18.745 Introduction to Lie Algebras October 28, 2010

Lecture 15 — Classical (Semi) Simple Lie Algebras and Root Systems

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Recall
$$O_{V,B}(\mathbb{F}) = \{a \in gl_V(\mathbb{F}) \mid B(au,v) + B(u,av) = 0, \text{ for all } u,v \in V\} \subset gl_V(\mathbb{F})$$
where V is a vector space over $\mathbb{F}$, B is a bilinear form : Vx$V \to \mathbb{F}$. Choosing a basis of V and denoting by B the matrix of the bilinear form in this basis, we proved we get the subalgebra

$$o_{n,B}(\mathbb{F}) = \{a \in gl_n(\mathbb{F}) \mid a^T B + Ba = 0\} \subset gl_n(\mathbb{F}).$$

For different choices of basis, we get isomorphic Lie algebras $o_{n,B}(\mathbb{F})$.

Now, consider the case where B is a symmetric non-degenerate bilinear form. If $\mathbb{F}$ is algebraically closed and char $\mathbb{F} \neq 2$, one can choose a basis in which the matrix of B is any symmetric non-degenerate matrix.

Example 15.1. I_N where $N = \dim V$.

We will choose a basis such that
$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 1 \\ 0 & 0 & & & \iddots & \\ 0 & & \ddots & 1 & & \vdots \\ \vdots & & 1 & \ddots & & 0 \\ & \iddots & & & 0 & 0 \\ 1 & & \cdots & 0 & 0 & 0 \end{pmatrix}$$

and denote by $so_N(\mathbb{F})$ the corresponding Lie algebra $o_{N,B}(\mathbb{F})$.

Exercise 15.1. Show $so_N(\mathbb{F}) = \{a \in gl_N(\mathbb{F}) | a + a' = 0\}$ where a' is the transposition of a with respect to the anti-diagonal.

Proof. $so_N(\mathbb{F}) = \{a \in gl_N \mathbb{F}) \mid a^T B + Ba = 0\}$ where B is the matrix consisting of ones along the anti-diagonal.

As $B = B^T$, we have $a^T B = a^T B^T = (Ba)^T$. Viewing B as a permutation matrix, we get Ba permutes the rows by $\text{row}_i \to \text{row}_{n-i}$. Transposing and reapplying B, we get $B(Ba)^T = a'$ and $BBa = a$. Hence we obtain the following sequence of implications

$$a^T B + Ba = 0$$
$$(Ba)^T + Ba = 0$$
$$B(Ba)^T + a = 0$$

and finally

$$a' + a = 0.$$

Therefore, $so_N(\mathbb{F}) = \{a \in gl_N(\mathbb{F}) | a + a' = 0\}$.

□

Example 15.2. $so_2(\mathbb{F}) = \{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \alpha \in \mathbb{F}\}$, which is one-dimensional abelian, hence not semisimple.

Proposition 15.1. *Assume $N \geq 3$, then $so_N(\mathbb{F})$ is semisimple.*

Proof. We show this by the study of the root space decomposition.

Case 1: $N = 2n + 1$ (odd). Let

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_n & & & & \\ & & & 0 & & & \\ & & & & -a_n & & \\ & & & & & \ddots & \\ & & & & & & -a_1 \end{pmatrix}. \subset so_{2n+1}(\mathbb{F})$$

This is a Cartan subalgebra since it contains a diagonal matrix with distinct entries.

Case 2: $N = 2n$ (even). Let

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_n & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}.$$

This is a Cartan subalgebra for the same reason.

In both cases, $\dim \mathfrak{h} = n$ and $\epsilon_1, ..., \epsilon_n$ form a basis $\mathfrak{h}^*$. Note that $\epsilon_{N+1-j}|_{\mathfrak{h}} = -\epsilon_j|_{\mathfrak{h}}$ and $\epsilon_{\frac{N+1}{2}}|_{\mathfrak{h}} = 0$ if N is odd.

Next, all eigenvectors for **ad** $\mathfrak{h}$ are elements $e_{i,j} - e_{N+1-j,N+1-i}, i, j \in \{1, 2, ..., N\}$ and the root is $\epsilon_i - \epsilon_j|_{\mathfrak{h}}$.

Hence the set of roots is:

$N = 2n + 1 : \Delta_{so_N(\mathbb{F})} = \{\epsilon_i - \epsilon_j, \epsilon_i, -\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$

$N = 2n : \Delta_{so_N(\mathbb{F})} = \{\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i, j \in \{1, ..., n\}, i \neq j\}$ □

Exercise 15.2. a) Using the root space decomposition, prove that $so_N(\mathbb{F})$ is semisimple if $N \geq 3$.

b) Show $so_N(\mathbb{F})$ is simple if $N = 3$ or $N \geq 5$ by showing that Δ is indecomposable.

