1. Prove that if a variety $X$ is covered by countably many constructible sets, then a finite number of those sets will cover $X$.

Let’s prove that if $X$ is a closed subset of a variety $W$, and $X$ is is covered by constructible sets, then it is covered by countably many of them. (Since $X$ will be a finite union of closed subvarieties, this assertion isn’t really stronger.)

We use noetherian induction. A constructible set $S$ is a finite union of locally closed sets. A countable union $\bigcup S_i$ of constructible sets is also a countable union of locally closed sets, say $L_{i,k}$, each one having the form $L_{i,k} = Y_{i,k} \cap U_{i,k}$ with $Y_{i,k}$ closed and $U_{i,k}$ open. Then $X$ is also the union of the closed sets $Y_{i,k}$. If $\dim X = n$, we can’t cover $X$ by countably many subsets of dimension $< n$. So at least one of the sets $Y_{i,k}$ must be equal to $X$. Then $S_i$ contains a nonempty open subset $X'$ of $X$. Let $Z$ be the closed complement of $X'$. This is a proper subset of $X$, so noetherian induction applies.

2. Let $G$ denote the Grassmanian $G(2,4)$ of lines in $\mathbb{P}^3$, and let $[\ell]$ denote the point of $G$ that corresponds to the line $\ell$. In the product variety $G \times G$ of pairs of lines, let $Z$ denote the set of pairs $[\ell_1],[\ell_2]$ whose intersection isn’t empty. Use the fact that projective space is proper to prove that $Z$ is a closed subset of $G \times G$.

Let $V$ be the four-dimensional vector space whose associated projective space is $\mathbb{P}^3$. A line $\ell$ is described by a decomposable element $u_1 u_2$ of $\Lambda^2 V$, and a point $p$ by an element $w$ of $\Lambda^1 V = V$. The point $p$ lies on $\ell$ if and only if the product $u_1 u_2 w$ in $\Lambda^3 V$ is zero, and this condition describes a closed subset of $G \times \mathbb{P}^3$.

The product $\Pi = G \times G \times \mathbb{P}^3$ parametrizes triples $([\ell_1],[\ell_2],p)$, where $\ell_i$ are lines and $p$ is a point. In $\Pi$, let $\Sigma$ be the set of triples such that $p \in \ell_1$ and $p \in \ell_2$. This is a closed set, so its projection to $G \times G$ is closed.
3. (a part of Theorem 5.8.2) Let \( f : Y \to X \) be a morphism of varieties. Suppose we know that the fibre dimension is a constructible function. Use the curve criterion to show that fibre dimension is semicontinuous.

We are to show that the fibre dimension \( \delta \) is semicontinuous. This means that the set of points at which \( \delta(p) \geq k \) is closed for every \( k \).

We may assume that \( X \) and \( Y \) are affine, say \( X = \text{Spec } A \) and \( Y = \text{Spec } B \).

**Case 1.** \( X \) is a smooth curve.

The kernel of the homomorphism \( \varphi : A \to B \) corresponding to the morphism \( f \) is a prime ideal \( P \) of \( A \), and since \( X \) is a curve, \( P \) is either the zero ideal or a maximal ideal. This gives us two possibilities:

(i) \( P \) is a maximal ideal, and The image of \( f \) is a point. Then there is just one fibre. This case is OK.

(ii) \( P \) is the zero ideal, and \( \varphi \) is injective. We may say that \( A \subset B \). Since \( A \) is a smooth curve, the maximal ideal \( M \) at a point \( x \) of \( X \) can be generated locally by a single element, say by the element \( \alpha \) of \( A \) (that depends on \( x \)). Then the fibre over \( x \) is the locus \( \alpha = 0 \) in \( Y = \text{Spec } B \). Krull’s Theorem tells us that if \( Y \) has dimension \( d \), the dimension of the fibre is \( d - 1 \). It is independent of the point \( x \).

**Case 2.** The general case.

We must show this: Let \( C \) be a smooth affine curve \( C \), let \( q \) be a point of \( C \), and let \( C' = C \setminus \{q\} \). If \( g : C \to Y \) is a morphism, and if the fibre dimension at each point of \( g(C') \) is \( \geq d \), then the fibre dimension at \( g(q) \) is also \( \geq d \).

Let \( h = fg \) be the composed map \( C \to X \):

\[
\begin{array}{ccc}
C & \xrightarrow{g} & Y \\
\downarrow & & \downarrow f \\
C & \xrightarrow{h} & X
\end{array}
\]

We form the fibred product \( Z = C \times_X Y \):

\[
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
C & \longrightarrow & X
\end{array}
\]

The fibre of the map \( Z \to C \) over a point \( q \) of \( C \) is the same as the fibre of \( Y \to X \) over \( p = h(q) \). So what must be shown is that the fibre of \( Z \) over \( q \) has dimension \( \geq d \).

What we know is that the fibred product \( Z \) is a closed subset of the product variety \( C \times Y \). So it is a finite union of closed subvarieties \( Z_1, \ldots, Z_k \) of \( C \times Y \). Case 1 applies to each of the varieties \( Z_i \).