As usual, k is a perfect field and $\bar{k}$ is a fixed algebraic closure of k. Recall that an affine (resp. projective) variety is an irreducible alebraic set in $\mathbb{A}^n = \mathbb{A}^n(\bar{k})$ (resp. $\mathbb{P}^n = \mathbb{P}^n(\bar{k})$).

14.1 Affine morphisms

We begin our discussion of maps between varieties with the simplest case, morphisms of affine varieties.

Definition 14.1. Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. A *morphism* $f \colon X \to Y$ is a map $f(P) := (f_1(P), \ldots, f_n(P))$ defined by polynomials $f_1, \ldots, f_n \in \bar{k}[x_1, \ldots, x_m]$ such that $f(P) \in Y$ for all points $P \in X$. We may regard $f_1, \ldots, f_n$ as representatives of elements of the coordinate ring $\bar{k}[X] = \bar{k}[x_1, \ldots, x_m]/I(X)$; we are evaluating $f_1, \ldots, f_n$ only at points in X, so there is no reason to distinguish them modulo the ideal $I(X)$.

As befits their name, morphisms can be composed: if $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms of varieties $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$, and $Z \subseteq \mathbb{A}^r$, then $(g \circ f) \colon X \to Z$ is defined by

$$(g \circ f)(P) := g(f(P)) = \big(g_1(f_1(P), \ldots, f_n(P)), \ \ldots, \ g_r(f_1(P), \ldots, f_n(P))\big).$$

Notice that in order for this composition to actually make sense, we need to pick particular representatives $g_1, \ldots, g_r \in \bar{k}[y_1, \ldots, y_m]$ modulo $I(Y)$ (of course it doesn't matter which). The rings $\bar{k}[x_1, \ldots, x_m]$ and $\bar{k}[X]$ are $\bar{k}$ algebras, so it makes sense to evaluate a polynomial with coefficients in $\bar{k}$ in either of these rings (depending on our perspective), but it does not make sense to "evaluate" an element of $\bar{k}[Y]$ at elements of $\bar{k}[X]$. We also have the identity morphism $f \colon X \to X$, which is defined by letting f_i be the polynomial x_i.[1]

Thus we have a category whose objects are affine varieties and whose morphisms are (no surpise) morphisms. Contrary to what you might expect (if you happened to be thinking of morphisms in the category of groups or rings), the image of a morphism is not necessarily a variety, or even an algebraic set.

Example 14.2. Consider the morphism $f \colon \mathbb{A}^2 \to \mathbb{A}^2$ defined by $f(x_1, x_2) = (x_1, x_1 x_2)$. Its image is the entire affine plane except for the points on the x_1-axis with $x_2 \neq 0$. This is not an algebraic set; this is obvious if $\bar{k} = \mathbb{C}$, and in general, if $g(y_1, y_2)$ vanishes on the image of f, then for any infinitely many $c \in \bar{k}$ the polynomial $h(t) = g(t, c)$ has infinitely many zeroes, hence is the zero polynomial, and this implies that g is the zero polynomial. Thus $I(\operatorname{im} f)$ is the zero ideal and the only algebraic set containing $\operatorname{im} f$ is all of $\mathbb{A}^2$.

On the other hand, if you were thinking of morphisms in the category of topological spaces (which is the better analogy), then morphisms of varieties behave as expected; indeed, they are continuous maps (and more), we just need to put the right topology on our varieties.

Definition 14.3. In the *Zariski topology* on $\mathbb{A}^n$ (resp. $\mathbb{P}^n$), the closed sets are precisely the algebraic sets. Any algebraic set in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$) then inherits the subspace topology.

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[1] Note that we are using the symbol x_i in three different ways: as an indeterminate used to define the polynomial ring $R = \bar{k}[x_1, \ldots, x_m]$, as an element of R (i.e., a polynomial), and as the function $x_i \colon \mathbb{A}^m(\bar{k}) \to \bar{k}$ that evaluates the polynomial x_i on a given input.

Let us verify that this actually defines a topology: the empty set and $\mathbb{A}^n$ are algebraic sets defined by the ideals (1) and (0), respectively (and similarly for $\mathbb{P}^n$), and algebraic sets are closed under arbitrary intersections (we can take the zero locus of an arbitrary sum of ideals), and finite unions (we can take the zero locus of a finite product of ideals).[2]

With a topology in place we can now use words like *open, closed, dense*, etc., when referring to subsets of $\mathbb{A}^n$ or $\mathbb{P}^n$, with the understanding that they refer to the Zariski topology. Note that our definition of the projective closure of an affine variety V embedded in $\mathbb{P}^n$ is consistent with this; we proved last time that the projective closure of V in $\mathbb{P}^n$ is a variety (hence closed), and it is clearly the smallest closed set that contains V: a homogeneous polynomial in $\bar{k}[x_0, \ldots, x_n]$ vanishes on V in $\mathbb{P}^n$ if and only if its dehomogenization vanishes on V in $\mathbb{A}^n$.

