Proof. Choose a set $\Pi$ of simple roots $\Delta$, satisfying $\Delta = \{ \alpha \in Q : (\alpha, \alpha) = 2 \}$. We will construct $\mathfrak{g}$, a semisimple Lie algebra, satisfying $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{F}E_{\alpha})$. We will think of $\mathfrak{h}$ as $\mathbb{F} \otimes \mathbb{Z} Q$. The brackets should be the following:

1. $[h, h'] = 0 \forall h, h' \in \mathfrak{h}$,
2. $[h, E_{\alpha}] = -[E_{\alpha}, h] = (\alpha, h)E_{\alpha}$,
3. $[E_{\alpha}, E_{-\alpha}] = -\alpha$ for $\alpha \in \Delta$,
4. $[E_{\alpha}, E_{\beta}] = \epsilon(\alpha, \beta)E_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Delta$,
5. $[E_{\alpha}, E_{\beta}] = 0$ if $\alpha, \beta \in \Delta, \alpha + \beta \notin \Delta \cup 0$

The problem is to find non-zero $\epsilon(\alpha, \beta) \in \mathbb{F}$ such that $\mathfrak{g}$ with the four brackets above is a Lie algebra (i.e. skew-symmetry, Jacobi identity). Then automatically $\mathfrak{g}$ will be simple with the root system $\Delta$, by our general criterion of simplicity.

**Proposition 20.1.** There exists $\epsilon : Q \times Q \to \pm 1$ with the following properties:

1. $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$
2. $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$
3. $\epsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}$

**Proof.** Choose a set $\Pi$ of simple roots $\{ \alpha_1, \ldots, \alpha_r \}$ (so $\Pi$ is a $\mathbb{Z}$-basis of $Q$). For each pair $i, j$, make a choice of $\epsilon(\alpha_i, \alpha_j)$ and $\epsilon(\alpha_j, \alpha_i)$ subject to the following constraints: $\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i, \alpha_j)}$ (for $i \neq j$) and $\epsilon(\alpha_i, \alpha_i) = -1$. Now extend $\epsilon$ bi-multiplicatively to all pairs of elements in $Q$. Now we can verify that the relation $\epsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}$ works, where $\alpha = \sum_{i} k_i \alpha_i$:

$$\epsilon(\alpha, \alpha) = \prod_{i,j} \epsilon(\alpha_i, \alpha_j)^{k_i k_j}$$

$$= \prod_{i} \epsilon(\alpha_i, \alpha_i)^{k_i^2} \prod_{i<j} \epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i)^{k_i k_j}$$

$$= (-1)^{\sum_{i} k_i^2 (\alpha_i, \alpha_i)/2} \prod_{i<j} (-1)^{k_i k_j (\alpha_i, \alpha_j)} = (-1)^{(\alpha, \alpha)/2}$$

**Remark.** $\epsilon(\alpha, \alpha) = -1$ if $\alpha$ is a root. Further, if $\alpha, \beta \in Q$, we can extend the identity $\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i, \alpha_j)}$ extends bi-multiplicatively to give $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}$. Alternatively, note that $\epsilon(\alpha + \beta, \alpha + \beta) = \epsilon(\alpha, \alpha)\epsilon(\beta, \beta)\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)$ gives us the following: $(-1)^{\frac{1}{2}(\alpha, \alpha) + \frac{1}{2}(\beta, \beta) + (\alpha, \beta)} = (-1)^{\frac{1}{2}(\alpha, \alpha) + \frac{1}{2}(\beta, \beta)\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)}$, which implies $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}$. 

1
Theorem 20.2. The brackets (1) – (4) above in $\mathfrak{g}$, with the form $\epsilon$ defined above, gives a simple Lie algebra of finite dimension with root system $(V = \mathbb{R} \otimes \mathbb{Z} Q, \Delta)$.

