As before, k is a perfect field, $\bar{k}$ is a fixed algebraic closure of k, and $\mathbb{A}^n = \mathbb{A}^n(\bar{k})$ is n-dimensional affine space.

13.1 Affine varieties

Definition 13.1. An algebraic set $Z \in \mathbb{A}^n$ is said to be *defined over* k if its ideal is generated by polynomials in $k[x_1, \ldots, k_n]$, that is, $I(Z)$ is equal to the ideal generated by $I(Z) \cap k[x_1, \ldots, x_n]$ in $\bar{k}[x_1, \ldots, k_n]$. We write Z/k to indicate that Z is an algebraic set that is defined over k and define the ideal

$$I(Z/k) = I(Z) \cap k[x_1, \ldots, x_n].$$

When Z is defined over k the action of the absolute Galois group G_k on $\mathbb{A}^n$ induces an action on Z, since for any $\sigma \in G_k$, any $f \in k[x_1, \ldots, x_n]$, and any $P \in \mathbb{A}^n$ we have

$$f(P^\sigma) = f(P)^\sigma.$$

In this case we have $Z(k) = \{P \in Z : P^\sigma = P \text{ for all } \sigma \in G_k\} = Z^{G_k}$.

Definition 13.2. Let Z be an algebraic set defined over k. The *affine coordinate ring* of Z/k is the ring

$$k[Z] = \frac{k[x_1, \ldots, x_n]}{I(Z/k)}.$$

We similarly define

$$\bar{k}[Z] = \frac{\bar{k}[x_1, \ldots, x_n]}{I(Z)}.$$

The coordinate ring $k[Z]$ may have zero divisors; it is an integral domain if and only if $I(Z/k)$ is a prime ideal. Even if $k[Z]$ has no zero divisors, $\bar{k}[Z]$ may still have zero divisors (the fact that $I(Z/k)$ is a prime ideal does not guarantee that $I(Z)$ is a prime ideal; the principal ideal $(x^2 + 1)$ is prime in $\mathbb{Q}$ but not in $\overline{\mathbb{Q}}$, for example). We want $k[Z]$ to be an integral domain so that we can work with its fraction field. Recall from last lecture that $I(Z)$ is a prime ideal if and only if Z is irreducible. This motivates the following definition.

Definition 13.3. An *affine variety* V is an irreducible algebraic set in $\mathbb{A}^n$.[1]

An algebraic set Z is a variety if and only if $I(Z)$ is a prime ideal; the one-to-one correspondence between algebraic sets and radical ideals restricts to a one-to-one correspondence between varieties and prime ideals (note that every prime ideal is necessarily a radical ideal). The set $\mathbb{A}^n$ is a variety since $I(\mathbb{A}^n)$ is the zero ideal, which is prime in the ring $\bar{k}[x_1, \ldots, k_n]$ because it is an integral domain (the zero ideal is prime in any integral domain).

Definition 13.4. Let V/k be an affine variety defined over k. The *function field* $k(V)$ of V is the fraction field of the coordinate ring $k[V]$.

We similarly define the function field of V over any extension of k on which V is defined. Every variety is defined over $\bar{k}$, so we can always refer to the function field $\bar{k}(V)$.

[1] Not all authors require varieties to be irreducible (but many do).

13.1.1 Dimension

Definition 13.5. The *dimension* of an affine variety V is the transcendence degree of the field extension $\bar{k}(V)/\bar{k}$.

Lemma 13.6. *The dimension of $\mathbb{A}^n$ is n, and the dimension of any point $P \in \mathbb{A}^n$ is 0.*

Proof. We have $\bar{k}[\mathbb{A}^n] = \bar{k}[x_1, \ldots, x_n]/(0) = \bar{k}[x_1, \ldots, x_n]$, so $\bar{k}(\mathbb{A}^n) = \bar{k}(x_1, \ldots, x_n)$ is a purely transcendental extension of $\bar{k}$ with transcendence degree n. For the point P, the ideal $I(P) = m_P$ is maximal, so the coordinate ring $\bar{k}[P] = \bar{k}[x_1, \ldots, x_n]/m_P$ is a field isomorphic to $\bar{k}$, as is $\bar{k}(P)$, and the transcendence degree of $\bar{k}/\bar{k}$ is obviously 0. □

Let us note an alternative definition of dimension using the *Krull dimension* of a ring.

