18.721 Assignment 6

This assignment is due Monday, April 6

1. Let $\alpha$ be an element of a domain $A$, and let $\beta = \alpha^{-1}$. Prove that if $\beta$ is integral over $A$, then it is an element of $A$.

2. Let $X = \text{Spec } A$, where $A = \mathbb{C}[x, y, z]/(y^2 - xz^2)$. Identify the normalization of $X$.

3. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism. Do this by showing that the fibres cannot have positive dimension.

4. The cyclic group $< \sigma >$ of order $n$ operates on the polynomial ring $R = \mathbb{C}[x, y]$, by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, $\zeta = e^{2\pi i/n}$. Let $A$ be the ring of invariants.
   (a) Describe the invariant polynomials.
   (b) Show that the polynomials $u_i = x^iy^{n-i}$, $i = 0, ..., n$, generate the ring $A$.
   (c) Find generators for the ideal of relations among the generators $u_i$ (the kernel of the homomorphism from the polynomial ring $\mathbb{C}[y_0, ..., y_n]$ to $A$ that sends $y_i$ to $u_i$).