18.721 Comments on Pset 8

1. Let \( X = \mathbb{P}^r \), and let \( Y \) be a closed subvariety of \( X \). Describe the sections of \( \mathcal{O}(n) \) on the complement \( U = X - Y \) of \( Y \).

By definition, the sections of \( \mathcal{O}(n) \) on \( U \) are the homogeneous fractions of degree \( n \) that are regular on \( U \). This means that if \( g/h \) is such a fraction, with \( g, h \) relatively prime, \( h \) cannot vanish at any point of \( U \). So there are three cases: If \( Y = X \), then \( U \) is empty and the only section of \( \mathcal{O}(n) \) is 0. If \( Y \) is a hypersurface, say \( f = 0 \), then \( h \) can be a scalar multiple of a power of \( f \). If \( Y \) isn’t \( X \) and isn’t a hypersurface, then it will have codimension at least 2. Since a nonconstant homogeneous polynomial vanishes on a subset of codimension 1, \( h \) must be a scalar. Then the sections of \( \mathcal{O}(n) \) on \( U \) are the global sections, which are homogeneous polynomials of degree \( n \).

2. Let \( X \) be the affine plane \( \text{Spec} \, A \), where \( A = \mathbb{C}[x, y] \), and let \( U \) be the complement of the origin in \( X \).

(a) Let \( \mathcal{M} \) be the \( \mathcal{O}_X \)-module that correponds to the \( A \)-module \( M = A/yA \). Show that \( \mathcal{M} \) is a finite \( \mathcal{O} \)-module, but that \( \mathcal{M}(U) \) isn’t a finite module over the ring \( \mathcal{O}(U) \).

(b) Show that, for any \( k \geq 1 \), the homomorphism

\[
\mathcal{O} \times \mathcal{O} \xrightarrow{(x,y)^i} \mathcal{O}
\]

is surjective on \( U \), though the associated map of sections on \( U \) isn’t surjective.

(a) The global sections of \( \mathcal{M} \) are the elements of the finite \( A \)-module \( M \). The module of sections of \( \mathcal{M} \) on an affine open subset \( V = \text{Spec} \, B \) of \( X \) is \( M \otimes_A B = B/yB \), which is a finite \( B \)-module. This shows that \( \mathcal{M} \) is a finite \( \mathcal{O} \)-module. To compute the sections of \( \mathcal{M} \) on \( U \), we cover \( U \) by the two affine open sets \( X_x = \text{Spec} \, A[x^{-1}] \) and \( X_y = A[y^{-1}] \), the complements of the two coordinate axes. The regular functions on \( U \) are the same as on \( X \): \( \mathcal{O}(U) = \mathcal{O}(X) = A \). The module of sections of \( \mathcal{M} \) on \( X_x \) is \( M[x^{-1}] \), and the module of sections on \( X_y \) is \( M[y^{-1}] \), which is the zero module. The sheaf property for this covering shows that \( \mathcal{M}(U) = M[x^{-1}] \). This isn’t a finite module over \( \mathcal{O}(U) = A \).

3. Let \( R \) be the polynomial ring \( \mathbb{C}[x, y, z] \), and let \( A = R/(f, g) \), where \( f(x, y, z) \) and \( g(x, y, z) \) are homogeneous polynomials of degrees \( m \) and \( n \), and with no common factor.

(a) Show that the sequence

\[
0 \to R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)} R \to A \to 0
\]
is exact.

(b) Because $f$ and $g$ are homogeneous, $A$ inherits a grading from the grading of $R$ by degree: $A = A_0 \oplus A_1 \oplus \cdots$. Prove that $\dim A_k = mn$ for all sufficiently large $k$.

(c) Explain in what way this is an algebraic version of Bézout’s Theorem.

(a) The only place where exactness isn’t obvious is at $R^2$. Let $(u, v)$ be an element of $R^2$ such that $(u, v)(f, g)^t = 0$, i.e., $uf = -vg$. Since $f, g$ are relatively prime, $f$ divides $v$, say $v = fw$. Then $uf = -wfg$, so $u = -wg$, and $(u, v)$ is the image of the element $w$ of $R$ via the map $(-g, f)$.

(b) The maps in the sequence change degrees. Multiplication by $f$ raises degree by $m$ etc.

So $A_d$ fits into a sequence

\[
\begin{align*}
0 & \to R_{r-m-n} \to R_{d-m} \oplus R_{d-n} \to R_d \to A_d \to 0
\end{align*}
\]

The alternating sum of the dimensions in an exact sequence is zero. The dimension of $R_d$ is the binomial coefficient $\binom{d+2}{2}$. Therefore

\[
\dim A_d = \binom{d+2}{2} - \binom{d-n+2}{2} - \binom{d-m+2}{2} + \binom{d-m-n+2}{2}
\]

This works out to $\dim A_d = mn$ when $d \geq m + n$.

