Summary of April 17 class

Review of characteristic properties of cohomology.

Let \( X \) be a variety. Cohomology is a sequence of functors \( H^0, H^1, \ldots \) from \( \mathcal{O} \)-modules to vector spaces, with these characteristic properties:

1. \( H^0(\mathcal{M}) \) is the space \( \mathcal{M}(X) \) of global sections of \( \mathcal{M} \).

2. Cohomology sequence: Let \( 0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0 \) be a short exact sequence of \( \mathcal{O} \)-modules. Then there is an exact sequence

\[
0 \to H^0(\mathcal{L}) \to H^0(\mathcal{M}) \to H^0(\mathcal{N}) \to H^1(\mathcal{L}) \to H^1(\mathcal{M}) \to \cdots
\]

3. If \( U \) is an affine open subset of \( X \), and let \( j \) denote the inclusion \( U \to X \). For any \( \mathcal{O}_U \)-module \( \mathcal{N} \) and for all \( q > 0 \), \( H^q(x, j_* \mathcal{N}) = 0 \).

The third property can be generalized:

**Theorem 1.** Let \( f : Y \to X \) be a morphism, where \( Y \) is an affine variety. For any \( \mathcal{O}_Y \)-module \( \mathcal{N} \) and any \( q > 0 \), \( H^q(X, f_* \mathcal{N}) = 0 \).

Let’s omit the proof.

**Affine Morphisms**

A morphism \( f : Y \to X \) is an affine morphism if for all affine open sets \( U \) of \( X \), the inverse image \( V = f^{-1}U \) is an affine variety.

For example, the inclusion of a closed subvariety is an affine morphism. So is the inclusion of an affine open subvariety. A finite morphism is an affine morphism.

On the other hand, if \( X \) is the affine plane and \( Y \) is the complement of the origin, the inclusion \( Y \to X \) isn’t an affine morphism. The map from \( \mathbb{P}^1 \) to a point isn’t an affine morphism.

**Theorem 2.** Let \( f : Y \to X \) be an affine morphism, and let \( \mathcal{N} \) be an \( \mathcal{O}_Y \)-module. Then \( H^q(X, f_* \mathcal{N}) \) is isomorphic to \( H^q(Y, \mathcal{N}) \).

**Corollary.** If \( i : Y \to X \) is the inclusion of a closed subvariety and \( \mathcal{M} \) is an \( \mathcal{O}_Y \)-module, then \( H^q(Y, \mathcal{M}) \approx H^q(X, i_* \mathcal{M}) \).

**proof** To simplify notation, we denote \( H^q(X, f_* \mathcal{N}) \) by \( F^q(\mathcal{N}) \) for the proof. So \( F^q \) are functors on \( \mathcal{O}_Y \)-modules. We are to show that \( F^q(\mathcal{N}) \) is isomorphic to \( H^q(Y, \mathcal{N}) \), and to do this, we verify the characteristic properties for \( F^q \).

First, \( F^0(\mathcal{N}) = H^0(X, f_* \mathcal{N}) = [f_* \mathcal{N}](X) \) which, by definition of \( f_* \), is equal to \( \mathcal{N}(Y) = H^0(Y, \mathcal{N}) \).

Next, we show that there is a cohomology sequence for \( F^q \). Let

\[
A : \quad 0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0
\]
be a short exact sequence of $O_Y$-modules. The sequence of $O_X$-modules

$$B : 0 \to f_*\mathcal{L} \to f_*\mathcal{M} \to f_*\mathcal{N} \to 0$$

is exact because, when $U$ is an affine open subset of $X$, its inverse image $V = f^{-1}U$ is affine open in $Y$. ($f$ is an affine morphism.) By definition, $f_*\mathcal{L}(U) = \mathcal{L}(V)$. So the sequence of sections of $B$ on $U$ is the same as the sequence of sections of $A$ on $V$, which is exact because $A$ is exact and $V$ is affine.

Finally, we verify the third property by showing that, if $j : V \to Y$ is the inclusion of an affine open set and $N$ is an $O_V$-module, then $F^q(j_*N) = 0$ for $q > 0$. Let $g$ be the composed map $fj : V \to X$. Then $F^q(j_*N) = H^q(X, f_*j_*N) = H^q(X, g_*N)$, which is zero by Theorem 1, because $V$ is affine. □

Cohomology of $O(n)$ on Projective Space

We denote the dimension of the cohomology group $H^q(X, \mathcal{M})$ by $h^q(\mathcal{M})$. We’ll use the next lemma, omitting the proof:

**Lemma.** Cohomology is compatible with limits. If $\mathcal{M}_0 \to \mathcal{M}_1 \to \cdots$ is a sequence of $O$-modules, then $\lim H^q(X, \mathcal{M}_k) \approx H^q(X, \lim \mathcal{M}_k)$.

**Theorem 3.** On the projective space $\mathbb{P}^d$ of dimension $d$

(i) If $n \geq 0$, $h^0(O(n)) = \binom{n+d}{d}$, and $h^q(O(n)) = 0$ when $q > 0$.

(ii) If $n = -r < 0$, $h^d(O(-r)) = \binom{r-1}{d}$ and $h^q(O(-r)) = 0$ if $q \neq d$.

**proof** We know that $H^0(\mathbb{P}, O(n))$ is the space of homogeneous polynomials of degree $n$, which has dimension $\binom{n+d}{d}$ when $n \geq 0$ and is zero when $n < 0$.

