Lecture 18 — Classification of Dynkin Diagrams

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1 Examples of Dynkin diagrams.

In the examples that follow, we will compute the Cartan matrices for the indecomposable root systems that we have encountered earlier. We record these as Dynkin diagrams, summarized in Figure 1. Later in the lecture, we will prove that these are actually the Dynkin diagrams of all possible indecomposable root systems. We also compute extended Dynkin diagrams specifically for the purposes of this proof.

In the following examples, the rank of the root system is always denoted by r, and we get simple roots $\alpha_1, \ldots, \alpha_r$. The largest root, used in the extended Dynkin diagrams, is denoted by θ.

Example 1.1. $A_r (r \geq 2)$: This corresponds to $\mathfrak{sl}_{r+1}(\mathbb{F})$. We have $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ and $\Delta_{\mathfrak{sl}_{r+1}(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, r+1\} \text{ and } i \neq j\} \subset V$, where V is the subspace of $\bigoplus_{i=1}^{r+1} \mathbb{R}\varepsilon_i$ on which the sum of coordinates (in the basis $\{\varepsilon_i \mid i \in \{1, \ldots, r+1\}\}$) is zero.

Take $f \in V^*$ given by $f(\varepsilon_1) = r+1, f(\varepsilon_2) = r, \ldots, f(\varepsilon_{r+1}) = 1$.

Then, $\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \ldots, r\}\}$ and $\theta = \varepsilon_1 - \varepsilon_{r+1}$.

For the Cartan matrix, recall that:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Hence, we obtain the Dynkin diagrams in Figure 1a.

Example 1.2. $B_r (r \geq 3)$: This corresponds to $\mathfrak{so}_{2r+1}(\mathbb{F})$. We have $\Delta_{\mathfrak{so}_{2r+1}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid i, j \in \{1, \ldots, r\} \text{ and } i \neq j\} \subset V = \bigoplus_{i=1}^{r} \mathbb{R}\varepsilon_i$.

Take f given by $f(\varepsilon_1) = r, \ldots, f(\varepsilon_r) = 1$.

Then, $\Delta_+ = \{\pm\varepsilon_i \pm \varepsilon_j, \varepsilon_i \mid i, j \in \{1, \ldots, r\} \text{ and } i \neq j\}$, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \ldots, r-1\}\} \sqcup \{\varepsilon_r\}$ and $\theta = \varepsilon_1 + \varepsilon_2$.

