18.745 Introduction to Lie Algebras September 14, 2010

Lecture 2 — Some Sources of Lie Algebras

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From Associative Algebras

We saw in the previous lecture that we can form a Lie algebra A_-, from an associative algebra A, with binary operation the commutator bracket $[a, b] = ab - ba$. We also saw that this construction works for algebras satisfying any one of a variety of other conditions.

As Algebras of Derivations

Lie algebras are often constructed as the algebra of derivations of a given algebra. This corresponds to the use of vector fields in geometry.

Definition 2.1. For any algebra A over a field $\mathbb{F}$, a derivation of A is an $\mathbb{F}$-vector space endomorphism D of A satisfying $D(ab) = D(a)b + aD(b)$. Let $\mathrm{Der}(A) \subset \mathfrak{gl}_A$ be denote the space of derivations of A.

For an element a of a Lie algebra $\mathfrak{g}$, define a map $\mathbf{ad}(a) : \mathfrak{g} \to \mathfrak{g}$, by $b \mapsto [a, b]$. This map is referred to as the adjoint operator. Rewriting the Jacobi identity as

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]], \tag{1}$$

we see that $\mathbf{ad}(a)$ is a derivation of $\mathfrak{g}$. Derivations of this form are referred to as inner derivations of $\mathfrak{g}$.

Proposition 2.1.

(a) $\mathrm{Der}(A)$ *is a subalgebra of* $\mathfrak{gl}_A$ *(with the usual commutator bracket).*

(b) The inner derivations of a Lie algebra $\mathfrak{g}$ *form an ideal of* $\mathrm{Der}(\mathfrak{g})$*. More precisely,*

$$[D, \mathbf{ad}(a)] = \mathbf{ad}(D(a)) \text{ for all } D \in \mathrm{Der}(\mathfrak{g}) \text{ and } a \in \mathfrak{g}. \tag{2}$$

Proof of (b): We check (2) by applying both sides to $b \in \mathfrak{g}$:

$$[D, \mathbf{ad}(a)]b = D[a, b] - [a, Db] = [Da, b] = \mathbf{ad}(Da)b,$$

where the second equality holds as D is a derivation. □

Exercise 2.1. Prove (a).

Solution: The derivations of A are those maps $D \in \mathfrak{gl}_A$ which satisfy $D(ab) - D(a)b - aD(b) = 0$ for all a and b in A. For fixed a and b, the left hand side of this equation is linear in D, so that the set of endomorphisms satisfying that single equation is a subspace. The set of dervations is the intersection over all a and b in A of these subspaces, which is a subspace.

We are only left to check that the bracket of two derivations is a derivation. For any $a, b \in A$ and $D_1, D_2 \in \mathrm{Der}(A)$ we calculate:

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b)) \\
&= D_1D_2(a)b + D_2(a)D_1(b) + D_1(a)D_2(b) + aD_1D_2(b) \\
&\quad - \{D_2D_1(a)b + D_1(a)D_2(b) + D_2(a)D_1(b) + aD_2D_1(b)\} \\
&= D_1D_2(a)b - D_2D_1(a)b + aD_1D_2(b) - aD_2D_1(b) \\
&= [D_1, D_2](a)b + a[D_1, D_2](b).
\end{aligned}$$

Thus the derivations are closed under the bracket, and so form a Lie subalgebra of $\mathfrak{gl}_A$. □

From Poisson Brackets

Exercise 2.2. Let $A = \mathbb{F}[x_1, \ldots, x_n]$, or let A be the ring of C^∞ functions on $x_1, \ldots, x_n$. Define a Poisson bracket on A by:

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}, \text{ for fixed choices of } \{x_i, x_j\} \in A. \tag{3}$$

Show that this bracket satisfies the axioms of a Lie algebra if and only if $\{x_i, x_i\} = 0$, $\{x_i, x_j\} = -\{x_j, x_i\}$ and any triple x_i, x_j, x_k satisfy the Jacobi identity.