Definition 13.7. The *Krull dimension* of a commutative ring R is the supremum of the set of integers d for which there exists a chain of distinct prime R-ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d.$$

The Krull dimension of a ring need not be finite, even when the ring is noetherian, but the Krull dimension of $\bar{k}[x_1, \ldots, x_n]$ is finite, equal to n, and this bounds the Krull dimension of the coordinate ring of any variety $V \subseteq \mathbb{A}^n$. The following theorem implies that dimension of a V is equal to the Krull dimension of $\bar{k}[V]$.

Theorem 13.8. *Let k be a field and let R be an integral domain finitely generated as a k-algebra. The Krull dimension of R is the transcendence degree of its fraction field over k.*

Proof. See [1, Theorem 7.22]. □

Now consider a chain of distinct prime ideals in $\bar{k}[V]$ of length d equal to the Krull dimension of $\bar{k}[V]$.

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d.$$

Since $\bar{k}[V]$ is an integral domain, the zero ideal is prime, so $\mathfrak{p}_0 = (0)$ (otherwise the chain would not be maximal). There is a one-to-one correspondence between ideals of the quotient ring $\bar{k}[V] = \bar{k}[x_1, \ldots, x_n]$ and ideals of $\bar{k}[x_1, \ldots, x_n]$ that contain $I(V)$, and this correspondence preserves prime ideals (this follows from the third ring isomorphism theorem). Thus we have a chain of distinct prime ideals in $\bar{k}[x_1, \ldots, x_n]$:

$$I(V) = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_d,$$

This corresponds to a chain of distinct varieties (with inclusions reversed):

$$V_d \subsetneq V_1 \subsetneq \cdots \subsetneq V_0 = V.$$

Conversely, we could have started with a chain of distinct varieties V and obtained a chain of distinct prime ideals in $\bar{k}[V]$. This one-to-one correspondence yields an alternative definition of the dimension of V.

Definition 13.9. The *geometric dimension* of a variety V is the largest integer d for which there exists a chain

$$V_0 \subsetneq \cdots \subsetneq V_d = V$$

of distinct varieties contained in V.

The discussion above shows that this agrees with our earlier definition. This notion of dimension also works for algebraic sets: the dimension of an algebraic set Z is the largest integer d for which there exists a chain of distinct varieties (irreducible algebraic sets) contained in Z.

13.1.2 Singular points

Definition 13.10. Let $V \subseteq \mathbb{A}^n$ be a variety, and let $f_1, \ldots, f_m \in \bar{k}[x_1, \ldots, x_n]$ be a set of generators for $I(V)$. A point $P \in V$ is a *nonsingular* (or *smooth*) if the $m \times n$ Jacobian matrix $M(P)$ with entries

$$M_{ij}(P) = \frac{\partial f_i}{\partial x_j}(P)$$

has rank $n - \dim V$; otherwise P is a *singular point* of V. If V has no singular points than we say that V is *smooth.*

A useful fact that we will not prove is that if one can show that the rank of $M(P)$ is equal to $n - d$ for every point $P \in \mathbb{A}^n$, then V is a smooth variety of dimension d.