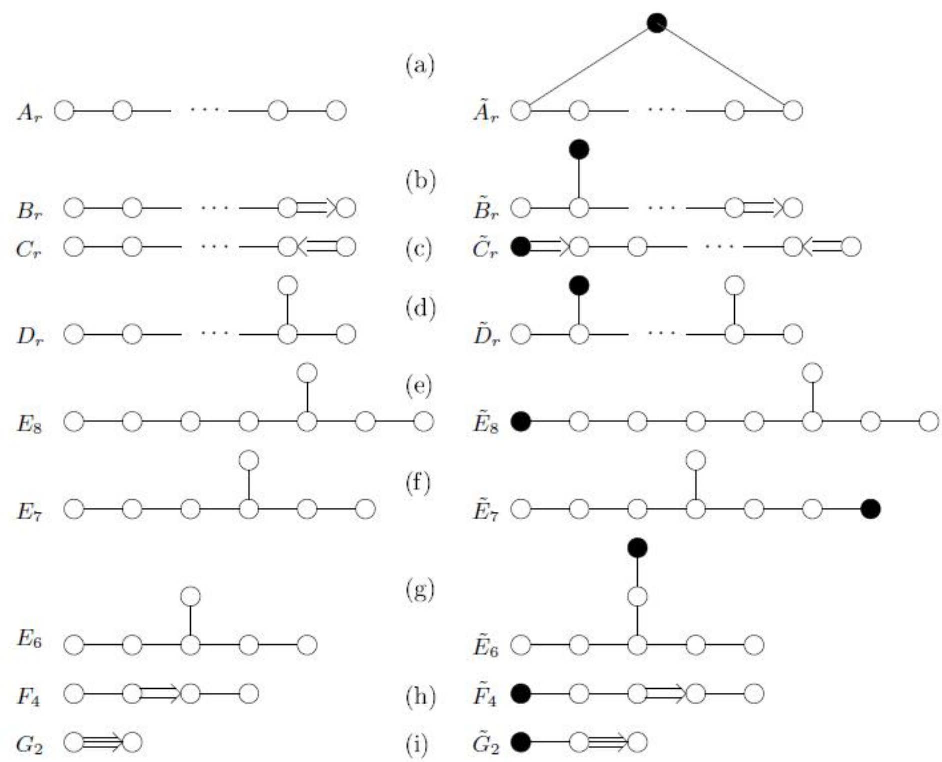

Figure 1: Dynkin diagrams and extended Dynkin diagrams of all the indecomposable root systems, from top to bottom, these correspond to: $A_r, B_r, C_r, D_r, E_8, E_7, E_6, F_4, G_2$.

The Cartan matrix is then:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}.$$

Hence, we get the Dynkin diagrams in Figure 1b.

Example 1.3. $C_r (r \geq 1)$: This corresponds to $\mathfrak{sp}_{2r}(\mathbb{F})$. We have $\Delta_{\mathfrak{sp}_{2r}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, r\}\} \subset V = \bigoplus_{i=1}^r \mathbb{R}\varepsilon_i$. We take the same f as in the previous case. Then, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i \in \{1, \ldots, r-1\}\} \sqcup \{2\varepsilon_r\}$ and $\theta = 2\varepsilon_1$. This gives the Cartan matrix:

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The corresponding Dynkin diagrams are shown in Figure 1c.

Example 1.4. $D_r (r \geq 4)$: This corresponds to $\mathfrak{so}_{2r}(\mathbb{F})$. We have $\Delta_{\mathfrak{so}_{2r}(\mathbb{F})} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, r\}$ and $i \neq j\}$. Define $f \in V^*$ by $f(\varepsilon_1) = r-1, \ldots, f(\varepsilon_r) = 0$.

Then $\Delta = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}$, $\Pi = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}$ and $\theta = \varepsilon_1 + \varepsilon_2$.

The Cartan matrix is:

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & 2 & -1 & -1 \\ \vdots & & 0 & -1 & 2 & 0 \\ 0 & \ldots & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Hence, we have the Dynkin diagrams in Figure 1d.

Example 1.5. E_8: We have $\Delta_{E_8} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \ldots, 8\}$ and $i \neq j\} \sqcup \{\underbrace{\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8)}_{\text{even number of + signs}}\}$ with $V = \bigoplus_{i=1}^8 \mathbb{R}\varepsilon_i$.

Let $f(\varepsilon_1) = 32, f(\varepsilon_2) = 6, f(\varepsilon_3) = 5, \ldots, f(\varepsilon_7) = 1, f(\varepsilon_8) = 0$.

Then, $\Delta_+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \sqcup \{\underbrace{\tfrac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_8)}_{\text{even number of + signs}}\}$. Also, $\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \ldots \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_7 + \varepsilon_8)\}$ and $\theta = \varepsilon_1 + \varepsilon_2$. We get the diagrams E_8 and $\tilde{E}_8$ in Figure 1e.

Exercise 1.1. E_6: We have:

$$\begin{aligned}\Delta_{E_6} =& \{\alpha \in \Delta_{E_8} \mid \alpha_1 + \cdots + \alpha_6 = 0 \text{ and } \alpha_7 + \alpha_8 = 0\}\\ =& \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, 6\} \text{ and } i \neq j\} \sqcup \{\pm(\varepsilon_7 - \varepsilon_8)\}\\ & \sqcup \{\underbrace{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \cdots \pm \varepsilon_6)}_{3\,+\text{ signs in the first 6 terms}} \pm \frac{1}{2}(\varepsilon_7 - \varepsilon_8)\}.\end{aligned}$$

Pick $f(\varepsilon_1) = 32, f(\varepsilon_2) = 7, f(\varepsilon_3) = 6, \dots, f(\varepsilon_7) = 2, f(\varepsilon_8) = 1$. We can then check that $\Pi = \{\varepsilon_2 - \varepsilon_3, \dots, \varepsilon_5 - \varepsilon_6, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8)\}$, in particular, $\varepsilon_1 - \varepsilon_2 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 - \varepsilon_8)$. Also, $\theta = \varepsilon_1 - \varepsilon_6$ for obvious reasons. We obtain the Dynkin diagrams in Figure 1g.

