As usual, all the rings we consider are commutative rings with an identity element.

18.1   Regular local rings

Consider a local ring R with unique maximal ideal $\mathfrak{m}$. The ideal $\mathfrak{m}$ is, in particular, an abelian group, and it contains $\mathfrak{m}^2$ as a normal subgroup, so we can consider the quotient group $\mathfrak{m}/\mathfrak{m}^2$, where the group operation is addition of cosets:

$$(m_1 + \mathfrak{m}^2) + (m_2 + \mathfrak{m}^2) = (m_1 + m_2) + \mathfrak{m}^2.$$

But $\mathfrak{m}$ is also an ideal, so it is closed under multiplication by R, and it is a maximal ideal, so $R/\mathfrak{m}$ is a field (the residue field). The quotient group $\mathfrak{m}/\mathfrak{m}^2$ has a natural structure as an $(R/\mathfrak{m})$-vector space. Scalars are cosets $r + \mathfrak{m}$ in the field $R/\mathfrak{m}$, and scalar multiplication is defined by

$$(r + \mathfrak{m})(m + \mathfrak{m}^2) = rm + \mathfrak{m}^2.$$

In practice one often doesn't write out the cosets explicitly (especially for elements of the residue field), but it is important to keep the underlying definitions in mind; they are a valuable compass if you ever start to feel lost.

The motivation for this discussion is the case where R is the local ring $\mathcal{O}_P$ of regular functions at a point P on a variety V. In this setting $\mathfrak{m}/\mathfrak{m}^2$ is precisely the vector space m_P/m_P^2 that is isomorphic to the $T_P^\vee$, the dual of the tangent space at P; recall from the previous lecture that P is a smooth point of V if and only if $\dim m_P/m_P^2 = \dim V$. We now give an algebraic characterization of this situation that does not involve varieties. We write $\dim \mathfrak{m}/\mathfrak{m}^2$ to indicate the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as an $(R/\mathfrak{m})$-vector space, and we write $\dim R$ to denote the (Krull) dimension of the ring R.

Definition 18.1. A Noetherian local ring R with maximal ideal $\mathfrak{m}$ is a *regular local ring* if $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$ (note that Noetherian is included in the definition of regular).[1]

We are particularly interested in regular local rings of dimension 1, these correspond to rings $\mathcal{O}_P$ of regular functions at a smooth point P on a curve (a variety of dimension one).

Theorem 18.2. *A ring R is a regular local ring of dimension one if and only if it is a discrete valuation ring.*

Proof. We prove the easier direction first. Let R be a discrete valuation ring (DVR) with maximal ideal $\mathfrak{m} = (t)$. Then R is a local ring, and it is certainly Noetherian, since it is a principal ideal domain (PID). Its prime ideals are (0) and (t), so it has dimension 1, and $t + \mathfrak{m}^2$ generates $\mathfrak{m}/\mathfrak{m}^2$, so $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$. Thus R is a regular local ring of dimension 1.

Let R be a regular local ring of dimension one. Its unique maximal ideal $\mathfrak{m}$ is not equal to $\mathfrak{m}^2$, since $\dim \mathfrak{m}/\mathfrak{m}^2 = 1 > 0$; in particular, $\mathfrak{m} \neq (0)$ and R is not a field. Let $t \in \mathfrak{m} - \mathfrak{m}^2$. Then $t + \mathfrak{m}^2$ generates $\mathfrak{m}/\mathfrak{m}^2$, since $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$. By Corollary 18.4 (proved below) of Nakayama's lemma, t generates $\mathfrak{m}$. So every $x \in R - (0)$ has the form $x = ut^n$, with $u \in R^\times$ and $n \in \mathbb{Z}_{\geq 0}$ (since R is a local ring with $\mathfrak{m} = (t)$), and every nonzero ideal is principal, of the form (t^n). It follows that the prime ideals in R are exactly (0) and (t), since R has dimension one. So $R = R/(0)$ is an integral domain, and therefore a PID, hence a DVR. $\square$

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[1] More generally, a Noetherian ring is *regular* if all of its localizations at prime ideals are regular.

