Massachusetts Institute of Technology

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NOTES FOR A COURSE IN

ALGEBRAIC GEOMETRY

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PREFACE

These are notes that have been used for an algebraic geometry course at MIT. I had thought of teaching such a course for quite a while, motivated partly by the fact that MIT didn't have very many courses suitable for students who had taken the standard theoretical math classes. I got around to thinking seriously about this twelve years ago, and have now taught the class seven times. I wanted to get to cohomology of $\mathcal{O}$-modules (aka quasicoherent sheaves) in one semester without presupposing a knowledge of sheaf theory or of much commutative algebra, so it has been a challenge. Fortunately, MIT has many outstanding students who are interested in mathematics. The students and I have made some progress, but much remains to be done. Ideally, one would like the development to be so natural as to seem obvious. Though I haven't tried to put in anything unusual, this has yet to be achieved. And there are too many pages for my taste. To paraphrase Pascal, we haven't had the time to make it shorter.

To cut the material down, I decided to work exclusively with varieties over the complex numbers, and to use that restriction freely. Schemes are not discussed. Some people will disagree with these decisions, but I feel that absorbing schemes and general ground fields won't be too difficult for someone who is familiar with complex varieties. Also, I don't go out of my way to state and prove things in their most general form.

If one plans to teach such a course in a single semester, it is essential to keep moving. One can't linger over the topics in the first Chapter. To save time, consider replacing some proofs with heuristic reasoning or omitting them. Proposition 1.9.11 on the order of vanishing of the discriminant is a candidate for some hand-waving, and Lemma 1.10.7 on flex points may be a proof to skip.

Indices can cloud the picture. When that happens, I recommend focussing attention on a low dimensional case. Schelter's neat proof of Chevalley's Finitness Theorem is a good example. Schelter discovered the proof while studying $\mathbb{P}^1$. That case demonstrates the main point, and is a bit easier to follow.

In Chapter 6 on $\mathcal{O}$-modules, all technical points about sheaves are eliminated when one sticks to affine open sets and localizations. Sections over other open sets are important, mainly because one wants the global sections, but the proof that a module extends to arbitrary open sets can safely be put on a back burner, as is done in the notes.

In Chapter 7, I decided to restrict to $\mathcal{O}$-modules when defining cohomology, and to characterize the cohomology axiomatically. This was in order to minimize technical points. Simplicial operations are eliminated, though they appear in disguise in the resolution (7.4.13).

The special topics at the ends of Chapters 2,3,4 enrich the subject. I don't recommend skipping them. And, without some of the applications at the end of Chapter 8, the Riemann Roch Theorem would be pointless.

When I last taught the subject in the spring of 2020, MIT semester had 39 class hours. I followed this schedule: Chapter 1, 6 hours, Chapters 2-7, roughly 4 hours each, Chapter 8, 7 hours, in-class quizzes, 2 hours. This was a brisk pace. The topics in the notes could be covered comfortably in a one-year course, and there would be time for some extra material.

Great thanks are due to the students who have been in my classes. Many of you contributed to these notes by commenting on the drafts or by creating figures. Though I remember you well, I'm not naming you individually because I'm sure I'd overlook someone important. I hope that you will understand.

A Note for the Student

The prerequisites are standard undergraduate courses in algebra, analysis, and topology, and the definitions of category and functor. I also suppose a familiarity with the implicit function theorem for complex variables. But don't worry too much about the prerequisites. You can look them up as needed, and many points are reviewed briefly in the notes as they come up.

Proofs of some lemmas and propositions are omitted. I have omitted a proof when I consider it simple enough that including it would just clutter up the text or, occasionally, when I feel that it is important for the reader to supply a proof.

As with all mathematics, working exercises and, most importantly, writing up the solutions carefully is, by far, the best way to learn the material well.

Chapter 1 PLANE CURVES

planecurves

We begin with plane curves. They were the first algebraic varieties to be studied, and they provide instructive examples. Chapters 2 – 7 are about varieties of arbitrary dimension. We will see in Chapter 5 how curves control higher-dimensional varieties, and we come back to curves in Chapter 8.

1.1 The Affine Plane

affine-plane

The n-dimensional *affine space* $\mathbb{A}^n$ is the space of n-tuples of complex numbers. The two-dimensional affine space $\mathbb{A}^2$ is the affine plane.

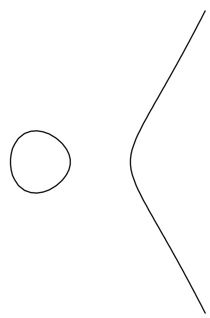
Let $f(x_1, x_2)$ be an irreducible polynomial in two variables, with complex coefficients. The set of points of the affine plane at which f vanishes, the *locus of zeros* of f, is called a *plane affine curve*. Let's denote that locus by X. Writing x for the vector (x_1, x_2),

affcurve (1.1.1)
$$X = \{x \,|\, f(x) = 0\}$$

The *degree* of the curve X is the degree of its irreducible defining polynomial f.

When it seems unlikely to cause confusion, we may abbreviate the notation for an indexed set, using a single letter, as here, where x stands for $(x_!, x_2)$.

goober8 **1.1.2.**



The Cubic Curve $y^2 = x^3 - x$ (real locus)

About figures. In algebraic geometry, the dimensions are too big to allow realistic figures. Even with an affine plane curve, one is dealing with a locus in the affine plane $\mathbb{A}^2$, whose topological dimension is 4. In some cases, such as in the figure above, depicting the real locus can be helpful, but in most cases, even the real locus is too big, and one must make do without a figure, or with a schematic diagram. □

We will get an understanding of the geometry of a plane curve as we go along, and we mention just one point here. A plane curve is called a *curve* because it is defined by one equation in two variables. Its *algebraic* dimension is 1. The only proper subsets of a curve X that can be defined by polynomial equations are the finite sets (see Proposition 1.3.12). But because our scalars are complex numbers, the affine plane $\mathbb{A}^2$ is a real space of dimension 4, and X will be a surface in that space. This is analogous to the fact that the *affine line* $\mathbb{A}^1$ is the plane of complex numbers.

One can see that a plane curve X has dimension 2, geometrically, by inspecting its projection to a line. To do this, one writes the defining polynomial as a polynomial in x_2:

$$f(x_1, x_2) = c_0 x_2^d + c_1 x_2^{d-1} + \cdots + c_d$$

whose coefficients c_i are polynomials in x_1. Let's suppose that d is positive, i.e., that f isn't a polynomial in x_1 alone. Let $X \xrightarrow{\pi} \mathbb{A}^1$ be the projection from the plane curve X to the affine x_1-line $\mathbb{A}^1$.

The *fibre* of a map $V \to U$ over a point p of U is the inverse image of p, the set of points of V that map to p. One can describe the fibre of the map π over the point $x_1 = a$, as the set of points (a, b) in which b is a root of the one-variable polynomial

$$f(a, x_2) = \overline{c}_0 x_2^d + \overline{c}_1 x_2^{d-1} + \cdots + \overline{c}_d$$

with $\overline{c}_i = c_i(a)$. There will be finitely many points in this fibre, and it won't be empty unless $f(a, x_2)$ is a constant. The plane curve X covers most of the x_1-line, a complex plane, finitely often.

1.1.3. Note. In contrast with complex polynomials in one variable, most polynomials in two or more variables are *irreducible* — they cannot be factored. This can be shown by a method called "counting constants". For instance, quadratic polynomials in x_1, x_2 depend on the six coefficients of the monomials $1, x_1, x_2, x_1^2, x_1x_2, x_2^2$ of degree at most two. Linear polynomials $ax_1 + bx_2 + c$ depend on three coefficients, but the product of two linear polynomials depends on only five parameters, because a scalar factor can be moved from one of the linear factors to the other. So the quadratic polynomials cannot all be written as products of linear polynomials. This reasoning is fairly convincing. It can be justified formally in terms of *dimension*, which will be discussed in Chapter 5. □ polyirred

(1.1.4) changing coordinates planecoords

We allow linear changes of variable and translations in the affine plane $\mathbb{A}^2$. When a point x is written as the column vector $(x_1, x_2)^t$, the coordinates $x' = (x_1', x_2')^t$ after such a change of variable will be related to x by a formula

$$x = Qx' + a \tag{1.1.5}$$ chgcoord

where Q is an invertible 2×2 matrix with complex coefficients and $a = (a_1, a_2)^t$ is a complex translation vector. This changes a polynomial equation $f(x) = 0$, to $f(Qx' + a) = 0$. One may also multiply a polynomial f by a nonzero complex scalar without changing its locus of zeros. Using these operations, all *lines*, plane curves of degree 1, become equivalent.

An *affine conic* is a plane affine curve of degree 2. Every affine conic is equivalent to one of the two loci

$$x_1^2 - x_2^2 = 1 \quad \text{or} \quad x_2 = x_1^2 \tag{1.1.6}$$ affconic

by a suitable linear change of variable, a translation, and scaling. The proof of this is similar to the one used to classify real conics. These loci might be called a complex 'hyperbola' and 'parabola', respectively. The complex 'ellipse' $x_1^2 + x_2^2 = 1$ becomes the 'hyperbola' when one multiplies the coordinate x_2 by i.

On the other hand, there are infinitely many inequivalent cubic curves. Cubic polynomials in two variables depend on the coefficients of the ten monomials $1,\ x_1, x_2,\ x_1^2, x_1x_2, x_2^2,\ x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$ of degree at most 3 in x_1, x_2. Linear changes of variable, translations, and scalar multiplication, give us only seven scalars to work with, leaving three essential parameters.

1.2 The Projective Plane

projplane

The n-dimensional *projective space* $\mathbb{P}^n$ is the set of equivalence classes of *nonzero* vectors $x = (x_0, x_1, ..., x_n)$, the equivalence relation being

equivrel (1.2.1)
$$(x_0', ..., x_n') \sim (x_0, ..., x_n) \quad \text{if} \quad (x_0', ..., x_n') = (\lambda x_0, ..., \lambda x_n) \qquad \big(\text{if} \quad x' = \lambda x\big)$$

for some nonzero complex number λ. The equivalence classes are the *points* of $\mathbb{P}^n$. One often refers to a point by giving a particular vector in its class.

When x is a nonzero vector, the one-dimensional subspace of $\mathbb{C}^{n+1}$ spanned by x consists of the vectors λx, together with the zero vector. So points of $\mathbb{P}^n$ correspond to one-dimensional subspaces of the complex vector space $\mathbb{C}^{n+1}$.

projline **(1.2.2) the projective line**

Points of the *projective line* $\mathbb{P}^1$ are equivalence classes of nonzero vectors $x = (x_0, x_1)$.

If the first coordinate x_0 of a vector $x = (x_0, x_1)$ isn't zero, we may multiply by $\lambda = x_0^{-1}$ to normalize the first entry to 1, and write the point that x represents in a unique way as $(1, u_1)$, with $u_1 = x_1/x_0$. There is one remaining point, the point represented by the vector $(0, 1)$. The projective line $\mathbb{P}^1$ can be obtained by adding this point, called the *point at infinity*, to the affine u_1-line, which is a complex plane. As u_1 tends to infinity in any direction, the point $(1, u_1)$ approaches $(0, 1)$. Topologically, $\mathbb{P}^1$ is a two-dimensional sphere.

projpl **(1.2.3) lines in projective space**

Let p and q be vectors that represent distinct points of the projective space $\mathbb{P}^n$. There is a unique *line* L in $\mathbb{P}^n$ that contains those points, the set of points $L = \{rp + sq\}$, with r, s in $\mathbb{C}$ not both zero. Points of L correspond bijectively to points of the projective line $\mathbb{P}^1$, by

pline (1.2.4)
$$rp + sq \quad \longleftrightarrow \quad (r, s)$$

A line in the projective plane $\mathbb{P}^2$ can also be described as the locus of solutions of a homogeneous linear equation

eqline (1.2.5)
$$s_0x_0 + s_1x_1 + s_2x_2 = 0$$

linesmeet **1.2.6. Lemma.** *In the projective plane, two distinct lines have exactly one point in common, and in a projective space of any dimension, a pair of distinct points is contained in exactly one line.* □

standcov **(1.2.7) the standard covering of the projective plane**

The *projective plane* $\mathbb{P}^2$ is the two-dimensional projective space. Its points are equivalence classes of nonzero vectors (x_0, x_1, x_2).

If the first entry x_0 of a point $p = (x_0, x_1, x_2)$ of the plane isn't zero, we may normalize it to 1 without changing the point: $(x_0, x_1, x_2) \sim (1, u_1, u_2)$, where $u_i = x_i/x_0$. We did the analogous thing for $\mathbb{P}^1$ above. The representative vector $(1, u_1, u_2)$ is uniquely determined by p, so points with $x_0 \neq 0$ correspond bijectively to points of the affine plane $\mathbb{A}^2$ with coordinates (u_1, u_2):

$$(x_0, x_1, x_2) \sim (1, u_1, u_2) \quad \longleftrightarrow \quad (u_1, u_2)$$

We regard the affine u_1, u_2-plane as a subset of $\mathbb{P}^2$ by this correspondence, and we denote that subset by $\mathbb{U}^0$. The points of $\mathbb{U}^0$, those with $x_0 \neq 0$, are the *points at finite distance*. The *points at infinity* of $\mathbb{P}^2$ are those of the form $(0, x_1, x_2)$. They are on the *line at infinity* L^0, the locus $\{x_0 = 0\}$ in $\mathbb{P}^2$. The projective plane is the union of the two sets $\mathbb{U}^0$ and L^0. When a point is given by a coordinate vector, one can assume that the first coordinate is either 1 or 0.

We may write the point (x_0, x_1, x_2) that is in $\mathbb{U}^0$ as $(1, u_1, u_2)$, with $u_i = x_i/x_0$ as above. The notation $u_i = x_i/x_0$ is important when the coordinate vector (x_0, x_1, x_2) has been given. When no coordinate vector of a point p has been given, one may simply assume that the first coordinate is 1 and write $p = (1, x_1, x_2)$.

There is an analogous correspondence between points $(x_0, 1, x_2)$ and points of an affine plane $\mathbb{A}^2$, and between points $(x_0, x_1, 1)$ and points of an affine plane. We denote the subsets $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$ by $\mathbb{U}^1$ and $\mathbb{U}^2$, respectively. The three sets $\mathbb{U}^0, \mathbb{U}^1, \mathbb{U}^2$ form the *standard covering* of $\mathbb{P}^2$ by three *standard affine open sets*. Since the vector $(0, 0, 0)$ has been ruled out, every point of $\mathbb{P}^2$ lies in at least one of the three standard open sets. Points whose three coordinates are nonzero lie in all of them.