It should be noted that the Zariski topology is extremely coarse. In $\mathbb{A}^1$, for example, every nonempty open set is the complements of finite sets, and in general every nonempty open set is dense in $\mathbb{A}^n$ (and in $\mathbb{P}^n$); the same applies in the subspace of a variety. And the Zariski topology is definitely *not* a Hausdorff topology; indeed, the intersection of any pair of nonempty open sets is not only nonempty, it must be dense!

Theorem 14.4. *Every morphism $f : X \to Y$ of affine varieties is continuous. That is, the inverse image $f^{-1}(Z)$ of any algebraic subset $Z \subseteq Y$ is an algebraic subset of X.*

Proof. Showing that the inverse image of a closed set is closed is the same thing as showing that the inverse of an open set is open, which is the definition of a continuous map. So let Z be an algebraic subset of Y defined by the ideal $(g_1, \ldots, g_r)$ (we include generators for $I(Y)$ in this list). Then $f^{-1}(Z)$ is the zero locus of $g_1(f_1, \ldots, f_n), \ldots, g_r(f_1, \ldots, f_n)$ (as compositions of polynomials) in X, hence an algebraic of subset of X. $\qquad\square$

Remark 14.5. It is not true that every continuous map between affine varieties is a morphism; the coarseness of the Zariski topology simply makes it too easy for a function to be continuous. The additional requirement that a morphism must satisfy is that it must also be a rational map, as we will see in the next lecture.

For affine varieties, an isomorphism is a bijective morphism whose inverse is a morphism, but we will use the more formal definition that applies in any category.

Definition 14.6. We say that two varieties $X \simeq Y$ are *isomorphic* if there exist morphisms $f \colon X \to Y$ and $g \colon Y \to X$ such that both $f \circ g$ and $g \circ f$ are the identity morphisms on X and Y, respectively. In this case we may refer to both f and g as *isomorphisms*.

Just as not every continuous map is an morphism, not every bicontinuous map (homemorphism) is an isomorphism. Indeed, not even a bicontinuous morphism is necessarily an isomorphism.

Example 14.7. Consider the map from $\mathbb{A}^1$ to $\mathbb{A}^2$ defined by $t \mapsto (t^2, t^3)$. The image of this map is a variety V (the polynomial $y^2 - x^3$ is irreducible in $\bar{k}[x, y]$, so the principal ideal $(y^2 - x^3)$ is prime). Thus we have a morphism $f \colon \mathbb{A}^1 \to V$, and it is clearly bijective; the inverse map can be defined as

$$f^{-1}(x, y) = \begin{cases} y/x & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

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[2]One needs to check that this also works for projective varieties and homogeneous ideals, but this is straight-forward; sums and products of homogeneous ideals are again homogeneous ideals and the rest follows from Problem 2 of Problem Set 6.

Moreover, f is a closed map (the only closed sets in $\mathbb{A}^1$ are points and $\mathbb{A}^1$ itself, and these are all mapped to closed sets in V), so it is bicontinuous and thus both a morphism and a homeomorphism. But it is not an isomorphism because its inverse is not a morphism; the function f^{-1} cannot be defined as a polynomial map.

The example above shows that two varieties may be isomorphic as topological spaces without being isomorphic as varieties; this should not be too surprising, the Zariski topology makes it very easy for varieties to be homoemorphic (indeed, every affine curve is homeomorphic to $\mathbb{A}^1$). On the other hand, in the example of the twisted cubic (see Lecture 13) we actually have an isomorphism of affine varieties.

We now come to a very important theorem that gives a one-to-one correspondence between morphisms of affine varieties $\phi\colon X \to Y$ and homomorphisms of their coordinate rings $\phi^*\colon \bar{k}[Y] \to \bar{k}[X]$ (note that the directions of the arrows are reversed). Actually, we want ϕ^* to be more than just a ring homomorphism, we also want it to fix the field $\bar{k}$. A compact way of saying this is to regard $\bar{k}[X]$ and $\bar{k}[Y]$ not as rings, but as algebras over $\bar{k}$, and require ϕ^* to be a homomorphism of $\bar{k}$-algebras. This means that that ϕ^* must commute with sums and products (in our setting this makes ϕ^* a ring homomorphism), and it must fix elements of $\bar{k}$.