Proof. The skew-symmetry follows by the Remark, since $\epsilon(\alpha, \beta) = -\epsilon(\beta, \alpha)$ if $\alpha + \beta \in \Delta$. It now suffices to check the Jacobi identity when $a, b, c \in \mathfrak{h}$ or $E_\alpha(\alpha \in \Delta)$. If $a \in E_\alpha, b \in E_\beta, c \in E_\gamma$ and $\alpha + \beta + \gamma \notin \Delta \cup 0$, then the Jacobi identity trivially holds since all three terms are 0. For the same, it trivially holds when $a, b, c \in \mathfrak{h}$. Otherwise we have the following cases:

Case 1: $a, b \in \mathfrak{h}, c = E_\alpha$. Then we have $[a, [b, c]] = (\alpha, b)[a, \alpha] = (\alpha, b)(\alpha, a)\epsilon_\alpha, [b, [c, a]] = -[b, [a, \alpha]] = -\epsilon_\alpha(\alpha, b)(\alpha, a)\epsilon_\alpha$, $[c, [a, b]] = 0$, so they sum up to 0, as required.

Case 2: $a \in \mathfrak{h}, b = E_\alpha, c = E_\beta$. Then we have: $[a, [b, c]] = (\alpha, \beta + \alpha)[b, c]; [b, [c, a]] = -\epsilon_\alpha(\alpha, b)[c, a] = -\epsilon(\alpha, a)[b, a]$, $[c, [a, b]] = 0$, so they sum up to 0, as required.

Case 3: $a = E_\alpha, b = E_\beta, c = E_\gamma, \alpha + \beta + \gamma = 0$. Then we have:

1. $[E_\alpha, [E_\beta, E_\gamma]] = \epsilon(\beta, -\alpha - \beta)[E_\alpha, E_\alpha] = -\epsilon(\beta, -\alpha)\epsilon(\beta, -\beta)\alpha$
2. $[E_\gamma, [E_\alpha, E_\beta]] = \epsilon(\alpha, \beta)[E_{\alpha - \beta}, E_{\alpha + \beta}] = \epsilon(\alpha, \beta)(\alpha + \beta)$
3. $[E_\beta, [E_\gamma, E_\alpha]] = \epsilon(\alpha - \beta, \alpha)[E_\beta, E_\gamma] = -\epsilon(\alpha, \alpha)\epsilon(\beta, -\alpha)\beta$

To note that they sum to 0, observe the following:

$\epsilon(\beta, -\beta)\epsilon(\beta, -\alpha)\alpha - \epsilon(\alpha, \alpha)\epsilon(-\beta, \alpha)\beta + \epsilon(\beta, \beta)(\alpha + \beta) = \epsilon(\beta, \alpha)\alpha + \epsilon(\beta, \beta)\epsilon(\alpha + \beta) = 0$

Exercise 20.1. Show that there are remaining two cases when $\alpha + \beta + \gamma \in \Delta$ (i) $\alpha = -\beta$ (ii) $(\alpha, \beta) = -1, (\beta, \gamma) = -1, (\alpha, \gamma) = 0$, and check the Jacobi identity in both of them.

Proof. (i) In this case, if $(\alpha, \gamma) = 0$, then since $\mathfrak{g}$ is simply laced, $\alpha + \gamma, \alpha - \gamma \notin \Delta$, so we have that $[E_\alpha, [E_{\alpha - \gamma}, E_\gamma]] = 0, [E_{\alpha - \gamma}, [E_\gamma, E_\alpha]] = 0, [E_\gamma, [E_\alpha, E_{\alpha - \gamma}]] = [E_\gamma, -\alpha] = (\alpha, \gamma)E_\gamma = 0$, so all three terms are 0.

WLOG, the other case is when $(\alpha, \gamma) = -1$ (since if $(\alpha, \gamma) = 1$ switch $\alpha$ with $-\alpha$), so since $\mathfrak{g}$ is simply laced, $\alpha + \gamma \in \Delta, \alpha - \gamma \notin \Delta$. Here we have that $[E_\alpha, [E_{\alpha - \gamma}, E_\gamma]] = 0, [E_{\alpha - \gamma}, [E_\gamma, E_\alpha]] = \epsilon(\gamma, \alpha)[E_{\alpha - \gamma}, E_{\alpha + \gamma}] = \epsilon(\gamma, \alpha)\epsilon(-\alpha, \alpha)\epsilon(-\gamma, \alpha)\epsilon(-\alpha, \gamma)E_\gamma$ while $[E_\gamma, [E_\alpha, E_{\alpha - \gamma}]] = -[E_\gamma, \alpha] = (\alpha, \gamma)E_\gamma$. So it suffices to prove that $\epsilon(\gamma, \alpha)\epsilon(-\alpha, \alpha)\epsilon(-\alpha, \gamma) + (\alpha, \gamma) = 0$, which follows from the fact that $\epsilon(\gamma, \alpha)\epsilon(\alpha, \gamma) = 1$ in this case.

