1 Finite dimensional representations of semisimple Lie algebras

Let \( g \) be a finite dimensional semisimple Lie algebra, over an algebraically closed field \( \mathbb{F} \) of characteristic 0. Choose a Cartan subalgebra \( h \subset g \) and a subset of positive roots \( \Delta_+ \subset h^* \). Let

\[
    g = \mathfrak{N}_- \oplus h \oplus \mathfrak{N}_+.
\]

be the triangular decomposition. Recall that \( \mathfrak{N}_+ \) (resp. \( \mathfrak{N}_- \)) is generated by the vectors \( E_1, \ldots, E_r \) (resp. \( F_1, \ldots, F_r \)) or, equivalently, that \( \mathfrak{N}_\pm = \oplus_{\alpha \in \Delta_+} g_{\pm \alpha} \). Let us define

\[
    b = h \oplus \mathfrak{N}_+.
\]

\( b \) is called a Borel subalgebra. Note that

\[
    [b, b] = \mathfrak{N}_+.
\]

Indeed, \([b, b] \subset \mathfrak{N}_+\), follows immediately by the definition of \( b \), while \([b, b] \supset \mathfrak{N}_+\) follows from the fact that \([h, g_\alpha] \neq 0\), if \( \alpha(h) \neq 0 \) and such \( h \) always exists, since \( \alpha \neq 0 \). As \( \mathfrak{N}_+ \) is a nilpotent subalgebra, we see that \( b \) is a solvable subalgebra. Moreover, \( b \) is a maximal solvable subalgebra (and all such subalgebras are conjugated).

Since by Weyl’s complete reducibility theorem, every finite dimensional \( g \)-module is a direct sum of irreducible ones, it suffices to study finite dimensional, irreducible \( g \)-modules.

**Proposition 1.1.** Let \( V \) be a finite dimensional, irreducible \( g \)-module. Then \( \exists \Lambda \in h^* \) and \( 0 \neq v_\Lambda \in V \) s.t. the following three properties hold:

\[
    \begin{align*}
        &i) \quad hv_\Lambda = \Lambda(h)v_\Lambda, \forall h \in h^*; \\
        &ii) \quad \mathfrak{N}_+v_\Lambda = 0; \\
        &iii) \quad \mathfrak{U}(g)v_\Lambda = V.
    \end{align*}
\]

It follows immediately that property \( iii) \) is equivalent to the following property

\[
    iii)' \quad \mathfrak{U}(\mathfrak{N}_-)
\]

**Proof.** By Lie’s Theorem, \( b \) has an eigenvector \( 0 \neq v \in V \) so that \( \forall b \in b, \tilde{\Lambda}(b)v, \) for some \( \tilde{\Lambda} \in h^* \). But, by the property illustrated in (1), we see that \( \tilde{\Lambda}(\mathfrak{N}_+) = 0 \), since \( \Lambda([b_1, b_2]) = \Lambda(b_1)\Lambda(b_2) - \Lambda(b_2)\Lambda(b_1) \). Let \( \Lambda = \tilde{\Lambda}|_h \in h^* \), then \( i) \) and \( ii) \) hold and \( iii) \) follows from the irreducibility of the \( g \)-module \( V \), since \( \mathfrak{U}(g)v_\Lambda \) (we are identifying \( v_\Lambda = v \)) is a non-zero submodule of \( V \) (it contains \( v_\Lambda \) since \( Id \in \mathfrak{U}(g) \)).

Definition 1.1. A \( \mathfrak{g} \)-module \( V \) (not necessarily finite dimensional) with the property that \( \exists \Lambda \in \mathfrak{h}^* \) and \( 0 \neq v_\Lambda \in V \) such that properties i), ii), iii) from the previous proposition hold, is called highest weight module with highest weight \( \Lambda \) and \( v_\Lambda \) is called a highest weight vector.

Let \( \Delta_+ = \{ \beta_1, \ldots, \beta_s \} \) be the set of positive roots for \( \mathfrak{g} \). Choose root vectors \( E_{\beta_i} \in \mathfrak{N}_+ \), \( E_{-\beta_i} \in \mathfrak{N}_- \) and let \( h_1, \ldots, H_n \) be a basis for \( \mathfrak{h} \), then vectors \( E_{\beta_i}, E_{-\beta_i} \) \((i = 1, \ldots, N)\), \( h_j \) \((j = 1, \ldots, n)\) form a basis for \( \mathfrak{g} \). By PBW theorem, monomials of the form

\[
E_{-\beta_1}^{m_1} \cdots E_{-\beta_s}^{m_s} H_1^{s_1} \cdots H_n^{s_n} E_{\beta_1}^{n_1} \cdots E_{\beta_s}^{n_s}, \quad m_i, n_j, s_k \in \mathbb{Z}_+.
\]

In particular

**Definition 1.2.** For an arbitrary \( \mathfrak{g} \)-module \( V \), let \( h \) be an element of \( \mathfrak{h}^* \), we denote \( V_\lambda = \{ v \in V \mid hv = \lambda(h)v, \forall h \in \mathfrak{h} \} \) the weight space for \( \mathfrak{h} \) attached to \( \lambda \). A non-zero vector \( v \in V_\lambda \) is called singular of weight \( \lambda \) if \( \mathfrak{N}_+ v = 0 \).

**Example 1.1.** Any \( \Lambda \in \mathfrak{h}^* \) is a singular weight of a highest weight \( \mathfrak{g} \)-module with highest weight \( \Lambda \).