In order to obtain an actual equivalence of categories, we want to specify objects that correspond to coordinate rings in a purely algebraic way that does not involve varieties. So consider an arbitrary integral domain R that is also a finitely generated $\bar{k}$-algebra; let us call such an R an *affine algebra*. If we denote the generators of R by $x_1, \ldots, x_n$, there is a canonical ring homomorphism

$$\bar{k}[x_1, \ldots, x_n] \to R$$

from the polynomial ring with indeterminates $x_1, \ldots, x_n$ onto R, and the kernel of this homomorphism is an ideal I for which $R = \bar{k}[x_1, \ldots, x_n]/I$. The ideal I is prime (since R is an integral domain), hence a radical ideal. Let V be the variety it defines in $\mathbb{A}^n$. Then by Hilbert's *Nullstellensatz*, we have $I = I(V)$ (note that here we use that $\bar{k}$ is algebraically closed). The coordinate ring of V is then $\bar{k}[V] \simeq \bar{k}[x_1, \ldots, x_n]/I \simeq R$.

Thus we have a one-to-one correspondence between affine varieties and affine algebras in which varieties correspond to their coordinate rings and affine algebras correspond to varieties as described above. By taking the morphisms to be k-algebra homomorphisms, we can consider the category of affine algebras. In order to prove that the category of affine varieties is equivalent to the category of affine algebras, we need to understand how their morphisms correspond.

Theorem 14.8. *The following hold:*

(i) *Every morphism $\phi\colon X \to Y$ of affine varieties induces a morphism $\phi^*\colon \bar{k}[Y] \to \bar{k}[X]$ of affine algebras such that $\phi^*(g) = g \circ \phi$.*

(ii) *Every morphism $\theta\colon R \to S$ of affine algebras induces a morphism $\theta^*\colon X \to Y$ of affine varieties with $R \simeq \bar{k}[Y]$ and $S \simeq \bar{k}[X]$ such that the image of $\theta(g)$ in $\bar{k}[X]$ is $g \circ \theta^*$.*

(iii) *If $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ are morphisms of affine varieties, then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.*

Before proving the theorem, let us comment on the notation $\phi^*(g) = g \circ \phi$. In order for this to make sense, we need to interpret it as follows: given $g \in \bar{k}[Y] = \bar{k}[y_1, \ldots, y_n]/I(Y)$, we pick a representative $\hat{g} \in \bar{k}[y_1, \ldots, y_n]$ (so g is the coset $\hat{g} + I(Y)$) and then $\phi^*(g)$ is the

reduction of the polynomial $\hat{g} \circ \phi = \hat{g}(\phi_1, \ldots, \phi_n) \in \bar{k}[x_1, \ldots, x_m]$ modulo $I(X)$ (i.e., its image under the quotient map). In short, $g \circ \phi$ means lift/compose/reduce.[3]

The key point is that for any f in $I(Y)$, the composition $f \circ \phi$ yields an element of $I(X)$, because ϕ maps points in X to points in Y and f vanishes at points in Y. Thus it does not matter which lift $\hat{g}$ we pick and our interpretation of $g \circ \phi$ is well defined.

Proof. We assume throughout that X and Y are varieties in $\mathbb{A}^m$ and $\mathbb{A}^n$, respectively.

(i) We first note that the operations of lifting from $\bar{k}[Y]$ to $\bar{k}[y_1, \ldots, y_n]$ and reducing from $\bar{k}[x_1, \ldots, x_m]$ to $\bar{k}[X]$ are both compatible with ring operations, and when lifting or reducing an element of $\bar{k}$ it remains fixed. Now if $g \in \bar{k}$ is a constant polynomial, then $g \circ \phi = g$, and for any $f, g \in \bar{k}[y_1, \ldots, y_n]$ we have

$$(f + g) \circ \phi = (f + g)(\phi_1, \ldots, \phi_n) = f(\phi_1, \ldots, \phi_n) + g(\phi_1, \ldots, \phi_n) = (f \circ \phi) + (g \circ \phi)$$

and

$$(fg) \circ \phi = (fg)(\phi_1, \ldots, \phi_n) = f(\phi_1, \ldots, \phi_n)g(\phi_1, \ldots, \phi_n) = (f \circ \phi)(g \circ \phi).$$

Thus ϕ^* is a ring homomorphism that fixes $\bar{k}$, hence a homomorphism of affine algebras.

