COHOMOLOGY

Let $X$ be a variety, and let

$$0 \longrightarrow \mathcal{L} \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{N} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}$-modules. So $f$ is injective, $\ker g = \text{im } f$, and $g$ is surjective. Then for any affine open subset $U = \text{Spec } A$, the sequence

$$0 \longrightarrow \mathcal{L}(U) \xrightarrow{f(U)} \mathcal{M}(U) \xrightarrow{g(U)} \mathcal{N}(U) \longrightarrow$$

is exact.

When $U$ isn’t affine, the map $g(U)$ needn’t be surjective, though the rest of the sequence is exact, as the next lemma asserts.

**Lemma.** If $0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$ is an exact sequence of $\mathcal{O}$-modules. For any open set $U$, the sequence $0 \rightarrow \mathcal{L}(U) \rightarrow \mathcal{M}(U) \rightarrow \mathcal{N}(U)$ is exact.

**proof** We choose an affine open covering $\{U^i\}$ of $U$ and form a diagram

$$000R \downarrow \downarrow R0 \longrightarrow \mathcal{L}(U) \longrightarrow \mathcal{M}(U) \longrightarrow \mathcal{N}(U) \mathcal{R} \downarrow \downarrow \mathcal{R}0 \longrightarrow \prod \mathcal{L}(U^i) \longrightarrow \prod \mathcal{M}(U^i)$$

in which the vertical rows are the exact sequences that express the sheaf property for the covering, and the second and third rows are exact because $U^i$ and $U^{ij}$ are affine. The exactness of the top row follows by inspection of the diagram. □

Now, given an exact sequence $0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$, the sequence of global sections $0 \rightarrow \mathcal{L}(X) \rightarrow \mathcal{M}(X) \rightarrow \mathcal{N}(X)$ is exact, but the right-hand arrow may not be surjective. Cohomology is a sequence of functors

$$\mathcal{O} - \text{modules} \xrightarrow{H^q} (\text{vector spaces})$$

$q = 0, 1, 2, \ldots$ that analyzes the failure of surjectivity in the following way: $H^0(\mathcal{M})$ is the space $\mathcal{M}(X)$ of global sections of $\mathcal{M}$, and associated to every short exact sequence $0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$ of $\mathcal{O}$-modules, there is a long exact cohomology sequence

$$0 \rightarrow H^0(\mathcal{L}) \rightarrow H^0(\mathcal{M}) \rightarrow H^0(\mathcal{N}) \rightarrow H^1(\mathcal{L}) \rightarrow H^1(\mathcal{M}) \rightarrow H^1(\mathcal{N}) \rightarrow H^2(\mathcal{L}) \rightarrow H^2(\mathcal{M}) \rightarrow H^2(\mathcal{N}) \rightarrow$$

that substitutes for right exactness.

**Characteristic Properties of Cohomology**

The cohomology $H^0, H^1, \ldots$ is characterized by three properties:

(1) $H^0(\mathcal{M})$ is the space of global sections $\mathcal{M}(X)$. 
(2) Associated to every short exact sequence $0 \to L \to M \to N \to 0$ there is a long exact cohomology sequence

$$0 \to H^0(L) \to H^0(M) \to H^0(N) \to H^1(L) \to H^1(M) \to H^1(N) \to H^2(L) \to H^2(M) \to H^2(N) \to \cdots$$

In this sequence, the maps $\delta^q : H^q(cn) \to H^{q+1}(L)$ are the coboundary maps. They are a part of the cohomology structure.

(3) Let $j : U \to X$ be the inclusion of an affine—open set. Then for every $\mathcal{O}_U$-module $N$ and for every $q > 0$, $H^q(j_\ast N) = 0$.

Theorem. There is a cohomology sequenc, and it is unique up to unique isomorphism.

Corollary. If $X$ is an affine variety, $H^q(\mathcal{M}) = 0$ for all $\mathcal{M}$ and all $q > 0$.

The corollary follows when the third characteristic property is applied to the identity map $X \to X$. When $X$ is an affine variety, the sequence $0 \to H^0(L) \to H^0(M) \to H^0(N) \to 0$ is exact, so the higher cohomology isn’t needed. The third property is an extension of this point to arbitrary varieties.

proof of uniqueness of cohomology:

Let a cohomology theory with the three characteristic properties be given. We choose an affine covering $\{U^\nu\}$ of our variety $X$, and we analyze cohomology using this covering. Let $j^\nu : U^\nu \to X$ be the inclusions maps. We can restrict an $\mathcal{O}_X$-module $\mathcal{M}$ to the open sets $U^\nu$, obtaining $\mathcal{O}_{U^\nu}$-modules $\mathcal{M}_{U^\nu}$. Then the direct image $j^\nu_\ast \mathcal{M}_{U^\nu}$ will be an $\mathcal{O}_X$-module. Let $R$ denote the product $\prod \mathcal{M}_{U^\nu}$

Lemma. (i) There is a canonical injective map $\mathcal{M} \to R$, and therefore a canonical short exact sequence $0 \to \mathcal{M} \to cr \to S \to 0$, where $S$ is the quotient $R/\mathcal{M}$.

(ii) $H^q(R) = 0$ for every $q > 0$.

proof (i) The sections of $j^\nu_\ast \mathcal{M}_{U^\nu}$ on an open set $V$ are identified as follows:

$$[j^\nu_\ast \mathcal{M}_{U^\nu}](V) = \mathcal{M}_{U^\nu}(V \cap U^\nu) = \mathcal{M}(V \cap U^\nu)$$

and there is a restriction map $\mathcal{M}(V) \to \mathcal{M}(V \cap U^\nu)$. This restriction map defines the map $\mathcal{M}(V) \to \prod j^\nu_\ast \mathcal{M}_{U^\nu}$. It is injective because the open sets $V \cap U^\nu$ cover $V$.

(ii) This follows from the third characteristic property. □

Now we look at the long cohomology sequence associated to the exact sequence $0 \to \mathcal{M} \to R \to S \to 0$. Because $H^q(R) = 0$ when $q > 0$, the long sequence breaks up into exact sequences

$$0 \to H^0(\mathcal{M}) \to H^0(\mathcal{R}) \to H^0(S) \to H^0(\mathcal{M}) \to 0$$

$$0 \to H^1(S) \to H^2(\mathcal{M}) \to 0$$
$0 \to H^2(S) \to H^3(M) \to 0$

and so on.

The zero-dimensional cohomology, being the space of global sections, is unique for every $\mathcal{O}$-module. Then the first of the above sequences determines $H^1(M)$ as the cokernel of a map $H^0(R) \to H^0(S)$ that is independent of the cohomology theory (though it depends on our chosen covering). Therefore $H^1(M)$ is determined uniquely, and this is true for every $\mathcal{O}$-module $M$, including for the module $S$. Then the second sequence determines $H^2(M)$ uniquely, and by induction on $q$, $H^q$ is determined uniquely.