(c) It is fairly difficult to make this precise, and there is more than one way to proceed.

first approach

Let $C$ and $D$ be the plane projective curves $f = 0$ and $g = 0$, and let’s suppose that $C$ and $D$ intersect transversally in $N$ points. In the affine space $\mathbb{A}^3$ with coordinates $x, y, z$, the locus $f = g = 0$ is a bouquet of $N$ lines $L_1, \ldots, L_N$ through the origin corresponding to the points of $C \cap D$. The coordinate ring of $L_i$ is a quotient of the coordinate algebra $R$ of $\mathbb{A}^3$, and it is a polynomial ring in one variable. Let’s call it $B_i$, and let $B = B_1 \times \cdots \times B_N$. Then $B$ is a finite module over $R$, and since $B_i$ is a quotient of $A$, we have a map $A \to B$.

Let $Y$ denote the disjoint union of the lines $L_1, \ldots, L_N$.

The point here is that the obvious map $Y \to X$ is bijective except at the origin. So the homomorphism $A \to B$ isn’t far from being an isomorphism. Each $B_i$ is graded, and $\dim B_i = 1$ for all $k \geq 0$. We grade $B$ accordingly, so that $\dim B_k = N$ for every $k \geq 0$. This grading is compatible with the grading of $A$.

Let $K$ and $C$ be the kernel and cokernel of the homomorphism $A \to B$:

\[
0 \to K \to A \to B \to C \to 0
\]

We grade $K$ using the grading of $A$, and $C$ using the grading of $B$.

**Lemma.** $K$ and $C$ are finite $R$-modules, and $K_k = C_k = 0$ when $k >> 0$.

It follows from the lemma that $A_k = B_k$ and therefore that $\dim A_k = N$, when $k >> 0$. 
proof of the lemma The kernel is a finite module because it is a submodule of the finite module \( A \). The cokernel is a finite module because it is a quotient of the finite module \( B \). To show that these modules are zero in large degree, we choose a homogeneous linear polynomial whose zero locus \( H \) doesn’t contain any of the lines \( L_i \), and we localize. In \( X_s \), origin is removed from the lines \( L_i \), and we localize. In \( X_s \), origin is removed from the lines \( L_i \). So \( L_i \) are closed disjoint subsets of \( X_s \) to which the Chinese Remainder Theorem applies. The ideals \( I_i \) of \( L_i \) are comaximal, and the map \( A_s \to \prod B_i = B_s \) is surjective. Since \( A \subset B \), that map is also bijective. This means that \( K_s = C_s = 0 \). Therefore if \( r > 0 \), \( s^r \) annihilates \( K \) and \( C \), which shows that \( A_k = B_k \) for all \( k \geq r \).

second approach

We will use the exact sequence of part (a). There is an analogous an exact sequence of \( O \)-modules:

\[
\begin{array}{ccccccccc}
\vdots & & & \to & O(-m-n) & \to & O(-m) \oplus O(-n) & \to & O & \to & A & \to & 0
\end{array}
\]

where \( A \) is the cokernel of the map \((f, g)^t \), which is zero except at the finite set of intersection points \( C \cap D \).

Lemma. Suppose that coordinates are chosen so that the hyperplane \( H : \{ x_0 = 0 \} \) (a line) doesn’t contain any point of \( C \cap D \). Then, for every \( n \), the \( O \)-modules \( A \) and \( A(n) = A \otimes_O O(n) \) are isomorphic.

Assuming that the lemma is proved, we twist the exact sequence (***) by \( d >> 0 \):

\[
0 \to O(d-n) \to O(d-m) \to O(d-n) \to O(d) \to A(d) \to 0
\]

The cohomology \( H^q O(k) \) is zero for all \( q > 0 \), and this implies that the sequence of global sections is exact. (This is an exercise.)

The sequence of global sections is

\[
0 \to R_{d-n} \to R_{d-m} \oplus R_{d-n} \to R_d \to H^0 A(d) \to 0
\]

Looking at the sequence (**), we see that \( H^0 A(d) \approx A_d \). Since \( A(d) \approx A \), the dimension of \( H^0 A \) is the same as that of \( A_d \), which, according to (b), is \( mn \). This is Bézout’s Theorem.

proof of the lemma We cover \( \mathbb{P}^2 \) by three open sets: the standard affine set \( U^0 \), the complement \( V^1 \) of \( C \) in \( \mathbb{P}^2 \), and the complement \( V^2 \) of \( D \). Here \( U^0 \) contains all points of \( C \cap D \), and \( V^1, V^2 \) don’t contain any of those points. Therefore \( A(V^1) = A(V^2) = 0 \). The sheaf property for the covering \( U^0, V^1, V^2 \) shows that \( A(\mathbb{P}^2) = A(U^0) \). Similarly, for every open set \( W \) of \( \mathbb{P}^2 \), \( A(W \cap V^1) = A(W \cap V^2) = 0 \), and \( A(W) = A(W \cap U^0) \). On \( U^0 \), \( A = A(nH) \). Therefore this is true on \( \mathbb{P}^2 \) as well, and since \( A(nH) \approx A(n) \), \( A \approx A(n) \).