18.745 Introduction to Lie Algebras September 9, 2010

Lecture 1 — Basic Definitions (I)

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Definition 1.1. An algebra A is a vector space V over a field $\mathbb{F}$, endowed with a binary operation which is bilinear:

$$\begin{aligned} a(\lambda b + \mu c) &= \lambda ab + \mu ac \\ (\lambda b + \mu c)a &= \lambda ba + \mu ca \end{aligned}$$

Example 1.1. The set of $n \times n$ matrices with the matrix multiplication, $\mathrm{Mat}_n(\mathbb{F})$ is an associative algebra: $(ab)c = a(bc)$.

Example 1.2. Given a vector space V, the space of all endomorphisms of V, End V, with the composition of operators, is an associative algebra.

Definition 1.2. A subalgebra B of an algebra A is a subspace closed under multiplication: $\forall a, b \in B, ab \in B$.

Definition 1.3. A Lie algebra is an algebra with product $[a, b]$ (usually called bracket), satisfying the following two axioms:

1. (skew-commutativity) $[a, a] = 0$
2. (Jacobi identity) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

Example 1.3.

1. Take a vector space $\mathfrak{g}$ with bracket $[a, b] = 0$. This is called an abelian Lie algebra;
2. $\mathfrak{g} = \mathbb{R}^3$,$[a, b] = a \times b$ (cross product);
3. Let A be an associative algebra with product ab. Then the space A with the bracket $[a, b] = ab - ba$ is a Lie algebra, denoted by A_-.

Exercise 1.1. Show the Jacobi identity holds in Example 1.3.3 for the following cases.

1. 2-member identity: $a(bc) = (ab)c$
2. 3-member identity: $a(bc) + b(ca) + c(ab) = 0$ and $(ab)c + (bc)a + (ca)b = 0$
3. 4-member identity: $a(bc) - (ab)c - b(ac) + (ba)c = 0$
4. 6-member identity: $[a, bc] + [b, ca] + [c, ab] = 0$

Proof. Expanding the Jacobi identity,

$$\begin{aligned}
& [a,[b,c]]+[b,[c,a]]+[c,[a,b]] \\
= \;& a(bc)-a(cb)-(bc)a+(cb)a+b(ca)-b(ac)-(ca)b+(ac)b+c(ab)-c(ba)-(ab)c+(ba)c \\
= \;& \quad [a(bc)-(ab)c]+[(cb)a-c(ba)] \\
& +[b(ca)-(bc)a]+[(ac)b-a(cb)] \\
& +[c(ab)-(ca)b]+[(ba)c-b(ac)] \\
= \;& \quad [a(bc)+b(ca)+c(ab)] \\
& -[(ab)c+(bc)a+(ca)b] \\
& -[a(cb)+c(ba)+b(ac)] \\
& +[(ac)b+(cb)a+(ba)c] \\
= \;& \quad [a(bc)-(ab)c-b(ac)+(ba)c] \\
& +[b(ca)-(bc)a-c(ba)+(cb)a] \\
& +[c(ab)-(ca)b-a(cb)+(ac)b] \\
= \;& \quad [a(bc)-(bc)a+b(ca)-(ca)b+c(ab)-(ab)c] \\
& -[a(cb)-(cb)a+b(ac)-(ac)b+c(ba)-(ba)c] \\
= \;& 0 \text{ if one of the identities is satisfied.}
\end{aligned}$$

□

Example 1.4. A special case of example 1.3.3: $\mathfrak{gl}_V = (\text{End } V)_{\text{-}}$, where V is a vector space, is a Lie algebra, called the general Lie algebra. In the case $V = \mathbb{F}^n$, we denote $\mathfrak{gl}_V = \mathfrak{gl}_n(\mathbb{F})$, the set of all $n \times n$ matrices with the bracket $[a,b] = ab - ba$.

Remark: Any subalgebra of a Lie algebra is a Lie algebra.

Example 1.5. The two most important classes of subalgebras of $\mathfrak{gl}_V$:

1. $sl_n(\mathbb{F}) = \{a \in \mathfrak{gl}_n(\mathbb{F}) | \text{tr } (a) = 0\}$;
2. Let B be a bilinear form on a vector space V.

   $o_{V,B} = \{a \in \mathfrak{gl}_V | B(a(u), v) = -B(u, a(v)) \forall u, v \in V\}$.

Exercise 1.2. Show that $tr[a,b] = 0 \ \forall a, b \in \text{Mat}_n(\mathbb{F})$. In particular, sl_n is a Lie algebra, called the special Lie algebra.

Proof.

$$\text{tr } [a,b] = \sum_i \sum_j (a_{ji}b_{ij} - b_{ji}a_{ij}) = 0$$

sl_n is trivially a subspace by the linearity of the trace, and we have shown it to be closed under the bracket operation. Hence, sl_n is a subalgebra and is therefore a Lie algebra. □

Exercise 1.3. Show that $o_{V,B}$ is a subalgebra of the Lie algebra $\mathfrak{gl}_V$

Proof. Consider $a, b \in o_{V,B}$. Then

$$B(ab(u), v) = -B(b(u), a(v)) = B(u, ba(v)))$$

from the property of a and b. Similarly,

$$B(ba(u), v) = -B(a(u), b(v)) = B(u, ab(v)))$$

subtracting the second of these from the first,

$$B([a, b](u), v) = B(u, -[a, b](v)) = -B(u, [a, b](v))$$

by the bilinearity of B. Hence, $o_{V,B}$ is closed under the bracket. As $o_{V,B}$ is trivially a subspace of $\mathfrak{gl}_V$, it is also a subalgebra and therefore a Lie algebra. □

Exercise 1.4. Let $V = \mathbb{F}^n$ and let B be the matrix of a bilinear form in the standard basis of $\mathbb{F}^n$. Show that

$$o_{\mathbb{F}^n,B} = \left\{a \in \mathfrak{gl}_n(\mathbb{F}) | a^T B + Ba = 0\right\}$$

where a^T denotes the transpose of matrix a.

