1. Prove that the elementary symmetric functions $s_1 = x_1 + \cdots + x_n, \ldots, s_n = x_1 \cdots x_n$ are algebraically independent.

This is easy using induction on $n$.

2. (i) Let $f(t, y) = ty^2 - 4y + t$. Solve $f = 0$ for $y$ by the quadratic formula, and sketch the real locus $f = 0$ in the $t, y$ plane.

(ii) What does Hensel’s Lemma say tell us? Factor $d$, modulo $t^4$.

$$f(t, y) = (y - t/4 - t^3/64)(ty - 4 + t^2/4) + O(t^4),$$ I think.

3. Let $C$ be a cubic curve with a node. Determine the degree of $C^*$, and the numbers of flexes, bitangents, nodes, and cusps of $C$ and of $C^*$.

Since it is a cubic, $C$ has no bitangents and therefore $C^*$ has no nodes. But $C$ has flexes, whose images in $C^*$ are cusps. The node of $C$ gives us a single bitangent to $C^*$.

We project $C$ to $\mathbb{P}^1$ from a generic point $q$ of $\mathbb{P}^2$. The since the degree of $C$ is 3, the discriminant will have degree $6 = 3(3 - 1)$. The discriminant has a double zero at the image of the node, and therefore four simple zeros at the images of tangent lines through $q$. This means that in $\mathbb{P}^*$, the dual curve $C^*$ meets the generic line $q^*$ in four points. So the degree of $C^*$ is 4. The Euler characteristic of the smooth curve $C'$ obtained by pulling apart the node can also be computed from the projection. Since the map $C \to \mathbb{P}^1$ has degree 3, the Euler characteristic is $e(C') = 3e(\mathbb{P}^1) - 4 = 6 - 4 = 2$. The genus is zero.

Next, we project $C^*$ to $\mathbb{P}^1$ from a generic point of $\mathbb{P}^*$. The discriminant has degree $12 = 4(4 - 1)$. It has a triple zero at the image of a cusp, and a simple zero at the image of a tangent line to $C^*$: $12 = 3\#(cusps) + \#(tangents)$. The degree of the dual curve $C^{**} = C$, which is 3, is the number of tangents. Therefore $C^*$ has 3 cusps and $C$ has 3 flexes.

4. Prove that a plane curve $X$ of degree 4 can have at most three singular points by showing that there is a conic $C$ that passes through any five points of $X$.

Let points $p_1, \ldots, p_5$ be given. Since $X$ has degree 4, at most four points can be on line. Perhaps I should have said that the conic may be reducible, i.e., the union of two lines. This occurs when three of the g points, say $p_1, p_2, p_3$, are on a line. The other line will be through the two remaining points.
Show that with a suitable choice of coordinates, one can reduce the defining polynomial $f$ to the form $z = 0$. So that the coefficients of $f$ are zero. Without spoiling the hypothesis, we can change coordinates as follows:

**5.** Let $C$ be a smooth cubic curve in $\mathbb{P}^2$, and let $p$ be a flex point of $C$. Choose coordinates so that $p$ is the point $(0,1,0)$ and the tangent line to $C$ at $p$ is the line $\{z = 0\}$.

**a)** Show that the coefficients of $x^2y, xy^2$, and $y^3$ in the defining polynomial $f$ of $C$ are zero.

**b)** Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form $f = y^2z + x^3 + axz^2 + bz^3$, where $x^3 + ax + b$ is a polynomial with distinct roots. Without spoiling the hypothesis, we can change coordinates as follows:

- replace $y$ by any combination $ax + by + cz$ with $b \neq 0$,
- replace $x$ by $dx + ez$ with $d \neq 0$,
- replace $z$ by $rz$ with $r \neq 0$.

Those are our allowed changes of coordinates.

Let’s set $z = 1$. We can put $z$ back when we are through:

$$f(x, y, 1) = a_0 y^2 + (a_1 x + a_2) y + (a_3 x^3 + a_4 x^2 + a_5 x + a_6)$$

One can check that, because $C$ is smooth, the coefficient $a + 0$ must be nonzero. We can normalize it to 1.

Next, because $f(x, y, 1)$ is quadratic in $y$, and we can “complete the square” to eliminate the coefficients $a_1$ and $a_2$. We will be left with $f(x, y, 1) = a_0 y^2 + a_3 x^3 + a_4 x^2 + a_5 x + a_6$. We can multiply $f$ by a scalar to make $a_3 = 1$, and then scale $y$ to make $a_0 = 1$. Because $C$ is smooth, the cubic $a_3 x^3 + a_4 x^2 + a_5 x + a_6$ has distinct roots $u_1, u_2, u_3$. So $a_3 \neq 0$ and we can normalize $a_3$ to 1. Then we can move the roots by a change of coordinates $x \rightarrow dx + e$ to achieve $u_1 + u_2 + u_3 = 0$. This eliminates the coefficients $a_4$. Then we can still multiply $x$ by a scalar to make $a_6 = 1$ unless by chance $a_6 = 0$. In that case, we can make $a_5 = 1$. We are left with $f = y^2 z + x^3 + axz^2 + z^3$ or $f = y^2 z + x^3 + xz^2$. 