Solution: If the Poisson bracket defines a Lie algebra structure for some choice of values $\{x_i, x_j\}$, then in particular, the axioms of a Lie algebra must be satisfied for brackets of terms x_i. The interesting question is whether the converse holds. We suppose then that the $\{x_i, x_j\}$ are chosen so that $\{x_i, x_j\} = -\{x_j, x_i\}$, and so that the Jacobi identity is satisfied for triples x_i, x_j, x_k.

The bilinearity of the bracket follows from the linearity of differentiation, and the skew-symmetry follows from the assumption of the skew symmetry on the x_i.

At this point we introduce some shorthands to simplify what follows. If f is any function, we write f_i for the derivative of f with respect to x_i. When we are discussing an expression e in terms of three functions f, g, h, we will write $\mathrm{CS}(e)$ for the 'cyclic summation' of e, the expression formed by summing those obtained from e by permuting the f, g, h cyclically. In particular, the Jacobi identity will be $\mathrm{CS}(\{f, \{g, h\}\}) = 0$.

First we calculate the iterated bracket of monomials x_i:

$$\{x_i, \{x_j, x_k\}\} = \sum_l \{x_j, x_k\}_l \{x_i, x_l\} \text{ (an example of the shorthands described).}$$

Now the iterated bracket of any three polynomials (or functions) f, g and h is:

$$\{h, \{f, g\}\} = \sum_{i,j,k,l} [f_{il} g_j h_k + g_{jl} f_i h_k] \{x_i, x_j\}\{x_k, x_l\} + \sum_{i,j,k,l} f_i g_j h_k \{x_i, x_j\}_l \{x_k, x_l\}.$$

By the assumption that the Jacobi identity holds on the x_i, we have (for any i, j, k):

$$\sum_l \mathrm{CS}(f_i g_j h_k)\{x_i, x_j\}_l \{x_k, x_l\} = 0,$$

for cyclicly permuting the f, g, h corresponds to cyclicly permuting the i, j, k (in the opposite order). Thus we have:

$$\mathrm{CS}\left(\{h, \{f, g\}\}\right) = \sum_{i,j,k,l} \mathrm{CS}(f_{il} g_j h_k + g_{jl} f_i h_k)\{x_i, x_j\}\{x_k, x_l\}.$$

The remaining task can be viewed as finding the $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$ coefficient in this expression, where we substitute all appearences of $\{x_\beta, x_\alpha\}$ for $-\{x_\alpha, x_\beta\}$, and so on. To do so, we tabulate all the appearances of terms which are multiples of $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$. We may as well assume here that $\alpha < \beta$ and $\gamma < \delta$.