1.2.8. Note. Which points of $\mathbb{P}^2$ are at infinity depends on which of the standard open sets is taken to be the one at finite distance. When the coordinates are (x_0, x_1, x_2), I like to normalize x_0 to 1, as above. Then the points at infinity are those of the form $(0, x_1, x_2)$. But when coordinates are (x, y, z), I may normalize z to 1. Then the points at infinity are the points $(x, y, 0)$. I hope this won't cause too much confusion. $\square$ pointatinfinity

(1.2.9) digression: the real projective plane realprojplane

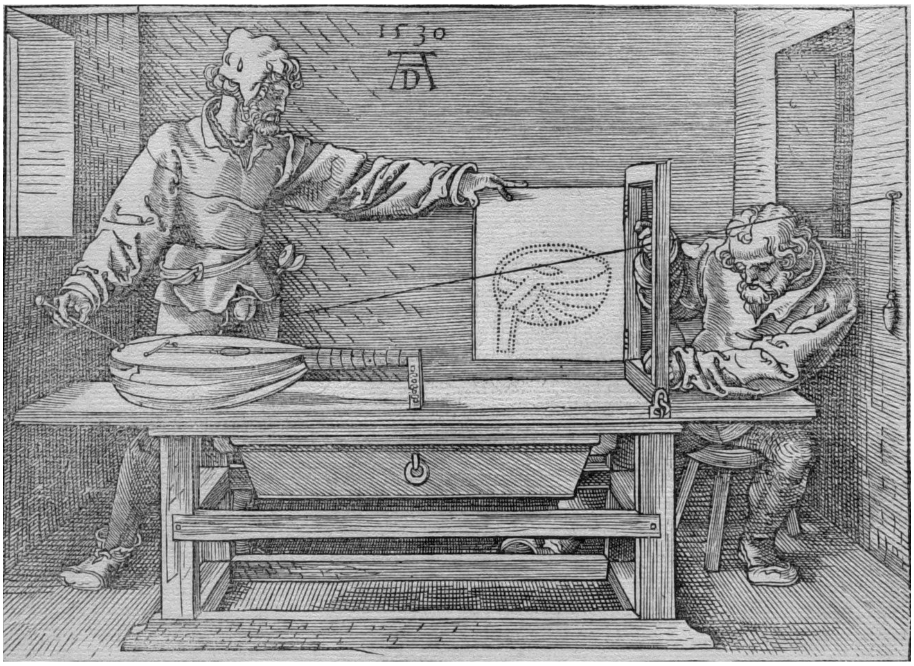
Points of the *real projective plane* $\mathbb{RP}^2$ are equivalence classes of nonzero real vectors $x = (x_0, x_1, x_2)$, the equivalence relation being $x' \sim x$ if $x' = \lambda x$ for some nonzero real number λ. The real projective plane can also be thought of as the set of one-dimensional subspaces of the real vector space $\mathbb{R}^3$.

Let's denote $\mathbb{R}^3$ by V. The plane U in V defined by the equation $x_0 = 1$ is analogous to the standard open subset $\mathbb{U}^0$ of the complex projective plane $\mathbb{P}^2$. We can project V from the origin $p_0 = (0, 0, 0)$ to U, sending a point $x = (x_0, x_1, x_2)$ of V to the point $(1, u_1, u_2)$, with $u_i = x_i/x_0$. The fibres of this projection are the lines through p_0 and x, with p_0 omitted.

The projection to U is undefined at the points $(0, x_1, x_2)$, which are orthogonal to the x_0-axis. The line connecting such a point to p_0 doesn't meet U. Those points are the points at infinity of $\mathbb{RP}^2$.

Looking from the origin, U becomes a "picture plane".

durerfig **1.2.10.**

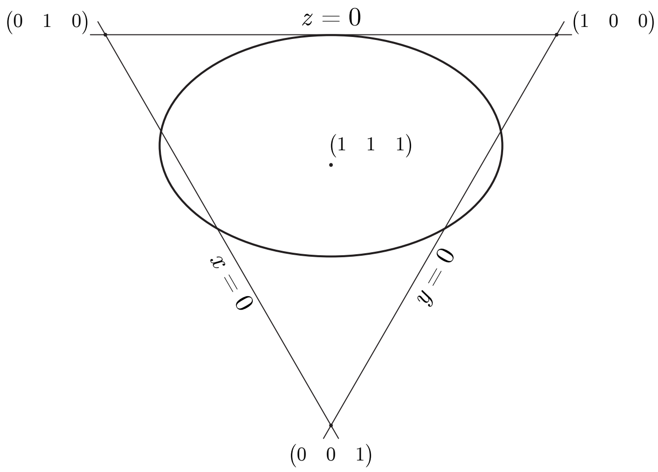


This is an illustration from a book on perspective by Albrecht Dürer

The projection from three-space to a picture plane goes back to the the 16th century, the time of Desargues and Dürer. Projective coordinates were introduced 200 years later, by Möbius.

The figure below shows the plane W: $x+y+z=1$ in the real vector space $\mathbb{R}^3$, together with its coordinate lines and a conic. The one-dimensional subspace spanned by a nonzero vector (x_0, y_0, z_0) in $\mathbb{R}^3$ will meet W in a single point unless that vector is on the line $L: x+y+z=0$. So W is a faithful representation of most of $\mathbb{RP}^2$. It contains all points except those on L.

goober4 **1.2.11.**



The Real Projective Plane

chgcoordssec (**1.2.12**) **changing coordinates in the projective plane**

An invertible 3×3 matrix P determines a linear change of coordinates in $\mathbb{P}^2$. With $x = (x_0, x_1, x_2)^t$ and $x' = (x'_0, x'_1, x'_2)^t$ represented as column vectors, that coordinate change is given by

(1.2.13) $$x = Px'$$ chg

The next proposition shows that four special points, the points

$$e_0 = (1,0,0)^t,\ e_1 = (0,1,0)^t,\ e_2 = (0,0,1)^t \quad \text{and} \quad \epsilon = (1,1,1)^t$$

determine the coordinates in $\mathbb{P}^2$.

1.2.14. Proposition. *Let p_0, p_1, p_2, q be four points of $\mathbb{P}^2$, no three of which lie on a line. There is, up to a scalar factor, a unique linear coordinate change $Px' = x$ such that $Pp_i = e_i$ and $Pq = \epsilon$.* *fourpoints*

proof. The hypothesis that the points p_0, p_1, p_2 don't lie on a line tells us that the vectors that represent those points are independent. They span $\mathbb{C}^3$. So q will be a combination $q = c_0p_0 + c_1p_1 + c_2p_2$, and because no three of the four points lie on a line, the coefficients c_i will be nonzero. We can *scale* the vectors p_i (multiply them by nonzero scalars) to make $q = p_0 + p_1 + p_2$ without changing the points. Next, the columns of P can be an arbitrary set of independent vectors. We let them be p_0, p_1, p_2. Then $Pe_i = p_i$, and $P\epsilon = q$. The matrix P is unique up to scalar factor. □

(1.2.15) conics projconics

A polynomial $f(x_0, ..., x_n)$ is *homogeneous*, *of degree* d, if all monomials that appear with nonzero coefficient have (total) degree d. For example, $x_0^3 + x_1^3 - x_0x_1x_2$ is a homogeneous cubic polynomial.

A homogeneous quadratic polynomal is a combination of the six monomials

$$x_0^2,\ x_1^2,\ x_2^2,\ x_0x_1,\ x_1x_2,\ x_0x_2$$

The locus of zeros of an irreducible homogeneous quadratic polynomial is a *conic*.

1.2.16. Proposition. *For any conic C, there is a choice of coordinates so that it becomes the locus* *classify-conic*

$$x_0x_1 + x_0x_2 + x_1x_2 = 0$$

proof. A conic will contain three points that aren't colinear. Let's leave the verification of this fact as an exercise. We choose three non-colinear points on the conic C, and adjust coordinates so that they become the points e_0, e_1, e_2. Let f be the homogeneous quadratic polynomial in those coordinates whose zero locus is C. Because e_0 is a point of C, $f(1,0,0) = 0$, and therefore the coefficient of x_0^2 in f is zero. Similarly, the coefficients of x_1^2 and x_2^2 are zero. So f has the form

$$f = ax_0x_1 + bx_0x_2 + cx_1x_2$$

Since f is irreducible, a, b, c aren't zero. By *scaling* appropriately (adjusting f, x_0, x_1, x_2 by scalar factors), we can make $a = b = c = 1$. We will be left with the polynomial $x_0x_1 + x_0x_2 + x_1x_2$. □

1.3 Plane Projective Curves

projcurve

The loci in projective space that are studied in algebraic geometry are the ones that can be defined by systems of *homogeneous* polynomial equations. The reason that we use homogeneous equations is this:

To say that a polynomial $f(x_0, ..., x_n)$ vanishes at a point of projective space $\mathbb{P}^n$ means that if the vector $a = (a_0, ..., a_n)$ represents a point p, then $f(a) = 0$. Perhaps this is obvious. Now, if a represents p, the other vectors that represent p are the vectors λa $(\lambda \neq 0)$. When f vanishes at p, $f(\lambda a)$ must also be zero. The polynomial $f(x)$ vanishes at p if and only if $f(\lambda a) = 0$ for every λ.

We write a polynomial $f(x_0, ..., x_n)$ as a sum of its *homogeneous parts*:

(1.3.1) $$f = f_0 + f_1 + \cdots + f_d$$ homparts

where f_0 is the constant term, f_1 is the linear part, etc., and d is the degree of f.

hompart-szero **1.3.2. Lemma.** *Let $f = f_0 + \cdots + f_d$ be a polynomial of degree d in $x_0, ..., x_n$, and let $a = (a_0, ..., a_n)$ be a nonzero vector. Then $f(\lambda a) = 0$ for every nonzero complex number λ if and only if $f_i(a) = 0$ for every $i = 0, ..., d$.*

This lemma shows that we may as well work with homogeneous equations.

proof of the lemma. We substitute into (1.3.1): $f(\lambda x) = f_0 + \lambda f_1(x) + \lambda^2 f_2(x) + \cdot + \lambda^d f_d(x)$. Evaluating at $x = a$, $f(\lambda a) = f_0 + \lambda f_1(a) + \lambda^2 f_2(a) + \cdot + \lambda^d f_d(a)$, and $f_i(a)$ are scalars (complex numbers). The right side of this equation is a polynomial of degree at most d in λ, with complex coefficients $f_i(a)$. Since a nonzero polynomial of degree at most d has at most d roots, $f(\lambda a)$ won't be zero for *every* λ unless that polynomial is zero — unless $f_i(a)$ is zero for every i. $\square$

fequalsgh **1.3.3. Lemma. (i)** *If the product $f = gh$ of two polynomials is homogeneous, then g and h are homogeneous.*
(ii) *The zero locus in projective space of a product gh of homogeneous polynomials is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.*
(iii) *The zero locus in affine space of a product gh of polynomials, not necessarily homogeneous, is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.* $\square$

locipone (1.3.4) loci in the projective line

Before going to plane projective curves, we describe the zero locus in $\mathbb{P}^1$ of a homogeneous polynomial in two variables.

factorhompoly **1.3.5. Lemma.** *A nonzero homogeneous polynomial $f(x, y) = a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d$ with complex coefficients is a product of homogeneous linear polynomials that are unique up to scalar factor.*

To prove this, one uses the fact that the field of complex numbers is algebraically closed. A one-variable complex polynomial factors into linear factors. To factor $f(x, y)$, one can factor the one-variable polynomial $f(1, y)$ into linear factors, substitute y/x for y, and multiply the result by x^d. When one adjusts scalar factors, one will obtain the expected factorization of $f(x, y)$. For instance, to factor $f(x, y) = x^2 - 3xy + 2y^2$, we substitute $x = 1$: $2y^2 - 3y + 1 = 2(y - 1)(y - \frac{1}{2})$. Substituting $y = y/x$ and multiplying by x^2, $f(x, y) = 2(y - x)(y - \frac{1}{2}x)$. The scalar 2 can be distributed arbitrarily among the linear factors. $\square$

When a homogeneous polynomial f is a product of linear factors, we can adjust the factors by scalars, to put f into the form

factorpolytwo (1.3.6)
$$f(x, y) = c(v_1x - u_1y)^{r_1} \cdots (v_kx - u_ky)^{r_k}$$

where no factor $v_ix - u_iy$ is a constant multiple of another, c is a nonzero scalar, and $r_1 + \cdots + r_k$ is the degree of f. The exponent r_i is the *multiplicity* of the linear factor $v_ix - u_iy$.

A linear polynomial $vx - uy$ determines the point (u, v) in the projective line $\mathbb{P}^1$, the unique *zero* of that polynomial, and changing the polynomial by a scalar factor doesn't change its zero. Thus the linear factors of the homogeneous polynomial (1.3.6) determine points of $\mathbb{P}^1$, the *zeros* of f. The points (u_i, v_i) are zeros of *multiplicity* r_i. The total number of those points, counted with multiplicity, will be the degree of f.

zerosandroots **1.3.7.** The zero (u_i, v_i) of f corresponds to a root $x = u_i/v_i$ of multiplicity r_i of the one-variable polynomial $f(x, 1)$, except when the zero is the point $(1, 0)$. This happens when the coefficient a_0 of f is zero, and y is a factor of f. One could say that $f(x, y)$ has a zero at infinity in that case.

This sums up the information contained in the locus of a homogeneous polynomial in the projective line. It will be a finite set of points with multiplicities.

intersectline (1.3.8) intersections with a line

Let Z be the zero locus of a homogeneous polynomial $f(x_0, ..., x_n)$ of degree d in projective space $\mathbb{P}^n$, and let L be a line in $\mathbb{P}^n$ (1.2.4). Say that L is the set of points $rp + sq$, where p and q are points that are represented by specific vectors $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$, respectively. So L corresponds to the projective line $\mathbb{P}^1$, by $rp + sq \leftrightarrow (r, s)$. Let's also assume that L isn't entirely contained in the zero locus Z. The intersection $Z \cap L$ corresponds to the zero locus in $\mathbb{P}^1$ of the polynomial $\overline{f}$ in r, s obtained by substituting $rp + sq$ into f. This substitution yields a homogeneous polynomial $\overline{f}(r, s)$ of degree d, and the zeros of $\overline{f}$ in $\mathbb{P}^1$ correspond to the points of $Z \cap L$. If f has degree d, there will be d zeros, counted with multiplicity.

For instance, let f be the polynomial $x_0x_1+x_0x_2+x_1x_2$. Then with $p = (a_0, a_1, a_2)$ and $q = (b_0, b_1, b_2)$, $\overline{f}$ is the following quadratic polynomial in r, s:

$$\begin{aligned}\overline{f}(r, s) = f(rp + sq) &= (ra_0 + sb_0)(ra_1 + sb_1) + (ra_0 + sb_0)(ra_2 + sb_2) + (ra_1 + sb_1)(ra_2 + sb_2)\\ &= (a_0a_1+a_0a_2+a_1a_2)r^2 + \big(\textstyle\sum_{i\neq j} a_ib_j\big)rs + (b_0b_1+b_0b_2+b_1b_2)s^2\end{aligned}$$

1.3.9. Definition. With notation as above, the *intersection multiplicity* of the zero locus Z and a line L at a point p is the multiplicity of zero of the polynomial $\overline{f}$. $\square$ intersect-linetwo

1.3.10. Corollary. *Let Z be the zero locus of a homogeneous polynomial f in projective space $\mathbb{P}^n$, and let L be a line in $\mathbb{P}^n$ that isn't contained in Z. The number of intersections of Z and L, counted with multiplicity, is equal to the degree of f.* $\square$ *XcapL*

(1.3.11) loci in the projective plane

lociptwo

The locus of zeros in $\mathbb{P}^2$ of a single irreducible homogeneous polynomial $f(x, y, z)$ is called a *plane projective curve*. The *degree* of a plane projective curve is the degree of its irreducible defining polynomial.

The next proposition shows that plane projective curves are the most interesting loci in the projective plane.

1.3.12. Proposition. *Homogeneous polynomials $f_1, ..., f_r$ in three variables with no common factor have finitely many common zeros in $\mathbb{P}^2$ if $r > 1$.* *fgzerofinite*

The proof of this proposition is below.

1.3.13. Note. Suppose that a homogeneous polynomial $f(x, y, z)$ is reducible, say $f = g_1 \cdots g_k$, that g_i are irreducible, and that no two of them are scalar multiples of one another. Then the zero locus C of f is the union of the zero loci V_i of the factors g_i. In this case, C may be called a *reducible curve*. redcurve

When there are multiple factors, say $f = g_1^{e_1} \cdots g_k^{e_k}$ and some e_i are greater than 1, it is still true that the locus $C : \{f = 0\}$ is the union of the loci $V_i : \{g_i = 0\}$, but the connection between the geometry of C and the algebra is weakened. In this situation, the structure of a *scheme* becomes useful We won't discuss schemes. The only situation in which we may need to keep track of multiple factors is when counting intersections with another curve D. For this purpose, one can use the *divisor* of f, which is defined to be the integer combination $e_1V_1 + \cdots + e_kV_k$. $\square$

A *rational function* is a fraction of polynomials. The polynomial ring $\mathbb{C}[x, y]$ embeds into its field of fractions, the field of rational functions in x, y. That field is often denoted by $\mathbb{C}(x, y)$, but let's denote it by F here. The polynomial ring $\mathbb{C}[x, y, z]$ in three variables becomes a subring of the one-variable polynomial ring $F[z]$. When one is presented with a problem about the ring $\mathbb{C}[x, y, z]$, it can be useful to begin by studying it in the ring $F[z]$, which is a principal ideal domain. The polynomial rings $\mathbb{C}[x, y]$ and $\mathbb{C}[x, y, z]$ are unique factorization domains, but not principal ideal domains.

