18.721 Assignment 8 (corrected)

This assignment is due Wednesday, April 22.

1. Let $X = \mathbb{P}^r$, and let $Y$ be a closed subvariety of $X$. Describe the sections of $\mathcal{O}(n)$ on the complement $U = X - Y$ of $Y$.

2. Let $X$ be the affine plane Spec $A$, where $A = \mathbb{C}[x, y]$, and let $U$ be the complement of the origin in $X$.

   (a) Let $\mathcal{M}$ be the $\mathcal{O}_X$-module that corresponds to the $A$-module $M = A/yA$. Show that $\mathcal{M}$ is a finite $\mathcal{O}$-module, but that $\mathcal{M}(U)$ isn’t a finite module over the ring $\mathcal{O}(U)$.

   (b) Show that the homomorphism

   $$
   \mathcal{O} \times \mathcal{O} \xrightarrow{(x,y)^t} \mathcal{O}
   $$

   is surjective on $U$, though the associated map of sections on $U$ isn’t surjective.

3. Let $R$ be the polynomial ring $\mathbb{C}[x, y, z]$, and let $A = R/(f, g)$, where $f(x, y, z)$ and $g(x, y, z)$ are homogeneous polynomials of degrees $m$ and $n$, and with no common factor.

   (a) Show that the sequence

   $$
   0 \rightarrow R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)^t} R \rightarrow A \rightarrow 0
   $$

   is exact.

   (b) Because $f$ and $g$ are homogeneous, $A$ inherits a grading from the grading of $R$ by degree: $A = A_0 \oplus A_1 \oplus \cdots$. Prove that $\dim A_k = mn$ for all sufficiently large $k$.

   (c) Explain in what way this is an algebraic version of Bézout’s Theorem.