To prove Corollary 18.4 used in the proof above we require a special case of what is known as Nakayama's lemma. The statement of the lemma may seem a bit strange at first, but it is surprisingly useful and has many applications.

Lemma 18.3 (Nakayama). *Let R be a local ring with maximal ideal $\mathfrak{m}$ and suppose that M is a finitely generated R-module with the property $M = \mathfrak{m}M$. Then M is the zero module.*

Proof. Let $b_1, \ldots, b_n$ be generators for M. By hypothesis, every b_i can be written in the form $b_i = \sum_j a_{ij} b_j$ with $a_{ij} \in \mathfrak{m}$. In matrix form we have $B = AB$, where $B = (b_1, \ldots, b_n)^t$ is a column vector and $A = (a_{ij})$ is an $n \times n$ matrix with entries in $\mathfrak{m}$. Equivalently, $(I - A)B = 0$, where I is the $n \times n$ identity matrix. The diagonal entries $1 - a_{ii}$ of $I - A$ are units, because $1 - a_{ii}$ cannot lie in $\mathfrak{m}$ (otherwise $1 \in \mathfrak{m}$, which is not the case), and every element of $R - \mathfrak{m}$ is a unit (since R is a local ring). However the off-diagonal entries of $I - A$ all lie in $\mathfrak{m}$. Expressing the determinant d of $I - A$ as a sum over permutations, it is clear that $d = 1 + a$ for some $a \in \mathfrak{m}$, hence d is a unit and $I - A$ is invertible. But then $(I - A)^{-1}(I - A)B = B = 0$, which means that M is the zero module. $\qquad \square$

Corollary 18.4. *Let R be a local Noetherian ring with maximal ideal $\mathfrak{m}$. Then $t_1, \ldots, t_n \in \mathfrak{m}$ generate $\mathfrak{m}$ if and only if their images generate $\mathfrak{m}/\mathfrak{m}^2$ as an $R/\mathfrak{m}$ vector space.*

Proof. The "only if" direction is clear. Let N be the ideal $(t_1, \ldots, t_n) \subseteq \mathfrak{m}$. If the images of $t_1, \ldots, t_n$ in $\mathfrak{m}/\mathfrak{m}^2$ generate $\mathfrak{m}/\mathfrak{m}^2$ as an $R/\mathfrak{m}$-vector space, then we have

$$N + \mathfrak{m}^2 = \mathfrak{m} + \mathfrak{m}^2$$
$$(N + \mathfrak{m}^2)/N = (\mathfrak{m} + \mathfrak{m}^2)/N$$
$$\mathfrak{m}(\mathfrak{m}/N) = \mathfrak{m}/N,$$

where we have used $N/N = 0$ and $\mathfrak{m} + \mathfrak{m}^2 = \mathfrak{m}$ (since $\mathfrak{m}^2 \subseteq \mathfrak{m}$). By Nakayama's lemma, $M = \mathfrak{m}/N$ is the zero module, so $\mathfrak{m} = N$ and $t_1, \ldots, t_n$ generate $\mathfrak{m}$. $\qquad \square$

18.2 Smooth projective curves

It follows from Theorem 18.2 that for a smooth curve C the local rings $\mathcal{O}_P = k[C]_{m_P}$ are all discrete valuation rings of $k(C)/k$. If C is a projective curve, then by Theorem 16.33 it is complete, and from the proof of Theorem 16.33 we know that it satisfies Chevalley's criterion: every valuation ring R of $k(C)/k$ contains a local ring $\mathcal{O}_P$. The fact that $\mathcal{O}_P$ is a discrete valuation ring actually forces $R = \mathcal{O}_P$; this is a consequence of the following theorem.

