18.745 Introduction to Lie Algebras	November 18, 2010

Lecture 20 — Explicitly constructing Exceptional Lie Algebras

Prof. Victor Kac — *Scribe: Vinoth Nandakumar*

First consider the simply-laced case: a symmetric Cartan matrix, root system Δ, root lattice $Q = \mathbb{Z}\Delta$, satisfying $\Delta = \{\alpha \in Q : (\alpha, \alpha) = 2\}$. We will construct $\mathfrak{g}$, a semisimple Lie algebra, satisfying $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{F} E_\alpha)$. We will think of $\mathfrak{h}$ as $\mathbb{F} \otimes_{\mathbb{Z}} Q$. The brackets should be the following:

1. $[h, h'] = 0 \ \forall \ h, h' \in \mathfrak{h}$,
2. $[h, E_\alpha] = -[E_\alpha, h] = (\alpha, h) E_\alpha$,
3. $[E_\alpha, E_{-\alpha}] = -\alpha$ for $\alpha \in \Delta$,
4. $[E_\alpha, E_\beta] = \epsilon(\alpha, \beta) E_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in \Delta$,
5. $[E_\alpha, E_\beta] = 0$ if $\alpha, \beta \in \Delta, \alpha + \beta \notin \Delta \cup 0$

The problem is to find non-zero $\epsilon(\alpha, \beta) \in \mathbb{F}$ such that $\mathfrak{g}$ with the four brackets above is a Lie algebra (i.e. skew-symmetry, Jacobi identity). Then automatically $\mathfrak{g}$ will be simple with the root system Δ, by our general criterion of simplicity.

Proposition 20.1. $\exists \epsilon : Q \times Q \to \pm 1$ *with the following properties:*

1. $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$
2. $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$
3. $\epsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}$

Proof. Choose a set Π of simple roots $\{\alpha_1, \cdots, \alpha_r\}$ (so Π is a $\mathbb{Z}$-basis of Q). For each pair i, j, make a choice of $\epsilon(\alpha_i, \alpha_j)$ and $\epsilon(\alpha_j, \alpha_i)$ subject to the following constraints: $\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i, \alpha_j)}$ (for $i \neq j$) and $\epsilon(\alpha_i, \alpha_i) = -1$. Now extend ϵ bi-multiplicatively to all pairs of elements in Q. Now we can verify that the relation $\epsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}$ works, where $\alpha = \sum_i k_i \alpha_i$:

$$\begin{aligned} \epsilon(\alpha, \alpha) &= \prod_{i,j} \epsilon(\alpha_i, \alpha_j)^{k_i k_j} \\ &= \prod_i \epsilon(\alpha_i, \alpha_i)^{k_i^2} \prod_{i<j} \epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i)^{k_i k_j} \\ &= (-1)^{\sum_i k_i^2 \frac{(\alpha_i, \alpha_i)}{2}} \prod_{i<j} (-1)^{k_j k_i (\alpha_i, \alpha_j)} = (-1)^{\frac{(\alpha, \alpha)}{2}} \end{aligned}$$

□

Remark. $\epsilon(\alpha, \alpha) = -1$ if α is a root. Further, if $\alpha, \beta \in Q$, we can extend the identity $\epsilon(\alpha_i, \alpha_j)\epsilon(\alpha_j, \alpha_i) = (-1)^{(\alpha_i, \alpha_j)}$ extends bi-multiplicatively to give $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}$. Alternatively, note that $\epsilon(\alpha + \beta, \alpha + \beta) = \epsilon(\alpha, \alpha)\epsilon(\beta, \beta)\epsilon(\alpha, \beta)\epsilon(\beta, \alpha)$ gives us the following: $(-1)^{\frac{1}{2}(\alpha, \alpha) + \frac{1}{2}(\beta, \beta) + (\alpha, \beta)} = (-1)^{\frac{1}{2}(\alpha, \alpha) + \frac{1}{2}(\beta, \beta)} \epsilon(\alpha, \beta)\epsilon(\beta, \alpha)$, which implies $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}$.

Theorem 20.2. *The brackets* (1) − (4) *above in* $\mathfrak{g}$*, with the form* ϵ *defined above, gives a simple Lie algebra of finite dimension with root system* $(V = \mathbb{R} \otimes_{\mathbb{Z}} Q, \Delta)$.