Thus we have another two series of simple Lie algebras: $so_{2n+1}(\mathbb{F})$ for $n \geq 1$ (type B) and $so_{2n}(\mathbb{F})$ for $n \geq 3$ (type D).

Proof. a) We must check (1), (2), and (3) of the semisimplicity criterion.

(1) is clear for B and D and (3) is clear for B. For (3) in case D, we have roots $\epsilon_i - \epsilon_j$ and $\epsilon_i + \epsilon_j$, adding and dividing by 2 (as char $\mathbb{F} \neq 2$) gives us ϵ_i, hence (3) holds.

(2) We compute $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = [e_{ij} - e_{N+1-j,N+1-i}, e_{ji} - e_{N+1-i,N+1-j}] = (e_{ii} - e_{N+1-i,N+1-i}) + (e_{N+1-j,N+1-j} - e_{jj}) = h_\alpha \in \mathfrak{h}$. As $\alpha(h_\alpha) \neq 0$, so_N is semisimple in $N \geq 3$.

b) To show simple for $N = 3$ and $N \geq 5$, we show that Δ is indecomposable. This is clear for $N = 3$. We list pairs and corresponding paths for $n \geq 3$. $N = 5$ is done separately. For ease of notation, we write ϵ_i as i and remark that any root is connected to its negative by the path of length one.

- $(i + j) \to (j + k)$ via $(i + j, -k - i, j + k)$
- $(i + j) \to (i - j)$ via $(i + j, k - i, j - k, i - j)$
- $(i + j) \to (i)$ via $(i + j, -j, i)$

  Therefore when $N > 5$, we may concatenate and find paths through $(i + j)$. When $N = 5$, we had the issue of connecting $(i + j)$ to $(i - j)$. As N is odd in this case, we may use the path $(i + j, -i, i - j)$. Therefore, $so_N(\mathbb{F})$ is simple for $N = 3$ and $N \geq 5$.

□

Exercise 15.3. Show $\Delta_{so_4(\mathbb{F})} = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\} \sqcup \{\epsilon_1 + \epsilon_2, -\epsilon_1 - \epsilon_2\}$ is the decomposition into decomposables. Deduce that $so_4(\mathbb{F})$ is isomorphic to $sl_2(\mathbb{F}) \oplus sl_2(\mathbb{F})$.

Proof. We have the decomposition

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1\} \sqcup \{\epsilon_1 + \epsilon_2, -\epsilon_1 - \epsilon_2\}.$$

This is in fact a decomposition since the collection of roots distance one from $\epsilon_1 - \epsilon_2$ is $\epsilon_2 - \epsilon_1$ as $\epsilon_1 - \epsilon_2 + \epsilon_1 + \epsilon_2 = 2\epsilon_1 \notin \Delta$ and $\epsilon_1 - \epsilon_2 + -\epsilon_1 - \epsilon_2 = -2\epsilon_2 \notin \Delta$. Hence, we have a decomposition.

To show isomorphic to $sl_2 \oplus sl_2$. Consider the basis of $\mathfrak{h}$, $x = e_{11} - e_{44}$ and $y = e_{22} - e_{33}$. If we change bases to from x, y to $x + y, x - y$. Let

$$e = e_{12} - e_{34}, f = e_{13} - e_{24}, g = e_{31} - e_{42}, h = e_{21} - e_{43}.$$

Then

$$[x + y, e] = 0, [x + y, f] = 2f, [x + y, g] = -2g, [x + y, h] = 0, [g + f] = x + y$$

and

$$[x - y, e] = 2e, [x - y, f] = 0, [x - y, g] = 0, [x - y, h] = -2h, [e, h] = x - y$$

and finally

$$[x+y, x-y] = [e,f] = [e,g] = [h,f] = [h,g] = 0.$$

Therefore, we have two copies of sl_2 formed by $\{f, g, x+y\}$ and $\{e, h, x-y\}$, and therefore an isomorphism. $\square$

Next consider the case where B is a skew-symmetric non-degenerate bilinear form. If $\mathbb{F}$ is of characteristic $\neq 2$, one can choose a basis in which the matrix of B is any skew-symmetric non-degenerate matrix where $N = \dim V = 2n$ (even). We get

$$sp_{n,B} = \{a \in gl_n(\mathbb{F}) \mid a^T B + B a^T = 0\} \subset gl_n(\mathbb{F}).$$

The best choice of B is

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 1 \\ 0 & 0 & & & \iddots & \\ 0 & & \ddots & 1 & & \vdots \\ \vdots & & -1 & \ddots & & 0 \\ & \iddots & & & 0 & 0 \\ -1 & & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 15.4. Repeat the discussion we've done for $so_N(\mathbb{F})$ in the case $sp_{2n}(\mathbb{F})$. First:

$$sp_{2n}(\mathbb{F}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ all } a,b,c,d \text{ are } n \text{ x } n \text{ such that } b = b', c = c', d = -a'\}$$