Theorem 18.5. *Let R_1 and R_2 be valuation rings with the same fraction field, let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be their respective maximal ideals, and suppose $R_1 \subsetneq R_2$. Then $\mathfrak{m}_2 \subsetneq \mathfrak{m}_1$ and $\dim R_2 < \dim R_1$. In particular, R_1 cannot be a discrete valuation ring.*

Proof. We first note $R_1 \subseteq R_2$ implies $R_1^\times \subseteq R_2^\times$. For $x \in R_2 - R_1$ we have $1/x \in R_1 \subseteq R_2$ and $x \in R_2^\times$, so $R_2 - R_1 \subseteq R_2^\times$. Thus $R_2 - R_2^\times \subseteq R_2 - R_1^\times = (R_2 - R_1) \sqcup (R_1 - R_1^\times)$, and this implies $\mathfrak{m}_2 = R_2 - R_2^\times \subseteq R_1 - R_1^\times = \mathfrak{m}_1$ since $R_2 - R_1$ and $R_2 - R_2^\times$ are disjoint. And for any $x \in R_2 - R_1$ we have $1/x \in R_2^\times = R_2 - \mathfrak{m}_2$ and $1/x \in R_1 - R_1^\times = \mathfrak{m}_1$, so $\mathfrak{m}_2 \subsetneq \mathfrak{m}_1$.

Every prime ideal of R_2 is contained in $\mathfrak{m}_2$, hence in $\mathfrak{m}_1$, and if $\mathfrak{p}$ is prime in R_2 then $\mathfrak{p} \cap R_1$ is clearly prime in R_1: if $ab \in \mathfrak{p}$ for some $a, b \in R_1 \subseteq R_2$ then one of a, b lies in $\mathfrak{p}$. Thus every chain of prime ideals in R_2 is also a chain of prime ideals in R_1, and in R_1 any

such chain can be extended by adding the prime ideal $\mathfrak{m}_1$. Thus $\dim R_2 < \dim R_1$. If R_1 is a DVR then $\dim R_2 < \dim R_1 = 1$, but $\dim R_2 \geq 1$, since R_2 is a valuation ring (not a field), therefore R_1 is not a DVR. □

Thus we have a one-to-one correspondence between the points on a smooth projective curve C and the discrete valuation rings of $k(C)/k$.

Theorem 18.6. *Let C be a smooth projective curve. Every rational map $\phi\colon C \to V$ from C to a projective variety V is a morphism.*

Proof. Let $\phi = (\phi_0 : \cdots : \phi_n)$ and consider any point $P \in C$. Let us pick a uniformizer t for the discrete valuation ring $\mathcal{O}_P$ (a generator for the maximal ideal m_P), and let

$$n = \min\{\operatorname{ord}_P(\phi_1), \ldots, \operatorname{ord}_P(\phi_n)\},$$

where $\operatorname{ord}_P\colon k(C) \to k(C)^\times/\mathcal{O}_P^\times \simeq \mathbb{Z}$ is the discrete valuation of $\mathcal{O}_P$. If $n = 0$ then ϕ is regular at P, since then all the ϕ_i are defined at P and at least one is a unit in $\mathcal{O}_P^\times$, hence nonzero at P. But in any case we have

$$\operatorname{ord}_P(t^{-n}\phi_i) = \operatorname{ord}_P(\phi_i) - n \geq 0$$

for $i = 0, \ldots, n$, with equality for at least one value of i. It follows that

$$\left(t^{-n}\phi_0 : \cdots : t^{-n}\phi_n\right) = (\phi_0 : \cdots : \phi_n)$$

is regular at P. This holds for every $P \in C$, so ϕ is a regular rational map, hence a morphism. □

Corollary 18.7. *Every rational map $\phi\colon C_1 \to C_2$ between smooth projective curves is either constant or surjective.*

Proof. Projective varieties are complete, so $\operatorname{im}(\phi)$ is a subvariety of C_2, and since $\dim C_2 = 1$ this is either a point (in which case ϕ is constant) or all of C_2. □

Corollary 18.8. *Every birational map between smooth projective curves is an isomorphism.*

It follows from Corollary 18.8 that if a curve C_1 is birationally equivalent to any smooth projective curve C_2, then all such C_2 are isomorphic. We want to show that such a C_2 always exists. Recall that birationally equivalent curves have isomorphic function fields. Thus it suffices to show that every function field of dimension one actually arises as the function field of a smooth projective curve.