1.3.14. Lemma. *Let $F = \mathbb{C}(x, y)$ be the field of rational functions in x, y.* *relprime*
(i) *If $f_1, ..., f_k$ are homogeneous polynomials in x, y, z with no common factor, their greatest common divisor in $F[z]$ is 1, and therefore they generate the unit ideal of $F[z]$. (The unit ideal of a ring R is the ring R itself.) So there is an equation of the form $\sum g_i' f_i = 1$, with g_i' in $F[z]$.*
(ii) *An irreducible element of $\mathbb{C}[x, y, z]$ that has positive degree in z is also an irreducible element of $F[z]$.*

proof. **(i)** This is a proof by contradiction. Let h' be a nonzero element of $F[z]$ that isn't a unit, i.e., isn't an element of F, and suppose that h' divides f_i in $F[z]$ for every i. Say that $f_i = u_i'h'$ with u_i' in $F[x]$. The coefficients of h' and u_i' are rational functions, whose denominators are polynomials in x, y. We multiply by a polynomial in x, y to clear the denominators from the coefficients of all of the elements h' and u'_i. This will give us equations of the form $d_i f_i = u_i h$, where d_i are polynomials in x, y, and h and u_i are polynomials in x, y, z. Since h' isn't in F, neither is h. So h will have positive degree in z. Let g be an irreducible factor of h of positive degree in z. Then g divides $d_i f_i$, but it doesn't divide d_i, which has degree zero in z. So g divides f_i, and this is true for every i. This contradicts the hypothesis that $f_1, ..., f_k$ have no common factor.

(ii) Say that a polynomial $f(x, y, z)$ factors in $F[z]$, $f = g'h'$, where g' and h' are polynomials of positive degree in z with coefficients in F. When we clear denominators from g' and h', we obtain an equation of the form $df = gh$, where g and h are polynomials in x, y, z of positive degree in z and d is a polynomial in x, y. Since neither g nor h divides d, f must be reducible. □

proof of Proposition 1.3.12. We are to show that homogeneous polynomials $f_1, ..., f_r$ in x, y, z with no common factor have finitely many common zeros. Lemma 1.3.14 tells us that we may write $\sum g'_i f_i = 1$, with g'_i in $F[z]$. Clearing denominators from the elements g'_i gives us an equation of the form

$$\sum g_i f_i = d$$

where g_i are polynomials in x, y, z and d is a polynomial in x, y. Taking suitable homogeneous parts of g_i and d produces an equation $\sum g_i f_i = d$ in which all terms are homogeneous.

Lemma 1.3.5 asserts that $d(x, y)$ is a product of linear polynomials, say $d = \ell_1 \cdots \ell_r$. A common zero of $f_1, ..., f_k$ is also a zero of d, and therefore it is a zero of ℓ_j for some j. It suffices to show that, for each j, $f_1, ..., f_r$ and ℓ_j have finitely many common zeros.

Since $f_1, ..., f_k$ have no common factor, there is at least one f_i that isn't divisible by ℓ_j. Then Corollary 1.3.10 shows that f_i and ℓ_j have finitely many common zeros. □

pointscurves

1.3.15. Corollary. *Every locus in the projective plane $\mathbb{P}^2$ that can be defined by a system of homogeneous polynomial equations is a finite union of points and curves.* □

The next corollary is a special case of the Strong Nullstellensatz, which will be proved in the next chapter.

idealprincipal

1.3.16. Corollary. *Let $f(x, y, z)$ be an irreducible homogeneous polynomial that vanishes on an infinite set S of points of $\mathbb{P}^2$. If another homogeneous polynomial $g(x, y, z)$ vanishes on S, then f divides g. Therefore, if an irreducible polynomial vanishes on an infinite set S, that polynomial is unique up to scalar factor.*

proof. If the irreducible polynomial f doesn't divide g, then f and g have no common factor, and therefore they have finitely many common zeros. □

classicaltopology

(1.3.17) the classical topology

The usual topology on the affine space $\mathbb{A}^n$ will be called the *classical topology*. A subset U of $\mathbb{A}^n$ is open in the classical topology if, whenever U contains a point p, it contains all points sufficiently near to p. We call this the classical topology to distinguish it from another topology, the *Zariski topology*, which will be discussed in the next chapter.

The projective space $\mathbb{P}^n$ also has a classical topology. A subset U of $\mathbb{P}^n$ is open if, whenever a point p of U is represented by a vector $(x_0, ..., x_n)$, all vectors $x' = (x'_0, ..., x'_n)$ sufficiently near to x represent points of U.

isopts

(1.3.18) isolated points

A point p of a topological space X is *isolated* if the set $\{p\}$ is both open and closed, or if both $\{p\}$ and its complement $X - \{p\}$ are closed. If X is a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$, a point p of X is isolated in the classical topology if X doesn't contain points p' distinct from p, but arbitrarily close to p.

noisolatedpoint

1.3.19. Proposition *Let n be an integer greater than one. In the classical topology, the zero locus of a polynomial in $\mathbb{A}^n$, or of a homogeneous polyomial in $\mathbb{P}^n$, contains no isolated points.*

1.3.20. Lemma. *Let f be a polynomial of degree d in $x_1, ..., x_n$. After a suitable coordinate change and scaling, $f(x)$ will become a monic polynomial of degree d in the variable x_n.* polymonic

proof. We write $f = f_0 + f_1 + \cdots + f_d$, where f_i is the homogeneous part of f of degree i. We choose a point p of $\mathbb{A}^n$ at which f_d isn't zero, and change variables so that p becomes the point $(0, ..., 0, 1)$. We call the new variables $x_1, .., .x_n$ and the new polynomial f. Then $f_d(0, ..., 0, x_n)$ will be equal to cx_n^d for some nonzero constant c, and f/c will be monic. □

proof of Proposition 1.3.19. The proposition is true for loci in affine space and also for loci in projective space. We look at the affine case.

Let $f(x_1, ..., x_n)$ be a polynomial, and let Z be its zero locus. If f is a product, say $f = gh$, then Z will be the union of the zero loci $Z_1 : \{g = 0\}$ and $Z_2 : \{h = 0\}$. A point p of Z will be in one of those two sets, say in Z_1. If p is an isolated point of Z, then its complement $U = Z - \{p\}$ in Z is closed. If so, then its complement $Z_1 - \{p\}$ in Z_1, which is the intersection $U \cap Z_1$, will be closed in Z_1, and therefore p will be an isolated point of Z_1. So it suffices to prove the proposition in the case that f is irreducible. Let p be a point of Z. We adjust coordinates and scale, so that p becomes the origin $(0, ..., 0)$ and f becomes monic in x_n. We relabel x_n as y, and write f as a polynomial in y:

$$\widetilde{f}(y) = f(x_1, ..., x_{n-1}, y) = y^d + c_{d-1}(x)y^{d-1} + \cdots + c_0(x)$$

where c_i is a polynomial in $x_1, ..., x_{n-1}$. Since f is irreducible, $c_0(x) \neq 0$. Since p is the origin and $f(p) = 0$, $c_0(0) = 0$. So $c_0(x)$, which is the product of the roots of $\widetilde{f}(y)$, will tend to zero with x. When $c_0(x)$ is small, at least one root of $\widetilde{f}$ will be small. So there are points of Z distinct from p, but arbitrarily close to p. □

1.3.21. Corollary. *Let C' be the complement of a finite set of points in a plane curve C. In the classical topology, a continuous function g on C that is zero at every point of C' is identically zero.* □ function-iszero

1.4 Tangent Lines

tanlines

(1.4.1) a notation for working locally

standnot

We will often want to inspect a plane projective curve $C : \{f(x_0, x_1, x_2) = 0\}$ in a neighborhood of a particular point p. To do this we may adjust coordinates so that p becomes the point $(1, 0, 0)$, and work with points $(1, x_1, x_2)$ in the standard open set $\mathbb{U}^0 : \{x_0 \neq 0\}$. When we identify $\mathbb{U}^0$ with the affine x_1, x_2-plane, p becomes the origin $(0, 0)$ and C becomes the zero locus of the nonhomogeneous polynomial $f(1, x_1, x_2)$. The loci $f(x_0, x_1, x_2) = 0$ and $f(1, x_1, x_2) = 0$ are the same on the subset $\mathbb{U}^0$.

This will be a standard notation for working locally. Of course, it doesn't matter which variable we set to 1. If the variables are x, y, z, we may prefer to take for p the point $(0, 0, 1)$ and work with the polynomial $f(x, y, 1)$.

(1.4.2) homogenizing and dehomogenizing

homde-homone

Let $f(x_0, x_1, ..., x_n)$ be a homogeneous polynomial. The polynomial $f(1, x_1, ..., x_n)$ is called the *dehomogenization* of f, with respect to the variable x_0. A simple procedure, *homogenization*, inverts this dehomogenization. Suppose given a nonhomogeneous polynomial $F(x_1, x_2)$ of degree d. To homogenize F, we replace the variables x_i, $i = 1, ..., n$, by $u_i = x_i/x_0$. Then since u_i have degree zero in x, so does $F(u_1, ..., u_n)$. When we multiply by x_0^d, the result will be a homogeneous polynomial of degree d in $x_0, ..., x_n$, that isn't divisible by x_0.

For example, let $F(x_1, x_2) = 1 + x_1 + x_2^2$. Then $x_0^2 F[u_1, u_2] = x_0^2 + x_0x_1 + x_2^2$.

1.4.3. Lemma. *A homogeneous polynomial $f(x_0, x_1, x_2)$ that isn't divisible by x_0 is irreducible if and only if $f(1, x_1, x_2)$ is irreducible.* □ foneirred

(1.4.4) smooth points and singular points

smsingpts

Let C be the plane curve defined by an irreducible homogeneous polynomial $f(x_0,x_1,x_2)$, and let f_i denote the partial derivative $\frac{\partial f}{\partial x_i}$, computed by the usual calculus formula. A point of C at which the partial derivatives f_i aren't all zero is a *smooth point* of C. A point at which all partial derivatives are zero is a *singular point*. A curve is *smooth*, or *nonsingular*, if it contains no singular point. Otherwise, it is a *singular curve*.

fermatcurve

The *Fermat curve*

(1.4.5) $$x_0^d + x_1^d + x_2^d = 0$$

is smooth because the only common zero of the partial derivatives $dx_0^{d-1}, dx_1^{d-1}, dx_2^{d-1}$, which is $(0,0,0)$, doesn't represent a point of $\mathbb{P}^2$. The cubic curve $x_0^3 + x_1^3 - x_0x_1x_2 = 0$ is singular at the point $(0,0,1)$.

The Implicit Function Theorem explains the meaning of smoothness. Suppose that $p = (1,0,0)$ is a point of C. We set $x_0 = 1$ and inspect the locus $f(1,x_1,x_2) = 0$ in the standard open set $\mathbb{U}^0$. If $f_2 = \frac{\partial f}{\partial x_2}$ isn't zero at p, the Implicit Function Theorem tells us that we can solve the equation $f(1,x_1,x_2) = 0$ for x_2 locally (for small x_1), as an analytic function φ of x_1, with $\varphi(0) = 0$, and then $f(1,x_1,\varphi(x_1))$ will be zero. (See (1.4.18) below.) Sending x_1 to $(1,x_1,\varphi(x_1))$ inverts the projection from C to the affine x_1-line locally. So at a smooth point, C is locally homeomorphic to the affine line.

eulerformula

1.4.6. Euler's Formula. If f is a homogeneous polynomial of degree d in the variables $x_0, ..., x_n$, then

$$\sum_i x_i \tfrac{\partial f}{\partial x_i} = d\, f$$

It suffices to check the formula when f is a monomial. You will be able to do this. For instance, if the variables are x,y,z, and $f = x^2y^3z$, then

$$xf_x + yf_y + zf_z = x(2xy^3z) + y(3x^2y^2z) + z(x^2y^3) = 6x^2y^3z = 6\, f \qquad \square$$

singpointoncurve

1.4.7. Corollary. **(i)** *If all partial derivatives of an irreducible homogeneous polynomial $f(x_0,x_1,x_2)$ are zero at a point p of $\mathbb{P}^2$, then f is zero at p, and therefore p is a point, a singular point, of the curve $\{f = 0\}$.*

(ii) *At a smooth point of the plane curve defined by an irreducible homogeneous polynomial f, at least two partial derivatives of f will be nonzero.*

(iii) *At a smooth point of the curve $\{f = 0\}$, the dehomogenization $f(1,u_1,u_2)$ will have a nonvanishing partial derivative.*

(iv) *The partial derivatives of an irreducible polynomial have no common (nonconstant) factor.*

(v) *A plane curve has finitely many singular points.* $\square$

tangent

(1.4.8) tangent lines and flex points

Let C be the plane projective curve defined by an irreducible homogeneous polynomial $f(x_0,x_1,x_2)$. A line L is *tangent* to C at a smooth point p if the intersection multiplicity of C and L at p is at least 2. (See (1.3.9).) A smooth point p of C is a *flex point* if the intersection multiplicity of C and its tangent line at p is at least 3, and p is an *ordinary* flex point if that intersection multiplicity is equal to 3.

Let L be a line through a point p and let q be a point of L distinct from p. We represent p and q by specific vectors (p_0,p_1,p_2) and (q_0,q_1,q_2), to write a variable point of L as $p + tq$, and we expand the restriction of f to L in a Taylor's series. The Taylor expansion carries over to complex polynomials because it is an identity. Let $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Taylor's formula is

taylor

(1.4.9) $$f(p+tq) \;=\; f(p) \;+\; \Big(\sum_i f_i(p)\, q_i\Big)t \;+\; \tfrac{1}{2}\Big(\sum_{i,j} q_i\, f_{ij}(p)\, q_j\Big)t^2 \;+\; O(3)$$

where the symbol $O(3)$ stands for a polynomial in which all terms have degree at least 3 in t. The point q is missing from this parametrization, but this won't be important.

The intersection multiplicity of C and L at a point p was defined in (**1.3.8**). It is equal to the lowest power of t that has nonzero coefficient in $f(p+tq)$. The point p lies on the curve C if $f(p)=0$. If so, and if p is a smooth point of C, then the line L with the parametrization $p+tq$ will be a tangent line to C at p, provided that the coefficient $\sum_i f_i(p)\,q_i$ of t is zero. If p is a smooth point and L is a tangent line, then p is a flex point if in addition, $\sum_{i,j} q_i f_{ij}(p)\,q_j$ is zero.

One can write the equation (1.4.9) in terms of the *gradient vector* $\nabla = (f_0, f_1, f_2)$ and the *Hessian matrix* H of f. The Hessian is the matrix of second partial derivatives f_{ij}:

(1.4.10)
$$H \;=\; \begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix}$$
hessian-matrix

Let ∇_p and H_p denote the evaluations of ∇ and H, respectively, at p.

When we regard p and q as the column vectors $(p_0,p_1,p_2)^t$ and $(q_0,q_1,q_2)^t$, Equation 1.4.9 can be written as

(1.4.11)
$$f(p+tq) = \; f(p) \;+\; \big(\nabla_p\, q\big)t \;+\; \tfrac{1}{2}(q^tH_p\,q)t^2 \;+\; O(3)$$
texp

in which q^t is the transpose of the column vector q, a row vector, and where $\nabla_p q$ and q^tH_pq are computed as matrix products.