Proof. The skew-symmetry follows by the Remark, since $\epsilon(\alpha, \beta) = -\epsilon(\beta, \alpha)$ if $\alpha + \beta \in \Delta$. It now suffices to check the Jacobi identity when $a, b, c \in \mathfrak{h}$ or $E_\alpha (\alpha \in \Delta)$. If $a \in E_\alpha, b \in E_\beta, c \in E_\gamma$ and $\alpha + \beta + \gamma \notin \Delta \cup 0$, then the Jacobi identity trivially holds since all three terms are 0. For the same, it trivially holds when $a, b, c \in \mathfrak{h}$. Otherwise we have the following cases:

Case 1: $a, b \in \mathfrak{h}, c = E_\alpha$. Then we have $[a,[b,c]] = (\alpha, b)[a, E_\alpha] = (\alpha, b)(\alpha, a)E_\alpha$, $[b,[c,a]] = -[b,[a, E_\alpha]] = -(\alpha, b)(\alpha, a)E_\alpha, [c,[a,b]] = 0$, so they sum up to 0, as required.

Case 2: $a \in \mathfrak{h}, b = E_\alpha, c = E_\beta$. Then we have: $[a,[b,c]] = (\alpha+\beta, a)[b,c]; [b,[c,a]] = -(\beta, a)[b,c]; [c,[a,b]] = -(\alpha, a)[b,c]$, so they sum up to 0, as required.

Case 3: $a = E_\alpha, b = E_\beta, c = E_\gamma$, $\alpha + \beta + \gamma = 0$. Then we have:

1. $[E_\alpha,[E_\beta, E_\gamma]] = \epsilon(\beta, -\alpha - \beta)[E_\alpha, E_{-\alpha}] = -\epsilon(\beta, -\alpha)\epsilon(\beta, -\beta)\alpha$
2. $[E_\gamma,[E_\alpha, E_\beta]] = \epsilon(\alpha, \beta)[E_{-\alpha-\beta}, E_{\alpha+\beta}] = \epsilon(\alpha, \beta)(\alpha + \beta)$
3. $[E_\beta,[E_\gamma, E_\alpha]] = \epsilon(-\alpha - \beta, \alpha)[E_\beta, E_{-\beta}] = -\epsilon(-\alpha, \alpha)\epsilon(-\beta, \alpha)\beta$

To note that they sum to 0, observe the following:

$$\epsilon(\beta, -\beta)\epsilon(\beta, -\alpha)\alpha - \epsilon(-\alpha, \alpha)\epsilon(-\beta, \alpha)\beta + \epsilon(\alpha, \beta)(\alpha + \beta) = \epsilon(\beta, \alpha)\alpha + \epsilon(\beta, \alpha)\beta + \epsilon(\alpha, \beta)(\alpha + \beta) = 0$$

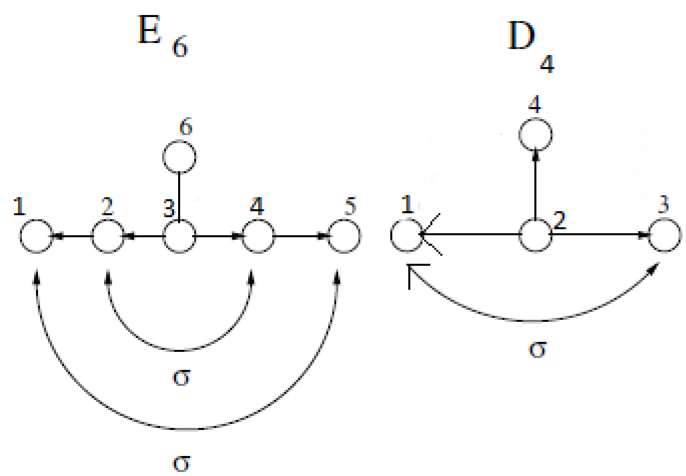
Exercise 20.1. Show that there are remaining two cases when $\alpha + \beta + \gamma \in \Delta$ (i) $\alpha = -\beta$ (ii) $(\alpha, \beta) = -1, (\beta, \gamma) = -1, (\alpha, \gamma) = 0$, and check the Jacobi identity in both of them.

Proof. (i) In this case, if $(\alpha, \gamma) = 0$, then since $\mathfrak{g}$ is simply laced, $\alpha + \gamma, \alpha - \gamma \notin \Delta$, so we have that $[E_\alpha,[E_{-\alpha}, E_\gamma]] = 0, [E_{-\alpha},[E_\gamma, E_\alpha]] = 0, [E_\gamma,[E_\alpha, E_{-\alpha}]] = [E_\gamma, -\alpha] = (\alpha, \gamma)E_\gamma = 0$, so all three terms are 0.