18.3 Function fields as abstract curves

Let F/k be a function field of dimension one, where k is an algebraically closed field. We know that if F is the function field of a smooth projective curve C, then there is a one-to-one correspondence between the points of C and the discrete valuation rings of F. Our strategy is to define an *abstract curve* C_F whose "points" correspond to the discrete valuation rings of F, and then show that it is actually isomorphic to a smooth projective curve.

So let $X = X_F$ be the set of all maximal ideals P of discrete valuation rings of F/k. The elements of $P \in X_F$ are called *points* (or *places*). Let $\mathcal{O}_{P,X} = \mathcal{O}_P$ denote the valuation

ring with maximal ideal P, and let ord_P denote its associated valuation. For any $U \subset X$ the *ring of regular functions* on U is the ring

$$\mathcal{O}_X(U) = \mathcal{O}(U) := \cap_{P \in U} \mathcal{O}_P = \{f \in F : \text{ord}_P(f) \geq 0 \text{ for all } P \in U\} \subseteq F,$$

and we call $\mathcal{O}(X)$ the *ring of regular functions* (or *coordinate ring*) of X. Note that $\mathcal{O}(X)$ is precisely the intersection of all the valuation rings of F/k.

For $f \in \mathcal{O}_P$ we define $f(P)$ to be the image of f in the residue field $\mathcal{O}_P/P \simeq k$; thus

$$f(P) = 0 \Longleftrightarrow f \in P \Longleftrightarrow \text{ord}_P(f) > 0.$$

For $f \in \mathcal{O}_X$ we have $f(P) = 0$ if and only if $\text{ord}_P(f) > 0$. We then give X the Zariski topology by taking as closed sets the zero locus of any subset of $\mathcal{O}(X)$.[2] If F is actually the function field of a smooth projective curve, all the definitions above agree with our usual notation, as we will verify shortly.

Definition 18.9. An *abstract curve* is the topological space $X = X_F$ with rings of regular functions $\mathcal{O}_{X,U}$ determined by the function field F/k as above. A morphism $\phi\colon X \to Y$ between abstract curves or projective varieties is a continuous map such that for every open $U \subseteq Y$ and $f \in \mathcal{O}_Y(U)$ we have $f \circ \phi \in \mathcal{O}_X(\phi^{-1}(U))$.

As you will verify in the homework, if X and Y are both projective varieties this definition of a morphism is equivalent to our earlier definition of a morphism between projective varieties. The identity map $X \to X$ is obviously a morphism, and we can compose morphisms: if $\phi\colon X \to Y$ and $\varphi\colon Y \to Z$ are morphisms, then $\varphi \circ \phi$ is continuous, and for any open $U \subseteq Z$ and $f \in \mathcal{O}_Z(U)$ we have $f \circ \varphi \in \mathcal{O}_Y(\varphi^{-1}(U))$, and then

$$f \circ (\varphi \circ \phi) = (f \circ \varphi) \circ \phi \in \mathcal{O}_X(\phi^{-1}(\varphi^{-1}(U))) = \mathcal{O}_X((\varphi \circ \phi)^{-1}(U)).$$

Thus we have a category whose objects include both abstract curves and projective varieties.

Let us verify that we have set things up correctly by proving that every smooth projective curve is isomorphic to the abstract curve determined by its function field. This follows immediately from our definitions, but it is worth unravelling them once just to be sure.

