Summary of May 8 class

Applications of Riemann-Roch

Let $Y$ be a smooth projective curve, let $D$ be a divisor on $Y$, and let $K$ be canonical divisor on $Y$. Riemann-Roch asserts that $h^0 \mathcal{O}(D) = h^1 \mathcal{O}(K - D)$ and $h^1 \mathcal{O}(D) = h^0 \mathcal{O}(K - D)$.

I. proof that $p_a = g$.

We put $D = 0$ into Riemann-Roch: $h^0 \mathcal{O} = h^1 \mathcal{O}(K)$ and $h^1 \mathcal{O} = h^0 \mathcal{O}(K)$. Moreover, $h^0 \mathcal{O} = 1$ and by definition, $h^1 \mathcal{O}$ is the arithmetic genus $p_a$. Thus $\chi \mathcal{O}(K) = h^0 - h^1 = p_a - 1$. By Riemann-Roch 1, $\chi \mathcal{O}(K) = \deg K + 1 - p_a$. Therefore $\deg K = 2p_a - 2$.

Next, we look at a branched covering $Y \xrightarrow{\pi} X = \mathbb{P}^1$. Let $n$ be the degree of $\pi$, and let's assume that the covering isn't branched above the point of $X$ at infinity. Let $x$ be a coordinate of $X$ on the standard affine open set $U^0$. If $q_i$ are the branch points of the covering, and $e_i$ are the ramification indices, then on $Y$, $dx$ has zeros of orders $e_i - 1$ at $q_i$. Also, on $X$, $dx$ has a pole of order 2 at $\infty$. Therefore $dx$ has poles of order 2 at each of the $n$ points of the fibre over $\infty$. Then if $K$ denotes the divisor of $dx$ on $Y$,

$$\deg K = \sum (e_i - 1) - 2n$$

Now we compute the topological Euler characteristic of $Y$. In terms of the covering,

$$e(Y) = ne(X) - \sum (e_i - 1) = 2n - \sum (e_i - 1) = -\deg K$$

On the other hand, $e(Y) = 2 - 2g$. Therefore $\deg K = 2g - 2$. So $g = p_a$.

II. Base Points.

Let $D$ be a divisor on $Y$. A point $p$ of $Y$ is a base point of $\mathcal{O}(D)$ if every global section of $\mathcal{O}(D)$ is also a global section of $\mathcal{O}(D - p)$, i.e., if $h^0 \mathcal{O}(D) = h^0 \mathcal{O}(D - p)$. The usual exact sequence $0 \rightarrow \mathcal{O}(D - p) \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O} \rightarrow 0$ shows that if $p$ is a base point, then $h^1 \mathcal{O}(D - p) = h^1 \mathcal{O}(D) + 1$, and if $p$ is not a base point, then $h^1 \mathcal{O}(D - p) = h^1 \mathcal{O}(D)$.

When $D$ is effective and $\mathcal{O}(D)$ has no base point, a generic global section of $\mathcal{O}(D)$ will have poles equal to $D$, not less than $D$.

Examples. $\mathcal{O}(K)$ has no base point if $g > 0$, because $h^1 \mathcal{O}(K - p) = h^0 \mathcal{O}(p)$, and there can be no function with just one simple pole, $h^0 \mathcal{O}(p) = h^0 \mathcal{O} = 1$.

On the other hand, $p$ is a base point of $\mathcal{O}(K + p)$ because $h^1 \mathcal{O}(K + p) = h^0 \mathcal{O}(-p) = 0$, which is less than $h^0 \mathcal{O}$.