So p is a smooth point of C if $f(p)=0$ and $\nabla_p \neq 0$. If p is a smooth point, then L is tangent to C at p when $\nabla_p\, q$ is zero, and p is a flex point when $\nabla_p\, q$ and $q^tH_p\,q$ are both zero.

The equation of the tangent line L at a smooth point p of C is $\nabla_p\, x = 0$, or

(1.4.12)
$$f_0(p)x_0 + f_1(p)x_1 + f_2(p)x_2 = 0$$
tanlineeq

The point q lies on the tangent line L if the coefficient of t in (1.4.11) is zero. So a line L is a tangent line at a smooth point p if it is orthogonal to the gradient ∇_p. There is a unique tangent line at a smooth point.

Note. Taylor's formula shows that the restriction of f to any line through a singular point has a multiple zero. However, we will speak of tangent lines only at smooth points of the curve.

1.4.13. Lemma. $\nabla_p\, p = d\,f(p)$ *and* $p^tH_p = (d-1)\nabla_p$. *applyeuler*

This lemma is obtained by applying Euler's Formula to the entries of ∇_p and H_p. □

We rewrite Equation 1.4.9 one more time, using the notation $\langle u,v\rangle$ to represent the symmetric bilinear form $u^tH_p\,v$ on the complex 3-dimenional vector space. It makes sense to say that this form vanishes on a pair of points of $\mathbb{P}^2$, because the condition $\langle u,v\rangle = 0$ doesn't change when u or v is multiplied by a nonzero scalar λ.

1.4.14. Proposition. *With notation as above,* *linewith-form*
(i) *Equation (1.4.9) can be written as*

$$f(p+tq) \;=\; \tfrac{1}{d(d-1)}\langle p,p\rangle \;+\; \tfrac{1}{d-1}\langle p,q\rangle t \;+\; \tfrac{1}{2}\langle q,q\rangle t^2 \;+\; O(3)$$

(ii) *A point p is a smooth point of C if and only if $\langle p,p\rangle = 0$ but $\langle p,v\rangle$ is not identically zero.*

proof. **(i)** This is obtained by applying Lemma 1.4.13 to (1.4.11).

(ii) $\langle p,v\rangle = \nabla_p v/(d-1)$ is identically zero if and only if $\nabla_p = 0$. □

1.4.15. Corollary. *Let p be a smooth point of C, let q be a point of $\mathbb{P}^2$ distinct from p, and let L be the line through p and q. Then* *bilinform*
(i) *L is tangent to C at p if and only if $\langle p,p\rangle = \langle p,q\rangle = 0$, and*
(ii) *p is a flex point of C with tangent line L if and only if $\langle p,p\rangle = \langle p,q\rangle = \langle q,q\rangle = 0$.* □

1.4.16. Theorem. *A smooth point p of the curve C is a flex point if and only if the Hessian determinant* $\det H_p$ *at p is zero.* *tangent-line* *proof.* Let p be a smooth point of C. If $\det H_p = 0$, the form $\langle u, v\rangle$ is degenerate, and there is a nonzero null vector q. Then $\langle p, q\rangle = \langle q, q\rangle = 0$. But p isn't a null vector, because $\langle p, v\rangle$ isn't identically zero at a smooth point. So q is distinct from p. Therefore p is a flex point.

Conversely, suppose that p is a flex point and let q be a point on the tangent line at p and distinct from p, so that $\langle p, p\rangle = \langle p, q\rangle = \langle q, q\rangle = 0$. The restriction of the form to the two-dimensional space spanned by p and q is zero, and this implies that the form is degenerate. If (p, q, v) is a basis of V, the matrix of the form will look like this:

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$$

□

hess-notzero **1.4.17. Proposition.**

(i) *Let $f(x, y, z)$ be an irreducible homogeneous polynomial of degree at least two. The Hessian determinant $\det H$ isn't divisible by f. In particular, the Hessian determinant isn't identically zero.*

(ii) *A plane curve that isn't a line has finitely many flex points.*

proof. **(i)** Let C be the plane curve defined by f. If f divides the Hessian determinant, every smooth point of C will be a flex point. We set $z = 1$ and look on the standard open set $\mathbb{U}^2$, choosing coordinates so that the origin p is a smooth point of C, and so that $\frac{\partial f}{\partial y} \neq 0$ at p. The Implicit Function Theorem tells us that we can solve the equation $f(x, y, 1) = 0$ for y locally, say $y = \varphi(x)$, where φ is an analytic function. The graph $\Gamma : \{y = \varphi(x)\}$ will be equal to C in a neighborhood of p. (See the review below.) A point of Γ is a flex point if and only if $\frac{d^2\varphi}{dx^2}$ is zero there. If this is true for all points near to p, then $\frac{d^2\varphi}{dx^2}$ will be identically zero, which implies that φ is linear, and since $\varphi(0) = 0$, that $\varphi(y)$ has the form ax. Then $y = ax$ solves $f = 0$, and therefore $y - ax$ divides $f(x, y, 1)$. But $f(x, y, z)$ is irreducible, and so is $f(x, y, 1)$. So $f(x, y, 1)$ and $f(x, y, z)$ are linear, contrary to hypothesis.

(ii) This follows from **(i)** and (1.3.12). The irreducible polynomial f and the Hessian determinant $\det H$ have finitely many common zeros. □

ifthm **1.4.18. Review.** (*about the Implicit Function Theorem*)

By *analytic function* $\varphi(x_1, ..., x_k)$, in one or more variables, we mean a complex-valued function, defined for small x, which can be represented as a power series that converges when x is small.

If $f(x, y)$ is a polynomial of two variables such that $f(0, 0) = 0$ and $\frac{df}{dy}(0, 0) \neq 0$, the Implicit Function Theorem asserts that there is a unique analytic function $\varphi(x)$ such that $\varphi(0) = 0$ and $f(x, \varphi(x))$ is identically zero.

Let $\mathcal{R}$ be the ring af analytic functions in x. In the ring $\mathcal{R}[y]$ of polynomials with coefficients in $\mathcal{R}$, the polynomial $y - \varphi(x)$ divides $f(x, y)$. To see this, we do division with remainder of f by the monic polynoial $y - \varphi(x)$ in y:

divrem (1.4.19)
$$f(x, y) = (y - \varphi(x))q(x, y) + r(x)$$

The quotient q and remainder r are in $\mathcal{R}[y]$, and $r(x)$ has degree zero in y, so it is in $\mathcal{R}$. Setting $y = \varphi(x)$ in the equation, one sees that $r(x) = 0$.

Let Γ be the graph of φ in a suitable neighborhood U of the origin in x, y-space. Since $f(x, y) = (y - \varphi(x))q(x, y)$, the locus $f(x, y) = 0$ in U has the form $\Gamma \cup \Delta$, where Γ is the zero locus of $y - \varphi(x)$ and Δ is the zero locus of $q(x, y)$. Differentiating, we find that $\frac{\partial f}{\partial y}(0, 0) = q(0, 0)$. So $q(0, 0) \neq 0$. Then Δ doesn't contain the origin, while Γ does. This implies that Δ is disjoint from Γ, locally. A sufficiently small neighborhood U of the origin won't contain any points of Δ. In such a neighborhood, the locus of zeros of f will be Γ.

If $\frac{\partial f}{\partial x}(0, 0)$ is also nonzero, one can also solve for x as an analytic function $\psi(y)$ of y. Then $\psi(y)$ will be a local inverse function of φ. □

1.5 Transcendence Degree

transcdeg

A domain that contains another domain R as a subring will be called an R-*algebra*. Domains that contain the complex numbers, $\mathbb{C}$-algebras, will occur frequently, so we refer to them simply as *algebras*.

If F is a field, we use the customary notation $F[\alpha_1, ..., \alpha_n]$ or $F[\alpha]$ for the F-algebra generated by a set of elements $\alpha = \{\alpha_1, ..., \alpha_n\}$, and we may denote the field of fractions of $F[\alpha_1, ..., \alpha_n]$ by $F(\alpha_1, ..., \alpha_n)$ or by $F(\alpha)$.

Let $F \subset K$ be a field extension. A set $\alpha = \{\alpha_1, ..., \alpha_n\}$ of elements of K is *algebraically dependent* over F if there is a nonzero polynomial $f(x_1, ..., x_n)$ with coefficients in F, such that $f(\alpha) = 0$. If there is no such polynomial, the set α is *algebraically independent* over F.

A set $\{\alpha_1, ..., \alpha_n\}$ is algebraically independent over F if and only if the surjective map from the polynomial algebra $F[x_1, ..., x_n]$ to $F[\alpha_1, ..., \alpha_n]$ that sends x_i to α_i is bijective. If so, we may say that $F[\alpha_1, ..., \alpha_n]$ is a polynomial algebra.

An infinite set is called algebraically independent over F if every finite subset is algebraically independent over F — if there is no polynomial relation among any finite set of its elements.

The set $\{\alpha_1\}$ consisting of a single element of K is algebraically dependent if α_1 is algebraic over F. Otherwise, it is algebraically independent, and then α_1 is said to be *transcendental* over F.

A *transcendence basis* for K over F is a finite algebraically independent set $\alpha = \{\alpha_1, ..., \alpha_n\}$ that isn't contained in a larger algebraically independent set. If there is a transcendence basis, its order is the *transcendence degree* of the field extension K. Proposition 1.5.3 below shows that all transcendence bases for K over F have the same order. If there is no (finite) transcendence basis, the transcendence degree of K over F is said to be infinite.

For example, when $K = F(x_1, ..., x_n)$ is the field of rational functions in n variables, the variables form a transcendence basis of K over F, and the transcendence degree of K over F is n. The elementary symmetric functions $s_1 = x_1 + \cdots + x_n\ , \ \ldots,\ s_n = x_1 \cdots x_n$ also form a transcendence basis.

An element a of a ring R is a *zero divisor* if there is a nonzero element b of R such that the product ab is zero. A *domain* is a nonzero ring with no zero divisors.

1.5.1. Proposition. *Let F be a field, let A be domain that is an F-algebra, and let K be its field of fractions. If K has transcendence degree n over F, then every algebaically independent set of elements of A is contained in an algebraically independent set of order n.* □ trdegalg

1.5.2. Proposition. *Let K/F be a field extension, let $\alpha = \{\alpha_1, ..., \alpha_n\}$ be a set of elements of K that is algebraically independent over F, and let $F(\alpha)$ be the field of fractions of the F-algebra that is generated by α.* algindtrivialities

(i) *Let β be another element of K. The set $\{\alpha_1, ..., \alpha_n, \beta\}$ is algebraically dependent if and only if β is algebraic over $F(\alpha)$.*

(ii) *The algebraically independent set α is a transcendence basis if and only if every element of K is algebraic over $F(\alpha)$.* □

1.5.3. Proposition. *Let K/F be a field extension. If K has a finite transcendence basis, then all algebraically independent subsets of K are finite, and all transcendence bases have the same order.* trdeg

proof. Let $\alpha = \{\alpha_1, ..., \alpha_r\}$ and $\beta = \{\beta_1, ..., \beta_s\}$ be subsets of K. Assume that K is algebraic over $F(\alpha)$ and that the set β is algebraically independent over F. We show that $s \le r$. The fact that all transcendence bases have the same order will follow: If both α and β are transcendence bases, then we can interchange α and β, so $r \le s$.

The proof that $s \le r$ proceeds by reducing to the trivial case that β is a subset of α. Suppose that some element of β, say β_s, isn't in the set α. The set $\beta' = \{\beta_1, ..., \beta_{s-1}\}$ is algebraically independent, but it isn't a transcendence basis. So K isn't algebraic over $F(\beta')$. Since K is algebraic over $F(\alpha)$, there is at least one element of α, say α_r, that isn't algebraic over $F(\beta')$. Then $\gamma = \beta' \cup \{\alpha_r\}$ will be an algebraically independent set of order s that contains more elements of the set α than β does. Induction shows that $s \le r$. □

1.5.4. Corollary. *Let $L \supset K \supset F$ be fields. If the degree $[L:K]$ of the field extension L/K is finite, then L and K have the same transcendence degree over F.* sametrdeg

This follows from Proposition 1.5.2. □

1.6 The Dual Curve

dualcurve

dualplanesect

(1.6.1) the dual plane

Let $\mathbb{P}$ denote the projective plane with coordinates x_0, x_1, x_2, let s_0, s_1, s_2 be scalars, not all zero, and let L be the line in $\mathbb{P}$ with the equation

lineequation

$$s_0x_0 + s_1x_1 + s_2x_2 = 0 \tag{1.6.2}$$

The solutions (x_0, x_1, x_2) of this equation, the points of L, are unchanged when we multiply (s_0, s_1, s_2) by a nonzero scalar λ. They determine the coefficients s_i up to a common nonzero factor. So L determines a point (s_0, s_1, s_2) in another projective plane $\mathbb{P}^*$ called the *dual plane*. We denote the point (s_0, s_1, s_2) of $\mathbb{P}^*$ by L^*. Moreover, a point $p = (x_0, x_1, x_2)$ in $\mathbb{P}$ determines a line in the dual plane, the line with the equation (1.6.2), when s_i are regarded as the variables and x_i as the scalar coefficients. We denote that line by p^*. The equation exhibits a duality between $\mathbb{P}$ and $\mathbb{P}^*$. A point p of $\mathbb{P}$ lies on the line L if and only if the equation is satisfied, and this means that, in $\mathbb{P}^*$, the point L^* lies on the line p^*.

As this duality shows, the dual $\mathbb{P}^{**}$ of the dual plane $\mathbb{P}^*$ is the plane $\mathbb{P}$.

dualcurvetwo

(1.6.3) the dual curve

Let C be the plane projective curve defined by an irreducible homogeneous polyomial f of degree at least two, and let U be the set of its smooth points. Corollary 1.4.7 tells us that U is the complement of a finite set in C. We define a map

$$U \xrightarrow{t} \mathbb{P}^*$$

as follows: Let p be a point of U and let L be the tangent line to C at p. The definition of the map is $t(p) = L^*$, where L^* is the point of $\mathbb{P}^*$ that corresponds to L. Thus the image $t(U)$ is the locus of the tangent lines at the smooth points of C. We assume that f has degree at least two because, if C were a line, the image $t(U)$ of U would be a point.

Let $\nabla f = (f_0, f_1, f_2)$ denote the gradient of f, with a $f_i = \frac{\partial f}{\partial x_i}$ as before. The tangent line L at a smooth point $p = (x_0, x_1, x_2)$ of C has the equation $f_0x_0 + f_1x_1 + f_2x_2 = 0$. Therefore $L^* = t(p)$ is the point

ellstarequation

$$(s_0, s_1, s_2) \sim \big(f_0(x), f_1(x), f_2(x)\big) = \nabla f(x) \tag{1.6.4}$$

phizerogzero

1.6.5. Lemma. *Let $\varphi(s_0, s_1, s_2)$ be a homogeneous polynomial of degree r, and let $g(x_0, x_1, x_2) = \varphi(\nabla f(x))$. Then $\varphi(s)$ is identically zero on the image $t(U)$ of the set U of smooth points if and only if $g(x)$ is identically zero on U, and this is true if and only if f divides g.*

proof. The point $s = (s_0, s_1, s_2)$ is in $t(U)$ if for some x in U and some $\lambda \neq 0$, $\nabla f(x) = \lambda s$. Then $g(x) = \varphi(\nabla f(x)) = \varphi(\lambda s) = \lambda^r \varphi(s)$. So $g(x) = 0$ if and only if $\varphi(s) = 0$. □

dualcurvethm

1.6.6. Theorem. *Let C be the plane curve defined by an irreducible homogeneous polynomial f of degree at least two. With notation as above, the image $t(U)$ is contained in a curve C^* in the dual plane $\mathbb{P}^*$.*

The curve C^* referred to in the theorem is the *dual curve*.

proof of Theorem 1.6.6. If an irreducible homogeneous polynomial $\varphi(s)$ vanishes on $t(U)$, it will be unique up to scalar factor (Corollary 1.3.16). We show first that there is a nonzero polynomial $\varphi(s)$, not necessarily irreducible or homogeneous, that vanishes on $t(U)$. The field $\mathbb{C}(x_0, x_1, x_2)$ has transcendence degree three over $\mathbb{C}$. Therefore the four polynomials f_0, f_1, f_2, and f are algebraically dependent. There is a nonzero polynomial $\psi(s_0, s_1, s_2, t)$ such that $\psi(f_0(x), f_1(x), f_2(x), f(x))$ is the zero polynomial. We can cancel factors of t, so we may assume that ψ isn't divisible by t. Let $\varphi(s) = \psi(s_0, s_1, s_2, 0)$. When t doesn't divide ψ, this isn't the zero polynomial. If a vector $x = (x_1, x_2, x_3)$ represents a point of U, then $f(x) = 0$, and therefore

$$\psi(f_0(x), f_1(x), f_2(x), f(x)) = \psi(\nabla f(x), 0) = \varphi(\nabla f(x))$$

Since the left side of this equation is identically zero, $\varphi(\nabla f(x)) = 0$ for every x that represents a point of U.

