Summary of April 24 class

I. Euler Characteristic

**Review:** Cohomology of a finite $\mathcal{O}$-module on projective space. Last time we had these three theorems:

1. If the support of $\mathcal{M}$ has dimension $k$, then $H^q\mathcal{M} = 0$ for all $q > k$.
2. $H^q\mathcal{M}(n) = 0$ if $q > 0$ and $n >> 0$.
3. For all $q$, $H^q\mathcal{M}$ is a finite-dimensional vector space.

I should have said last time: These theorems are from Serre’s classical 1955 paper “Faisceaux algébriques cohérents,” in which he introduced finite $\mathcal{O}$-modules, calling them “coherent algebraic sheaves”.

If $X$ is a projective variety and $\mathcal{M}$ is a finite $\mathcal{O}_X$-module, then $H^q(X, \mathcal{M})$ is isomorphic to $H^q(\mathbb{P}, i_*\mathcal{M})$, where $i$ is the inclusion $X \subset \mathbb{P}$. So Theorems 1 and 3 apply to any projective variety.

As before, we denote by $h^q\mathcal{M}$ the dimension of $H^q\mathcal{M}$.

**Definition.** The *Euler characteristic* $\chi(\mathcal{M})$ of a finite $\mathcal{O}$-module on a projective variety $X$ is the alternating sum $\sum_q h^q\mathcal{M}$. This makes sense because $h^q\mathcal{M}$ is finite, and the sum is finite.

**Lemma.** Let $0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \cdots \to \mathcal{M}_k \to 0$ be an exact sequence of $\mathcal{O}$-modules on a projective variety. The alternating sum $\sum_i \chi(\mathcal{M}_i)$ is zero.

To prove this, one first shows that when $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of finite-dimensional vector spaces, the alternating sum $\sum (-1)^q \dim V_q$ is zero. Then one proves the lemma for a short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$. For this sequence, the statement is that $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P}) = 0$. This follows from the cohomology sequence. Then the case of a longer sequence of $\mathcal{O}$-modules is proved by induction.

II. Divisors on a smooth projective curve

The local rings of a smooth projective curve $X$ are valuation rings. The valuation that corresponds to $p$ is denoted by $v_p$.

A divisor on $X$ is a (finite) combination of points: $D = \sum r_i p_i$. The degree of the divisor $D = \sum r_i p_i$ is the sum of the coefficients: $\deg D = \sum r_i$.

A divisor $D = \sum r_i p_i$ is effective if $r_i \geq 0$ for all $i$. We may indicate that $D$ is effective by writing $D \geq 0$.

Also, we say that a divisor $D$ is effective on an open set $U$ of $X$ if $D = \sum r_i p_i$ and $r_i \geq 0$ for all $p_i$ that are points of $U$, ignoring points that aren’t in $U$.

A rational function $f$ on $X$ has a zero of order $r > 0$ at a point $p$ if $v_p(f) = r$, and it has a pole of order $r$ if $v_p(f) = -r$. 


The divisor of a function $f$ is

$$\text{div } f = \sum_p v_p(f)p = (\text{zeros}) - (\text{poles})$$

**a classical problem.** A classical problem of algebraic geometry is to determine the functions that have prescribed poles. This is usually difficult, and one must be satisfied if one can determine the dimension of the space of such functions. Cohomology is the main tool for this.

To help with the problem we introduce an $\mathcal{O}$-module $\mathcal{O}(D)$ associated to a divisor $D$ on a smooth projective curve $X$. By definition, the sections of $\mathcal{O}(D)$ on an open set $U$ are the rational functions $\alpha$ such that the divisor $\text{div } f + D$ is effective on $U$ (together with 0). Thus the global sections are rational functions $\alpha$ such that $\text{div } \alpha + D$ is effective on the whole variety $X$ (together with 0). They are the solutions to the classical problem.

For example, sections of $\mathcal{O}(-p)$ on an open set $U$ that contains $p$ are the rational functions with a zero at $p$, and that are regular at other points of $U$. The sections on an open set $U$ that doesn’t contain $p$ are the regular functions on $U$. So $\mathcal{O}(-p)$ is the maximal ideal $m_p$ at $p$.

The sections of $\mathcal{O}(p)$ on an open set $U$ that contains $p$ are the rational functions with a pole of order at most 1 at $p$ and that are regular at other points of $U$. The sections on an open set $U$ that doesn’t contain $p$ are the regular functions on $U$.

**Lemma.** (i) $\mathcal{O}(D)$ is a locally free $\mathcal{O}$-module of rank one.

(ii) $\mathcal{O}(D) \otimes_{\mathcal{O}} \mathcal{O}(E) \approx \mathcal{O}(D + E)$.

Since $\mathcal{O}(-p) = m_p$, there is an exact sequence

$$0 \rightarrow \mathcal{O}(-p) \rightarrow \mathcal{O} \rightarrow \kappa_p \rightarrow 0$$

where $\kappa_p$ is the $\mathcal{O}$-module defined by the residue field $k(p)$. Its sections on any open set $U$ that contain $p$ are the elements of the residue field $k(p)$, and its only section on an open set that doesn’t contain $p$ is 0.

