18.745 Introduction to Lie Algebras	December 7, 2010
Lecture 24 — Finite dimensional $\mathfrak{g}$-modules over a s.s. Lie algebra.	
Prof. Victor Kac	*Scribe: Mario De Franco, Roberto Svaldi*

1 Finite dimensional representations of semisimple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional semisimpe Lie algebra, over an algebraically closed field $\mathbb{F}$ of characteristic 0. Choose a Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and a subset of positive roots $\Delta_+ \subset \mathfrak{h}^*$. Let

$$\mathfrak{g} = \mathfrak{N}_- \oplus \mathfrak{h} \oplus \mathfrak{N}_+$$

be the triangular decomposition. Recall that $\mathfrak{N}_+$ (resp. $\mathfrak{N}_-$) is generated by the vectors $E_1, \dots, E_r$ (resp. $F_1, \dots, F_r$) or, equivanlenty, that $\mathfrak{N}_\pm = \oplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}$. Let us define

$$\mathfrak{b} \doteqdot \mathfrak{h} \oplus \mathfrak{N}_+.$$

$\mathfrak{b}$ is called a Borel subalgebra. Note that

$$[\mathfrak{b}, \mathfrak{b}] = \mathfrak{N}_+. \tag{1}$$

Indeed, $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{N}_+$, follows immediately by the definition of $\mathfrak{b}$, while $[\mathfrak{b}, \mathfrak{b}] \supset \mathfrak{N}_+$ follows from the fact that $[h, \mathfrak{g}_\alpha] \neq 0$, if $\alpha(h) \neq 0$ and such h always exists, since $\alpha \neq 0$. As $\mathfrak{N}_+$ is a nilpotent subalgebra, we see that $\mathfrak{b}$ is a solvable subalgebra. Moreover, $\mathfrak{b}$ is a maximal solvable subalgebra (and all such subalgebras are conjugated).
Since by Weyl's complete reducibility theorem, every finite dimensional $\mathfrak{g}$-module is a direct sum of irreducible ones, it suffices to study finite dimensional, irreducible $\mathfrak{g}$-modules.

Proposition 1.1. *Let V be afinite dimensional, irreducile $\mathfrak{g}$-module. Then $\exists \Lambda \in \mathfrak{h}^*$ and $0 \neq v_\Lambda \in V$ s.t. the following three properties hold:*

i) $hv_\Lambda = \Lambda(h)v_\Lambda$, $\forall h \in \mathfrak{h}^*$;

ii) $\mathfrak{N}_+ v_\Lambda = 0$;

iii) $\mathfrak{U}(\mathfrak{g})v_\Lambda = V$.

It follows immediately that property iii) is equivalent to the following property

$$iii)' \ \mathfrak{U}(\mathfrak{N}_-)$$

Proof. By Lie's Theorem, $\mathfrak{b}$ has an eigenvector $0 \neq v \in V$ so that $\forall b \in \mathfrak{b}$, $\tilde{\Lambda}(b)v$, for some $\tilde{\Lambda} \in \mathfrak{h}^*$. But, by the property illustrated in (1), we see that $\tilde{\Lambda}(\mathfrak{N}_+) = 0$, since $\tilde{\Lambda}([b_1, b_2]) = \Lambda(b_1)\Lambda(b_2) - \Lambda(b_2)\Lambda(b_1)$. Let $\Lambda = \tilde{\Lambda}_{|\mathfrak{h}} \in \mathfrak{h}^*$, then *i)* and *ii)* hold and *iii)* follows from the irreducibility of the $\mathfrak{g}$-module V, since $\mathfrak{U}(g)v_\lambda$ (we are identifying $v_\Lambda = v$) is a non-zero submodule of V (it contains v_Λ since $Id \in \mathfrak{U}(\mathfrak{g})$). □

Definition 1.1. A $\mathfrak{g}$-module V (not necessarily finite dimensional) with the property that $\exists \Lambda \in \mathfrak{h}^*$ and $0 \neq v_\Lambda \in V$ such that properties $i), ii), iii)$ from the previous proposition hold, is called highest weight module with heighest weight Λ and v_λ is called a heighest weight vector.

Let $\Delta_+ = \{\beta_1, \ldots, \beta_r\}$ be the set of positive roots for $\mathfrak{g}$. Choose root vectors $E_{\beta_i} \in \mathfrak{N}_+,\ E_{-\beta_i} \in \mathfrak{N}_-$ and let $h_1, \ldots, H_n$ be a nasis for $\mathfrak{h}$, then vectors $E_{\beta_i}, E_{-\beta_i}$ $(i = 1, \ldots, N),\ h_j$ $(j = 1, \ldots, n)$ form a basis for $\mathfrak{g}$. By PBW theorem, monomials of the form

$$E_{-\beta_1}^{m_1} \ldots E_{-\beta_N}^{m_N} H_1^{s_1} \ldots H_r^{s_r} E_{\beta_1}^{n_1} \ldots E_{\beta_N}^{n_N},\ m_i, n_j, s_k \in \mathbb{Z}_+.$$

In particular

Definition 1.2. For an arbitrary $\mathfrak{g}$-module V, let h be an element of $\mathfrak{h}^*$, we denote $V_\lambda = \{v \in V \mid hv = \lambda(h)v,\ \forall h \in \mathfrak{h}\}$ the weight space for $\mathfrak{h}$ attached to λ. A non-zero vector $v \in V_\lambda$ is called singular of weight λ if $\mathfrak{N}_+ v = 0$.