Next, say that f has degree d. Then the partial derivatives f_i have degree d–1. Therefore $\nabla f(\lambda x) = \lambda^{d-1}\nabla f(x)$ for all λ, and because the vectors x and λx represent the same point of U, $\varphi(\nabla f(\lambda x)) = \varphi(\lambda^{d-1}\nabla f(x))) = 0$ for all λ, when x is in U. Writing $\nabla f(x) = s$, $\varphi(\lambda^{d-1}s) = 0$ for all λ when x is in U. Since λ^{d-1} can be any complex number, Lemma 1.3.2 tells us that the homogeneous parts of $\varphi(s)$ vanish at s, when $s = \nabla f(x)$ and x is in U. So the homogeneous parts of $\varphi(s)$ vanish on $t(U)$. This shows that there is a nozero, homogeneous polynomial $\varphi(s)$ that vanishes on $t(U)$. We choose such a polynomial $\varphi(s)$. Let its degree be r.

Let $g(x) = \varphi(\nabla f(x))$. If f has degree d, then g will be homogeneous, of degree $r(d-1)$. It will vanish on U, and therefore on C (1.3.21). So f will divide g. If $\varphi(s)$ factors, then $g(x)$ factors accordingly, and because f is irreducible, it will divide one of the factors of g. The corresponding factor of φ will vanish on $t(U)$ (1.6.5). So we may replace the homogeneous polynomial φ by one of its irreducible factors. □

In principle, the proof of Theorem 1.6.6 gives a method for finding a polynomial that vanishes on the dual curve. That method is to find a polynomial relation among f_x, f_y, f_z, f, and set $f = 0$. But it is usually painful to determine the defining polynomial of C^* explicitly. Most often, the degrees of C and C^* will be different. Moreover, several points of the dual curve C^* may correspond to a singular point of C, and vice versa.

We give two examples in which the computation is easy.

1.6.7. Examples. exampled-ualone

(i) (*the dual of a conic*) Let $f = x_0x_1 + x_0x_2 + x_1x_2$ and let C be the conic $f = 0$. Let $(s_0, s_1, s_2) = (f_0, f_1, f_2) = (x_1{+}x_2,\ x_0{+}x_2,\ x_0{+}x_1)$. Then

(1.6.8) $$s_0^2 + s_1^2 + s_2^2 - 2(x_0^2 + x_1^2 + x_2^2) = 2f \qquad \text{and} \qquad s_0s_1 + s_1s_2 + s_0s_2 - (x_0^2 + x_1^2 + x_2^2) = 3f$$ exampled-ualtwo

We eliminate $(x_0^2 + x_1^2 + x_2^2)$ from the two equations:

(1.6.9) $$(s_0^2 + s_1^2 + s_2^2) - 2(s_0s_1 + s_1s_2 + s_0s_2) = -4f$$ equationd-ualthree

Setting $f = 0$ gives us the equation of the dual curve. It is another conic.

(ii) (*the dual of a cuspidal cubic*) The dual of a smooth cubic is a curve of degree 6. It is too much work to compute that dual here. We compute the dual of a singular cubic instead. The curve C defined by the irreducible polynomial $f = y^2z + x^3$ has a singularity, a cusp. at the point $(0, 0, 1)$. The Hessian matrix of f is

$$H = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

and the Hessian determinant $\det H$ is $h = -24xy^2$. The common zeros of f and h are the singular point $(0, 0, 1)$, and a single flex point $(0, 1, 0)$.

We scale the partial derivatives of f to simplify notation. Let $u = f_x/3 = x^2$, $v = f_y/2 = yz$, and $w = f_z = y^2$. Then

$$v^2w - u^3 = y^4z^2 - x^6 = (y^2z + x^3)(y^2z - x^3) = f(y^2z - x^3)$$

The zero locus of the irreducible polynomial $v^2w - u^3$ is the dual curve, another singular cubic. □

(1.6.10) a local equation for the dual curve equa-tionofcstar

We label the coordinates in $\mathbb{P}$ and $\mathbb{P}^*$ as x, y, z and u, v, w, respectively, and we work in a neighborhood of a smooth point p_0 of the curve C defined by a homogeneous polynomial $f(x, y, z)$. We choose coordinates so that $p_0 = (0, 0, 1)$, and that the tangent line L_0 at p_0 is the line $\{y = 0\}$. The image of p_0 in the dual curve C^* is the point L_0^* at which $(u, v, w) = (0, 1, 0)$.

Let $\widetilde{f}(x, y) = f(x, y, 1)$. In the affine x, y-plane, the point p_0 becomes the origin $(0, 0)$. So $\widetilde{f}(p_0) = 0$, and since the tangent line is L_0, $\frac{\partial \widetilde{f}}{\partial x}(p_0) = 0$, while $\frac{\partial \widetilde{f}}{\partial y}(p_0) \neq 0$. We can solve the equation $\widetilde{f} = 0$ for y as an analytic function $y(x)$, with $y(0) = 0$. Let $y'(x)$ denote the derivative $\frac{dy}{dx}$. Differentiating the equation $f(x, y(x)) = 0$ shows that $y'(0) = 0$.

Let $\widetilde{p}_1 = (x_1, y_1)$ be a point of C_0 near to $\widetilde{p}_0$, so that $y_1 = y(x_1)$, and let $y_1' = y'(x_1)$. The tangent line L_1 at $\widetilde{p}_1$ has the equation

localtangent (1.6.11)
$$y - y_1 = y_1'(x - x_1)$$

Putting z back, the homogeneous equation of the tangent line L_1 at the point $p_1 = (x_1, y_1, 1)$ is

$$-y_1'x + y + (y_1'x_1 - y_1)z = 0$$

The point L_1^* of the dual plane that corresponds to L_1 is

projlocaltangent (1.6.12)
$$(u_1, v_1, w_1) = (-y_1', 1, y_1'x_1 - y_1)$$

bidualone **(1.6.13) the bidual**

The *bidual* C^{**} of C is the dual of the curve C^*. It is a curve in the space $\mathbb{P}^{**}$, which is $\mathbb{P}$.

bidualC **1.6.14. Theorem.** *A plane curve C of degree greater than one is equal to its bidual C^{**}.*

We use the following notation for the proof:

- U is the set of smooth points of the curve C, and U^* is the set of smooth points of the dual curve C^*.
- $U^* \xrightarrow{t^*} \mathbb{P}^{**} = \mathbb{P}$ is the map analogous to the map $U \xrightarrow{t} \mathbb{P}^*$.
- V is the set of points p of C such that p is a smooth point of C and also $t(p)$ is a smooth point of C^*, and V^* is the image tV.

Then $V \subset U \subset C$ and $V^* \subset U^* \subset C^*$.

vopen **1.6.15. Lemma.**
(i) *V is the complement of a finite set in C.*
(ii) *Let p_1 be a point near to a smooth point p of a curve C, let L_1 and L be the tangent lines to C at p_1 and p, respectively, and let q be intersection point $L_1 \cap L$. Then $\lim_{p_1 \to p} q = p$.*
(iii) *If L is the tangent line to C at a point p of V, then p^* is the tangent line to C^* at the point L^*, and $t^*(L^*) = p$.*
(iv) *V^* is the complement of a finite set in C^*, and the map $V \xrightarrow{t} V^*$ is bijective.*

The points and lines that appear in **(ii)** are displayed in the figure below.
proof. **(i)** Let S and S^* denote the finite sets of singular points of C and C^*, respectively. The set V is obtained from C by deleting points of S and points in the inverse image of S^*. The fibre of the map $U \xrightarrow{t} \mathbb{P}^*$ over a point L^* of C^* is the set of smooth points of C whose tangent line is L. Since L meets C in finitely many points, the fibre is finite. So the inverse image of the finite set S^* is finite.

(ii) We work analytically in a neighborhood of p, choosing coordinates so that $p = (0, 0, 1)$ and that L is the line $\{y = 0\}$. Let $(x_q, y_q, 1)$ be the coordinates of the point q. Since q is a point of L, $y_q = 0$. The coordinate x_q can be obtained by substituting $x = x_q$ and $y = 0$ into the local equation (1.6.11) for L_1: $x_q = x_1 - y_1/y_1'$.

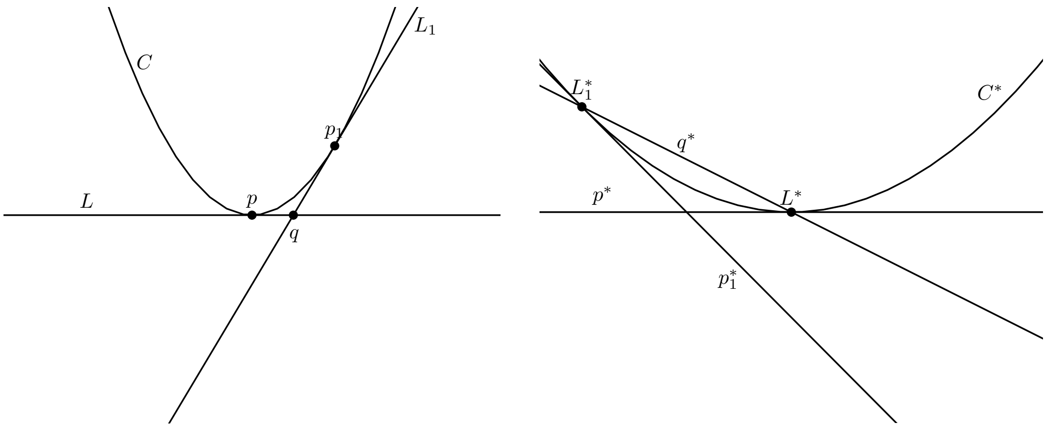
Now, when a function has an nth order zero at the point $x = 0$, i.e, when it has the form $y = x^n h(x)$, where $n > 0$ and $h(0) \neq 0$, the order of zero of its derivative at that point is $n-1$. This is verified by differentiating $x^n h(x)$. Since the function $y(x)$ has a zero of positive order at p, $\lim_{p_1 \to p} y_1/y_1' = 0$. We also have $\lim_{p_1 \to p} x_1 = 0$. Therefore $\lim_{p_1 \to p} x_q = 0$, and $\lim_{p_1 \to p} q = \lim_{p_1 \to p} (x_q, y_q, 1) = (0, 0, 1) = p$.

(iii) Let p_1 be a point of C near to p, and let L_1 be the tangent line to C at p_1. The image L_1^* of p_1 is the point $(f_0(p_1), f_1(p_1), f_2(p_1))$ of C^*. Because the partial derivatives f_i are continuous,

$$\lim_{p_1 \to p} L_1^* = (f_0(p), f_1(p), f_2(p)) = L^*$$

With $q = L \cap L_1$ as above, q^* is the line through the points L^* and L_1^*. As p_1 approaches p, L_1^* approaches L^*, and therefore q^* approaches the tangent line to C^* at L^*. On the other hand, it follows from **(ii)** that q^* approaches p^*. Therefore the tangent line to C^* at L^* is p^*. By definition, $t^*(L^*)$ is the point of C that corresponds to the tangent line p^* at L^*. So $t^*(L^*) = p^{**} = p$. □

1.6.16. curveand-dual



A Curve and its Dual

In this figure, the curve C on the left is the parabola $y = x^2$. We used the local equation (1.6.11) to obtain a local equation $u^2 = 4w$ of the dual curve C^*.

proof of theorem 1.6.14. Let p be a point of V, and let L be the tangent line at p. The map t^* is defined at L^*), and $t^*(L^*) = p$. Since $L^* = t(p)$, $t^*t(p) = p$. It follows that the restriction of t to V is injective, and that it defines a bijective map from V to its image V^*, whose inverse function is t^*. So V is contained in the bidual C^{**}. Since V is dense in C and since C^{**} is a closed set, C is contained in C^{**}. Since C and C^{**} are curves, $C = C^{**}$. □

1.6.17. Corollary. **(i)** *Let U be the set of smooth points of a plane curve C, and let t denote the map from U to the dual curve C^*. The image $t(U)$ of U is the complement of a finite subset of C^*.* *cstarisim-age*

(ii) *If C is a smooth curve, the map $C \xrightarrow{t} C^*$, is defined at all points of C, and it is a surjective map.*

proof. **(i)** With U, U^*, and V as above, $V = t^*t(V) \subset t^*(U^*) \subset C^{**} = C$. Since V is the complement of a finite subset of C, so is $t^*(U^*)$. The assertion to be proved follows when we interchange C and C^*.

(ii) The map t is continuous, so its image $t(C)$ is a compact subset of C^*, and by **(i)**, its complement S is a finite set. Therefore S is both open and closed. It consists of isolated points of C^*. Since a plane curve has no isolated point, S is empty. □

1.6.18. Corollary. *Let C be a smooth curve, and suppose that the tangent line L at a point p of C isn't tangent to C at another point, i.e., that L isn't a bitangent. Then the path defined by the local equation (1.6.12) traces out the dual curve C^* near to $L^* = (0, 1, 0)$.* *notbitan-gent*

proof. Let D be an open neighborhood of p in C in the classical topology, such that the equation (1.6.12) describes the point L_1^* when p_1 is in D. The complement of D in C is compact, and so is its image tZ. If $L^* = t(p)$ isn't in tZ, then p has a neighborhood U whose image is disjoint from tZ. In that neighborhood, the local equation traces out the dual curve.

The reasoning breaks down when C is singular, because the locus of smooth points won't be compact. □

1.7 Resultants and Discriminants

resultant

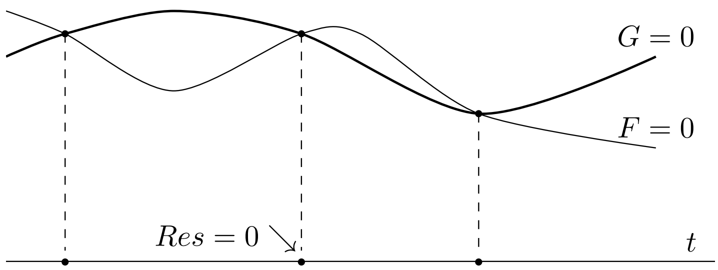
Let

polys (1.7.1) $$F(x) = x^m + a_1 x^{m-1} + \cdots + a_m \quad \text{and} \quad G(x) = x^n + b_1 x^{n-1} + \cdots + b_n$$

be monic polynomials. The *resultant* $\mathrm{Res}(F, G)$ of F and G is a certain polynomial in the undetermined coefficients a_i, b_j. Its important property is that, when the coefficients are given values in a field, the resultant is zero if and only if F and G have a common factor.

For instance, suppose that $F(x) = x + a_1$ and $G(x) = x^2 + b_1 x + b_2$. The root $-a_1$ of F is a root of G if $G(a_1) = a_1^2 - b_1 a_1 + b_2$ is zero. The resultant of F and G is $a_1^2 - b_1 a_1 + b_2$.

restwovar **1.7.2. Example.** Suppose that the coefficients a_i and b_j in (1.7.1) are polynomials in t, so that F and G become polynomials in two variables. Let C and D be (possibly reducible) curves $F = 0$ and $G = 0$ in the affine plane $\mathbb{A}^2_{t,x}$. The resultant $\mathrm{Res}(F, G)$, computed regarding x as the variable, will be a polynomial in t, whose roots are the t-coordinates of the intersections of C and D.