Given a divisor $D$, we tensor this sequence with the module $\mathcal{O}(D)$. The tensor product sequence is exact because $\mathcal{O}(D)$ is locally free:

$$(*) \quad 0 \rightarrow \mathcal{O}(D - p) \rightarrow \mathcal{O}(D) \rightarrow \epsilon \rightarrow 0$$

where $\epsilon = \kappa_p \otimes_{\mathcal{O}} \mathcal{O}(D)$, which is another one-dimensional $\mathcal{O}$-module. Theorem 1 shows that $h^q \epsilon = 0$ for all $q > 0$, while $h^0 \epsilon = 1$.

The table showing the dimensions of the cohomology for the sequence $(*)$ looks like this:

<table>
<thead>
<tr>
<th>$\mathcal{O}(D-p)$</th>
<th>$\mathcal{O}(D)$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^0$</td>
<td>*</td>
<td>1</td>
</tr>
<tr>
<td>$h^1$</td>
<td>*</td>
<td>0</td>
</tr>
<tr>
<td>$h^2$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Looking at this table, we see that there are just two possibilities: Either
\[ h^0 \mathcal{O}(D) = h^0 \mathcal{O}(D-p) + 1 \] and \[ h^1 \mathcal{O}(D) = h^1 \mathcal{O}(D-p), \]
or
\[ h^0 \mathcal{O}(D) = h^0 \mathcal{O}(D-p) \] and \[ h^1 \mathcal{O}(D) = h^1 \mathcal{O}(D-p) - 1. \]

And in either case,
\[ (**) \quad \chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D-p)) + 1. \]

**Riemann-Roch Theorem** (version 1) Let \( D \) be a divisor on a smooth projective curve \( X \). Then
\[ \chi(\mathcal{O}(D)) = \deg D + \chi(\mathcal{O}) \]
\( \deg D \) being the degree of \( D \).

This theorem follows from (**) because we can get from the divisor 0 to any divisor \( D \) by a finite sequence of operations, each of which adds or subtracts a point.

**Corollary 1.** The divisor \( \text{div} f \) of a rational function \( f \) on a smooth projective curve has degree zero: \( \#(\text{zeros}) = \#(\text{poles}) \). The number of points at which \( f \) takes a given value \( c \) is also equal to the number of poles.

**proof.** Let \( D = \text{div} f \). Multiplication by \( f \) defines an isomorphism \( \mathcal{O}(D) \rightarrow \mathcal{O} \). For, the sections of \( \mathcal{O}(D) \) on an open set \( U \) are the rational functions \( \alpha \) such that \( \text{div} \alpha + D \) is effective on \( U \) (together with 0). Then the divisor of \( f \alpha \) is \( \text{div} \alpha + \text{div} f = \text{div} \alpha + D \). So the divisor of \( f \alpha \) is effective, which means that \( f \alpha \) is a section of \( \mathcal{O} \). Multiplication by \( f^{-1} \) defines the inverse map.

Riemann-Roch tells us that \( \chi(\mathcal{O}(D)) = \deg D + \chi(\mathcal{O}) \), and since \( \mathcal{O}(D) \) is isomorphic to \( \mathcal{O} \), \( \chi(\mathcal{O}(D)) = \chi(\mathcal{O}) \). So \( \deg D = 0 \).

The last assertion comes from the fact that \( f \) and \( f - c \) have the same poles. \( \square \)

**Corollary 2.** The only functions regular on all of a smooth projective curve are the constant functions: \( H^0 \mathcal{O} = \mathbb{C} \).

**proof** According to the previous corollary, a nonconstant function would have a pole.

The dimension of \( H^1 \mathcal{O} \) is called the arithmetic genus of \( X \), and is denoted by \( p_a \). Thus Riemann-Roch can be written as
\[ \chi(\mathcal{O}(D)) = \deg D + 1 - p_a \]

**Corollary 3.** Let \( p \) be a point of \( X \). The dimension of \( H^0 \mathcal{O}(np) \) tends to infinity with \( n \).

Thus there is a rational function with a pole of any sufficiently large order \( n \) at a point, and with no other pole.

**Corollary 4.** A smooth projective curve \( X \) is connected in the classical topology.
**proof** We need a bit of complex analysis for this. Let’s suppose we know that $X$ is a compact, orientable, two-dimensional manifold (see notes). Then, suppose that $X$ is the union $X_1 \cup X_2$ of two disjoint nonempty parts. They will also be compact, orientable manifolds. We choose a point $p$ in $X_1$. There is a rational function $f$ on $X$ with a pole of some order at $p$ and no other pole. It will be an analytic function with no pole on $X_2$, and since $X_2$ is compact, $f$ will be bounded there. The *maximum principle* asserts that an analytic function takes on its maximum at the boundary of any region. Since $X_2$ has no boundary, $f$ must be constant on $X_2$, say $f = c$. Then the rational function $f - c$ on $X$ is zero on $X_2$. However, the zero set of $f - c$ is a closed subset of $X$, and since $X$ is a curve, a closed set is either finite, or the whole curve $X$. Since $X_2$ isn’t a finite set, $f - c$ must be identically zero. This contradicts the fact that $f$ has a pole at $p$. □