18.721 Assignment 9

This assignment is due Wednesday, April 29.

1. On the projective line $\mathbb{P}^1$, when are two direct sums $\mathcal{O}(m) \oplus \mathcal{O}(n)$ and $\mathcal{O}(r) \oplus \mathcal{O}(s)$ isomorphic?

2. There is an error in the statement of this problem. See web page.

   Let $f$ and $g$ be homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2, x_3]$, of degrees $d$ and $e$ respectively, and with no common factor. Let $X$ be the locus of common zeros of $f$ and $g$ in the projective space $\mathbb{P}^3$ with coordinates $x$, and let $i$ be the inclusion $X \rightarrow \mathbb{P}$.

   (a) Construct an exact sequence

   $$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d - e) \rightarrow \mathcal{O}_{\mathbb{P}}(-d) \oplus \mathcal{O}_{\mathbb{P}}(-e) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow i_* \mathcal{O}_X \rightarrow 0$$

   (b) Prove that $X$ is connected, i.e., that it is not the union of two proper disjoint Zariski-closed subsets of $\mathbb{P}$.

   (c) Determine the cohomology of $\mathcal{O}_X$.

3. (a) Let $K$ be the function field of a projective space $X$, and let $\mathcal{F}$ be the constant function field $\mathcal{O}_X$-module, whose module of sections on any open set $U$ is $\mathcal{F}(U) = K$ (see notes). Prove that $H^q(X, \mathcal{F}) = 0$ for all $q > 0$.

   (b) Let $\{U^i\}$ be an open covering of $X$. Suppose that rational functions $f_i$ are given such that $f_i - f_j$ is a regular function on $U^i \cap U^j$ for all $i, j$. The Cousin Problem asks for a regular function $f$ on $\mathbb{P}$ such that $f - f_i$ is regular on $U^i$ for each $i$. Analyze the Cousin Problem, making use of the exact sequence

   $$0 \rightarrow \mathcal{O} \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$$

   where $\mathcal{Q}$ is the quotient $\mathcal{F}/\mathcal{O}$. 