WLOG, the other case is when $(\alpha, \gamma) = -1$ (since if $(\alpha, \gamma) = 1$ switch α with $-\alpha$), so since $\mathfrak{g}$ is simply laced, $\alpha + \gamma \in \Delta, \alpha - \gamma \notin \Delta$. Here we have that $[E_\alpha,[E_{-\alpha}, E_\gamma]] = 0, [E_{-\alpha},[E_\gamma, E_\alpha]] = \epsilon(\gamma, \alpha)[E_{-\alpha}, E_{\alpha+\gamma}] = \epsilon(\gamma, \alpha)\epsilon(-\alpha, \alpha)\epsilon(-\alpha, \gamma)E_\gamma$ while $[E_\gamma,[E_\alpha, E_{-\alpha}]] = -[E_\gamma, \alpha] = (\alpha, \gamma)E_\gamma$. So it suffices to prove that $\epsilon(\gamma, \alpha)\epsilon(-\alpha, \alpha)\epsilon(-\alpha, \gamma) + (\alpha, \gamma) = 0$, which follows from the fact that $\epsilon(\gamma, \alpha)\epsilon(\alpha, \gamma) = (\alpha, \gamma) = -1$ in this case.

(ii) If no two of α, β, γ sum to 0, then using the fact that $(\alpha+\beta+\gamma, \alpha+\beta+\gamma) = 2$, one deduces that $(\alpha, \beta) + (\alpha, \gamma) + (\beta, \gamma) = -2$, so since none of them can be -2 (if $(\alpha, \beta) = -2, \alpha = -\beta$), after re-ordering $(\alpha, \beta) = -1, (\beta, \gamma) = -1, (\alpha, \gamma) = 0$. Then $[E_\alpha,[E_\beta, E_\gamma]] = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)E_{\alpha+\beta+\gamma}$; $[E_\beta,[E_\gamma, E_\alpha]] = 0; [E_\gamma,[E_\alpha, E_\beta]] = \epsilon(\alpha, \beta)\epsilon(\gamma, \alpha)\epsilon(\gamma, \beta)E_{\alpha+\beta+\gamma}$. So it suffices to show that we have: $\epsilon(\beta, \gamma)\epsilon(\alpha, \beta)\epsilon(\alpha, \gamma) + \epsilon(\alpha, \beta)\epsilon(\gamma, \alpha)\epsilon(\gamma, \beta) = 0$, which is true since $\epsilon(\alpha, \gamma) = \epsilon(\gamma, \alpha), \epsilon(\beta, \gamma) = -\epsilon(\gamma, \beta)$ in this case. □

Above was the simply-laced case. For the non-simply laced case, note by the following two exercises that each non-simply laced Lie algebra can be expressed as a sub-algebra of a simply-laced one. More precisely, type $B_r \subset D_{r+1}, C_r \subset A_{2r-1}, F_4 \subset E_6, G_2 \subset D_4$. To see this, put the following orientations on the Dynkin diagrams of E_6 and D_4 and define automorphisms of their respective Dynkin diagram (σ_2 for E_6 and σ_3 for D_4) switching the indicated vertices:



Exercise 20.2. Check that the map $E_i \to E_{\sigma(i)}, F_i \to F_{\sigma(i)}, H_i \to H_{\sigma(i)}$ defines an automorphism of $\tilde{\mathfrak{g}}(A)$, and hence of $\mathfrak{g}(A)$.

Proof. The relation $[H_{\sigma(i)}, H_{\sigma(j)}] = 0$ holds trivially, as does the relation $[E_i, F_j] = \sigma_{i,j} H_j$ (since the map σ is a bijection). It remains to check the relation $[H_{\sigma(i)}, E_{\sigma(j)}] = a_{ij} E_{\sigma(j)}$ and the analogous relation for the F's; this is exactly equivalent to $a_{ij} = a_{\sigma(i),\sigma(j)}$, which follows from the fact that σ is an automorphism of the Dynkin diagram and preserves the inner products of its roots. Since $\tilde{\mathfrak{g}}(A)$ has a unique maximal ideal, it is invariant under σ, hence σ induces an automorphism of $\mathfrak{g}(A)$. □

Exercise 20.3. For σ_2, in E_6 the eleemtns $\{X_1 + X_5, X_2 + X_4, X_3, X_6\}$ where $X = E, F$ or H lie in a fixed point sub-algebra $E_6^{\sigma_2}$ of σ_2 in E_6, and satisfy all Chevalley relations of $\tilde{\mathfrak{g}}(F_4)$. Likewise, for σ_3 and D_4, the elements $\{X_1 + X_3 + X_4, X_2\}$ satisfy all Chevalley relations of $\tilde{\mathfrak{g}}(G_2)$.

