Summary of May 4 class

Review from last time. There is an \( \mathcal{O} \)-module of differentials \( \Omega_X \) on a variety \( X \). On an affine variety \( \text{Spec } A \), the module \( \Omega_A \) of differentials is generated by elements \( da, a \in A \), with the relations

\[
d(a + b) = da + db \quad , \quad d(ab) = a \, db + b \, da \quad , \quad dc = 0 \text{ if } c \in \mathbb{C}
\]

**Proposition.** On a smooth curve \( Y \), the module \( \Omega_Y \) of differentials is locally free rank one – an invertible module.

Thus \( \Omega_Y \) is isomorphic to \( \text{co}(K) \) for some canonical divisor \( K \) which is determined up to linear equivalence: If \( K_1 \) and \( K_2 \) are two canonical divisors, \( K_1 - K_2 \) is the divisor of a function.

**Riemann-Roch Theorem 2** Let \( Y \) be a smooth projective curve, and let \( D \) be a divisor on \( Y \). Then

\[
h^0 \mathcal{O}(D) = h^1 \mathcal{O}(K - D) \quad \text{and} \quad h^1 \mathcal{O}(D) = h^0 \mathcal{O}(K - D)
\]

The two assertions are equivalent because \( K - (K - D) = D \).

In particular, \( h^0 \mathcal{O}(K) = h^1 \mathcal{O} \), which is the genus \( g \) of \( Y \), and \( h^1 \text{co}(k) = h^0 \mathcal{O} = 1 \). Thus \( \chi \mathcal{O}(K) = -\chi(\text{co}) = g - 1 \), and by Riemann-Roch 1, \( \chi \mathcal{O}(K) = \deg K + 1 - g \).

**Corollary.** The degree \( \deg K \) of a canonical divisor is \( 2g - 2 \).

curves of genus 2. Let \( Y \) be such a curve. The degree of \( K \) is \( 2g - 2 = 2 \), and \( h^0 \mathcal{O}(K) = 2 \). There is a global section \( \alpha \) of \( \mathcal{O}(K) \), and \( \text{div } \alpha + K \), which is another canonical divisor, is effective. So we may suppose that \( K \) is effective, say \( K = p_1 + p_2 \), where \( p_i \) might be equal. A basis \( (1, \alpha) \) for \( H^0 \mathcal{O}(K) \) defines a morphism

\[
Y \longrightarrow \pi
\]

that sends a point \( q \) to \((1, \alpha(q))\) if \( q \) isn’t one of the poles \( p_i \) of \( \alpha \), and its sends the points \( p_i \) to \((\alpha^{-1}(q), 1) = (0, 1)\). (There can’t be a function with just one pole of order 1.) So \( \pi \) sends two points \( p_1, p_2 \) to infinity, and therefore it takes every value twice. It is a double covering.

A curve \( Y \) that can be represented as a double covering of \( \mathbb{P}^1 \) is called a hyperelliptic curve. This is a strange term, but every elliptic curve can be represented in this way. More about hyperelliptic curves will follow in a couple of days.

**Corollary.** Every curve of genus 2 is hyperelliptic.

curves of genus 3

Let \( Y \) be a curve of genus 3. Then \( \deg K = 2g - 2 = 4 \), and we may assume that \( K \) is effective, say \( K = p_1 + p_2 + p_3 + p_4 \), with possible repetitions of points. Here \( H^0 \mathcal{O}(K) \)
has dimension 3. Let \((1, \alpha, \beta)\) be a basis. The poles of the functions \(\alpha\) and \(\beta\) are among the points \(p_1 + \cdots + p_4\). Let's check that all of those points are poles of one or the other of those functions.

If \(p = p_4\), say, were not a pole of \(\alpha\) or \(\beta\), then those functions would be sections of \(\mathcal{O}(p_1 + p_2 + p_3)\), and we would have \(H^0\mathcal{O}(K) = H^0\mathcal{O}(K - p) = h^1\mathcal{O}(p)\). This isn't possible. We've seen that \(\mathcal{O}(p)\) can't have a nonconstant global section. So \(h^0\mathcal{O}(p) = 1\) and by RR1, \(\chi\mathcal{O}(p) = 1 + 1 - g = -1\), so \(h^1\mathcal{O}(p) = 2\), while \(h^0\mathcal{O}(K) = 3\).