(ii) Let $\theta \colon R \to S$ be a morphism of affine algebras. As described above, there exist varieties X and Y for which $R \simeq \bar{k}[Y]$ and $S \simeq \bar{k}[X]$, and any morphism $R \to S$ induces a morphism $\bar{k}[Y] \to \bar{k}[X]$ that commutes with these isomorphisms.[4] So without loss of generality we assume $\theta \colon \bar{k}[Y] \to \bar{k}[X]$ is an affine algebra morphism of coordinate rings. We now define a morphism $\theta^* \colon X \to Y$ by letting $\theta^* = (\theta(y_1), \ldots, \theta(y_n))$, where $\theta(y_i)$ denotes the image under θ of the image of the polynomial y_i in $\bar{k}[Y]$ under the quotient map from $\bar{k}[y_1, \ldots, y_n]$. For any $g \in \bar{k}[Y]$ we have

$$g \circ \theta^* = \hat{g}(\theta(y_1), \ldots, \theta(y_n)) = \theta(g(y_1, \ldots, y_n)) = \theta(g),$$

where the middle equality follows from the fact that θ is a ring homomorphism that fixes $\bar{k}$. We also note that for any $f \in I(Y)$ we have $f \circ \theta^* = \theta(f) = \theta(0) = 0$, so $f(\theta^*(P)) = 0$ for all $f \in I(Y)$ and $P \in X$, which implies that the image of θ^* lies in Y. Thus θ^* is indeed a morphism from X to Y as claimed.

(iii) For any $g \in \bar{k}[Z]$ we have

$$(\psi \circ \phi)^*(g) = g \circ (\psi \circ \phi) = (g \circ \psi) \circ \phi = \phi^*(g \circ \psi) = (\phi^*(\psi^*(g)) = (\phi^* \circ \psi^*)(g). \qquad \square$$

Corollary 14.9. *The categories of affine varieties and affine algebras are contravariantly equivalent.*[5]

Proof. The only thing that remains to be shown is that the two functors arising from (i) and (ii) of Theorem 14.8 are inverses, that is, we need to show that $(\phi^*)^* = \phi$ and $(\theta^*)^* = \theta$, up to isomorphism.[6] The second equality is clear from the statement of the theorem and the first is clear from its proof. $\qquad \square$

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[3]If we view $\phi_1, \ldots, \phi_n$ as elements of $\bar{k}[X]$, we also need to lift the ϕ_i in order to compute $g \circ \phi$.

[4]One says that the induced morphism is *natural*; more precisely, the functor from the category of function fields to the category of function fields of varieties is a *natural transformation* (in fact, a natural isomorphism). If you think this is just a fancy way of stating the obvious, you are right; but the same phenomenon occurs in more general situations where it is not always so obvious.

[5]Contravariantly equivalent categories are also called *dual* categories; they are also said to be *anti-equivalent*, but we won't use this term.

[6]Up to isomorphism means that the domains and codmains of the morphisms on either side of the equality need not be precisely equal, they just need to be isomorphic, and the isomorphisms and the morphisms must form a commutative diagram; in other words, $(\phi^*)^*$ is naturally isomorphic to ϕ (and similarly for θ).

Corollary 14.10. *All the nonempty affine parts of a projective variety are isomorphic.*

Proof. We proved in Lecture 13 (see Corollary 13.26) that the nonempty affine parts of a projective variety all have the same coordinate ring (up to isomorphism). □

Definition 14.11. If $\phi = (\phi_1, \ldots, \phi_n)$ is a morphism of varieties $X \to Y$ that are defined over k, we say that ϕ is *defined over k* if $\phi_1, \ldots, \phi_n \in k[Y]$. Equivalently, ϕ is defined over k if $\phi^\sigma = (\phi_1^\sigma, \ldots, \phi_n^\sigma) = \phi$ for all $\sigma \in G_k$.[7] If ϕ is an isomorphism defined over k and it has an inverse isomorphism defined over k, then we say that X and Y are *isomorphic over k*.

Corollary 14.12. *Let X and Y be affine varieties defined over k. If $\phi \colon X \to Y$ is a morphism defined over k then the affine algebra morphism $\phi^* \colon \bar{k}[Y] \to \bar{k}[X]$ restricts to an affine algebra morphism from $k[Y]$ to $k[X]$.*

Proof. This follows immediately from the definition $\phi^*(g) = g \circ \phi$. □

References

[1] J.H. Silverman, *The arithemetic of elliptic curves*, 2nd edition, Springer, 2009.

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[7]Proving this equivalence is not completely trivial; see [1, Ex. I.1.12a].