**Notation 1.1.** Given \( \Lambda \in \mathfrak{h}^* \), let \( D(\Lambda) = \{ \Lambda - \sum_{i=1}^r k_i \alpha_i : k_i \in \mathbb{Z}_+ \} \subset \mathfrak{h}^* \), where \( \Pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \} \) is the set of simple roots of \( \mathfrak{g} \).

**Theorem 1.2.** Let \( V \) be a highest weight \( \mathfrak{g} \)-module with highest weight \( \Lambda \in \mathfrak{h}^* \). Then,

(a) \( V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda \)

(b) \( V_\Lambda = \mathbb{F}v_\Lambda \) and \( \dim V_\Lambda < \infty \)

(c) \( V \) is an irreducible \( \mathfrak{g} \)-module if and only if \( \mathbb{F}^* v_\Lambda \) are the only singular vectors.

(d) \( V \) contains a unique proper maximal submodule.

(e) If \( v \) is a singular vector with weight \( \lambda \), then \( \Omega(v) = (\lambda + 2\rho, \lambda)v \). Here \( (\cdot, \cdot) \) is a non-degenerate symmetric invariant bilinear form on \( \mathfrak{g} \) and \( \Omega \) is the corresponding Casimir operator, and 

\[
2\rho = \sum_{\alpha \in \Delta_+} \alpha.
\]

(f) \( \Omega|_V = (\Lambda + 2\rho, \Lambda)Id_V \)

(g) If \( \lambda \) is a singular weight, then \( (\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho) \).

Proof: Pf a), b)

By iii), \( V = U(\mathfrak{n}_-)v_\Lambda = \sum \mathbb{F} E_{-\beta_1}^{m_1} \cdots E_{-\beta_N}^{m_N} v_\Lambda \in V_\Lambda = \sum_{i=1}^N m_i \beta_i \in D(\Lambda) \), proving a) and b).

Pf c)

We know

\[
(*) \quad U = \bigoplus_{\lambda \in D(\Lambda)} (U \cap V_\lambda)
\]

for a submodule \( U \) by a previous lecture. So choose \( \lambda \in D(\Lambda) \) to be of minimal height with \( U \cap V_\Lambda \neq 0 \). Then \( E_\alpha v = 0 \) for any \( v \in U \cap V_\Lambda \), so \( v \) is a singular vector. And if \( v \) is a singular vector of weight \( \lambda \), then \( U(\mathfrak{g})v = U(\mathfrak{n}_-)v \) which is a proper submodule of \( V \) unless \( \lambda = \Lambda \).
Pf d) 

The sum of proper submodules of $V$ is again a proper submodule because it does not contain $v_\Lambda$. Thus this sum is a unique maximal submodule.

Pf e) 

Take a a basis $\{E_{\beta_i}, E_{-\beta_i}, H_i\}$ and its dual $\{E_{-\beta_i}, E_{\beta_i}, H^i\}$ and compute Casimir operator $\Omega = \sum_i H_i H^i + \sum_{i}^N E_{\beta_i} E_{-\beta_i} + E_{-\beta_i} E_{\beta_i} = \sum_i H_i H^i + 2 \sum_{i}^N E_{-\beta_i} E_{\beta_i} + 2\nu^{-1}\alpha$. Apply this to a singular vector $v_\lambda$ to get

$$\Omega v_\lambda = \sum_i \lambda(H_i)v_\lambda + \sum_1^N (\lambda, \beta_i)v_\lambda + 0$$

The right hand side is $(\lambda, \lambda) + 2(\lambda, \rho)$.

Pf f) 

$$\Omega v_\lambda = (\Lambda + 2\rho, \Lambda)v_\lambda$$ by e) and since $\Omega$ commutes with $U(\mathfrak{g})$ we get $\Omega(E^m_{-\beta_1}....E_{-\beta_N}v_\lambda) = (\Lambda + 2\rho, \Lambda)E^m_{-\beta_1}....E_{-\beta_N}v_\lambda$

Pf g) 

follows from f) and e).

Pf h) 

If $\lambda$ is singular weight, then $(\lambda + 2\rho, \lambda) = (\Lambda + 2\rho, \Lambda)$ by g). This describes a compact set in which the singular weights must lie. But $\lambda \in D(\Lambda)$, a discrete set. As the intersection of a discrete set and compact set is finite, we have that the singular weights must be finite in number.

A Verma module $M(\Lambda)$ is highest weight module with highest weight $\Lambda$ such that any other module with highest weight $\Lambda$ is quotient of $M(\Lambda)$. We construct $M(\Lambda)$ as $U(\mathfrak{g})/U(\mathfrak{g})(n_+; h - \Lambda(h), h \in \mathfrak{h})$

By Theorem 1 d), $M(\Lambda)$ has unique maximum submodule $J(\Lambda)$ such that $L(\Lambda) = M(\Lambda)/J(\Lambda)$ is unique highest weight module with highest weight $\Lambda$.

**Theorem 1.3.** (a) For any $\Lambda \in \mathfrak{h}^*$, there exists a Verma module $M(\Lambda)$, unique up to isomorphism.

(b) $M(\Lambda)$ has unique irreducible quotient $L(\Lambda)$

(c) $M(\Lambda) = M(\Lambda')$ (resp. $L(\Lambda) = L(\Lambda')$) iff $\Lambda = \Lambda'$

(d) $E^m_{-\beta_1}....E^m_{-\beta_N}v_\lambda$ form basis of $M(\Lambda)$.

**Proof:** a), b), c) are clear. d) follows from the PBW theorem because $E^m_{-\beta_1}....E^m_{-\beta_N}$ never lies in $J(\Lambda)$. 

3