Next, let $\mathfrak{h}$ be the set of all diagonal matrices in $sp_{2n}(\mathbb{F})$

$$\mathfrak{h} = \{\begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_r & & & \\ & & & -a_r & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}, a_i \in \mathbb{F}\}$$

Find all eigenvectors for **ad** $\mathfrak{h}$. Show that the set of roots is

$$\Delta_{sp_{2n}(\mathbb{F})} = \{\epsilon_i - \epsilon_j, 2\epsilon_i, -2\epsilon_i, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j \mid i,j \in \{1,...,n\}, i \neq j\}$$

Show always indecomposable and deduce that $sp_{2n}(\mathbb{F})$ simple for all $n \geq 1$.

These Lie algebras are called type C simple Lie algebras.

Proof. We must check (1),(2), and (3) of the semisimplicity criterion. (1) is clear.

For (2), split a matrix M into its four quadrants. Label the upper left quadrant A. The upper left half of the upper right quadrant B, with that portion of the anti-diagonal X. Finally, the upper left half of the lower left quadrant C, with that portion of the anti-diagonal Y. We then have the following eigenvectors.

- $e_{ij} - e_{N+1-j,N+1-i}$ if $e_{ij} \in A$
- $e_{ij} + e_{N+1-j,N+1-i}$ if $e_{ij} \in B \cup C$
- e_{ij} if $e_{ij} \in X \cup Y$

with eigenvalues $\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, -\epsilon_i - \epsilon_j, 2\epsilon_i, -2\epsilon_i$ for $e_{ij} \in A, B, C, X, Y$ respectively.

To compute $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, we have

$$[e_{ij} - e_{N+1-j,N+1-i}, e_{ji} - e_{N+1-i,N+1-j}] = (e_{ii} - e_{N+1-i,N+1-i}) + (e_{N+1-j,N+1-j} - e_{jj}) = h_\alpha \in \mathfrak{h}$$

$$[e_{ij} + e_{N+1-j,N+1-i}, e_{ji} + e_{N+1-i,N+1-j}] = (e_{ii} - e_{N+1-i,N+1-i}) - (e_{N+1-j,N+1-j} - e_{jj}) = h_\alpha \in \mathfrak{h}$$

$$[e_{ij}, e_{jj}] = e_{ii} - e_{jj} = h_\alpha \in \mathfrak{h}.$$

In each case $\alpha(h_\alpha) \neq 0$, and therefore (2) holds.

Finally, (3) is clear, so we only must show in-decomposablility to get simplicity.

From Exercise 15.2, when $n \geq 3$, have that all pairs of the form $\pm\epsilon_i \pm \epsilon_j$ are connected to $\epsilon_i - \epsilon_j$. We may connect $2\epsilon_j$ to $\epsilon_i - \epsilon_j$ as $2\epsilon_j + \epsilon_i - \epsilon_j = \epsilon_i + \epsilon_j \in \Delta$ and therefore through concatenation we are done.

When $n = 2$ we may connect $\epsilon_i - \epsilon_j$ to $\epsilon_i + \epsilon_j$ as $\epsilon_i - \epsilon_j + \epsilon_i + \epsilon_j = 2\epsilon_i \in \Delta$, so we have in-decomposability.

When $n = 1$, indecomposability is clear.

□

Remark 1. Thus we get four series of simple Lie algebras $A_n = sl_n(\mathbb{F})(n \geq 1), B_n = so_{2n+1}(\mathbb{F})(n \geq 1), C_n = sp_{2n}(\mathbb{F})(n \geq 1), D_n = so_{2n}(\mathbb{F}), (n \geq 4)$ called the classical simple Lie algebras.

Proposition 15.2. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. Then*

a) Any symmetric invariant bilinear form is either non-degenerate or identically zero.

b) Any two non-degenerate such bilinear forms are proportional: $(a, b)_1 = \lambda(a, b)_2$.

