1. **Classify algebras that are complex vector spaces of dimensions two or three.**

Let’s do the case of dimension 3. Let $\alpha$ be an element of $A$, not a constant. Then $1, \alpha, \alpha^2, \alpha^3$ are dependent, so $\alpha$ is the root of a polynomial $f(x)$ of degree 2 or 3. We take $f$ of minimal degree and we factor: $f(x) = (x - a)(x - b)$ or $f(x) = (x - a)(x - b)(x - c)$. Suppose that $f$ has two or more distinct roots, say $a \neq b$. Let $u = (\alpha - a)$, $v = (\alpha - b)$, and $w = (\alpha - c)$ if applicable. These elements aren’t zero because $\alpha \notin \mathbb{C}$, and they aren’t units because $uv = 0$ or $uvw = 0$. So $u$ and $v$ are contained in maximal ideals $m_1$ and $m_2$, which are distinct because $u - v$ is the nonzero scalar $b - a$.

The Chinese Remainder Theorem (CRT) tells us that, if $m_1, \ldots, m_k$ are the maximal ideals of $A$, then $A$ maps surjectively to $(A/m_1) \times \cdots \times (A/m_k)$, with kernel the product ideal $m_1 \cdots m_k$. Also, $A/m_i = \mathbb{C}$. So there are at most three maximal ideals, and if there are three, then $A$ is isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

Suppose that there are two maximal ideals $m_1, m_2$. They give us a surjective map $A \to \mathbb{C} \times \mathbb{C}$. The kernel $I$ of this map has dimension 1, and $I = m_1 m_2$. Since $I$ is in every maximal ideal, it is nilpotent, and since $\dim I = 1$, $I^2 = 0$. The squares $m_1^2$ and $m_2^2$ are comaximal too, so $A$ maps surjectively to $(A/m_1^2) \times (A/m_2^2)$ with kernel $m_1^2 m_2^2 = I^2 = 0$. The map is not only surjective, it is bijective. Therefore $A$ is the product $A_1 \times A_2$ of two algebras, whose dimensions have to be 1 and 2. This puts us back in the case of dimension two.

Suppose that there is just one maximal ideal. Then every elements $\alpha$ of $A$ satisfies a polynomial relation $f = 0$, where $f$ is a power of $x - a$ for some $a$. When $\alpha$ is in $m$, $a = 0$. Every element of $m$ is nilpotent. If $m$ contains an element $\alpha$ such that $\alpha^2 \neq 0$, then $1, \alpha, \alpha^2$ will be a basis, and $A = \mathbb{C}[\alpha] \approx \mathbb{C}[x]/(x^3)$.

Suppose finally that $\alpha^2 = 0$ for every element $\alpha$ in $m$. We choose a basis $(1, \alpha, \beta)$ for $A$ with $\alpha^2 = \beta^2 = 0$. Let $(\alpha)$ denote the ideal $\alpha A$. Since $A > m > (\alpha) > 0$, $\dim(\alpha) = 1$. We must have $(\alpha) > (\alpha \beta)$ because $(\alpha \beta^2) = 0$. Therefore $\alpha \beta = 0$. The algebra is isomorphic to $\mathbb{C}[x, y]/(x^2, y^2, xy)$.

Thus any algebra of dimension 3 is isomorphic to one of the following:

- $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$,
- $\mathbb{C} \times (\mathbb{C}[x]/(x^2))$,
- $\mathbb{C}[x]/(x^3)$,  
- $\mathbb{C}[x, y]/(x^2, xy, y^2)$. 

**18.721 Comments on Assignment 3**
2. Find generators for the ideal of \( \mathbb{C}[x, y] \) that vanish on the three points \((0, 0), (0, 1), (1, 0)\).

The maximal ideals at the three points are \(M_1 = (x, y), M_2 = (x, y-1), \) and \(M_3 = (x-1, y)\), and, according to CRT, the kernel of the surjective map \( \mathbb{C}[x, y] \to \mathbb{C} \times \mathbb{C} \times \mathbb{C} \) is the product ideal \(M_1M_2M_3\). It can be generated by the three elements \(xy, x^2 - x, y^2 - y\).

3. Let \(B\) be a finite type domain, and let \(p\) and \(q\) be points of the affine variety \(Y = \text{Spec } B\). Let \(A\) be the set of elements \(f \in B\) such that \(f(p) = f(q)\). Adapt the method of proof of Theorem 2.7.5 to show

(a) \(A\) is a finite type domain.
(b) \(B\) is a finite \(A\)-module.
(c) \(\varphi : \text{Spec } B \to \text{Spec } A\) be the morphism obtained from the inclusion \(A \subset B\). Show that \(\varphi(p) = \varphi(q)\), and that \(\varphi\) is bijective everywhere else.

(a), (b) Say that \(B\) is generated as algebra by \(y = y_1, ..., y_n\). The Nullstellensatz tells us that the points are given by evaluating \(y\), Let’s change of variable so that \(p = (0, ..., 0)\) and \(q = (1, ..., 1)\). The maximal ideals at these points are \(m_p = (y_1, ..., y_n)\) and \(m_q = (y_1 - 1, ..., y_n - 1)\).

The elements of the ring \(A\) can be written in the form \(f = c + g\), where \(g\) is an element of \(I := m_p \cap m_q\). Since the maximal ideals are comaximal, the intersection is the product \(I = m_p m_q\), which is the ideal generated by the products \(z_i = y_i(y_i - 1)\). Let \(R\) be the algebra generated by \(z_1, ..., z_n\). Suppose that an element \(\beta\) of \(B\) is written as a polynomial \(g(y)\) in \(y\). Then, allowing coefficients in \(R\), we can use the equations \(y_i^2 = y_i + z_i\) to reduce the degree in each \(y_i\) to 1, to obtain an expression for \(\beta\) as polynomial \(h(y)\) with coefficients in \(R\) that has degree at most 1 in each \(y_i\). Therefore \(B\) is generated, as \(R\)-module, by the finite set of monomials of degree at most 1 in each \(y_i\). It is a finite \(R\)-module. Since \(R\) is finitely generated, it is noetherian. Therefore the \(R\)-submodule \(A\) of \(B\) is also finitely generated. Then is is generated as algebra, by the elements \(z_1, ..., z_n\) together with a finite set of \(R\)-module generators. And since \(R \subset A\) and \(B\) is a finite \(R\)-module, it is also a finite \(A\)-module.

(c) Let \(Y = \text{Spec } B\), \(X = \text{Spec } A\), and let \(u : Y \to X\) be the morphism that corresponds to the inclusion \(A \subset B\). A point \(s\) of \(Y\) corresponds to a homomorphism \(\pi_s : B \to \mathbb{C}\), and its image in \(X\) corresponds to the composed homomorphism \(A \subset B \to \mathbb{C}\). Since \(\pi_p\) and \(\pi_q\) have the same restriction to \(A\), \(p\) and \(q\) have the same image in \(X\). Suppose that \(s\) and \(t\) are points of \(Y\), and that \(s\) is not \(t\), \(p\) or \(q\). We choose an element \(\beta\) of \(B\) whose value at \(t, p,\) and \(q\) is zero and whose value at \(s\) is 1. We can do this because \(B\) is a quotient of a polynomial ring \(\mathbb{C}[y_1, ..., y_n]\), and there are polynomials taking arbitrary values at finite sets of points. Then \(\beta\) is in \(A\), and because \(\beta(s) \neq \beta(t), u(s) \neq u(t)\).