The analogous statement is true when there are more variables. If F and G are relatively prime polynomials in x, y, z, the loci $C : \{F = 0\}$ and $D : \{G = 0\}$ in $\mathbb{A}^3$ will be surfaces, and the intersection $S = C \cap D$ will be a curve. The resultant $\mathrm{Res}_z(F, G)$, computed regarding z as the variable, is a polynomial in x, y. Its zero locus is the projection of S to the x, y-plane. □

The formula for the resultant is nicest when one allows leading coefficients different from 1. We work with homogeneous polynomials in two variables to prevent the degrees from dropping when a leading coefficient happens to be zero. The common zeros of two homogeneous polynomials $f(x, y)$ and $g(x, y)$ correspond to the common roots of the polynomials $F(x) = f(x, 1)$ and $G(x) = g(x, 1)$, except when the common zero is the point $(0, 1)$ at infinity.

Let

hompolys (1.7.3) $$f(x, y) = a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m \quad \text{and} \quad g(x, y) = b_0 x^n + b_1 x^{n-1} y + \cdots + b_n y^n$$

be homogeneous polynomials in x and y, of degrees m and n, respectively, and with complex coefficients. If these polynomials have a common zero $(x, y) = (u, v)$ in $\mathbb{P}^1_{xy}$, then $vx - uy$ divides both g and f (see (1.3.6)). Then the polynomial $h = fg/(vx - uy)$, which has degree $r = m + n - 1$, will be divisible by f and also by g. Suppose that this is so, and that $h = pf = qg$, where p and q are homogeneous polynomials of degrees $n - 1$ and $m - 1$, respectively. Then p will be a linear combination of the polynomials $x^i y^j$, with $i + j = n - 1$ and q will be a linear combination of the polynomials $x^k y^\ell$, with $k + \ell = m - 1$. The fact that the two combinations pf and qg are equal tells us that the $r + 1$ polynomials

mplus-npolys (1.7.4) $$x^{n-1} f,\ x^{n-2} y f, ..., \ y^{n-1} f \ ;\ x^{m-1} g,\ x^{m-2} y g, ..., \ y^{m-1} g$$

of degree r are (linearly) dependent. For example, if f has degree 3 and g has degree 2, and if f and g have a common zero, then the polynomials

$$\begin{aligned}
xf &= a_0 x^4 + a_1 x^3 y + a_2 x^2 y^2 + a_3 x y^3 \\
yf &= \qquad\quad a_0 x^3 y + a_1 x^2 y^2 + a_2 x y^3 + a_3 y^4 \\
x^2 g &= b_0 x^4 + b_1 x^3 y + b_2 x^2 y^2
\end{aligned}$$

$$\begin{aligned} xyg &= \qquad b_0x^3y + b_1x^2y^2 + b_2xy^3 \\ y^2g &= \qquad\qquad bx^2y^2 + b_1xy^3 + b_2y^4 \end{aligned}$$

will be dependent. Conversely, if the polynomials (1.7.4) are dependent, there will be an equation of the form $pf - qg = 0$, with p of degree $n-1$ and q of degree $m-1$. Then since g has degree n while p has degree $n-1$, at least one zero of g must be a zero of f.

The polynomials (1.7.4) have degree $r = m+n-1$. We form a square $(m+n)\times(m+n)$ matrix $\mathcal{R}$, the *resultant matrix*, whose columns are indexed by the monomials $x^r, x^{r-1}y, ..., y^r$ of degree r, and whose rows list the coefficients of those monomials in the polynomials (1.7.4). The matrix is illustrated below for the cases $m,n = 3,2$ and $m,n = 1,2$, with dots representing entries that are zero:

(1.7.5) $$\mathcal{R} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{pmatrix} \quad \text{or} \quad \mathcal{R} = \begin{pmatrix} a_0 & a_1 & \cdot \\ \cdot & a_0 & a_1 \\ b_0 & b_1 & b_2 \end{pmatrix}$$ resmatrix

The *resultant* of f and g is defined to be the determinant of $\mathcal{R}$.

(1.7.6) $$\mathrm{Res}(f, g) = \det \mathcal{R}$$ resequals-det

In this definition, the coefficients of f and g can be in any ring.

The resultant $\mathrm{Res}(F, G)$ of the monic, one-variable polynomials $F(x) = x^m + a_1x^{m-1} + \cdots + a_m$ and $G(x) = x^n + b_1x^{n-1} + \cdots + b_n$ is the determinant of the matrix obtained from $\mathcal{R}$ by setting $a_0 = b_0 = 1$.

1.7.7. Corollary. *Let f and g be homogeneous polynomials in two variables, or monic polynomials in one variable, of degrees m and n, respectively, and with coefficients in a field. The resultant* $\mathrm{Res}(f, g)$ *is zero if and only if f and g have a common factor. If so, there will be polynomials p and q of degrees $n-1$ and $m-1$ respectively, such that $pf = qg$. If the coefficients are complex numbers, the resultant is zero if and only if f and g have a common zero.* $\square$ *homogre-sult*

When the leading coefficients a_0 and b_0 of f and g are both zero, the point $(1, 0)$ of $\mathbb{P}^1$ will be a zero of f and of g. In that case, one could say that f and g have a common zero at infinity.

1.7.8. Aside. *(the entries of the resultant matrix)* Define $a_i = 0$ when i isn't in the range $0, ..., m$, and $b_i = 0$ when i isn't in the range $0, ..., n$. The resultant matrix has two parts. For rows 1 to n, the (i, j)-entry R_{ij} of $\mathcal{R}$ is the coefficient of $x^{m+n-j}y^{j-1}$ in the polynomial $x^{n-i}y^i f$, which is equal to the coefficient a_{j-i} of $x^{m+i-j}y^{j-i}$ in f. So $R_{ij} = a_{j-i}$. The computation for the bottom part of $\mathcal{R}$ is similar, except that one needs to adjust the indices. For rows $n+1, .., m+n$, let $k = n - i$. The (i, j)-entry of $\mathcal{R}$ is the coefficient of $x^{m+n-k}y^{k-1}$ in $x^{n-k}y^k g$, which is equal to the coefficient b_{j-k} of $x^{m+k-j}y^{j-k}g$. The i, j-entry of $\mathcal{R}$ is resentries

$$R_{ij} = a_{j-i} \text{ when } i = 1, ..., n\,, \text{ and } R_{ij} = b_{j-k} \text{ when } i = n+k \text{ and } k = 1, ..., m$$

entries with negative subscripts being zero. $\square$

(1.7.9) weighted degree

weights

When defining the degree of a polynomial, one may assign an integer called a *weight* to each variable. If one assigns weight w_i to the variable x_i, the monomial $x_1^{e_1} \cdots x_n^{e_n}$ gets the *weighted degree*

$$e_1w_1 + \cdots + e_nw_n$$

For instance, one may assign weight k to the coefficient a_k of the polynomial $f(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots \pm a_n$. This is natural because, if f factors into linear factors, $f(x) = (x - \alpha_1) \cdots (x - \alpha_n)$, then a_k will be the kth elementary symmetric function in the roots $\alpha_1, ..., \alpha_n$. When a_k written as a polynomial in the roots, its degree will be k.

degresult **1.7.10. Lemma.** *Let $f(x,y)$ and $g(x,y)$ be homogeneous polynomials of degrees m and n respectively, with variable coefficients a_i and b_j, as in (1.7.3). When one assigns weight i to a_i and to b_i, the resultant* $\text{Res}(f,g)$ *becomes a weighted homogeneous polynomial of degree mn in the variables $\{a_i, b_j\}$.* □

For example, when the degrees of f and g are 1 and 2, respectively, the resultant $\text{Res}(f,g)$ is the determinant of the 3×3 matrix depicted in (1.7.5), which is $a_0^2b_2 + a_1^2b_0 - a_0a_1b_1$. Its weighted degree is $1\cdot2 = 2$.

resroots **1.7.11. Proposition.** *Let F and G be products of monic linear polynomials, say $F = \prod_{i=1}^m (x-\alpha_i)$ and $G = \prod_{j=1}^n (x-\beta_j)$. Then*

$$\text{Res}(F,G) \quad = \quad \prod_{i,j}(\alpha_i - \beta_j) \quad = \quad \prod_i G(\alpha_i)$$

proof. The equality of the second and third terms is obtained by substituting α_i for x into the formula $G = \prod(x-\beta_j)$. We prove the first equality. Let the polynomials F and G have variable roots α_i and β_j, let R denote the resultant $\text{Res}(F,G)$, and let $\Pi = \prod_{i.j}(\alpha_i - \beta_j)$. Lemma 1.7.10 tells us that, when we write the coefficients of F and G as symmetric functions in the roots, α_i and β_j, the resultant R will be homogeneous. Its (unweighted) degree in $\{\alpha_i, \beta_j\}$ will be mn. This is also the degree of Π. To show that $R = \Pi$, we choose i and j. We view R as a polynomial in the variable α_i, and divide by $\alpha_i - \beta_j$, which is monic in α_i:

$$R = (\alpha_i - \beta_j)q + r$$

where r has degree zero in α_i. Corollary 1.7.7 tells us that the resultant R vanishes when we make the substitution $\alpha_i = \beta_j$, because the coefficients of F and G are in the field of rational functions in $\{\alpha_i, \beta_j\}$. Looking at the equation above, we see that the remainder r also vanishes when $\alpha_i = \beta_j$. On the other hand, the remainder is independent of α_i. It doesn't change when we make that substitution. Therefore the remainder is zero, and $\alpha_i - \beta_j$ divides R. This is true for all i and j, so Π divides R, and since these two polynomials have the same degree, $R = c\Pi$ for some scalar c. One can show that $c = 1$ by computing R and Π for some particular polynomials. We suggest making the computation with $F = x^m$ and $G = x^n - 1$. □

restriviali-ties **1.7.12. Corollary.** *Let F, G, and H be monic polynomials and let c be a scalar. Then*
(i) $\text{Res}(F, GH) = \text{Res}(F,G)\,\text{Res}(F,H)$, *and*
(ii) $\text{Res}(F(x-c), G(x-c)) = \text{Res}(F(x), G(x))$. □

discrim-sect

(1.7.13) the discriminant

The *discriminant* $\text{Discr}(F)$ of a polynomial $F = a_0x^m + a_1x^{n-1} + \cdots a_m$ is the resultant of F and its derivative F':

discrdef (1.7.14)
$$\text{Discr}(F) = \text{Res}(F, F')$$

It is computed using the formula for the resultant of a polynomial of degree m, and it will be a weighted polynomial of degree $m(m-1)$. The definition makes sense when the leading coefficient a_0 is zero, but the discriminant will be zero in that case.

When F is a polynomial of degree n with complex coefficients, the discriminant is zero if and only if F and F' have a common factor, which happens when F has a multiple root.

Note. The formula for the discriminant is often normalized by a scalar factor. We won't make this normalization, so our formula is slightly different from the usual one.

The discriminant of the quadratic polynomial $F(x) = ax^2 + bx + c$ is

discrquadr (1.7.15)
$$\det\begin{pmatrix} a & b & c \\ 2a & b & \cdot \\ \cdot & 2a & b \end{pmatrix} = -a(b^2 - 4ac)$$

and the discriminant of a monic cubic $x^3 + px + q$ whose quadratic coefficient is zero is

$$(1.7.16) \qquad \det \begin{pmatrix} 1 & \cdot & p & q & \cdot \\ \cdot & 1 & \cdot & p & q \\ 3 & \cdot & p & \cdot & \cdot \\ \cdot & 3 & \cdot & p & \cdot \\ \cdot & \cdot & 3 & \cdot & p \end{pmatrix} = 4p^3 + 27q^2$$ discrcubic

As mentioned, these formulas differ from the usual ones by a scalar factor. The usual formula for the discriminant of the quadratic $ax^2 + bx + c$ is $b^2 - 4ac$, and the discriminant of the cubic $x^3 + px + q$ is usually written as $-4p^3 - 27q^2$.

Though it conflicts with our definition, we'll follow tradition and continue writing the discriminant of a quadratic as $b^2 - 4ac$.

1.7.17. Example. Suppose that the coefficients a_i of F are polynomials in t, so that $F = F(t, x)$ becomes a polynomial in two variables. Let's suppose that F is an irreducible polynomial. Let C be the curve $F = 0$ in the t, x-plane. The discriminant $\mathrm{Discr}_x(F)$, computed regarding x as the variable, will be a polynomial in t. At a root t_0 of the discriminant, $F(t_0, x)$ will have a multiple root. Therefore the vertical line $\{t = t_0\}$ will be tangent to C, or pass though a singular point of C. □ discrtwovar

1.7.18. Proposition. *Let K be a field of characteristic zero. The discriminant of an irreducible polynomial F with coefficients in K isn't zero. Therefore F has no multiple root.* *discrnotzero*

proof. When F is irreducible, it cannot have a factor in common with the derivative F', which has lower degree. □

This proposition is false when the characteristic of K isn't zero. In characteristic p, the derivative F' might be the zero polynomial.

1.7.19. Proposition. *Let $F = \prod(x - \alpha_i)$ be a polynomial that is a product of monic linear polynomials $x - \alpha_i$. Then* *discrformulas*

$$\mathrm{Discr}(F) = \prod_i F'(\alpha_i) = \prod_{i \neq j} (\alpha_i - \alpha_j) = \pm \prod_{i<j} (\alpha_i - \alpha_j)^2$$

proof. The fact that $\mathrm{Discr}(F) = \prod F'(\alpha_i)$ follows from (1.7.11). We prove the second equality by showing that $F'(\alpha_i) = \prod_{j, j\neq i} (\alpha_i - \alpha_j)$. By the product rule for differentiation,

$$F'(x) = \sum_k (x - \alpha_1) \cdots \widehat{(x - \alpha_k)} \cdots (x - \alpha_n)$$

where the hat $\widehat{\ }$ indicates that that term is deleted. When we substitute $x = \alpha_i$, all terms in this sum, except the one with $k = i$, become zero. □

1.7.20. Corollary. $\mathrm{Discr}(F(x)) = \mathrm{Discr}(F(x - c))$. □ *translatediscr*

1.7.21. Proposition. *Let $F(x)$ and $G(x)$ be monic polynomials. Then* *discrprop*

$$\mathrm{Discr}(FG) = \pm \,\mathrm{Discr}(F)\,\mathrm{Discr}(G)\mathrm{Res}(F, G)^2$$

proof. This proposition follows from Propositions 1.7.11 and 1.7.19 for polynomials with complex coefficients. It is true for polynomials with coefficients in any ring because it is an identity. For the same reason, Corollary 1.7.12 is true when the coefficients of the polynomials F, G, H are in any ring. □

When f and g are polynomials in several variables, one of which is z, $\mathrm{Res}_z(f, g)$ and $\mathrm{Discr}_z(f)$ will denote the resultant and the discriminant, computed regarding f and g as polynomials in z. They will be polynomials in the other variables.

1.7.22. Lemma. *Let f be an irreducible polynomial in $\mathbb{C}[x, y, z]$ of positive degree in z, and not divisible by z. The discriminant $\mathrm{Discr}_z(f)$, regarding f as a polynomial in z, is a nonzero polynomial in x, y.* *firroverK*

proof. This follows from Lemma 1.3.14 **(ii)** and Proposition 1.7.18. □

1.8 Nodes and Cusps

nodes

singmult **(1.8.1) the multiplicity of a singular point**

Let C be the projective curve defined by an irreducible homogeneous polynomial $f(x,y,z)$ of degree d, and let p be a point of C. We choose coordinates so that $p=(0,0,1)$, and we set $z=1$. This gives us an affine curve C_0 in $\mathbb{A}^2_{x,y}$, the zero set of the polynomial $\widetilde{f}(x,y)=f(x,y,1)$, and p becomes the origin. We write

seriesf (1.8.2)
$$\widetilde{f}(x,y)=f_0+f_1+f_2+\cdots+f_d$$

where f_i is the homogeneous part of $\widetilde{f}$ of degree i. The homogeneous part f_i is also the coefficient of z^{d-i} in $f(x,y,z)$. If the origin p is a point of C_0, the constant term f_0 will be zero, and the linear term f_1 will define the tangent direction to C_0 at p, If f_0 and f_1 are both zero, p will be a singular point of C. It seems permissible to drop the tilde and the subscript 0 in what follows, denoting $f(x,y,1)$ by $f(x,y)$, and C_0 by C.