13.2 Projective space

Definition 13.11. n-dimensional *projective space* $\mathbb{P}^n$ over k is the set of all points in $\mathbb{A}^{n+1} - \{\mathbf{0}\}$ modulo the equivalence relation

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$$

for all $\lambda \in \bar{k}^\times$. We use the ratio notation $(a_0 : \ldots : a_n)$ to denote the equivalence class of $(a_0, \ldots, a_n)$, and call it a *projective point* or a *point* in $\mathbb{P}^n$. The set of k-rational points in $\mathbb{P}^n$ is

$$\mathbb{P}^n(k) = \{(a_0 : \ldots : a_n) \in \mathbb{P}^n : a_0, \ldots, a_n \in k\}$$

(and similarly for any extension of k in $\bar{k}$).

Remark 13.12. Note that $(a_0 : \ldots : a_n) \in \mathbb{P}^n(L)$ does not necessarily imply that all a_i lie in L, it simply means that there exists some $\lambda \in \bar{k}^\times$ for which all λa_i lie in L. However we do have $a_i/a_j \in L$ for all $0 \le i, j \le n$.

The absolute Galois group G_k acts on $\mathbb{P}^n$ via

$$(a_0 : \ldots : a_n)^\sigma = (a_0^\sigma : \ldots : a_n^\sigma).$$

This action is well defined, since $(\lambda P)^\sigma = \lambda^\sigma P^\sigma \sim P^\sigma$ for any $\lambda \in \bar{k}^\times$ and $P \in \mathbb{A}^{n+1} - \{\mathbf{0}\}$. We then have

$$\mathbb{P}^n(k) = (\mathbb{P}^n)^{G_k}.$$

13.3 Homogeneous polynomials

Definition 13.13. A polynomial $f \in \bar{k}[x_0, \ldots, x_n]$ is *homogenous of degree* d if

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$$

for all $\lambda \in \bar{k}$. Equivalently, every monomial in f has total degree d. We say that f is *homogeneous* if it is homogeneous of some degree.

Fix an integer $i \in [0, n]$. Given any polynomial $f \in \bar{k}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ in n variables, let d be the total degree of f and define the *homegenization* of f (with respect to x_i) to be the polynomial

$$F(x_0, \ldots, x_n) = x_i^d f\left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right).$$

Conversely, given any homogenous polynomial $F \in \bar{k}[x_0, \dots, x_n]$, the polynomial

$$f(x_0, \dots, x_{i-1}, x_{i+1}, \dots) = f(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)$$

is the *dehomegenization* of F (with respect to x_i).

Let $P = (a_0 : \dots : a_n)$ be a point in $\mathbb{P}^n$ and let and f be a homogeneous polynomial in $\bar{k}[x_0, \dots, x_n]$. The value $f(a_0, \dots, a_n)$ will depend, in general, on our choice of representative $(a_0, \dots, a_n)$ for P. However,

$$f(a_0, \dots, a_n) = 0 \quad \Longleftrightarrow \quad f(\lambda a_0, \dots, \lambda a_n) = 0 \text{ for all } \lambda \in \bar{k}^{\times}.$$

Thus it makes sense to write $f(P) = 0$ (or $f(P) \neq 0$), and the zero locus of a homogeneous polynomial is a well-defined subset of $\mathbb{P}^n$.

13.3.1 Affine covering of projective space

For $0 \leq i \leq n$, the zero locus of the homogeneous polynomial x_i is the *hyperplane*

$$H_i = \{(a_0 : \dots : a_{i-1} : 0 : a_{i+1} : \dots : a_n) \in \mathbb{P}^n\},$$

which corresponds to a copy of $\mathbb{P}^{n-1}$ embedded in $\mathbb{P}^n$.

Definition 13.14. The complement of H_i in $\mathbb{P}^n$ is the *affine patch* (or *affine chart*)

$$U_i = \{(a_0 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n) \in \mathbb{P}^n\},$$

which corresponds to a copy of $\mathbb{A}^n$ embedded in $\mathbb{P}^n$ (note that fixing $a_i = 1$ fixes a choices of representative for the projective point $(a_0 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n)$).