Proof. The condition for members of $o_{V,B}$,

$$B(a(u), v) + B(u, a(v)) = 0$$

reads, in terms of the standard basis, employing summation convention (a repeated index is summed over):

$$B(a_{ij}u_j\vec{e_i}, v_k\vec{e_k}) + B(u_j\vec{e_j}, a_{ik}v_k\vec{e_i}) = 0.$$

Hence, from the bilinearity of B,

$$a_{ij}B_{ik}u_jv_k + B_{ji}a_{ik}u_jv_k = 0.$$

This is true $\forall u_j, v_k$. Therefore,

$$(a^T)_{ji}B_{ik} + B_{ji}a_{ik} = 0.$$

□

Remark Special cases of $o_{\mathbb{F}^n,B}$ are the following:

1. $so_{n,B}(\mathbb{F})$ if B is a non-degenerate symmetric matrix; this is called the orthogonal Lie algebra.
2. $sp_{n,B}(\mathbb{F})$ if B is a non-degenerate skew-symmetric matrix; this is called the symplectic Lie algebra.

The three series of Lie algebras $sl_n(\mathbb{F})$, $so_{n,B}(\mathbb{F})$ and $sp_{n,B}(\mathbb{F})$ are the most important for this course's examples.

Convenient notation: If X,Y are subspaces of a Lie algebra $\mathfrak{g}$, then $[X, Y]$ denotes the span of all vectors $[x, y]$, where $x \in X$,$y \in Y$.

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra. In the above notation, a subspace $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called an ideal if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

Definition 1.5. A derived subalgebra of a Lie algebra $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}]$.

Proposition 1.1. $[\mathfrak{g}, \mathfrak{g}]$ *is an ideal of a Lie algebra* $\mathfrak{g}$

Proof. Let $a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. Then $[a, b] \in [\mathfrak{g}, \mathfrak{g}]$. □

We now classify Lie algebras in 1 and 2 dimensions.

Dim 1. $\mathfrak{g} = \mathbb{F}a$, $[a, a] = 0$ so the Abelian Lie algebra is the only one.

Dim 2. Consider $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. Let $\mathfrak{g} = \mathbb{F}x + \mathbb{F}y$, then $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}[x, y]$. Therefore, $\dim[\mathfrak{g}, \mathfrak{g}] \leq 1$.

Case 1. $\dim[\mathfrak{g}, \mathfrak{g}] = 0$, Abelian Lie algebra.

Case 2. $\dim[\mathfrak{g}, \mathfrak{g}] = 1$, $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}b, b \neq 0$. Take $a \in \mathfrak{g} \backslash \mathbb{F}b$. Then $[a, b] \in [\mathfrak{g}, \mathfrak{g}]$, hence $[a, b] = \lambda b$ and $\lambda \neq 0$, otherwise $[\mathfrak{g}, \mathfrak{g}] = 0$. So, replacing a by $\lambda^{-1}a$, we get $[a, b] = b$. Hence, we have found a basis of $g : g = \mathbb{F}a + \mathbb{F}b$ with bracket $[a, b] = b$. So this Lie algebra is isomorphic to the subalgebra $\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \subset \mathfrak{gl}_2(\mathbb{F}) \right\}$, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $[a, b] = b$.

Exercise 1.5. Let $f : \mathrm{Mat}_n(\mathbb{F}) \to \mathbb{F}$ be a linear function such that $f([a, b]) = 0$. Show that $f(a) = \lambda \mathrm{tr}\ (a)$, for some λ independent of $a \in \mathrm{Mat}_n(\mathbb{F})$.

Proof. The condition $f([a, b]) = 0$ means

$$f(a_{ij}b_{jk}e_{ik} - b_{ij}a_{jk}e_{ik}) = 0.$$

By linearity of f,

$$(a_{ij}b_{jk} - b_{ij}a_{jk})f(e_{ik}) = 0 \ \forall a, b \in \mathrm{Mat}_n(\mathbb{F})$$

where summation convention has been used. Let $a = e_{mn}, b = e_{nn}$ for some $m \neq n$. Then $f(e_{mn}) = 0$. Hence

$$f([a, b]) = (a_{ij}b_{ji} - b_{ij}a_{ji})f(e_{ii}) = 0.$$

But $f(e_{ii}) = f(e_{jj})\ \forall i, j$ as $f(e_{ii}) - f(e_{jj}) = f(e_{ij}e_{ji} - e_{ji}e_{ij}) = 0$. Hence $f(e_{ii}) = \lambda$ for some constant λ, and $f(a) = \mathrm{tr}\ (a)f(e_{ii}) = \lambda \mathrm{tr}\ (a)$ □