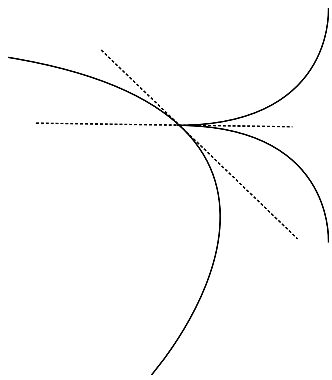
We use analogous notation for an analytic function $f(x,y)$ (1.4.18), denoting the homogeneous part of degree i of the series f by f_i:

multr (1.8.3)
$$f(x,y)=f_0+f_1+\cdots$$

Let C denote the locus of zeros of f in a neighborhood of the origin p. To describe the singularity of C at the origin, we look at the part of f of lowest degree. The smallest integer r such that $f_r(x,y)$ isn't zero is called the *multiplicity* of C at p. When the multiplicity is r, f will have the form $f_r+f_{r+1}+\cdots$.

Let L be the line $\{vx=uy\}$ through p, and suppose that $u\neq 0$. In analogy with Definition 1.3.9, the intersection multiplicity (1.3.9) of C and L at p is the order of zero of the series in x obtained by substituting $y=vx/u$ into f. The intersection multiplicity will be r unless $f_r(u,v)$ is zero. If $f_r(u,v)=0$, the intersection multiplicity will be greater than r. A line L through p whose intersection multiplicity with C at p is greater than the multiplicity of C will be called a *special line*. The special lines correspond to the zeros of f_r in $\mathbb{P}^1$. Because f_r has degree r, there will be at most r special lines.

goober14 **1.8.4.**



a Singular Point, with its Special Lines (real locus)

###please make point a visible dot in this figure###

dpt **(1.8.5) double points**

To analyze a singularity at the origin, one may *blow up* the plane. The map $W\xrightarrow{\pi}X$ from the (x,w)-plane to the (x,y)-plane defined by $\pi(x,w)=(x,xw)$ is called an *affine blowup* because the fibre over the origin in X is the line $\{x=0\}$ in W : $\pi(0,w)=(0,0)$ for all w. (It might seem more appropriate to call the inverse of π the blowup, but the inverse isn't a map.)

The blowup is bijective at points (x, y) of X at which $x \neq 0$, and points $(x, 0)$ of X with $x \neq 0$ aren't in its image.

Suppose that the origin p is a *double point*, a point of multiplicity 2, and let the quadratic part of f be

$$f_2 = ax^2 + bxy + cy^2$$ quadraticterm

To blow up the plane, we adjust coordinates so that c isn't zero, and normalize c to 1. Writing

$$f(x, y) = ax^2 + bxy + y^2 + dx^3 + \cdots$$

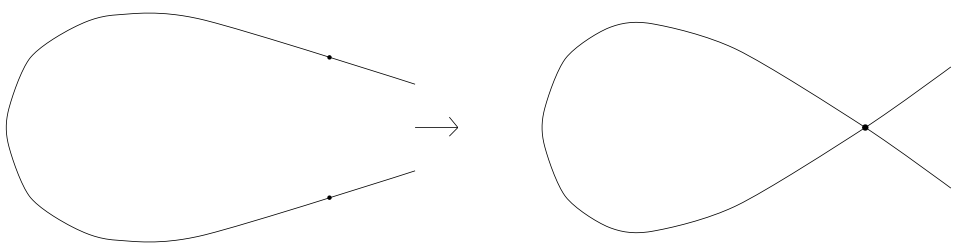
we make the substitution $y = xw$ and cancel x^2. This gives us a polynomial

$$g(x, w) \;=\; f(x, xw)/x^2 \;=\; a + bw + w^2 + dx + \cdots$$

in which all of the terms represented by $\cdots$ are divisible by x. Let D be the locus $\{g = 0\}$ in W. The blowup map π restricts to a map $D \xrightarrow{\overline{\pi}} C$. Since π is bijective at points at which $x \neq 0$, so is $\overline{\pi}$.

Suppose first that the quadratic polynomial $y^2 + by + a$ has distinct roots α, β, so that $ax^2 + bxy + y^2 = (y - \alpha x)(y - \beta x)$. Then $g(x, w) = (w - \alpha)(w - \beta) + dx + \cdots$. The fibre of D over the origin $p = (0, 0)$ in X is obtained by substituting $x = 0$ into g. It consists of the two points $(x, w) = (0, \alpha)$ and $(x, w) = (0, \beta)$. The partial derivative $\frac{\partial g}{\partial w}$ isn't zero at either of those points, so they are smooth points of D. At each of the points, we can solve $g(x, w) = 0$ for w as analytic functions of x, say $w = u(x)$ and $w = v(x)$, with $u(0) = \alpha$ and $v(0) = \beta$. So the curve C has two analytic branches $y = xu(x)$ and $y = xv(x)$ through the origin, with distinct tangent directions α and β. This singularity is called a *node*. A node is the simplest singularity that a curve can have.

1.8.6. maptonode



a Map to a Nodal Curve

###This figure is ugly. Curves aren't smooth.###

When the discriminant $b^2 - 4ac$ is zero, f_2 will be a square, and f will have the form

$$f(x, y) = (y - \alpha x)^2 + dx^3 + \cdots$$

Let's change coordinates, substituting $y + \alpha x$ for y, so that

(1.8.7) $$f(x, y) = y^2 + dx^3 + \cdots$$ cuspeq

The blowup substitution $y = xw$ gives

$$g(x, w) = w^2 + dx + \cdots$$

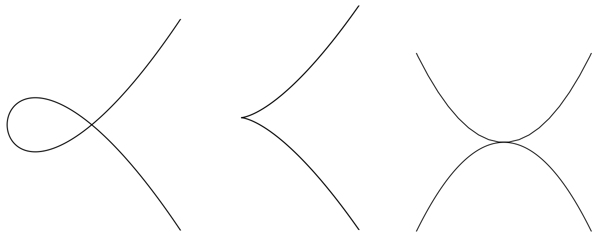
Here the fibre over $(x, y) = (0, 0)$ is the point $(x, w) = (0, 0)$, and $g_w(0, 0) = 0$. However, if $d \neq 0$, then $g_x(0, 0) \neq 0$, and if so, then D will be smooth at $(0, 0)$, and the equation of C will have the form $y^2 + dx^3 + \cdots$. This singularity is called a *cusp*.

The *standard cusp* is the locus $y^2 = x^3$. All cusps are analytically equivalent with the standard cusp.

nodeor-cusp **1.8.8. Corollary.** *A double point p of a curve C is a node or a cusp if and only if the blowup of C is smooth at the points that lie over p.* □

The simplest example of a double point that isn't a node or cusp is a *tacnode*, a point at which two smooth branches of a curve intersect with the same tangent direction.

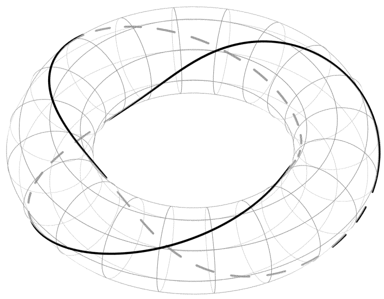
goober23



1.8.9. **a Node, a Cusp, and a Tacnode** (real locus)

Cusps have an interesting geometry. Let $\overline{x}$ denote the complex conjugate of x. The intersection of the standard cusp $X : \{y^2 = x^3\}$, with a small 3-sphere $S : \{\overline{x}x + \overline{y}y = \epsilon\}$ in $\mathbb{C}^2$ is a *trefoil knot*, as is illustrated below.

trefoil **1.8.10.**



Intersection of a Cusp Curve with a Three-Sphere

This nice figure was made by Jason Chen and Andrew Lin.

The standard cusp X, the locus $y^2 = x^3$, can be parametrized as $(x,y) = (t^2,t^3)$. The trefoil knot is the locus of points $(x,y) = (e^{2i\theta}, e^{3i\theta})$, the set of points of X of absolute value $\sqrt{2}$. It embeds into the product of a unit x-circle and a unit y-circle in $\mathbb{C}^2$, a torus that we denote by T. The figure depicts T as the usual torus T_0 in $\mathbb{R}^3$, but the mapping from T to T_0 distorts the torus. The circumference of T_0 represents the x-coordinate, and the loop through the hole represents y. As θ runs from 0 to 2π, the point (x,y) goes around the circumference twice, and it loops through the hole three times, as is illustrated. □

tracecusp **1.8.11. Proposition.** *Let $x(t) = t^2 + \cdots$ and $y(t) = t^3 + \cdots$ be analytic functions of t, whose orders of vanishing are 2 and 3, as indicated. For small t, the path $(x,y) = (x(t), y(t))$ in the x,y-plane traces out a curve with a cusp at the origin.*

proof. We show that there are analytic functions $b(x) = b_2x^2 + \cdots$ and $c(x) = x^3 + \cdots$ that vanish to orders 2 and 3 at $x = 0$, such that $x(t)$ and $y(t)$ solve the equation $y^2 + b(x)y + c(x) = 0$. The locus of such an equation has a cusp at the origin.

We solve for b and c. The function $x = t^2 + \cdots = t^2(1 + \cdots)$ has an analytic square root of the form $z = t + \cdots$. This follows from the Implicit Function Theorem, which also tells us that t can be written as an analytic function of z. So the function z is a coordinate equivalent to t, and we may replace t by z. Then we will have $x = t^2$, and y will still have a zero of order 3, $y = t^3 + \cdots$, though the series for y is changed.

Let's call the *even part* of a series $\sum a_nt^n$ the sum of the terms a_nt^n with n even, and the *odd part* the sum of terms with n odd. We write $y(t) = u(t) + v(t)$, where u an v are the even and the odd parts of y, respectively. The convergent series $y(t)$ is absolutely convergent its radius of convergence. Therefore $u(t)$ and $v(t)$ are convergent series too. Since y has a zero of order 3, v has a zero of order 3 and u has a zero of order at least 4.

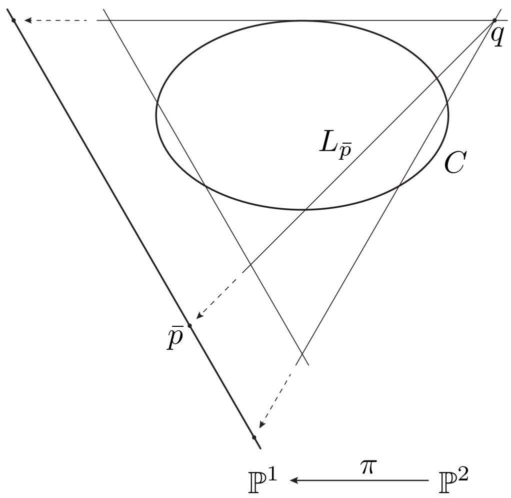
Now $y^2 = (u^2 + v^2) + 2uv$, $uy = u^2 + uv$, and $y^2 - 2uy + (u^2 - v^2) = 0$. The terms $-2u$ and $u^2 + v^2$ in this last equation are even series. They can be written as convergent series in $x = t^2$, say $-2u = b(x)$ and $u^2 - v^2 = c(x)$. Then b will have a zero of order at least 2, c will have a zero of order 3, and $y^2 + b(x)y + c(x) = 0$. $\square$

(1.8.12) projection to a line coverline

Let π denote the *projection* $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ that drops the last coordinate, sending a point (x, y, z) to (x, y). The projection is defined at all points of $\mathbb{P}^2$ except at the *center of projection*, the point $q = (0, 0, 1)$.

The fibre of π over a point $\overline{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the line through $p = (x_0, y_0, 0)$ and $q = (0, 0, 1)$, with the point q omitted — the set of points (x_0, y_0, z_0). We denote that line by L_{pq} or $L_{\overline{p}}$.

1.8.13. projtoline



Projection from the Plane to a Line

###The letters in this figure should be same size as in the text. Also, I'm not happy with the dashed arrows. I'd like more space between the dashes###

Let $f(x, y, z)$ be an irreducible homogeneous polynomial whose zero locus C is a plane curve that doesn't contain the center of projection q, and let d be the degree of f. the projection π will be defined at all points of the curve. We write f as a polynomial in z,

$$f = c_0 z^d + c_1 z^{d-1} + \cdots + c_d \tag{1.8.14}$$ poly-inztwo

with c_i homogeneous, of degree i in x, y. When q isn't in C, the scalar $c_0 = f(0, 0, 1)$ won't be zero, and we can normalize c_0 to 1, so that f becomes a monic polynomial of degree d in z.

Let's assume that C is a smooth curve. The fibre of C over a point $\overline{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the intersection of C with the line L_{pq} described above. Its points are (x_0, y_0, α), where α is a root of the one-variable polynomial

$$\widetilde{f}(z) = f(x_0, y_0, z) \tag{1.8.15}$$ effp

We call the smooth curve C a *branched covering* of $\mathbb{P}^1$, of degree d. All but finitely many fibres of C over $\mathbb{P}^1$ consist of d points.

The fibres of π with fewer than d points are those above the zeros of the discriminant (see Lemma 1.7.22). Those zeros are the *branch points* of the covering. We use the same term for points of C, calling a point of C a *branch point* if its tangent line is L_{pq}, in which case its image in $\mathbb{P}^1$ will also be a branch point.

flexbitan **1.8.16. Proposition.** *Let C be a smooth plane curve, let q be a generic point of the plane, and let p be a branch point of C, so that the tangent line L at p contains q. The intersection multiplicity of L and C at p is 2, and L and C have $d-2$ other intersections of multiplicity 1.*

The proof is below, but we explain the word *generic* first.

genpos **(1.8.17) generic and general position**

In algebraic geometry, the word *generic* is used for an object, a point for instance, that has no special 'bad' properties. Typically, the object will be parametrized somehow, and the adjective generic indicates that the parameter representing that particular object avoids a proper closed subset of the parameter space, which may be described explicitly or not. The phrase *general position* has a similar meaning. It indicates that an object is not in a special 'bad' position. In Proposition 1.8.16, what is required of the generic point q is that it shall not lie on a flex tangent line or on a *bitangent line* — a line that is tangent to C at two or more points. We have seen that a smooth curve C has finitely many flex points (1.4.17), and Lemma 1.8.18 below states that it has finitely many bitangents. So q must avoid a finite set of lines. Most points of the plane will be generic in this sense. □

proof of Proposition 1.8.16. The intersection multipicity of the tangent line L with C at p is at least 2 because L is a tangent line. It will be equal to 2 unless p is a flex point. The generic point q won't lie on any of the finitely many flex tangents, so the intersection multiplicity at p is 2. Next, the intersection multiplicity at another point p' of $L \cap C$ will be 1 unless L is tangent to C at p' as well as at p, i.e., unless L is a bitangent. The generic point q won't lie on a bitangent. □

finbitan **1.8.18. Lemma.** *A plane curve has finitely many bitangent lines.*

proof. We use the map $U \xrightarrow{t} C^*$ from the set U of smooth points of C to the dual curve C^*. If L is tangent to C at distinct smooth points p and p', then t will be defined at those points, and $t(p) = t(p') = L^*$. Therefore L^* will be a singular point of C^*. Since C^* has finitely many singular points, C has finitely many bitangents. □

genus **(1.8.19) the genus of a plane curve**

We describe the topological structure of a smooth plane curve in the classical topology here.

curveshomeomorphic **1.8.20. Theorem.** *A smooth projective plane curve of degree d is a compact, orientable and connected two-dimensional manifold.*

The fact that a smooth curve is a two-dimensional manifold follows from the Implicit Function Theorem. (See the discussion (**1.4.4**)).

orientability: A two-dimensional manifold is orientable if one can choose one of its two sides (as in front and back of a sheet of paper) in a continuous, consistent way. A smooth curve C is orientable because its tangent space at a point, the affine line with the equation (1.4.11), is a one-dimensional complex vector space. Multiplication by i orients the tangent space by defining the counterclockwise rotation, and the right-hand rule tells us which side of C is "up".

compactness: A plane projective curve is compact because it is a closed subset of the compact space $\mathbb{P}^2$.

connectedness: The fact that a plane curve is connected is subtle. It mixes topology and algebra. Unfortunately, I don't know a proof that fits into our discussion here. It will be proved later (see Theorem 8.2.11).