Theorem 18.10. *Let C be a smooth projective curve and let $X = X_{k(C)}$ be the abstract curve associated to its function field. Then C and X are isomorphic.*

Proof. For the sake of clarity, let us identify the points (discrete valuation rings) of X as maximal ideals m_P corresponding to points $P \in C$. As noted above there is a one-to-one correspondence between $P \in C$ and $m_P \in X$, we just need to show that this induces an isomorphism of curves. So let $\phi\colon C \to X$ be the bijection that sends P to m_P.

For any $U \subseteq C$ we have, by definition, $\mathcal{O}_C(U) = \cap_{P \in U}\mathcal{O}_{P,C}$ and $\mathcal{O}_X(V) = \cap_{m_P \in V}\mathcal{O}_{m_P,X}$, so $\mathcal{O}_C(U) = \mathcal{O}_X(\phi(U))$ In particular,

$$\mathcal{O}(C) = \mathcal{O}(\phi(C)) = \mathcal{O}(X),$$

hence the rings of regular functions of C and X are actually identical (not just in bijection). Moreover, for any open $U \subseteq X$ and $f \in \mathcal{O}_X(U)$ we have $f \circ \phi = f \in \mathcal{O}_C(\phi^{-1}(U))$, and for any open $U \subseteq C$ and $f \in \mathcal{O}_{C,U}$ we have $f \circ \phi^{-1} = f \in \mathcal{O}_X(\phi(U))$.

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[2]As we will prove in this next lecture, this is just the cofinite topology: the open sets are the empty set and complements of finite sets.

A set $U \subseteq C$ is closed if and only if it is the zero locus of some subset of $\mathcal{O}(C)$, and for any $P \in C$, equivalently, any $\phi(P) \in X$, we have

$$f(P) = 0 \Longleftrightarrow \mathrm{ord}_P(f) > 0 \Longleftrightarrow f(\phi(P)) = 0,$$

where we are using the definition of $f(\phi(P)) = f(m_P)$ for $m_P \in X$ on the right. It follows that ϕ is a topology isomorphism from C to X; in particular, both ϕ and ϕ^{-1} are continuous. Thus ϕ and ϕ^{-1} are both morphisms, and $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are the identity maps. □

One last ingredient before our main result; we want to be able to construct smooth affine curves with a specified function field that contain a point whose local ring is equal to a specific discrete valuation ring.

Lemma 18.11. *Let R be a discrete valuation ring of a function field F/k of dimension one. There exists a smooth affine curve C with $k(C) = F$ such that $R = \mathcal{O}_P$ for some $P \in C$.*

Proof. The extension F/k is finitely generated, so let $\alpha_1, \ldots, \alpha_n$ be generators, and replace α_i with $1/\alpha_i$ as required so that $\alpha_1, \ldots, \alpha_n \in R$. Let S be the intersection of all discrete valuation rings of F/k that contain the subalgebra $k[\alpha_1, \ldots, \alpha_n] \subseteq F$. Then $S \subseteq R$ is an integral domain with fraction field F. The kernel of the map from the polynomial ring $k[x_1, \ldots, x_n]$ to S that sends each x_i to α_i is a prime ideal I for which $S = k[x_1, \ldots, x_n]/I$. The variety $C \subseteq \mathbb{A}^n$ defined by I has coordinate ring $k[C] = S \subseteq R$ and function field $k(C) = F$, so it has dimension one and is a curve

Moreover, the curve C is smooth; its coordinate ring S is integrally closed (it is an intersection of discrete valuation rings, each of which is integrally closed), and by Lemma 18.12 below, all its local rings $\mathcal{O}_P$ are discrete valuation rings, hence regular, and therefore every point $P \in C$ is smooth.