(ii) If no two of $\alpha, \beta, \gamma$ sum to 0, then using the fact that $(\alpha + \beta + \gamma, \alpha + \beta + \gamma) = 2$, one deduces that $(\alpha, \beta) + (\alpha, \gamma) + (\beta, \gamma) = -2$, so since none of them can be $-2$ (if $(\alpha, \beta) = -2, \alpha = -\beta)$, after reordering $(\alpha, \beta) = -1, (\beta, \gamma) = -1, (\alpha, \gamma) = 0$. Then $[E_\alpha, [E_\beta, E_\gamma]] = \epsilon(\gamma, \alpha)\epsilon(\alpha, \gamma)\epsilon(\alpha, \gamma)E_{\alpha + \beta + \gamma}; [E_\beta, [E_\gamma, E_\alpha]] = 0; [E_\gamma, [E_\alpha, E_\beta]] = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)\epsilon(\gamma, \beta)E_{\alpha + \beta + \gamma}$. So it suffices to show that we have: $\epsilon(\beta, \gamma)\epsilon(\alpha, \beta)\epsilon(\alpha, \gamma) + \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)\epsilon(\gamma, \beta) = 0$, which is true since $\epsilon(\alpha, \gamma) = \epsilon(\gamma, \alpha), \epsilon(\beta, \gamma) = -\epsilon(\gamma, \beta)$ in this case.

Above was the simply-laced case. For the non-simply laced case, note by the following two exercises that each non-simply laced Lie algebra can be expressed as a sub-algebra of a simply-laced one. More precisely, type $B_r \subset D_{r+1}, C_r \subset A_{2r-1}, F_4 \subset E_6, G_2 \subset D_4$. To see this, put the following
orientations on the Dynkin diagrams of $E_6$ and $D_4$ and define automorphisms of their respective Dynkin diagram ($\sigma_2$ for $E_6$ and $\sigma_3$ for $D_4$) switching the indicated vertices:

![Dynkin Diagrams](image)

**Exercise 20.2.** Check that the map $E_i \rightarrow E_{\sigma(i)}$, $F_i \rightarrow F_{\sigma(i)}$, $H_i \rightarrow H_{\sigma(i)}$ defines an automorphism of $\tilde{g}(A)$, and hence of $g(A)$.

**Proof.** The relation $[H_{\sigma(i)}, H_{\sigma(j)}] = 0$ holds trivially, as does the relation $[E_i, F_j] = \sigma_{i,j} H_j$ (since the map $\sigma$ is a bijection). It remains to check the relation $[H_{\sigma(i)}, E_{\sigma(j)}] = a_{i,j} E_{\sigma(j)}$ and the analogous relation for the $F$'s; this is exactly equivalent to $a_{i,j} = a_{\sigma(i),\sigma(j)}$, which follows from the fact that $\sigma$ is an automorphism of the Dynkin diagram and preserves the inner products of its roots. Since $\tilde{g}(A)$ has a unique maximal ideal, it is invariant under $\sigma$, hence $\sigma$ induces an automorphism of $g(A)$. □

**Exercise 20.3.** For $\sigma_2$, in $E_6$ the elements $\{X_1 + X_5, X_2 + X_4, X_3, X_6\}$ where $X = E, F$ or $H$ lie in a fixed point sub-algebra $E_6^{\sigma_2}$ of $\sigma_2$ in $E_6$, and satisfy all Chevalley relations of $\tilde{g}(F_4)$. Likewise, for $\sigma_3$ and $D_4$, the elements $\{X_1 + X_3 + X_4, X_2\}$ satisfy all Chevalley relations of $\tilde{g}(G_2)$.