If we pick a hyperplane, say H_0, we can partition $\mathbb{P}^n$ as

$$\mathbb{P}^n = U_0 \sqcup H_0 \simeq \mathbb{A}^n \sqcup \mathbb{P}^{n-1}.$$

We can now apply the same procedure to $H_0 \simeq \mathbb{P}^{n-1}$, and repeating this yields

$$\mathbb{P}^n \simeq \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^1 \sqcup \mathbb{P}^0,$$

where the final $\mathbb{P}^0$ corresponds a single projective point in $\mathbb{P}^n$.

Alternatively, we can view $\mathbb{P}^n$ as the union of $n+1$ (overlapping) affine patches, each corresponding to a copy of $\mathbb{A}^n$ embedded in $\mathbb{P}^n$. Note that every projective point P lies in at least one affine patch.

Remark 13.15. Just as a manifold is locally defined in terms of an atlas of overlapping charts (each of which maps the neighborhood of a point to an open set in Euclidean space), we can view $\mathbb{P}^n$ as being locally defined in terms of its overlapping affine patches, viewing each as mapping a neighborhood of $\mathbb{P}^n$ to $\mathbb{A}^n$ (this viewpoint can be made quite rigorous, but we will not do so here).

13.4 Projective varieties

For any set S of polynomials in $\bar{k}[x_0, \ldots, x_n]$ we define the (*projective*) *algebraic set*

$$Z_S = \{P \in \mathbb{P}^n : f(P) = 0 \text{ for all homogeneous } f \in S\}.$$

Definition 13.16. A *homogeneous ideal* in $\bar{k}[x_0, \ldots, x_n]$ is an ideal that is generated by a set of homogeneous polynomials.

Note that not every polynomial in a homogeneous ideal I is homogeneous (the sum of homogeneous polynomials of different degrees is not homogeneous), but this has no impact on the algebraic set Z_I, since our definition of Z_I ignores elements of I that are not homogeneous.

Definition 13.17. Let Z be an algebraic set in $\mathbb{P}^n$, the (*homogeneous*) *ideal of* Z is the ideal $I(Z)$ generated by all the homogeneous polynomials in $\bar{k}[x_0, \ldots, x_n]$ that vanish at every point in Z.

We say that Z is *defined over* k if its ideal can be generated by homogeneous polynomials in $k[x_0, \ldots, x_n]$, and write Z/k to indicate this. If Z is defined over k the set of k-*rational points* on Z is

$$Z(k) = Z \cap \mathbb{P}^n(k) = Z^{G_k},$$

and similarly for any extension of k in $\bar{k}$.

As with affine varieties, we say that an algebraic set in $\mathbb{P}^n$ is irreducible if it is nonempty and not the union of two smaller algebraic sets in $\mathbb{P}^n$.

Definition 13.18. A (*projective*) *variety* is an irreducible algebraic set in $\mathbb{P}^n$.

As you will show on the problem set, an algebraic set $Z \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Z)$ is prime. One can then define the coordinate ring $k[V]$ and function field $k(V)$ of a projective variety exactly as in the affine case. Here we take a different approach using affine patches, which yields the same result.

Definition 13.19. Let V be a projective variety with homogeneous ideal $I = (f_1, \ldots, f_m)$. Let I_i be the ideal generated by the dehomegenizations of $f_1, \ldots, f_m$ at x_i. Then I_i is a prime ideal (since I is) and the ith *affine part* of V is the affine variety $V_i = V \cap U_i$ whose ideal is I_i. We can then write $V = \bigcup_i V_i$ as the union of its affine parts.

Definition 13.20. The *dimension* of a projective variety V is the maximum of the dimensions of its affine parts, and V is *smooth* if and only if all its affine parts are.

Finally, we define the *coordinate ring* $k[V]$ of a projective variety V/k to be the coordinate ring of any of its nonempty affine parts (we will prove below that it doesn't matter which one we pick), and the *function field* $k(V)$ of V is the fraction field of its coordinate ring, and similarly for any extension of k in $\bar{k}$.