i	j	k	l	multiple of $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$	i	j	k	l	multiple of $\{x_\alpha, x_\beta\}\{x_\gamma, x_\delta\}$
α	β	γ	δ	$+ f_{\alpha\delta} g_\beta h_\gamma + g_{\beta\delta} f_\alpha h_\gamma$	γ	δ	α	β	$+ f_{\gamma\beta} g_\delta h_\alpha + g_{\delta\beta} f_\gamma h_\alpha$
β	α	γ	δ	$- f_{\beta\delta} g_\alpha h_\gamma - g_{\alpha\delta} f_\beta h_\gamma$	δ	γ	α	β	$- f_{\delta\beta} g_\gamma h_\alpha - g_{\gamma\beta} f_\delta h_\alpha$
α	β	δ	γ	$- f_{\alpha\gamma} g_\beta h_\delta - g_{\beta\gamma} f_\alpha h_\delta$	γ	δ	β	α	$- f_{\gamma\alpha} g_\delta h_\beta - g_{\delta\alpha} f_\gamma h_\beta$
β	α	δ	γ	$+ f_{\beta\gamma} g_\alpha h_\delta + g_{\alpha\gamma} f_\beta h_\delta$	δ	γ	β	α	$+ f_{\delta\alpha} g_\gamma h_\beta + g_{\gamma\alpha} f_\delta h_\beta$
α	β	γ	δ	$+ g_{\alpha\delta} h_\beta f_\gamma + h_{\beta\delta} g_\alpha f_\gamma$	γ	δ	α	β	$+ g_{\gamma\beta} h_\delta f_\alpha + h_{\delta\beta} g_\gamma f_\alpha$
β	α	γ	δ	$- g_{\beta\delta} h_\alpha f_\gamma - h_{\alpha\delta} g_\beta f_\gamma$	δ	γ	α	β	$- g_{\delta\beta} h_\gamma f_\alpha - h_{\gamma\beta} g_\delta f_\alpha$
α	β	δ	γ	$- g_{\alpha\gamma} h_\beta f_\delta - h_{\beta\gamma} g_\alpha f_\delta$	γ	δ	β	α	$- g_{\gamma\alpha} h_\delta f_\beta - h_{\delta\alpha} g_\gamma f_\beta$
β	α	δ	γ	$+ g_{\beta\gamma} h_\alpha f_\delta + h_{\alpha\gamma} g_\beta f_\delta$	δ	γ	β	α	$+ g_{\delta\alpha} h_\gamma f_\beta + h_{\gamma\alpha} g_\delta f_\beta$
α	β	γ	δ	$+ h_{\alpha\delta} f_\beta g_\gamma + f_{\beta\delta} h_\alpha g_\gamma$	γ	δ	α	β	$+ h_{\gamma\beta} f_\delta g_\alpha + f_{\delta\beta} h_\gamma g_\alpha$
β	α	γ	δ	$- h_{\beta\delta} f_\alpha g_\gamma - f_{\alpha\delta} h_\beta g_\gamma$	δ	γ	α	β	$- h_{\delta\beta} f_\gamma g_\alpha - f_{\gamma\beta} h_\delta g_\alpha$
α	β	δ	γ	$- h_{\alpha\gamma} f_\beta g_\delta - f_{\beta\gamma} h_\alpha g_\delta$	γ	δ	β	α	$- h_{\gamma\alpha} f_\delta g_\beta - f_{\delta\alpha} h_\gamma g_\beta$
β	α	δ	γ	$+ h_{\beta\gamma} f_\alpha g_\delta + f_{\alpha\gamma} h_\beta g_\delta$	δ	γ	β	α	$+ h_{\delta\alpha} f_\gamma g_\beta + f_{\gamma\alpha} h_\delta g_\beta$

Of course, if $\alpha = \gamma$ and $\beta = \delta$, there is repetition, so that the right hand columns of this table must be ignored. Whether or not this is the case, the reader will be able to arrange the entries of this table into cancelling pairs. □

Example 2.1. Let $A = \mathbb{F}[p_1, \ldots, p_n, q_1, \ldots, q_n]$. Let $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_i, q_j\} = -\{q_i, p_j\} = \delta_{i,j}$. Both conditions clearly hold, and explicitly:

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}$$

is a Lie algebra bracket, important in classical mechanics.

Via Structure Constants

Given a basis $e_1, \ldots, e_n$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, the bracket is determined by the structure constants $c_{ij}^k \in \mathbb{F}$, defined by:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

The structure constants must satisfy the obvious skew-symmetry condition ($c_{ii}^k = 0$ and $c_{ij}^k = -c_{ji}^k$), and a more complicated (quadratic) condition corresponding to the Jacobi identity.

Definition 2.2. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$, be two Lie algebras over $\mathbb{F}$ and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ a linear map. We say that φ is a homomorphism if it preserves the bracket: $\varphi([a,b]) = [\varphi(a), \varphi(b)]$, and an isomorphism if it is bijective. If there exists an isomorphism φ, we say that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, written $\mathfrak{g}_1 \cong \mathfrak{g}_2$.

Exercise 2.3. Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be homomorphism. Then:

(a) $\ker \varphi$ is an ideal of $\mathfrak{g}_1$.

(b) im φ is a subalgebra of $\mathfrak{g}_2$.

(c) im $\varphi \cong g_1/\ker \varphi$.