E_7 : Also, $\Delta_{E_7} = \{\alpha \in \Delta_{E_8} \mid \alpha_1 + \alpha_2 + \cdots + \alpha_8 = 0\}$, so:

$$\begin{aligned}\Delta_{E_7} =& \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, 8\} \text{ and } i \neq j\}\\ & \sqcup \{\underbrace{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \cdots \pm \varepsilon_8)}_{4\,+\text{ signs}}\}.\end{aligned}$$

Again, pick $f(\varepsilon_1) = 32, f(\varepsilon_2) = 7, f(\varepsilon_3) = 6, \dots, f(\varepsilon_7) = 2, f(\varepsilon_8) = 1$. Then, $\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \dots, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)\}$ and $\theta = \varepsilon_1 - \varepsilon_8$ for obvious reasons. We obtain the Dynkin diagrams in Figure 1f.

Exercise 1.2. F_4: We have $\Delta_{F_4} = \{\pm\varepsilon_i, \pm(\varepsilon_i - \varepsilon_j), \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i, j \in \{1, \dots, 4\} \text{ and } i \neq j\}$. Pick $f(\varepsilon_1) = 30,\ f(\varepsilon_2) = 3,\ f(\varepsilon_3) = 2,\ f(\varepsilon_4) = 1$. Then, $\Pi = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$ and so $\theta = \varepsilon_1 + \varepsilon_2$. This produces the Dynkin diagrams of Figure 1h.

G_2: Finally, $\Delta_{G_2} = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\}$. Pick $f(\varepsilon_1) = 2, f(\varepsilon_2) = 1, f(\varepsilon_3) = 4$. Then, $\Pi = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$, and so $\theta = 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2$. We obtain the Dynkin diagrams of Figure 1i.

2 Classification of Dynkin Diagrams.

Theorem 2.1. *A complete non-redundant list of connected Dynkin diagrams is the following:* $D(A_r)(r \geq 2)$*,* $D(B_r)(r \geq 3)$*,* $D(C_r)(r \geq 1)$*,* $D(D_r)(r \geq 4)$*,* $D(E_6)$*,* $D(E_7)$*,* $D(E_8)$*,* $D(F_4)$ *and* $D(G_2)$*. (See Figure 1 for these Dynkin diagrams and their extended versions.)*

Remark 2.2. *Note that* $D(A_1) = D(B_1) = D(C_1)$*,* $D(B_2) = D(C_2)$ *and* $D(D_3) = D(A_3)$*.*

Proof. We prove that there are no other Dynkin diagrams. To do this, we find all connected graphs with connections of the 4 types depicted in Figure 2, such that the matrix of any subgraph has a positive determinant, in particular any subgraph must be a Dynkin diagram. Equivalently, we require that the determinant of all principal minors of the corresponding Cartan matrix are positive. In particular, our graphs contain no extended Dynkin diagrams as induced subgraphs, since these have determinant 0.

Figure 2: The four Dynkin diagram connection types, corresponding to the four types of 2×2 Cartan matrix minors.

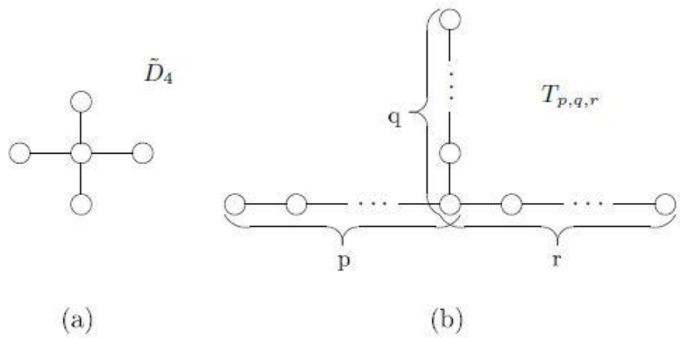

Figure 3: Some diagrams for the simply laced case.