13.5 Projective closure

Definition 13.21. If $Z \subseteq \mathbb{A}^n$ is any affine algebraic set, we can embed it in $\mathbb{P}^n$ by identifying $\mathbb{A}^n$ with the affine patch U_0 of $\mathbb{P}^n$; we write $Z \subseteq \mathbb{A}^n \subset \mathbb{P}^n$ to indicate this embedding. The *projective closure* of Z in $\mathbb{P}^n$, denoted $\overline{Z}$, is the projective algebraic set defined by the ideal generated by all the homogenizations (with respect to x_0) of all the polynomials in $I(Z)$.

When the ideal of an algebraic set $Z \subseteq \mathbb{A}^n$ is principal, say $I(Z) = (f)$, then $I(\overline{Z})$ is generated by the homogenization of f. But in general the homegenizations of a set of generators for $I(Z)$ *do not* generate $I(\overline{Z})$, as shown by the following example.

Example 13.22. Consider the *twisted cubic* $C = \{(t, t^2, t^3) : t \in \bar{k}\} \subseteq \mathbb{A}^3 \subset \mathbb{P}^3$. It is the zero locus of the ideal

$$(x^2 - y, x^3 - z)$$

in $\bar{k}[x, y, z]$, hence an algebraic set, in fact, an affine variety of dimension 1 (an *affine curve*). To see this note that $\bar{k}[C] = \bar{k}[x, y, z]/I(C) \simeq \bar{k}[x]$ is obviously an integral domain, so $I(C)$ is prime, and the function field $\bar{k}(C) \simeq \bar{k}(x)$ has transcendence degree 1.

If we homogenize the generators of $I(C)$ by introducing a new variable w, we get the homogeneous ideal $I = (x^2 - wy, x^3 - w^2z)$. The zero locus of this ideal in $\mathbb{P}^3$ is

$$\{(1 : t : t^2 : t^3) : t \in \bar{k}\} \cup \{(0 : 0 : y : z) : y, z \in \bar{k}\},$$

which ought to strike you as a bit too large to be the projective closure of C; indeed, the homogeneous polynomial $y^2 - xz$ is not in I even though $y^2 - x$ is in $I(C)$, so this cannot be $\overline{C}$. But if we instead consider the homogeneous ideal

$$(x^2 - wy,\ xy - wz,\ y^2 - xz),$$

we see that its zero locus is

$$\{(1 : t : t^2 : t^3) : t \in \bar{k}\} \cup \{(0 : 0 : 0 : 1)\},$$

and we claim this is $\overline{C}$. There are many ways to prove this, but here is completely elementary argument: Suppose that $f \in \bar{k}[w, x, y, z]$ is homogeneous of degree d, with C in its zero locus. Then the polynomial $g(t) = f(1, t, t^2, t^3)$ must be the zero polynomial (here we use that $\bar{k}$ is infinite). If $f(0, 0, 0, 1) \neq 0$, then f must contain a term of the form cz^d with $c \in \bar{k}^\times$. But then $g(t) = ct^{3d} + h(t)$ with $\deg h \leq 3(d-1) + 2 = 3d - 1 < 3d$, which means that g cannot be the zero polynomial, a contradiction. The claim follows.

Theorem 13.23. *If $V \in \mathbb{A}^n \subset \mathbb{P}^n$ is an affine variety then its projective closure $\overline{V}$ is a projective variety, and $V = \overline{V} \cap \mathbb{A}^n$ is an affine part of $\overline{V}$.*

Proof. For any polynomial $f \in \bar{k}[x_1, \ldots, x_n]$, let $\overline{f} \in \bar{k}[x_0, x_1, \ldots, x_n]$ denote its homogenization with respect to x_0. For any $f \in \bar{k}[x_1, \ldots, x_n]$ and any point $P \in \mathbb{A}^n$, we have $f(P) = 0$ if and only if $\overline{f}(\overline{P}) = 0$, where $\overline{P} = (1 : a_1 : \ldots : a_n)$ is the projective closure of P (viewing points as singleton algebraic sets). It follows that $V = \overline{V} \cap \mathbb{A}^n$.