Let $\phi \colon R \to R/\mathfrak{m} = k$ be the quotient map and consider the point $P(\phi(x_1), \ldots, \phi(x_n))$. Every f in the maximal ideal m_P of $\mathcal{O}_P$ satisfies

$$\phi(f) = \phi(f(x_1, \ldots, x_n)) = f(\phi(x_1), \ldots, \phi(x_n)) = f(P) = 0$$

and therefore lies in $\mathfrak{m}$. By Theorem 18.5, $R = \mathcal{O}_P$ as desired. □

The following lemma is a standard result of commutative algebra (so feel free to skip the proof on a first reading), but it is an essential result that has a reasonably straight-forward proof (using Theorem 18.2), so we include it here.[3]

Lemma 18.12. *If A is an integrally closed Noetherian domain of dimension one then all of its localizations at nonzero prime ideals are discrete valuation rings.*[4]

Proof. Let F be the fraction field of A and let $\mathfrak{p}$ be a nonzero prime ideal. We first note that $A_\mathfrak{p}$ is integrally closed. Indeed, if $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ is an equation with $a_i \in A_\mathfrak{p}$ and $x \in F$, then we may pick $s \in A - \mathfrak{p}$ so that all the sa_i lie in A (let s be the product of all the denominators $c_i \notin \mathfrak{p}$ of $a_i = b_i/c_i$). Multiplying through by s^n yields an equation $(sx)^n + sa_{n-1}(sx)^{n-1} + \cdots + s^n a_0 = 0$ in $y = sx$ with coefficients in A. Since A is integrally closed, $y \in A$, therefore $x = y/s \in A_\mathfrak{p}$ as desired.

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[3]There are plenty of shorter proofs, but they tend to use facts that we have not proved.

[4]Such rings are called *Dedekind domains*. They play an important role in number theory where they appear as the ring of integers of a number field. The key property of a Dedekind domain is that ideals can be uniquely factored into prime ideals, although we don't use this here.

Let $\mathfrak{m} = \mathfrak{p}A_\mathfrak{p}$ be the maximal ideal of $A_\mathfrak{p}$. The ring $A_\mathfrak{p}$ has dimension one, since $(0) \subsetneq \mathfrak{m}$ are all the prime ideals in $A_\mathfrak{p}$ (otherwise we would have a nonzero prime $\mathfrak{q}$ properly contained in $\mathfrak{m}$, but then $\mathfrak{q} \cap A$ would be a nonzero prime properly contained in $\mathfrak{p}$, contradicting $\dim A = 1$). Thus $R = A_\mathfrak{p}$ is a local ring of dimension one. By Theorem 18.2, to show that R is a DVR it suffices to prove that R is regular; it is clear that R is Noetherian (since A is), we just need to show $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R = 1$. By Nakayama's lemma, $\mathfrak{m}^2 \neq \mathfrak{m}$, so $\dim \mathfrak{m}/\mathfrak{m}^2 \neq 0$. To show $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$ it suffices to prove that $\mathfrak{m}$ is principal. To do this we adapt an argument of Serre from [1, §I.1].

Let $S = \{y \in F : y\mathfrak{m} \subseteq R\}$, and let $\mathfrak{m}S$ denote the R-ideal generated by all products xy with $x \in \mathfrak{m}$ and $y \in S$ (just like an ideal product). Then $\mathfrak{m} \subseteq \mathfrak{m}S \subseteq R$, so either $\mathfrak{m}S = \mathfrak{m}$ or $\mathfrak{m}S = R$. We claim that the latter holds. Assuming it does, then $1 = \sum x_i y_i$ for some $x_i \in \mathfrak{m}$ and $y_i \in S$. The products $x_i y_i$ all lie in R but not all can lie in $\mathfrak{m}$, so some $x_j y_j$ is invertible. Set $x = x_j/(x_j y_j)$ and $y = y_j$ so that $xy = 1$, with $x \in \mathfrak{m}$ and $y \in S$. We can then write any $z \in \mathfrak{m}$ as $z = 1 \cdot z = xy \cdot z = x \cdot yz$. But $yz \in R$, since $y \in S$, so every $z \in \mathfrak{m}$ actually lies in (x). Thus $\mathfrak{m} = (x)$ is principal as desired, assuming $\mathfrak{m}S = R$.