Proof. It is clear that the elements in question lie in the fixed point sub-algebra. In either case, the first Chevalley relations (that the Cartan subalgebra is abelian) is trivial. The third Chevalley relation (about the commutator of an E and an F) follows from the third Chevelley relation for E_6 and F_2, combined with the fact that in both sets $\{X_1+X_5, X_2+X_4, X_3, X_6\}$ and $\{X_1+X_3+X_4, X_2\}$, the indices of different elements are distinct. The second Chevalley relation is equivalent to saying that the in E_6, the four elements $\{X_1+X_5, X_2+X_4, X_3, X_6\}$ correspond (in terms of inner products) to the four simple roots of F_4; and that in D_4, the two elements $\{X_1 + X_3 + X_4, X_2\}$ correspond (in terms of inner products) to the two simple roots of G_2. Both of these assertions are trivial to verify. □

By these exercises, we have homomorphisms $\tilde{\mathfrak{g}}(F_4) \to \mathfrak{g}(E_6)^{\sigma_2}$, and $\tilde{\mathfrak{g}}(G_2) \to \mathfrak{g}(D_4)^{\sigma_3}$. This proves that $\mathfrak{g}(F_4)$ and $\mathfrak{g}(G_2)$ are finite dimensional, completing the proof. Soon we will show that in fact, $\mathfrak{g}(E_6)^{\sigma_3} = \mathfrak{g}(F_4), \mathfrak{g}(D_4)^{\sigma_3} = \mathfrak{g}(G_2)$. □

Using this explicit construction of simply-laced algebras, we can easily construct a symmetric invariant bilinear form (which is unique up to constant factor). We have a bilinear form $(\cdot, \cdot)$ on Q;

extend it by bilinearity to $\mathfrak{h}$. We let $(\mathfrak{h}, E_\alpha) = 0, (E_\alpha, E_\beta) = 0$ if $\alpha + \beta \neq 0, (E_\alpha, E_{-\alpha}) = -1$.

Exercise 20.4. Check that this bilinear form is invariant.

Proof. It is sufficient to prove that $([a, b], c) = (a, [b, c])$ when a, b, c are each either in $\mathfrak{h}$ or of the form E_α. If $a, b, c \in \mathfrak{h}$ clearly both sides are 0. If $a, b \in \mathfrak{h}, c = E_\alpha$, then the LHS is clearly 0, while the RHS is also 0 since $(\mathfrak{h}, E_\alpha) = 0$; a similar situation happens if $b, c \in \mathfrak{h}, a = E_\alpha$. If $a, c \in \mathfrak{h}, b = E_\alpha$, then both sides are again 0 since $(\mathfrak{h}, E_\alpha) = 0$. If $a = E_\alpha, b = E_\beta, c \in \mathfrak{h}$, the LHS is 0 unless $\alpha + \beta = 0$, and $[b, c] \in \mathbb{F}E_\beta$, so the RHS is also 0 unless $\alpha + \beta = 0$. If $\alpha + \beta = 0$, then the LHS is $([E_\alpha, E_{-\alpha}], c) = -(\alpha, c)$, while the RHS is $(E_\alpha, [E_{-\alpha}, c]) = (\alpha, c)(E_\alpha, E_{-\alpha}) = (\alpha, c)$. By symmetry, the case where $c = E_\alpha, b = E_\beta, a \in \mathfrak{h}$ follows. If $a = E_\alpha, b \in \mathfrak{h}, c = E_\beta$, the LHS is $([E_\alpha, b], E_\gamma) = -(\alpha, b)(E_{\alpha, E_\gamma})$, and the RHS is $(E_\alpha, [b, E_\gamma]) = (b, \gamma)(E_\alpha, E_\gamma)$. Clearly both quantities are equal if $\alpha + \gamma = 0$, and both quantities are 0 otherwise. The final case is when $a = E_\alpha, b = E_\beta, c = E_\gamma$; here both sides are clearly 0 unless $\alpha + \beta + \gamma = 0$. If this quantity is 0, then the LHS is $-\epsilon(\alpha, \beta)$, while the RHS is $\epsilon(\beta, -\alpha - \beta) = -\epsilon(\beta, \alpha) = \epsilon(\alpha, \beta)$, where in the last equality we use the fact that $\alpha + \beta$ is a root. □