Example 1.1. Any $\Lambda \in \mathfrak{h}^*$ is a singular weight of a highest weight $\mathfrak{g}$-module with highest weight Λ.

Notation 1.1. Given $\Lambda \in \mathfrak{h}^*$, let $D(\Lambda) = \{\Lambda - \sum_{i=1}^r k_i\alpha_i : k_i \in \mathbb{Z}_+\} \subset \mathfrak{h}^*$, where $\Pi = \{\alpha_1,\ \alpha_2,\ \ldots\ \alpha_r\}$ is the set of simple roots of $\mathfrak{g}$.

Theorem 1.2. *Let V be a highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$. Then,*

(a) $V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda$

(b) $V_\Lambda = \mathbb{F}v_\Lambda$ *and* $\dim V_\lambda < \infty$

(c) V *is an irreducible* $\mathfrak{g}$*-module if and only if* $\mathbb{F}^* v_\Lambda$ *are the only singular vectors.*

(d) V *contains a unique proper maximal submodule.*

(e) If v *is a singular vector with weight* λ*, then* $\Omega(v) = (\lambda + 2\rho, \lambda)v$*. Here* $(\cdot, \cdot)$ *is a non-degenerate symmetric invariant bilinear form on* $\mathfrak{g}$ *and* Ω *is the corresponding Casimir operator, and* $2\rho = \sum_{\alpha \in \Delta_+} \alpha$.

(f) $\Omega|_V = (\Lambda + 2\rho, \Lambda) Id_V$

(g) If λ *is a singular weight, then* $(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho)$.

Proof: Pf a, b)

By iii), $V = U(\mathfrak{n}_-)v_\Lambda = \sum \mathbb{F}E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N}^{m_N} v_\Lambda \in V_{\Lambda} - \sum_1^N m_i\beta_i \in D(\Lambda)$, proving a) and b).

Pf c)

We know

$$(*) \qquad U = \oplus_{\lambda \in D(\Lambda)} (U \cap V_\lambda)$$

for a submodule U by a previous lecture. So choose $\lambda \in D(\Lambda)$ to be of minimal height with $U \cap V_\lambda \neq 0$. Then $E_\alpha v = 0$ for any $v \in U \cap V_\lambda$, so v is a singular vector. And if v is a singular vector of weight λ, then $U(\mathfrak{g})v = U(\mathfrak{n}_-)v$ which is a proper submodule of V unless $\lambda = \Lambda$.

Pf d)

The sum of proper submodules of V is again a proper submodule because it does not contain v_Λ. Thus this sum is a unique maximal submodule.

Pf e)

Take a a basis $\{E_{\beta_i}, E_{-\beta_i}, H_i\}$ and its dual $\{E_{-\beta_i}, E_{\beta_i}, H^i\}$ and compute Casimir operator $\Omega = \sum_1^r H_i H^i + \sum_1^N E_{\beta_i} E_{-\beta_i} + E_{-\beta_i} E_{\beta_i} = \sum_1^r H_i H^i + 2\sum_1^N E_{-\beta_i} E_{\beta_i} + 2\nu^{-1}\alpha$. Apply this to a singular vector v_λ to get

$$\Omega v_\lambda = \sum_1^r \lambda(H_i)\lambda(H^i)v_\lambda + \sum_1^N (\lambda, \beta_i)v_\lambda + 0$$

The right hand side is $(\lambda, \lambda) + 2(\lambda, \rho))$.

Pf f)

$\Omega v_\Lambda = (\Lambda + 2\rho, \Lambda)v_\Lambda$ by e) and since Ω commutes with $U(\mathfrak{g})$ we get $\Omega(E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N} v_\Lambda v_\lambda) = (\Lambda + 2\rho, \Lambda)E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N} v_\Lambda$

Pf g)

follows from f) and e).

Pf h)

If λ is singular weight, then $(\lambda + 2\rho, \lambda) = (\Lambda + 2\rho, \Lambda)$ by g). This describes a compact set in which the singular weights must lie. But $\lambda \in D(\Lambda)$, a discrete set. As the intersection of a discrete set and compact set is finite, we have that the singular weights must be finite in number.

A **Verma module** $M(\Lambda)$ is highest weight module with highest weight Λ such that any other module with highest weight Λ is quotient of $M(\Lambda)$. We construct $M(\Lambda)$ as $U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{n}_+; h - \Lambda(h), h \in \mathfrak{h})$

By Theorem 1 d), $M(\Lambda)$ has unique maximum submodule $J(\Lambda)$ such that $L(\Lambda) = M(\Lambda)/J(\Lambda)$ is unique highest weight module with highest weight Λ.

Theorem 1.3. *(a) For any $\Lambda \in \mathfrak{h}*$, there exists a Verma module $M(\Lambda)$, unique up to isomorphism.*

(b) $M(\Lambda)$ has unique irreducible quotient $L(\Lambda)$

(c) $M(\Lambda)] = M(\Lambda')$ (resp. $L(\Lambda) = L(\Lambda')$) iff $\Lambda = \Lambda'$

(d) $E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N}^{m_N} v_\lambda$ form basis of $M(\Lambda)$.

Proof: a), b), c) are clear. d) follows from the PBW theorem because $E_{-\beta_1}^{m_1} \ldots . E_{-\beta_N}^{m_N}$ never lies in $J(\Lambda)$.