**Proof.** It is clear that the elements in question lie in the fixed point sub-algebra. In either case, the first Chevalley relations (that the Cartan subalgebra is abelian) is trivial. The third Chevalley relation (about the commutator of an $E$ and an $F$) follows from the third Chevalley relation for $E_6$ and $F_2$, combined with the fact that in both sets $\{X_1 + X_5, X_2 + X_4, X_3, X_6\}$ and $\{X_1 + X_3 + X_4, X_2\}$, the indices of different elements are distinct. The second Chevalley relation is equivalent to saying that the in $E_6$, the four elements $\{X_1 + X_5, X_2 + X_4, X_3, X_6\}$ correspond (in terms of inner products) to the four simple roots of $F_4$; and that in $D_4$, the two elements $\{X_1 + X_3 + X_4, X_2\}$ correspond (in terms of inner products) to the two simple roots of $G_2$. Both of these assertions are trivial to verify. □

By these exercises, we have homomorphisms $\tilde{g}(F_4) \rightarrow g(E_6)^{\sigma_2}$, and $\tilde{g}(G_2) \rightarrow g(D_4)^{\sigma_3}$. This proves that $g(F_4)$ and $g(G_2)$ are finite dimensional, completing the proof. Soon we will show that in fact, $g(E_6)^{\sigma_3} = g(F_4)$, $g(D_4)^{\sigma_3} = g(G_2)$. □

Using this explicit construction of simply-laced algebras, we can easily construct a symmetric invariant bilinear form (which is unique up to constant factor). We have a bilinear form $(\cdot, \cdot)$ on $Q$;
We can write $\alpha$, $\beta$ true because $(\alpha, \beta = 0)$. If $\alpha + \beta \neq 0$, $(E_\alpha, E_\beta) = -1$.

**Exercise 20.4.** Check that this bilinear form is invariant.

**Proof.** It is sufficient to prove that $([a, b], c) = (a, [b, c])$ when $a, b, c$ are each either in $\mathfrak{h}$ or of the form $E_\alpha$. If $a, b, c \in \mathfrak{h}$ clearly both sides are 0. If $a, b \in \mathfrak{h}, c = E_\alpha$, then the LHS is clearly 0, while the RHS is also 0 since $(\mathfrak{h}, E_\alpha) = 0$; a similar situation happens if $b, c \in \mathfrak{h}, a = E_\alpha$. If $a, c \in \mathfrak{h}, b = E_\alpha$, then both sides are again 0 since $(\mathfrak{h}, E_\alpha) = 0$. If $a = E_\alpha, b = E_\beta, c \in \mathfrak{h}$, the LHS is 0 unless $\alpha + \beta = 0$, and $[b, c] \in \mathbb{F}E_\beta$, so the RHS is also 0 unless $\alpha + \beta = 0$. If $\alpha + \beta = 0$, then the LHS is $([E_\alpha, E_\beta, c]) = -(\alpha, c)$, while the RHS is $(E_\alpha, [E_\beta, c]) = (\alpha, c)E_\alpha E_\beta = (\alpha, c)$. By symmetry, the case where $c = E_\alpha, b = E_\beta, a \in \mathfrak{h}$ follows. If $a = E_\alpha, b \in \mathfrak{h}, c = E_\beta$, the LHS is $([E_\alpha, b], E_\gamma) = -(\alpha, b)(E_\alpha, E_\gamma)$, and the RHS is $(E_\alpha, [b, E_\gamma]) = (b, \gamma)(E_\alpha, E_\gamma)$. Clearly both quantities are equal if $\alpha + \gamma = 0$, and both quantities are 0 otherwise. The final case is when $a = E_\alpha, b = E_\beta, c = E_\gamma$; here both sides are clearly 0 unless $\alpha + \beta + \gamma = 0$. If this quantity is 0, then the LHS is $-\epsilon(\alpha, \beta)$, while the RHS is $\epsilon(\beta, -\alpha - \beta) = -\epsilon(\beta, \alpha) = -\epsilon(\alpha, \beta)$, where in the last equality we use the fact that $\alpha + \beta$ is a root.