We now prove that $\mathfrak{m}S = R$ by supposing the contrary and deriving a contradiction. We will do this by proving that $\mathfrak{m}S = \mathfrak{m}$ implies both $S \subseteq R$ and $S \not\subseteq R$. So assume $\mathfrak{m}S = \mathfrak{m}$.

We first prove $S \subseteq R$. Since $\mathfrak{m}S = \mathfrak{m}$, for any $\lambda \in S$ we have $\lambda \mathfrak{m} \subseteq \mathfrak{m}$. The ring R is Noetherian, so let $m_1, \ldots, m_k$ be generators for $\mathfrak{m}$. We then have k equations of the form $\sum_{i,j} a_{ij} m_j = \lambda m_i$ with $a_{ij} \in R$. Thus λ is an eigenvalue of the matrix (a_{ij}) and therefore a root of its characteristic polynomial, which is monic, with coefficients in R. Since R is integrally closed, $\lambda \in R$, and therefore $S \subseteq R$ as claimed.

We now prove $S \not\subseteq R$, thereby obtaining a contradiction. Let $x \in \mathfrak{m} - \{0\}$, and consider the ring $T_x = \{y/x^n : y \in R, n \geq 0\}$. We claim $T_x = F$: if not, it contains a nonzero maximal ideal $\mathfrak{q}$ with $x \notin \mathfrak{q}$ (since x is a unit in T_x), so $\mathfrak{q} \cap R \neq \mathfrak{m}$, and clearly $\mathfrak{q} \cap R \neq (0)$, but then $\mathfrak{q} \cap R$ is a prime ideal of R strictly between (0) and $\mathfrak{m}$, which contradicts $\dim R = 1$. So every element of $T_x = F$ can be written in the form y/x^n, and this holds for any $x \in \mathfrak{m}$. Applying this to a fixed $1/z$ with $z \in \mathfrak{m} - \{0\}$, we see that every $x \in \mathfrak{m} - \{0\}$ satisfies $x^n = yz$ for some $y \in R$ and $n \geq 0$, thus $x^n \in (z)$ for all $x \in \mathfrak{m}$ and sufficiently large n. Applying this to our generators $m_1, \ldots, m_k$ for $\mathfrak{m}$, choose n so that $m_1^n, \ldots, m_k^n \in (z)$, and then let $N = kn$ so that $(\sum_i r_i m_i)^N \in (z)$ for all choices of $r_i \in R$. Thus $\mathfrak{m}^n \subseteq (z)$ for all $n \geq N$, and there is some minimal $n \geq 1$ for which $\mathfrak{m}^n \subseteq (z)$. If $n = 1$ then $\mathfrak{m} = (z)$ is principal and we are done. Otherwise, choose $y \in \mathfrak{m}^{n-1}$ so that $y \notin (z)$ but $y\mathfrak{m} \subseteq (z)$. Then $(y/z)\mathfrak{m} \in R$, so $y/z \in S$, but $y/z \notin R$ (since $z \in \mathfrak{m}$), so $S \not\subseteq R$ as claimed. $\qquad \square$

We are now ready to prove our main theorem.

Theorem 18.13. *Every abstract curve is isomorphic to a smooth projective curve.*

Proof. Let $X = X_F$ be the abstract curve associated to the function field F/k. Then $\mathcal{O}(X)$ is an affine algebra, and there is a corresponding affine curve A. The curve A is smooth, since all its local rings $\mathcal{O}_P$ are discrete valuation rings, but it is not complete, so not every point on X (each corresponding to a discrete valuation rings of F/k) corresponds to a point on A. So let C be the projective closure of A; the curve C need not be smooth, but it is complete, and it satisfies Chevalley's criterion. Thus for each point $P \in X$, the associated discrete valuation ring $\mathcal{O}_{P,X}$ contains the local ring $\mathcal{O}_{Q,C}$ of a point $Q \in C_1$. The point Q is certainly unique; if $\mathcal{O}_{P,X}$ contained two distinct local rings it would contain the entire function field, which is not the case (to see this, note that for any distinct $P, Q \in C$ the zero locus of $m_P + m_Q$ is empty).