Solution: Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be homomorphism of Lie algebras. Then:

(a) The kernel $\ker \varphi$ is an subspace of $\mathfrak{g}_1$, as in particular φ is $\mathbb{F}$-linear. Furthermore, if $x \in \ker \varphi$ and $y \in \mathfrak{g}_1$, we have $\varphi([x,y]) = [\varphi(x), \varphi(y)] = [0, \varphi(y)] = 0$. Thus $[x,y] \in \ker \varphi$. This shows that $\ker \varphi$ is an ideal of $\mathfrak{g}$.

(b) The image im φ is a subspace, again as φ is $\mathbb{F}$-linear. Now for any $u, v \in$ im φ, we may write $u = \varphi(x)$ and $v = \varphi(y)$ for elements $x, y \in \mathfrak{g}_1$. Then $[u,v] = [\varphi(x), \varphi(y)] = \varphi([x,y]) \in$ im φ. Thus the image is a subalgebra.

(c) Consider the map $\psi : \mathfrak{g}_1/\ker \varphi \to$ im φ given by $x + \ker \varphi \mapsto \varphi(x)$. We must first see that ψ is well defined. If $x + \ker \varphi = x' + \ker \varphi$, then $x' - x \in \ker \varphi$, so that:

$$\varphi(x) = \varphi(x) + \varphi(x' - x) = \varphi(x) + \varphi(x') - \varphi(x) = \varphi(x').$$

Thus our definition of ψ does not depend on choice of representative, and ψ is well defined. It is trivial to see that ψ is a homomorphism. Now suppose that $x + \ker \varphi \in \ker \psi$. Then $\varphi(x) = 0$, so that $x \in \ker \varphi$, and $x + \ker \varphi = 0 + \ker \varphi$. Thus ψ is injective, and that ψ is surjective is obvious. Thus ψ is an isomorphism $\mathfrak{g}_1/\ker \varphi \to$ im φ. □

As the Lie Algebra of an Algebraic (or Lie) Group

Definition 2.3. An algebraic group G over a field $\mathbb{F}$ is a collection $\{P_\alpha\}_{\alpha \in I}$ of polynomials on the space of matrices $\mathrm{Mat}_n(\mathbb{F})$ such that for any unital commutative associative algebra A over $\mathbb{F}$, the set

$$G(A) := \{g \in \mathrm{Mat}_n(A) \mid g \text{ is invertible, and } P_\alpha(g) = 0 \text{ for all } \alpha \in I\}$$

is a group under matrix multiplication.

Example 2.2. The general linear group GL_n. Let $\{P_\alpha\} = \emptyset$, so that $\mathrm{GL}_n(A)$ is the set of invertible matrices with entries in A. This is a group for any A, so that GL_n is an algebraic group.

Example 2.3. The special linear group SL_n. Let $\{P_\alpha\} = \{\det(x_{ij}) - 1\}$, so that $\mathrm{SL}_n(A)$ is the set of invertible matrices with entries in A and determinant 1. This is a group for any A, so that SL_n is an algebraic group.

Exercise 2.4. Given $B \in \mathrm{Mat}_n(\mathbb{F})$, let $O_{n,B}(A) = \{g \in \mathrm{GL}_n(A) : g^{\mathrm{T}} B g = B\}$. Show that this family of groups is given by an algebraic group.

Solution: For any unital commutative associative algebra A over $\mathbb{F}$, the set

$$O_{n,B}(A) = \{g \in \mathrm{GL}_n(A) : g^{\mathrm{T}} B g = B\}$$

is a group under matrix multiplication, as if $g, h \in O_{n,B}(A)$ we have:

$$(gh)^{\mathrm{T}} B (gh) = h^{\mathrm{T}} g^{\mathrm{T}} B g h = h^{\mathrm{T}} B h = B, \text{ and } \left(g^{-1}\right)^{\mathrm{T}} B g^{-1} = \left(g^{\mathrm{T}}\right)^{-1} \left(g^{\mathrm{T}} B g\right) g^{-1} = B.$$

We then only have to show that the condition $g^{\mathrm{T}} B g = B$ can be written as a collection of polynomial equations in the entries g_{ij} of the matrix g, with coefficients in $\mathbb{F}$. This is obvious — we have one polynomial equation for each of the n^2 entries of the matrix, and the coefficients depend only on the entries of B. □

