg D + 1 - \ell(D)$ is bounded, independent of D .

Theorem 21.15. *There is a non-negative integer g such that*

$$\deg(D) + 1 - \ell(D) \leq g$$

holds for all $D \in \text{Div}_k C$.

The proof below is adapted from [1, Prop. 1.4.14].

Proof. Let $f \in k(C)$ be transcendental over k , and let $A = \text{div}_\infty f \geq 0$. Let v_1, \dots, v_d be a basis for $k(C)/k(f)$, where $d = \deg A = [k(C) : k(f)]$ (by Corollary 19.24). Choose a divisor $B \geq 0$ so that $\text{div} v_i \geq -B$ for each v_i (this is clearly possible).

¹Modulo annoying special cases like genus 0 curves that are not rational (and genus 1 curves that are not elliptic curves). Such annoyances can be eliminated by insisting on at least one rational point.

For any integer $n \geq 0$, the set of functions $S = \{v_i f^j : 1 \leq i \leq d, 0 \leq j \leq n\}$ is clearly linearly independent over k , since v_1, \dots, v_d are linearly independent over $k(f)$ and f is transcendental over k . And $S \subseteq \mathcal{L}(nA + B)$, since $\text{div}(v_i f^j) \geq -nA - B$ for all $v_i f^j \in S$. Therefore

$$\ell(nA + B) \geq d(n + 1) = (n + 1) \deg A \quad (2)$$

for all $n \geq 0$. But we also have $nA \leq nA + B$, since $B \geq 0$, and Corollary 21.11.c implies

$$\ell(nA + B) - \ell(nA) \leq \deg(nA + B) - \deg(nA) = \deg B. \quad (3)$$

Combining (2) and (3) yields

$$\ell(nA) \geq \ell(nA + B) - \deg B \geq (n + 1) \deg A - \deg B = \deg(nA) + (\deg A - \deg B).$$

It follows that

$$\deg(nA) + 1 - \ell(nA) \leq \deg A - \deg B + 1$$

for all $n \geq 0$. Let $g = \deg A - \deg B + 1$ so that

$$\deg(nA) + 1 - \ell(nA) \leq g, \quad (4)$$

where we note that $g \geq 0$, by Corollary 21.11.d.

Now let D be any divisor in $\text{Div}_k C$ and write $D = D_0 - D_\infty$ as the difference of two effective divisors. We claim that D_0 is equivalent to an effective divisor $D' \leq nA$, for some n . By Corollary 21.11.c, we have

$$\ell(nA) - \ell(nA - D_0) \leq \deg(nA) - \deg(nA - D_0) = \deg D_0,$$

and applying (4) yields

$$\ell(nA - D_0) \geq \ell(nA) - \deg D_0 \geq \deg(nA) + 1 - g + \deg D_0.$$

The RHS is clearly positive for sufficiently large n , so pick n so that $\ell(nA - D_0) > 0$ and let $f \in \mathcal{L}(nA - D_0)$ be nonzero. Now define $D' := D_0 - \text{div } f$ so that

$$D' = D_0 - \text{div } f \leq D_0 - (D_0 - nA) = nA,$$

as claimed. We have $D \leq D_0$, so $\ell(D_0) - \ell(D) \leq \deg D_0 - \deg D$, by Corollary 21.11.c, and

$$\begin{aligned} \deg D + 1 - \ell(D) &\leq \deg D_0 + 1 - \ell(D_0) \\ &= \deg D' + 1 - \ell(D') \\ &\leq \deg(nA) + 1 - \ell(nA) \\ &\leq g, \end{aligned}$$

where we used $D' \sim D_0$ equality, $D' \leq nA$ to get the second inequality, and then (4). \square

References

- [1] H. Stichtenoth, *Algebraic function fields and codes*, Springer, 2009.