So let $\phi\colon X \to C$ map each $P \in X$ to the unique $Q \in C_1$ for which $\mathcal{O}_{Q,C} \subseteq \mathcal{O}_{P,X}$. It is easy to see that ϕ is continuous; indeed, since we are in dimension one it suffices to note that it is surjective, and this is so: every local ring $\mathcal{O}_{Q,C}$ is contained in a discrete valuation ring $\mathcal{O}_{P,X}$ (possibly more than one, this can happen if Q is singular).[5] To check that it is a morphism, if $U \subseteq C$ is open and $f \in \mathcal{O}_C(U) = \cap_{Q \in U} \mathcal{O}_{Q,C}$ then we have $\mathcal{O}_X(\phi^{-1}(U)) = \cap_{\phi(P) \in U} \mathcal{O}_{P,X} \supseteq \cap_{Q \in U} \mathcal{O}_{Q,C}$ and therefore $f \circ \phi \in \mathcal{O}_X(V)$ as required.

Now let $C_1 = C$ and $\phi_1 = \phi$. There are finitely many singular points $Q \in C$ (the singular locus has dimension 0), and for each such Q the inverse image $\phi^{-1}(Q) \subseteq X$ is closed and not equal to X (since ϕ is surjective and C has more than one point), so finite. Let $P_2, \ldots, P_n \in X$ be the finite list of points whose images under ϕ_1 are singular in C.

For each P_i we now let C_i be the projective closure of the smooth affine curve with function field F/k and a local ring $\mathcal{O}_{P,C_i}$ equal to $\mathcal{O}_{P_i,X}$, given by Lemma 18.11. Then $k(C_i) = F$ and the point on C_i corresponding to P_i is smooth by construction, since its local ring is precisely the discrete valuation ring $\mathcal{O}_{P_i}$. Define a surjective morphism $\phi_i\colon X \to C_i$ exactly as we did for ϕ_1.

We now consider the product variety $Y = \prod_i C_i$ and define the morphism $\varphi\colon X \to Y$ by $\varphi(P) = (\phi_1(P), \ldots, \phi_n(P))$. The variety Y is a product of projective varieties and can be smoothly embedded in a single projective space.[6] The image of φ in Y is a projective curve C whose function field is isomorphic to F, and C is smooth because, by construction, every point $P \in C$ is smooth in one of its affine parts. By Theorem 18.10, the smooth projective curve C is isomorphic to the abstract curve associated to its function field, namely, X. $\square$

Corollary 18.14. *Every curve C is birationally equivalent to a smooth projective curve that is unique up to isomorphism.*

Proof. By 18.10 there exists an abstract curve corresponding to the function field $k(C)$, and by Theorem 18.13 this abstract curve is isomorphic to a smooth projective curve. Uniqueness follows from Corollary 18.8. $\square$

The smooth projective curve to which a given curve C is birationally equivalent is called the *desingularization C*. Henceforth, whenever we write down an equation for a curve (which may be affine and/or have singularities) we can always assume that we are referring to its desingularization.

Remark 18.15. In the proof of Theorem 18.13 we made no attempt to control the dimension of the projective space into which we embedded the smooth projective curve C isomorphic to our abstract curve X. Using more concrete methods, one can show that it is always possible to embed C in $\mathbb{P}^3$. In general, one can do no better than this; indeed we will see plenty of examples of smooth projective curves that cannot be embedded in $\mathbb{P}^2$.

References

[1] J. P. Serre, *Local fields*, Springer, 1979.

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[5]This follows from Problem 2 part 5 on Problem Set 8.
[6]Using the Segre morphism, as proved in problem 1 part 4 on Problem Set 8.