Summary of March 30 class

Projective double planes

$X = \mathbb{P}^2$, coordinates $x_0, x_1, x_2$. $f(x)$ a homogeneous, square-free polynomial of even degree $2d$. $y$ a variable of weight $d$.

Then

$$y^2 = fx$$

is a homogeneous equation of degree $2d$ that has a locus $Y$ in the weighted projective space $\mathbb{WP}$ with coordinates $(x_0, x_1, x_2, y)$. The equivalence relation that defines points of $\mathbb{WP}$ is

$$(x_0, x_1, x_2, y) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda^d y)$$

The locus $Y$ is the double plane.

The projection $\pi : Y \rightarrow X$ has fibres \{$(x, y), (x, -y)$\}. Two points in the fibre unless $f(x) = 0$. Then fibre is \{(x, 0)\} and has one point.

The branch locus $\Delta$ is the (possibly reducible) curve $f(x) = 0$ in the plane $X$.

There is an obvious automorphism $\sigma$ of $Y$ that sends $(x, y) \rightarrow (x, -y)$.

A curve in $Y$ is a subvariety of dimension 1. Its projection will be a curve in $X$. Conversely, let $C$ be a curve in $X$, say $h(x) = 0$ where $h$ is an irreducible homogeneous polynomial. There are 3 possibilities for the inverse image $\pi^{-1}C$ in $Y$:

- **$C$ ramifies**: if $C$ is a component of $\Delta$. Then the map $\pi^{-1}C = D \rightarrow C$ is bijective. This happens when $h$ divides $f$. Then modulo $h$, the equation of $Y$ becomes $y^2 = 0$.
- **$C$ splits**: $\pi^{-1}C = D_1 \cup D_2$ where $D_i$ are curves in $Y$, and $D_2 = \sigma D_1$. The map $D_1 \rightarrow C$ will be generically bijective. Some fibres through singular points may consist of two points.
- **$C$ doesn’t split** (and doesn’t ramify): $\pi^{-1}C = D$ is irreducible. The fibres of the map $D \rightarrow C$ consist of two points, except for fibres over the points of $C \cap \Delta$.

This occurs when the ideal $I$ of $\mathbb{C}[x, y]$ generated by $y^2 - f$ and $h$ is a prime ideal. If $C$ splits, the ideal $I$ is the intersestion of two prime ideals.

analog from algebraic number theory: $A = \mathbb{Z}$, $B = \mathbb{Z}[\delta]$, $\delta^2 = -5$. The prime integer $p = 5$ ramifies.

- The prime $p = 3$ splits: $3B = (3, 1 + \delta)(3, 1 - \delta)$
- The primes $p = 7$ doesn’t split.

Note that in the example $p = 3$, though (3) splits, its ideal factors aren’t principal ideals.

Back to the double plane $Y$. Let’s say that $d = 2$, so that the degree of $f(x)$ is 4, and that $f$ is a generic quartic polynomial. Then $Y$ is a quartic double plane. The branch locus is the generic quartic curve $\Delta : f = 0$ in the plane $X$.

Does a line $L$ in $X$ split, or not, in $Y$. Let’s choose coordinates so that $L$ is the line $x_0 = 0$. This is the same as asking about the ideal $(y^2 - f, x_0)$ in $\mathbb{C}[x, y]$ Is it is or is it not, a prime ideal? $L$ splits if this ideal isn’t prime. We can work modulo $x_0$. Let $\overline{f}(x_1, x_2) = f(0, x_1, x_2)$. Modulo $x_0$, the question becomes: Is the principal ideal of the ring $\mathbb{C}[x_1, x_2, y]$? generated by $n y^2 - \overline{f}$ a prime ideal? Since $\mathbb{C}[x_1, x_2, y]$ is a polynomial ring,
the ideal will be prime if the polynomial \( y^2 - \overline{f} \) is irreducible. So \( L \) split if \( y^2 - \overline{f} \) factors. If it fattors, say \( y^2 - \overline{f} = (x + \alpha)(x + \beta) \), then \( \beta = -\alpha \), and \( y^2 - \overline{f} = y^2 - \alpha^2 \). So \( y^2 - \overline{f} \) factors if and only if \( \overline{f} \) is a square polynomial in \( x_1, x_2 \).

Is the rational function obtained from \( f \) by restriction to \( L \) a square? The rational functions on \( L \) are functions of one variable. A rational function is a square if and only if its zeros have even order. The zeros of \( \overline{f} \) are the intersections of \( L \) with the branch locus \( \Delta \). Those intersections have to have even multiplicity. Therefore \( L \) splits if and only if its a bitangent to the quartic curve \( \Delta \). (When \( \Delta \) is generic, it won’t have a fourfold tangent.) There are 28 bitangents, so 28 lines that split.

(If \( \Delta \) had degree 6, a line would split if it was a tritangent to \( \Delta \). This won’t occur when \( \Delta \) is generic.)

Quartic Double Planes and Cubic Surfaces

We use coordinates \((x_0, x - 1, x_2, z)\) in projective 3-space \( \mathbb{P}^3 \). Let \( S \) be a generic cubic surface, the zero locus of a homogeneous cubic polynomial \( g(x, z) \), but let’s choose coordinates so that \( q = (0, 0, 0, 1) \) is a generic point that lies on \( S \). Then the coefficient of \( z^3 \) in \( g \) is zero, and \( g \) has the form

\[
a z^2 + b z + c
\]

where \( a, b, c \) are homogeneous polynomials in \( x \) of degrees 1, 2, 3. Let \( f = b^2 - 4ac \) be the discriminant of this quadratic polynomial in \( z \), a homogeneous quartic curve. Let \( Y \) be the quartic double plane \( y^2 = f \).

**Lemma:** If \( g \) is generic, so is \( f \). \( \square \)

The quadratic formula defines a bijection almost everywhere \( S \leftrightarrow Y: z = (-b + y)/2a \) and \( y = 2az + b \). The bijection is undefined on the line \( L_0: a = 0 \).

**the line** \( L_0 \): Modulo \( a \), the equation of \( Y \) becomes \( y^2 = b^2 \). So \( L_0 \) splits in \( Y \). It is a bitangent.

The equation \( a = 0 \) defines a plane \( H_0 \) in \( \mathbb{P}^3 \) as well as the line \( L_0 \) in \( X = \mathbb{P}^2 \). The plane \( H_0 \) contains \( q = (0001) \), and it projects to \( L_0 \). The intersection \( S \cap H_0 \) is the cubic curve in the plane \( H_0 \) obtained from the cubic equation for \( S \) by setting \( a = 0 \). That equation is \( \overline{b}z + \overline{c} = 0 \), where \( \overline{b}, \overline{c} \) are the restrictions of \( b, c \) to \( H_0 \). It is singular at the point \( q \). Since \( q \) is supposed to be generic, it won’t lie on a line in \( S \). So the cubic is irreducible.

Next, let \( L \) be one of the 27 remaining bitangents to \( \Delta \). Then \( L \) meets \( L_0 \) in just one point \( p \). the bijection \( S \leftrightarrow Y \) is defined above all points of \( Y \) except \( p \). The linear equation that defines \( L \) also defines a plane \( H \) in \( \mathbb{P}^3 \) that contains \( q \), and \( S \cap H \) is a cubic curve \( C \) that maps to \( L \) when the point \( q \) is removed. Since \( S \leftrightarrow Y \) is defined except at one point of \( L, C \) must have at least two components, one of which contains \( q \). Since there is no line through the generic point \( q \), that component must be a conic, and the other component must be a line. This gives us 27 lines in the generic cubic surface \( S \).