We map \(Y\) to \(\mathbb{P}^2\) by the basis \((1, \alpha, \beta)\). Let \(L\) be the line \(x_0 = 0\) in \(\mathbb{P}^2\). The points \(q\) of \(Y\) whose images are in \(L\) are those such that \(\alpha\) or \(\beta\) has a pole at \(q\). Say that \(\alpha/\beta\) is regular at a point \(q\) of \(Y\). (Otherwise, \(\beta/\alpha\) will be regular there.) Then \((1, \alpha, \beta) \approx (1/\beta, \alpha/\beta, 1)\), and the image of \(q\) will be \((0, *, 1)\), a point of \(L\). Therefore the points whose images are in \(L\) are the points \(p_1, \ldots, p_4\). Four points (counted with multiplicity) map to the line. The image \(X\) of \(Y\) in \(\mathbb{P}^2\) is a plane curve whose degree must be a divisor of 4. The image can't have degree 1 because \(9, \alpha \beta\) are independent. We have proved

**Proposition.** With notation as above, the image \(X\) of \(Y\) is either a conic, or a curve of degree 4.

If \(X\) is a conic, isomorphic to \(\mathbb{P}^1\), then \(Y\) will be a double cover of \(X\). In this case, \(Y\) is **hyperelliptic**.

If the image \(X\) has degree 4, then the morphism \(Y \to X\) will be generically injective as well as surjective. Then \(Y\) will be the normalization of \(X\). Since a plane curve of degree 4 has arithmetic genus 3, \(X\) must be smooth and isomorphic to \(Y\).

**Corollary.** A curve of genus 3 is isomorphic to a plane curve of degree \(e\), or else it is hyperelliptic.

**proof of RR2 for the projective line**

When \(X = \mathbb{P}^1\), the proof of RR2 is a simple computation that, however, doesn't give much insight.

According to the Birkhoff-Grothendieck Theorem, every locally free \(\mathcal{O}_X\)-module is a direct sum of twisting modules. It is overkill to apply this theorem here, but it tells us that if \(D\) is a divisor on \(X\), \(\mathcal{O}(D) = \mathcal{O}(n)\) for some \(n\), and as you will be able to check, \(n\) is the degree of the divisor. So when we determine the degree of the module \(\Omega_X\) of differentials, we can use the known cohomology of the twisting modules.

For review: if \(n \geq 0\), then \(h^0\mathcal{O}(n) = n + 1\), and \(h^1 = 0\). If \(r > 0\), then \(h^0\mathcal{O}(-r) = 0\) and \(h^1\mathcal{O}(-r) = r - 1\).

To identify \(\Omega_X\) as a twisting module, we look on the standard open cover \(U^0 = \text{Spec} \mathbb{C}[u]\) and \(U^1 = \text{Spec} \mathbb{C}[v]\), where \(u = x_1/x_0\) and \(v = x_0/x_1 = u^{-1}\). As we know, the module of differentials on the polynomial ring \(\mathbb{C}[u]\) and on its spectrum \(U^0\) is free, with basis \(du\). And on \(U^1\) the module of differentials has basis \(dv\).
We note that $du = dv^{-1} = -v^2 dv$. So the differential $du$ has a pole of order 2 at the point $p$ at infinity. It is a section of $\Omega_X(2p) = \Omega_X \otimes_{\mathcal{O}} \mathcal{O}(2p)$, and as a section of that $\mathcal{O}$-module, it is nowhere zero. This means that multiplication by $du$ defines an isomorphism $\mathcal{O} \to \Omega_X(2p)$. So $\Omega_X(2p) \approx \mathcal{O}$, and $\Omega_X \approx \mathcal{O}(-2p) \approx \mathcal{O}(-2)$.

This being so, what has to be shown for RR2 is that $h^0\mathcal{O}(n) = h^1\mathcal{O}(-2 - n)$ and $h^1\mathcal{O}(n) = h^1\mathcal{O}(-2 - n)$. This is true.