Part 1. We first classify all simply-laced Dynkin diagrams $D(A)$, i.e. diagrams using only ◯ ◯ or ◯ − ◯ connections (which correspond to a symmetric Cartan matrix A). Such a diagram contains no cycles, since otherwise it contains $D(\tilde{A}_r)$. It has simple edges and it may or may not contain branching vertices. If there are no branching vertices, we get $D(A_r)$. If there are two branching vertices, the diagram contains $D(\tilde{D}_r)$, which is not possible. If there is precisely one branching vertex, it has at most 3 branches, since $D(\tilde{D}_4)$ is the 4-star in Figure 3a.

Therefore, it remains to consider the case when our graph $D(A)$ is of the form $T_{p,q,r}$, $p \geq q \geq r \geq 2$, depicted in Figure 3b.

If $r = 3$, then $T_{p,q,r}$ contains $D(\tilde{E}_6) = T_{3,3,3}$, which is impossible, so $r = 2$.

If $q = 2$, we have type D.

Now, assume $q \geq 3$. If $q > 3$, then $D(A)$ contains $D(\tilde{E}_7) = T_{4,4,2}$, which is impossible, so we are left with the case $T_{p,3,2}$ with $p \geq 3$.

The case $p = 3$ is E_6, the case $p = 4$ is E_7 and the case $p = 5$ is E_8. The case $p = 6$ is $\tilde{E}_8$, so $T_{p,3,2}$ with $p > 5$ is impossible.

Part 2. We now classify all non-simply-laced diagrams $D(A)$, i.e. those containing double- or triple-edge connections (corresponding to a non-symmetric Cartan matrix A). This can be done by computing many large determinants, but we would rather argue using the following result:

Exercise 2.1. Let A be an $r \times r$ matrix, and let B (resp. C) be the submatrices of A obtained by removing the first row and column (resp. the first two rows and two columns). We have:

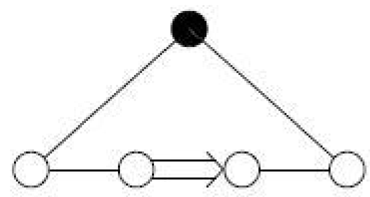

Figure 4: A loop with one double-connection.

a) If

$$A = \left(\begin{array}{c|ccc} 2 & -a & 0 \cdots & 0 \\ \hline -b & & & \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right),$$

then, $\det(A) = 2\det(B) - ab\det(C)$.

b) If

$$A = \begin{pmatrix} c_1 & -a_1 & 0 & \cdots & 0 & -b_r \\ -b_1 & c_2 & -a_2 & 0 & & 0 \\ 0 & -b_2 & c_3 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -a_{r-2} & 0 \\ 0 & & \ddots & -b_{r-2} & c_{r-1} & -a_{r-1} \\ -a_r & 0 & \cdots & 0 & -b_{r-1} & c_r \end{pmatrix},$$

then, $\det(A - \varepsilon e_{12}) = \det(A) - \varepsilon\,(b_1\det(C) + \prod_{i=2}^{r} a_i)$. In particular, if $b_1 > 0$, $a_i > 0$ for $i = 2, \ldots, r$, $\det(C) > 0$ and $\varepsilon > 0$, then $\det(A - \varepsilon e_{12}) < \det(A)$.

Proof. We check these items separately.

a) This is clear, just do cofactor expansion along the first row of A to obtain directly $\det(A) = 2\det(B) - ab\det(C)$.

b) This is also clear. Consider the first row of $A - \varepsilon e_{12}$, write $(c_1, -a_1 - \varepsilon, 0, 0, \ldots, 0, -b_r) = (c_1, -a_1, 0, \ldots, 0, -b_r) + (0, -\varepsilon, 0, \ldots, 0)$, use the multilinearity of det to write $\det(A - \varepsilon e_{12}) = \det(A) + \varepsilon \det D$, where D is obtained from A removing column 2 and row 1, and then do cofactor expansion along the first column of D to get $\det(D) = -b_1\det(C) + (-1)^r(-1)^{r-1}\prod_{i=2} a_i = -b_1 - \prod_{i=2} a_i$, where the second term comes directly from a lower triangular matrix.