Next we define the compact form $g_C$ of $g$ when $\mathbb{F} = \mathbb{C} \supset \mathbb{R}$. Suppose $g$ is simply-laced, and $g_\mathbb{R} = \mathfrak{h}_\mathbb{R} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{R}E_\alpha)$ be the Lie algebra over $\mathbb{R}$. Define an automorphism $\omega_\mathbb{R}$ of $g_\mathbb{R}$ by letting it act as $-1$ on $\mathfrak{h}$, and let $\omega_\mathbb{R}(E_\alpha) = E_{-\alpha}$.

**Exercise 20.5.** Check that this is an automorphism.

**Proof.** It suffices to prove that $\omega([a, b]) = [\omega(a), \omega(b)]$ when $a, b$ are either in $\mathfrak{h}$ or of the form $E_\alpha$. If both $a, b$ are in $\mathfrak{h}$, then both sides are 0. If $a \in \mathfrak{h}, b = E_\alpha$, then the LHS is $\omega((\alpha, a)E_\alpha) = (\alpha, a)E_{-\alpha}$, while the RHS is $[-a, E_{-\alpha}] = (\alpha, a)E_{-\alpha}$. Finally, if $a = E_\alpha, b = E_\beta$ then both sides are 0 unless $\alpha + \beta$ is a root; if it is a root then both sides are clearly $\epsilon(\alpha, \beta)E_{\alpha + \beta}$ since $\epsilon(\alpha, \beta) = \epsilon(-\alpha, -\beta)$.

Now extend $\omega_\mathbb{R}$ from $g_\mathbb{R}$ to $g = \mathbb{C} \otimes \mathbb{R} g_\mathbb{R}$ to be an anti-linear automorphism $\omega$, by $\omega(\lambda a) = \lambda \omega(a)$.

**Definition 20.1.** The fixed point set of $\omega$ is a Lie algebra over $\mathbb{R}$, $g_C$, called the compact form of $g$.

**Exercise 20.6.** If $g = \mathfrak{sl}_n(\mathbb{C})$, then $g_C = \mathfrak{su}_n = \{ A \in \mathfrak{sl}_n(\mathbb{C}) | A = -\bar{A}^t \}$, and $\omega(A) = -\bar{A}^t$.

**Proof.** In this case, it is clear that $E_{\alpha_i - \alpha_j} = E_{ij}$ if $i < j$, and $-E_{ij}$ if $i > j$ (this is to fulfill the condition $[E_\alpha, E_{-\alpha}] = -\alpha$). Then it is clear that the automorphism $\omega_\mathbb{R}$ sends $A$ to $-\bar{A}^t$, and consequently $\omega$ sends $A$ to $-\bar{A}^t$, as required.

**Proposition 20.3.** The restriction of the invariant symmetric bilinear form $(\cdot, \cdot)$ from $g$ to $g_C$ is negative definite.

**Proof.** We can write $g_C = i\mathfrak{h}_\mathbb{R} + \sum_{\alpha \in \Delta_+} \mathbb{R}(E_\alpha + E_{-\alpha}) + \sum_{\alpha \in \Delta_+} i\mathbb{R}(E_\alpha - E_{-\alpha})$ and these 3 spaces are orthogonal to each other. It remains to show that it is negative-definite one each space. This is true because $(ih, ih) = -(h, h) < 0; (E_\alpha + E_{-\alpha}, E_\alpha + E_{-\alpha}) = -2 < 0; (i(E_\alpha - E_{-\alpha}), i(E_\alpha - E_{-\alpha})) = -2 < 0$. 


Finally, the restriction of the invariant bilinear form (Killing form) from $\mathfrak{g}(E_6)$ or $\mathfrak{g}(D_4)$ to $\mathfrak{g}^{\sigma_i}$ is non-degenerate, hence $\mathfrak{g}^{\sigma_i}$ is semi-simple and thus simple. To see this, just take $\mathfrak{g}_c \cap \mathfrak{g}^{\sigma_i}$, where $\mathfrak{g} = E_6$ or $D_4$. Since the Killing form is negative definite on $\mathfrak{g}_c$, it is negative definite on $\mathfrak{g}_c \cap \mathfrak{g}^{\sigma_i}$, and thus also on its complexification $\mathfrak{g}^{\sigma_i}$. It follows that $E_6^{\sigma_2} = F_4, D_4^{\sigma_3} = G_2$. 