Proof. a) If $(\cdot, \cdot)$ is an invariant bilinear form and I is its kernel, then I is an ideal, hence $\mathfrak{g}$ simple implies that either $I = 0$ or $I = \mathfrak{g}$.

b) Choose a basis of $\mathfrak{g}$ and let B_i be the matrix of $(\cdot, \cdot)_i$ in the basis. $\mathrm{Det}(B_i) \neq 0$. Consider $\det(B_1 - \lambda B_2) = \det(B_2)\det(B_1 B_2^{-1} - \lambda I) = 0$ if λ is an eigenvalue of $B_1 B_2^{-1}$. Hence the form $(a, b)_1 - \lambda(a, b)_2$ is a degenerate, invariant, bilinear form as det is 0. Hence the form $(a, b)_1 - \lambda(a, b)_2$ is identically zero by (a), which implies $(a, b)_1 = \lambda(a, b)_2$. □

Corollary 15.3. *If $\mathfrak{g} \subset gl_N(\mathbb{F})$ is a simple Lie algebra, then the Killing from on $\mathfrak{g}$ is proportional to the trace form $(a, b) = tr\ ab$ on $\mathfrak{g}$.*

Example 15.3. On $gl_N(\mathbb{F})$: (1) tr $e_{ii}e_{ij} = \delta_{ij}$ (with e_{ii} basis of D), hence the induced bilinear form on $D^* = (\epsilon_i, \epsilon_j) = \delta_{ij}$ (2). Hence for all classical simple Lie algebras A,B,C,D, the Killing form is a positive constant multiple of (1) and on $\mathfrak{h}^*$ is a positive constant multiple of (2).

Definition 15.1. Let V be a finite dimensional real Euclidean space, i.e. V finite dimensional vector space over $\mathbb{R}$ with symmetric positive definite bilinear form $(\cdot, \cdot)$.

Let $\Delta \subset V$ be a subset of V. Then the pair (V, Δ) is called a root system if:

i) Δ finite, $0 \notin \Delta$, Δ spans V over $\mathbb{R}$;

ii) (String Condition) For any $\alpha, \beta \in \Delta$, the set $\{\beta + j\alpha \mid j \in \mathbb{Z}\} \bigcap (\Delta \cup \{0\})$ is a string $\beta + p\alpha, \beta + (p-1)\alpha, ..., \beta - q\alpha$ where $p, q \in \mathbb{Z}$, and $p - q = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$;

iii) For all $\alpha \in \Delta$, we have $k\alpha \in \Delta$ if and only if $k = 1$ or $k = -1$.

Example 15.4. The basic example: Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, $\mathfrak{h}$ a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*_{\mathbb{Q}}$ the set of roots, $(\cdot, \cdot)$ the Killing form on $\mathfrak{h}^*_{\mathbb{Q}}$ which is $\mathbb{Q}$-valued and positive definite.

Let $V = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}$, i.e. linear combinations of roots with real coefficients and extend the Killing form by bilinearity. Then the pair (V, Δ) is a root system, called the $\mathfrak{g}$ root system.

Remark 2. This construction is independent of the choice of the Cartan subalgebra $\mathfrak{h}$ due to Chevalley's Theorem.

Exercise 15.5. Let (V, Δ) be a root system. Then Δ is indecomposable if and only if there does not exist non-trivial decomposition $(V, \Delta) = (V_1, \Delta_1) \oplus (V_2, \Delta_2)$ where $V = V_1 \oplus V_2$, $V_1 \perp V_2$, $\Delta_i \subset V_i$, and $\Delta = \Delta_1 \cup \Delta_2$. (Hint: Use String Condition)

Moreover, the decomposition of $\Delta = \bigsqcup \Delta_i$ into indecomposable sets corresponds to decomposition of the root system in the orthogonal direct sum of indecomposable root systems.

Proof. For the first direction, suppose we have a decomposition $\Delta = \Delta_1 \sqcup \Delta_2, \Delta_i \subset V_i, \alpha \in \Delta_i, \beta \in \Delta_2$. Therefore, $\alpha + \beta \notin \Delta \cup \{0\}$, hence $q = 0$. As well, clearly $-\alpha \in \Delta_2$, so $p = 0$. Therefore, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 0$, hence $(\alpha, \beta) = 0$. Let $V_i = span(\Delta_i)$, then by above $V_1 \perp V_2$, and thus $V = V_1 \oplus V_2$.

On the other hand, suppose $V = V_1 \oplus V_2, V_1 \perp V_2$. Choose $\Delta_i = \Delta \cap V_i$. We show $\Delta_1 \sqcup \Delta_2$ is a decomposition. In the contrary case, choose $\alpha \in \Delta_i, \beta \in \Delta_2$ and suppose $\alpha + \beta \in \Delta \cup \{0\}$. Then, $\alpha + \beta \neq 0$ since $(\alpha, -\alpha) \neq 0$, so $\alpha + \beta \in \Delta$. Without loss of generality, $\alpha + \beta \in \Delta_1 \subset V_1$, as $\beta \in \Delta_2 \subset V_2$ and $V_1 \perp V_2$, we have $0 = (\alpha + \beta, \beta) = (\alpha, \beta) + (\beta, \beta) = (\beta, \beta)$. This is a contradiction, hence $\alpha + \beta \notin \Delta$, so we have a decomposition.

By the above argument, it is clear that the decomposition into indecomposables corresponds to the orthogonal decomposition with respect to $(\cdot, \cdot)$.

□