$\square$

Exercise 2.1 implies that in the case of a non-simply laced diagram $D(A)$, there are no cycles since then $\det(A) < \det(\tilde{A}_r) = 0$, a contradiction. For example, the diagram in Figure 4 has $A = \tilde{A}_4 - E_{12}$, so $\det A < 0$.

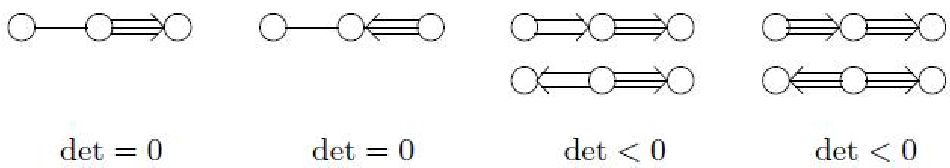

Figure 5: Possible neighbors of G_2 in a diagram.

Next, looking at the extended Dynkin diagrams, if $D(A)$ contains G_2 (Figure 1i), then it must be exactly G_2 by Exercise 2.1. Indeed, otherwise, using Figure 5 we see that there is a principal submatrix of A of the form:

$$M = \begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \text{ with } a, b > 0. \text{ Let } M' = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \text{ and } M'' = \begin{pmatrix} 2 \end{pmatrix}.$$

Hence, by Exercise 2.1, $\det(M) = 2\det(M') - ab\det(M'') = 2(1 - ab) \leq 0$, a contradiction. (The matrix of the second graph in the figure is actually A^T, but all the determinants are the same.)

It remains to consider the case when $D(A)$ has only simple or double connections. Looking at the extended Dynkin diagrams, $D(\tilde{C}_r)$ (in Figure 1c) cannot be a subdiagram. By Exercise 2.1, the variants with flipped arrow directions also do not work. Indeed, using the notation of the exercise, they are obtained from the Cartan matrix A of $D(\tilde{C}_r)$ by replacing some of A, B or C by their transposes, which does not change any of the determinants in the calculation. Thus, Exercise 2.1a shows that each of these variants have also determinant 0.

Therefore, $D(A)$ may contain only one double connection. But then, we cannot have branching points, since $D(\tilde{B}_r)$ contains a double edge and a branching point. So, the only remaining case is a line with a left-right double edge, having p single edges to the left, and q single edges to the right. If $p = 0$, we get $D(C_r)$. If $q = 0$, we get $D(B_r)$. If $p = q = 1$, we get $D(F_4)$. The diagram $D(\tilde{F}_4)$ has $p = 2, q = 1$; its transpose has $p = 1, q = 2$. Hence, if $p > 1$ or $q > 1$, $D(A)$ contains $D(\tilde{F}_4)$, which is impossible.

□

We have now shown that any finite-dimensional simple Lie algebra yields one of a very restricted set of Dynkin diagrams (and hence Cartan matrices). The next step in the classification of semisimple Lie algebras will be to give a construction associating an abstract Cartan matrix to a Lie algebra, and hence to prove that these four classes plus five exceptional algebras are in fact the only finite-dimensional simple Lie algebras.

We depict the plan of our actions:

$$\begin{tikzcd} \text{s.s. Lie alg. } \mathfrak{g} \arrow[r, "\text{Cartan } \mathfrak{h}"] & \underset{\text{root system}}{(V = \mathfrak{h}^*_{\mathbb{R}}, \Delta)} \arrow[r, "\text{choose } f \in V^*"] & \Pi \arrow[r] & \underset{\text{Cartan matrix}}{A} & \subseteq & \text{abstract Cartan matrix } A \arrow[lllll] \\ \mathfrak{g}(A) \arrow[u] \end{tikzcd}$$