Let $H$ be the hyperplane $x_0 = 0$. This is a projective space of dimension $d - 1$, and by the Corollary above, $H^q(H, O_H(n)) \approx H^q(\mathbb{P}, i_* O(n))$, $i$ being the inclusion $H \to \mathbb{P}$.

There are exact sequences

$$0 \longrightarrow \mathcal{O}_\mathbb{P}(n-1) \overset{x_0}{\longrightarrow} \mathcal{O}_\mathbb{P}(n) \overset{r}{\longrightarrow} i_*\mathcal{O}_H(n) \longrightarrow 0$$

where $r$ is the restriction, which is obtained by setting $x_0 = 0$. This gives us a cohomology sequence

$$\cdots \to H^q(\mathbb{P}, O(n-1)) \to H^q(\mathbb{P}, O(n)) \to H^q(H, O(n)) \to H^{q+1}(\mathbb{P}, O(n-1)) \to \cdots$$
The Case \( n \geq 0 \)

Instead of exhibiting the cohomology sequence, we make a table showing the dimensions of the cohomology. We denote the dimension of \( H^q(X, \mathcal{M}) \) by \( h^q(\mathcal{M}) \).

The table looks like this:

\[
\begin{array}{ccc}
\h_0 & (n-1+d) & (n+d) & (n+d-1) \\
\h_1 & * & * & * \\
\h_2 & * & * & * \\
\h_3 & \\
\end{array}
\]

where the columns contain the dimensions of cohomology of \( \mathcal{O}_\mathbb{P}(\mathcal{O}(n-1)), \mathcal{O}_\mathbb{P}(\mathcal{O}(n)), \mathcal{O}_H(\mathcal{O}(n-1)) \), the asterisks being entries not yet determined.

The top row contains the dimensions of the spaces of global sections, which form a short exact sequence. The exactness is easy to verify directly, and it corresponds to the standard combinatorial formula

\[
(n+d) = (n-1+d) + (n+d-1)
\]

By induction on \( d \), the entries * in the third column are zero. Looking at the table, we see that the maps \( H^q(bbp, \mathcal{O}((n-1)) \to H^q(\mathbb{P}, \mathcal{O}(n)) \) are bijective for all \( q > 0 \). We remember that the limit \( \mathcal{O}(nH) \) is the module \( j_*\mathcal{O}_\mathbb{U}^0 \), where \( j \) is the inclusion of the standard affine \( \mathbb{U}^0 \) into \( \mathbb{P} \), and \( \mathcal{O}(n) \approx \mathcal{O}(nH) \). By the lemma, \( \lim H^q(\mathbb{P}, \mathcal{O}(nH)) \) is isomorphic to \( H^q(\mathbb{P}, j_*\mathcal{O}_\mathbb{U}^0) \), which is zero when \( q > 0 \), because \( \mathbb{U}^0 \) is affine. Since the maps \( H^q(bbp, \mathcal{O}((n-1)) \to H^q(\mathbb{P}, \mathcal{O}(n)) \) are bijective for all \( q > 0 \) and the limit is zero, \( H^q(\mathbb{P}, \mathcal{O}(n)) = 0 \) for all \( n \geq 0 \).

Note. I neglected to refer to the lemma in class. This was a mistake.

The Case \( n = -r < 0 \)

We compute the cohomology of \( \mathcal{O}(-1) \) using the exact sequence

\[
0 \to \mathcal{O}_\mathbb{P}(-1) \to \mathcal{O}_\mathbb{P} \to \mathcal{O}_H \to 0
\]

Since \( h^q(\mathcal{O}_\mathbb{P}) \) and \( h^q(\mathcal{O}_H) \) are zero when \( q > 0 \) and equal to 1 when \( q = 0 \), we find that \( h^q(\mathcal{O}(-1)) = 0 \) for all \( q \). Then the exact sequence

\[
0 \to \mathcal{O}_\mathbb{P}(-r - 1) \to \mathcal{O}_\mathbb{P}(-r) \to \mathcal{O}_H(-r) \to 0
\]

verifies the assertion for \( n < 0 \) by induction on \( r \) and \( d \). \( \square \)
Cohomology of a Plane Curve

Let $Y$ be a curve of degree $d$ in $\mathbb{P}^2$, the zero locus of an irreducible polynomial $f$ of degree $r$. Then one has an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-r) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_Y \to 0$$

where $i$ is the inclusion $Y \to \mathbb{P}$. The previous theorem computes the cohomology of $\mathcal{O}_{\mathbb{P}}(-r)$ and $\mathcal{O}_{\mathbb{P}}$:

- $h^0: 0, 1, 1$
- $h^1: 0, 0, *$
- $h^2: (r-1), 0, *
- $h^3: 0, 0, *$

The exact cohomology sequence shows that $h^1(\mathcal{O}_Y) = (r-1)$ and that $h^q(\mathcal{O}_Y) = 0$ for $q > 1$.

The dimension $h^1(\mathcal{O}_Y)$ is called the \textit{arithmetic genus} of $Y$ and is denoted by $p_a$. When $Y$ is smooth, it is equal to the geometric genus $g$. We’ll provethat later. But the arithmetic genus is $(r-1)$ also when $Y$ is singular, even when it is a reducible curve.

The values of $p_a$ when the degrees of $Y$ are 1, 2, 3, 4, 5... are 0, 0, 1, 3, 6... respectively. A plane curve cannot have arithmetic genus 2.