Next we define the compact form $\mathfrak{g}_C$ of $\mathfrak{g}$ when $\mathbb{F} = \mathbb{C} \supset \mathbb{R}$. Suppose $\mathfrak{g}$ is simply-laced, and $\mathfrak{g}_\mathbb{R} = \mathfrak{h}_\mathbb{R} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{R}E_\alpha)$ be the Lie algebra over $\mathbb{R}$. Define an automorphism $\omega_\mathbb{R}$ of $\mathfrak{g}_\mathbb{R}$ by letting it act as -1 on $\mathfrak{h}$, and let $\omega_\mathbb{R}(E_\alpha) = E_{-\alpha}$.

Exercise 20.5. Check that this is an automorphism.

Proof. It suffices to prove that $\omega([a, b]) = [\omega(a), \omega(b)]$ when a, b are either in $\mathfrak{h}$ or of the form E_α. If both a, b are in $\mathfrak{h}$, then both sides are 0. If $a \in \mathfrak{h}, b = E_\alpha$, then the LHS is $\omega((\alpha, a)E_\alpha) = (\alpha, a)E_{-\alpha}$, while the RHS is $[-a, E_{-\alpha}] = (\alpha, a)E_{-\alpha}$. Finally, if $a = E_\alpha, b = E_\beta$ then both sides are 0 unless $\alpha + \beta$ is a root; if it is a root then both sides are clearly $\epsilon(\alpha, \beta)E_{\alpha+\beta}$ since $\epsilon(\alpha, \beta) = \epsilon(-\alpha, -\beta)$. □

Now extend $\omega_\mathbb{R}$ from $\mathfrak{g}_\mathbb{R}$ to $\mathfrak{g} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}_\mathbb{R}$ to be an anti-linear automorphism ω, by $\omega(\lambda a) = \bar{\lambda}\omega(a)$.

Definition 20.1. The fixed point set of ω is a Lie algebra over $\mathbb{R}$, $\mathfrak{g}_\mathbb{C}$, called the compact form of $\mathfrak{g}$.

Exercise 20.6. If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then $\mathfrak{g}_\mathbb{C} = \mathfrak{su}_n = \{A \in \mathfrak{sl}_n(\mathbb{C}) | A = -\bar{A}^t\}$, and $\omega(A) = -\bar{A}^t$.

Proof. In this case, it is clear that $E_{\alpha_i - \alpha_j} = E_{ij}$ if $i < j$, and $-E_{ij}$ if $i > j$ (this is to fulfill the condition $[E_\alpha, E_{-\alpha}] = -\alpha$). Then it is clear that the automorphism $\omega_\mathbb{R}$ sends A to $-A^t$, and consequently ω sends A to $-\bar{A}^t$, as required. □

Proposition 20.3. *The restriction of the invariant symmetric bilinear form $(\cdot, \cdot)$ from $\mathfrak{g}$ to $\mathfrak{g}_c$ is negative definite.*

Proof. We can write $\mathfrak{g}_c = i\mathfrak{h}_\mathbb{R} + \sum_{\alpha \in \Delta_+} \mathbb{R}(E_\alpha + E_{-\alpha}) + \sum_{\alpha \in \Delta_+} i\mathbb{R}(E_\alpha - E_{-\alpha})$ and these 3 spaces are orthogonal to each other. It remains to show that it is negative-definite one each space. This is true because $(ih, ih) = -(h, h) < 0$; $(E_\alpha + E_{-\alpha}, E_\alpha + E_{-\alpha}) = -2 < 0, (i(E_\alpha - E_{-\alpha}), i(E_\alpha - E_{-\alpha})) = -2 < 0$. □

Finally, the restriction of the invariant bilinear form (Killing form) from $\mathfrak{g}(E_6)$ or $\mathfrak{g}(D_4)$ to $\mathfrak{g}^{\sigma_i}$ is non-degenerate, hence $\mathfrak{g}^{\sigma_i}$ is semi-simple and thus simple. To see this, just take $\mathfrak{g}_c \cap \mathfrak{g}^{\sigma_i}$, where $\mathfrak{g} = E_6$ or D_4. Since the Killing form is negative definite on $\mathfrak{g}_c$, it is negative definite on $\mathfrak{g}_c \cap \mathfrak{g}^{\sigma_i}$, and thus also on its complexification $\mathfrak{g}^{\sigma_i}$. It follows that $E_6^{\sigma_2} = F_4, D_4^{\sigma_3} = G_2$.