To show that $\overline{V}$ is a projective variety, we just need to show that it is irreducible, equivalently (by Problem Set 6), that its ideal is prime. So let $fg \in I(\overline{V})$. Then fg vanishes on $\overline{V}$, hence it vanishes on V, as does the dehomegenization $f(1, x_1, \ldots, x_n)g(1, x_1, \ldots, x_n)$ But $I(V)$ is prime (since V is a variety), so either $f(1, x_1, \ldots, x_n)$ of $g(1, x_1, \ldots, x_n)$ lies in $I(V)$, and therefore one of f and g lies in $I(\overline{V})$. Thus $I(\overline{V})$ is prime. □

Theorem 13.24. *Let V be a projective variety and let V_i be any of its nonempty affine parts. Then V_i is an affine variety and V is its projective closure.*

Proof. Without loss of generality we assume $i = 0$ and use the notation introduced in the proof above, identifying $\mathbb{A}^n$ with U_0. As above, for any $f \in \bar{k}[x_1, \ldots, x_n]$ and any point $P \in \mathbb{A}^n$, we have $f(P) = 0$ if and only if $\overline{f}(\overline{P}) = 0$. It follows that V_0 is an algebraic set defined by the ideal generated by the dehomegenization of all the homogeneous polynomials in $I(V)$, and therefore $V = \overline{V_0}$.

To show that V_0 is an affine variety, we just need to check that $I(V_0)$ is a prime ideal. So let $fg \in I(V_0)$. Then $\overline{fg} \in I(V)$ and therefore either $\overline{f}$ or $\overline{g}$ is in $I(V)$ (since $I(V)$ is prime), and then either f or g must lie in $I(V_0)$. Thus $I(V_0)$ is prime. □

Remark 13.25. Theorem 13.23 is still true if "variety" is replaced by "algebraic set", but Theorem 13.24 is not.

Corollary 13.26. *The dimension, coordinate ring, and function field of an affine variety are equal to those of its projective closure. The dimension, coordinate ring, and function field of a projective variety are equal to those of each of its nonempty affine parts.*

Remark 13.27. One can define the function field of a projective variety V directly in terms of its homogeneous ideal $I(V)$ rather than identifying it with the function field of its nonempty affine pieces (all of which are isomorphic), but some care is required. The function field $\bar{k}(V)$ is *not* the fraction field of $\bar{k}[x_0, \ldots, x_n]/I(V)$, it is the subfield of $\bar{k}[x_0, \ldots, x_n]/I(V)$ consisting of all fractions g/h where g and h are both homogeneous polynomials (modulo $I(V)$) of the *same degree*, with $h \neq 0$. This restriction is necessary in order for us to sensibly think of elements of $\bar{k}(V)$ as functions from V to $\bar{k}$. In order to evaluate a function $f(x_0, \ldots, x_n)$ at a projective point $P = (a_0 : \ldots : a_n)$ in a well-defined way we must require that

$$f(\lambda a_0, \ldots, \lambda a_n) = f(a_0, \ldots, a_n)$$

for any $\lambda \in \bar{k}^\times$. If $f = g/h$ with g and h homogeneous of degree d, then

$$f(\lambda a_0, \ldots, \lambda a_n) = \frac{g(\lambda a_0, \ldots, \lambda a_n)}{h(\lambda a_0, \ldots, \lambda a_n)} = \frac{\lambda^d g(a_0, \ldots, a_n)}{\lambda^d h(a_0, \ldots, a_n)} = \frac{g(a_0, \ldots, a_n)}{h(a_0, \ldots, a_n)} = f(a_0, \ldots, a_n),$$

as required. With this definition the function field $\bar{k}(V)$ is isomorphic to the function field of each of its nonempty affine parts.

References

[1] A. Knapp, *Advanced Algebra*, Springer, 2007.