Let $D$ be an effective divisor. Assume that $h^0 \mathcal{O}(D) \geq 2$, and that $\mathcal{O}(D)$ has no base point. We use a basis $(\alpha_0, ..., \alpha_r)$ of $H^0 \mathcal{O}(d)$ to map $Y$ to projective space, say $Y \xrightarrow{\pi} \mathbb{P}$. The degree of $\pi$ is defined to be the number of points of $Y$ in the inverse image of a generic hyperplane $H$. 
Proposition. With the above notation, the degree of $\pi$ is equal to the degree of $D$.

proof Say that $H$ is the hyperplane $\sum c_ix_i = 0$. Then its inverse image is the set of points $q$ of $Y$ such that $\sum c_i\alpha_i(q) = 0$. The rational function $\beta = \sum c_i\alpha_i$ is a generic global section of $O(D)$, so its polar divisor is equal to $D$. It has the same number of zeros as poles.

The Canonical Map.

Here $Y$ is a curve of genus at least 2. The canonical map is the map $Y \xrightarrow{\pi} \mathbb{P}^{g-1}$ defined by a basis of $O(K)$. Since $O(K)$ has no base point, the degree of this map is $\deg K = 2g - 2$.

If $g = 2$, $\pi$ maps $Y$ to $\mathbb{P}^1$, and its degree is 2.

Recall that $Y$ is called hyperelliptic if it can be represented as a double cover of the projective line $X$. Thus every curve of genus 2 is hyperelliptic. Though this isn’t obvious, most curves of genus 3 or more aren’t hyperelliptic.

Theorem. Let $Y$ be curve of genus at least 2. Then either $Y$ is hyperelliptic, or else the canonical map $\pi$ is an embedding of $Y$ into projective space.

Sketch of the proof We suppose that the canonical map $\pi$ is not injective, and we show that then $Y$ is hyperelliptic. Let $q_1$ an $q_2$ be points of $Y$ such that $\pi(q_1) = \pi(q_2)$, and let’s assume that $K$ is chosen so that $q_i$ are not in its support. Then a section of $O(K)$ that vanishes at $q_1$ also vanishes at $q_2$. So $H^0O(K - q_1 - q_2) = H^0O(K - q_1)$, and since $O(K)$ has no base point, $h^0O(K - q_1 - q_2) = h^0O(K - q_1) = h^0O(K) - 1 = g - 1$. Then $h^1O(q_1 + q_2) = g - 1$. By RR, $\chi(q_1 + q_2) = \deg(q_1 + q_2) + 1 - g = 3 - g$. Then $h^0O(q_1 + q_2) = 2$. Since $h^0O(q_i) = 1$, $O(q_1 + q_2)$ has no base points. A basis of $H^0O(q_1 + q_2)$ defines a map $Y \rightarrow \mathbb{P}^1$ of degree 2. So $Y$ is hyperelliptic.

Thus, if $Y$ isn’t hyperelliptic, the map $\pi : Y \rightarrow \mathbb{P}^{g-1}$ is injective. This isn’t quite enough to show that $Y$ is isomorphic to its image $X$. When $Y \rightarrow X$ is bijective, the function fields will be equal, and we can conclude that $Y$ is the normalization of $X$. But, could $X$ have a cusp?

To show that this doesn’t occur, we go over the argument above, replacing the point pair $q_1 + q_2$ by a divisor of the form $2q$. The same reasoning shows that if $h^0O(K - 2q) = O(K - q)$, then $h^0O(2q) = 2$, and $Y$ is hyperelliptic. Consequently, if $Y$ isn’t hyperelliptic, then for every point $q$ of $Y$, there is a section $\alpha$ of $O(K)$ that vanishes at $q$, but doesn’t have a zero of order 2 there. We can use $\alpha$ as one element of the basis of $H^0O(K)$, say $\alpha = \alpha_0$. Then $\pi^{-1}(x_0)$ contains the point $q$ but doesn’t contain $2q$. This means that a nonzero tangent vector to $Y$ at $q$ doesn’t map to zero in $X$, and it implies that the map $Y \rightarrow X$ is an isomorphism. There is a clumsy proof in the notes.

There is one more remarkable fact about hyperelliptic curves:

Theorem. A smooth projective curve $Y$ of genus $g \geq 2$ can be represented as a double covering of $X = \mathbb{P}^1$ in at most one way.

I ran out of time to do the proof.