Definition 2.4. Over a given field $\mathbb{F}$, define the algebra of dual numbers D to be

$$D := \mathbb{F}[\epsilon]/(\epsilon^2) = \{a + b\epsilon \mid a, b \in \mathbb{F},\ \epsilon^2 = 0\}.$$

We then define the Lie algebra **Lie** G of an algebraic group G to be

$$\textbf{Lie } G := \{X \in \mathfrak{gl}_n(\mathbb{F}) \mid I_n + \epsilon X \in G(D)\}.$$

Example 2.4. (1) **Lie** $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F})$, since $(I_n + \epsilon X)^{-1} = I_n - \epsilon X$. ($I_n - \epsilon X$ approximates the inverse to order two, but over dual numbers, order two is ignored).

(2) **Lie** $\mathrm{SL}_n = \mathfrak{sl}_n(\mathbb{F})$.

(3) **Lie** $O_{n,B} = o_{\mathbb{F}^n, B}$.

Exercise 2.5. Prove (2) and (3) from example 2.4.

Solution: For (2), We need only prove the formula $\det(I_n + \epsilon X) = 1 + \epsilon \operatorname{tr}(X)$. It is trivial when $n = 1$, and we proceed by induction on n. Consider the matrix $I_n + \epsilon X$, and the cofactor expansion of the determinant along the final column. If $i < n$, the (i, n) entry is a multiple of ϵ, and so is every entry of the i^{th} column of the matrix obtained by removing the row and column containing (i, n). Thus the corresponding cofactor has no contribution to the overall determinant. The determinant is therefore $1 + \epsilon X_{nn}$ multiplied by the minor corresponding to (n, n). By induction, the determinant is $(1 + \epsilon X_{nn})(1 + \epsilon(\operatorname{tr}(X) - X_{nn}))$. The result follows.

For (3), the following calculation gives the result:

$$(1 + \epsilon X)^{\mathrm{T}} B (1 + \epsilon X) = B + \epsilon X^{\mathrm{T}} B + \epsilon X^{\mathrm{T}} B + \epsilon^2 X^{\mathrm{T}} B X = B + \epsilon(X^{\mathrm{T}} B + X^{\mathrm{T}} B).$$ □

Theorem 2.2. ***Lie*** *G is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{F})$.*

Proof. We first show that **Lie** G is a subspace. Indeed, $X \in$ **Lie** G iff $P_\alpha(I_n + \epsilon X) = 0$ for all α. Using the Taylor expansion:

$$P_\alpha(I_n + \epsilon X) = P_\alpha(I_n) + \sum_{i,j} \frac{\partial P_\alpha}{\partial x_{ij}}(I_n)\, \epsilon\, x_{ij},$$

as $\epsilon^2 = 0$. Now as $P_\alpha(I_n) = 0$ (every group contains the identity), this condition is linear in the x_{ij}, so that **Lie** G is a subspace.

Now suppose that $X, Y \in \textbf{Lie } G$. We wish to prove that $XY - YX \in \textbf{Lie } G$. We have:

$$I_n + \epsilon X \in G(\mathbb{F}[\epsilon]/(\epsilon^2)), \text{ and } I_n + \epsilon' Y \in G(\mathbb{F}[\epsilon']/((\epsilon')^2)).$$

Viewing these as elements of $G(\mathbb{F}[\epsilon, \epsilon']/(\epsilon^2, (\epsilon')^2))$, we have

$$(I_n + \epsilon X)(I_n + \epsilon' Y)(I_n + \epsilon X)^{-1}(I_n + \epsilon' Y)^{-1} = I_n + \epsilon\epsilon'(XY - YX) \in G(\mathbb{F}[\epsilon, \epsilon']/(\epsilon^2, (\epsilon')^2)).$$

Hence $I_n + \epsilon\epsilon'(XY - YX) \in G(\mathbb{F}[\epsilon\epsilon']/((\epsilon\epsilon')^2)) = G(D)$, so that $XY - YX \in \textbf{Lie } G$. $\square$