The topological *Euler characteristic* of a compact, orientable two-dimensional manifold M is the alternating sum $b^0 - b^1 + b^2$ of its Betti numbers. The Betti numbers are the dimensions of the homology groups of M. The Euler characteristic, which we denote by e, can be computed using a *topological triangulation*, a subdivision of M into topological triangles, called *faces*, by the formula

vef (1.8.21) $$e = |\text{vertices}| - |\text{edges}| + |\text{faces}|$$

For example, a sphere is homeomorphic to a tetrahedron, which has four vertices, six edges, and four faces. Its Euler characteristic is $4-6+4=2$. Any other topological triangulation of a sphere, such as the one given by the icosahedron, yields the same Euler characteristic.

Every compact, connected, orientable two-dimensional manifold is homeomorphic to a sphere with a finite number of "holes" or "handles". Its *genus* is the number of handles. A torus has one handle. Its genus is one. The projective line $\mathbb{P}^1$, a two-dimensional sphere, has genus zero.

The Euler characteristic and the genus are related by the formula

(1.8.22) $$e = 2 - 2g$$ genuseuler

The Euler characteristic of a torus is zero, and the Euler characteristic of $\mathbb{P}^1$ is two.

To compute the Euler characteristic of a smooth curve C of degree d, we analyze a generic projection (a projection from a generic point q of the plane), to represent C as a branched covering of the projective line: $C \xrightarrow{\pi} \mathbb{P}^1$ (see **1.8.17**). We choose generic coordinates x, y, z in $\mathbb{P}^2$ and project from the point $q = (0, 0, 1)$. When the defining equation of C is written as a monic polynomial in z: $f = z^d + c_1 z^{d-1} + \cdots + c_d$ where c_i is a homogeneous polynomial of degree i in the variables x, y, the discriminant $\mathrm{Discr}_z(f)$ with respect to z will be a homogeneous polynomial of degree $d(d-1) = d^2-d$ in x, y.

Let $\widetilde{p}$ be the image in $\mathbb{P}^1$ of a point p of C. The covering $C \xrightarrow{\pi} \mathbb{P}^1$ will be branched at $\widetilde{p}$ when the tangent line at p is the line L_{pq} through p and q. Proposition 1.8.16 tells us that if L_{pq} is a tangent line, there will be one intersection of multiplicity 2 and $d-1$ simple intersections. The discriminant will have a simple zero at such a point $\widetilde{p}$. This is proved in Proposition 1.9.12 below. Let's assume it for now.

Since the discriminant has degree d^2-d, there will be d^2-d points $\widetilde{p}$ of $\mathbb{P}^1$ at which the discriminant vanishes, and the fibre over such a point will contain $d-1$ points. Those points $\widetilde{p}$ are the *branch points* of the covering. All other fibres consist of d points.

We triangulate the sphere $\mathbb{P}^1$ in such a way that the branch points are among the vertices, and we use the inverse images of the vertices, edges, and faces to triangulate C. Then C will have d faces and d edges lying over each face and each edge of $\mathbb{P}^1$, respectively. There will also be d vertices of C lying over a vertex of $\mathbb{P}^1$, except when the vertex is one of the branch points. In that case the the fibre will contain only $d-1$ vertices. So the Euler characteristic of C can be obtained by multiplying the Euler characteristic of $\mathbb{P}^1$ by d and subtracting the number d^2-d of branch points:

(1.8.23) $$e(C) \;=\; d\,e(\mathbb{P}^1) - (d^2-d) \;=\; 2d - (d^2-d) \;=\; 3d - d^2$$ eulercover

This is the Euler characteristic of any smooth curve of degree d, so we denote it by e_d :

(1.8.24) $$e_d = 3d - d^2$$ equatione

Formula (1.8.22) shows that the genus g_d of a smooth curve of degree d is

(1.8.25) $$g_d \;=\; \tfrac{1}{2}(d^2 - 3d + 2) = \binom{d-1}{2}$$ equationg

Thus smooth curves of degrees $1, 2, 3, 4, 5, 6, \ldots$ have genus $0, 0, 1, 3, 6, 10, \ldots$, respectively. A smooth plane curve cannot have genus 2.

The generic projection to $\mathbb{P}^1$ with center q can also be used to compute the degree of the dual curve of a smooth curve C of degree d. The degree of the dual C^* is the number of its intersections with the generic line q^* in $\mathbb{P}^*$. The intersections of C^* and q^* are the points L^*, where L is a tangent line that contains q. As we saw above, there are d^2-d such lines.

1.8.26. Corollary. *Let C be a smooth plane projective curve of degree d. The degree d^* of the dual curve C^* is the number of tangent lines to C that pass through a generic point q of the plane. It is equal to d^2-d.* □ *degdual*

When C is a singular curve, the degree of C^* will be less than d^2-d.

When $d = 2$, C will be a conic, and $d^* = d$. As we have seen, the dual curve of a conic is also a conic. But when $d > 2$, $d^* = d^2 - d$ will be greater than d. Then the dual curve C^* must be singular. If it were smooth, the degree of its dual curve C^{**} would be ${d^*}^2 - d^*$, which would be greater than d. This would contradict the fact that $C^{**} = C$. For instance, when $d = 3$, $d^* = 3^2 - 3 = 6$, and ${d^*}^2 - d^* = 30$. The dual curve C^* is singular enough to account for the discrepancy between 30 and 3. (See (1.11.2)

1.9 Hensel's Lemma

hensel The resultant matrix (1.7.5) arises in a second context that we explain here.

Suppose given a product $P = FG$ of two polynomials, say

multiply-polys (1.9.1) $$\left(c_0x^{m+n}+c_1x^{m+n-1}+\cdots+c_{m+n}\right) \;=\; \left(a_0x^m+a_1x^{m-1}+\cdots+a_m\right)\left(b_0x^n+b_1x^{n-1}+\cdots+b_n\right)$$

We call the relations among the coefficients that are implied by this polynomial equation the *product equations*. The product equations are

$$c_i = a_ib_0 + a_{i-1}b_1 + \cdots + a_0b_i = \sum_{j=0}^{i} a_{i-j}b_j$$

for $i = 0, ..., m+n$. For instance, when $m = 3$ and $n = 2$, the product equations are

prodeqns **1.9.2.**

$$\begin{aligned}
c_0 &= a_0b_0 \\
c_1 &= a_1b_0 + a_0b_1 \\
c_2 &= a_2b_0 + a_1b_1 + a_0b_2 \\
c_3 &= a_3b_0 + a_2b_1 + a_1b_2 \\
c_4 &= \qquad\;\; a_3b_1 + a_2b_2 \\
c_5 &= \qquad\qquad\quad\;\; a_3b_2
\end{aligned}$$

Let J denote the Jacobian matrix of partial derivatives of $c_1, ..., c_{m+n}$ with respect to the variables $b_1, ..., b_n$ and $a_1, ..., a_m$, treating a_0, b_0 and c_0 as constants. Since $\frac{\partial c_i}{\partial b_j} = a_{i-j}$ and $\frac{\partial c_i}{\partial a_j} = b_{i-j}$, the i, j-entry of J is

$$J_{ij} = a_{i-j} \text{ when } j = 1, ..., n \text{ and } J_{ij} = b_{i-k} \text{ when} j = n + k \text{ and} k = 1, ..., m$$

entries with negative subscripts being set to 0.

When $m, n = 3, 2$,

prodjacob (1.9.3) $$J = \frac{\partial(c_i)}{\partial(b_j, a_k)} = \begin{pmatrix} a_0 & . & b_0 & . & . \\ a_1 & a_0 & b_1 & b_0 & . \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & . & b_2 & b_1 \\ . & a_3 & . & . & b_2 \end{pmatrix}$$

jacres **1.9.4. Lemma.** *The Jacobian matrix J is the transpose of the resultant matrix $\mathcal{R}$ (1.7.5).*

See (1.7.8). But this seems like an occasion to quote Cayley. While discussing the the Cayley-Hamilton Theorem, he wrote: 'I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case.' □

jacobian-notzero **1.9.5. Corollary.** *Let F and G be polynomials with complex coefficients. The Jacobian matrix is singular if and only if, either F and G have a common root, or $a_0 = b_0 = 0$.* □

This corollary has an application to polynomials with analytic coefficients. Let

poly-forhensel (1.9.6) $$P(t, x) = c_0(t)x^d + c_1(t)x^{d-1} + \cdots + c_d(t)$$

be a polynomial in x whose coefficients c_i are analytic functions of t, and let $\overline{P} = P(0, x) = \overline{c}_0x^d + \overline{c}_1x^{d-1} + \cdots + \overline{c}_d$ be the evaluation of P at $t = 0$, so that $\overline{c}_i = c_i(0)$. Suppose given a factorization $\overline{P} = \overline{F}\,\overline{G}$, where $\overline{F} = x^m+\overline{a}_1x^{m-1}+\cdots+\overline{a}_m$ and $\overline{G} = \overline{b}_0x^n+\overline{b}_1x^{n-1}+\cdots+\overline{b}_n$ are polynomials with complex coefficients, and $\overline{F}$ is monic. Are there polynomials $F(t, x) = x^m+a_1x^{m-1}+\cdots+a_m$ and $G(t, x) = b_0x^n+b_1x^{n-1}+\cdots+b_n$, with F monic, whose coefficients a_i and b_j are analytic functions of t, such that $F(0, x) = \overline{F}$, $G(0, x) = \overline{G}$, and $P = FG$?

1.9.7. Hensel's Lemma. *With notation as above, suppose that $\overline{F}$ and $\overline{G}$ have no common root. Then P factors: $P = FG$, where F and G are polynomials in x, whose coefficients are analytic functions of t, and F is monic.* *hensellemma*

proof. We look at the product equations. Since F is supposed to be monic, we set $a_0(t) = 1$. The first product equation tells us that $b_0(t) = c_0(t)$. Corollary 1.9.5 tells us that the Jacobian matrix for the remaining product equations is nonsingular at $t = 0$, so according to the Implicit Function Theorem, the product equations have a unique solution in analytic functions $a_i(t), b_j(t)$. $\square$

Note that P isn't assumed to be monic. If $\overline{c}_0 = 0$, the degree of $\overline{P}$ will be less than the degree of P. In that case, $\overline{G}$ will have lower degree than G.

1.9.8. Example. Let $P = c_0(t)x^2 + c_1(t)x + c_2(t)$. The product equations $P = FG$ with $F = x + a_1$ monic and $G = b_0x + b_1$, are henselex

(1.9.9) $$c_0 = b_0\,, \quad c_1 = a_1b_0 + b_1\,, \quad c_2 = a_1b_1$$ quadfactor

and the Jacobian matrix is
$$\frac{\partial(c_1, c_2)}{\partial(b_1, a_1)} = \begin{pmatrix} 1 & b_0 \\ a_1 & b_1 \end{pmatrix}$$

Suppose that $\overline{P} = P(0, x)$ factors: $\overline{c}_0x^2 + \overline{c}_1x + \overline{c}_2 = (x + \overline{a}_1)(\overline{b}_0x + \overline{b}_1) = \overline{F}\,\overline{G}$. The determinant of the Jacobian matrix at $t = 0$ is $\overline{b}_1 - \overline{a}_1\overline{b}_0$. It is nonzero if and only if the factors $\overline{F}$ and $\overline{G}$ are relatively prime, in which case P factors too.

On the other hand, the one-variable Jacobian criterion allows us to solve the equation $P(t, x) = 0$ for x as function of t with $x(0) = -\overline{a}_1$, provided that $\frac{\partial P}{\partial x} = 2c_0x + c_1$ isn't zero at the point $(t, x) = (0, -\overline{a}_1)$. If $\overline{P}$ factors as above, then when we substitute (1.9.9) into $\overline{P}$, we find that $\frac{\partial P}{\partial x}(0, -\overline{a}_1) = -2\overline{c}_0\overline{a}_1 + \overline{c}_1 = \overline{b}_1 - \overline{a}_1\overline{b}_0$. Not surprisingly, $\frac{\partial P}{\partial x}(0, -\overline{a}_1)$ is equal to the determinant of the Jacobian matrix at $t = 0$. $\square$

(**1.9.10**) order of vanishing of the discriminant vandisc

We introduce some terminology for use in the next proposition. Let X be the affine x-line, let Y be the affine x, y-plane and let p be the origin in Y. Two curves are said to intersect *transversally* at a point p if they are smooth at p and if their tangent lines there are distinct.

Let C be the plane affine curve defined by a polynomial $f(x, y)$ with no multiple factors, and suppose that C contains the origin p. Let L be the y-axis $\{x = 0\}$ in Y. Suppose that all intersections of C with L are transversal, except for the point p. This will be true when the coordinates x, y are generic.

1.9.11. Proposition. *affdvan*

(i) a) *Let p be a smooth point of C with tangent line L. If p isn't a flex point of C, the discriminant* $\mathrm{Discr}_y(f)$ *has a simple zero at the origin.*

b) *If p is a node of C and L is not a special line at p,* $\mathrm{Discr}_y(f)$ *has a double zero at the origin.*

c) *If p is a cusp of C and L is not its special line at p,* $\mathrm{Discr}_y(f)$ *has a triple zero at the origin.*

(ii) *If p is an ordinary flex point of C and L is its tangent line, then* $\mathrm{Discr}_y(f)$ *has a double zero at the origin.*

make figure ??###

proof. **(i)** Let $\overline{f}(y) = f(0, y)$. In each of the three cases, $\overline{f}(y)$ will have a double zero at $y = 0$. We will have $\overline{f}(y) = y^2\overline{h}(y)$, with $\overline{h}(0) \neq 0$. So y^2 and $\overline{h}(y)$ have no common root. We apply Hensel's Lemma: $f(x, y) = g(x, y)h(x, y)$, where g and h are polynomials in y whose coefficients are analytic functions of x, g is monic, $g(0, y) = y^2$, and $h(0, y) = \overline{h}$. Then $\mathrm{Discr}_y(f) = \pm\,\mathrm{Discr}_y(g)\,\mathrm{Discr}_y(h)\,\mathrm{Res}_y(g, h)^2$ (1.7.21).

Since C is tranversal to L except at q, $\overline{h}$ has simple zeros (1.8.16). Then $\mathrm{Discr}_y(h)$ doesn't vanish at $y = 0$. Neither does $\mathrm{Res}_y(g, h)$. So the orders of vanishing of $\mathrm{Discr}_y(f)$ and $\mathrm{Discr}_y(g)$ at p are equal.

We replace f by g, so that f becomes a monic quadratic polynomial in y, of the form

$$f(x, y) = y^2 + b(x)y + c(x)$$

where the coefficients b and c are now analytic functions of x, and $f(0,y) = y^2$. The discriminant $\mathrm{Discr}_y(f) = b^2 - 4c$ is unchanged when we complete the square by the substitution of $y - \frac{1}{2}b$ for y, and if p is a smooth point, a node or a cusp, that property isn't affected by this operation. So we may assume that f has the form $y^2 + c(x)$. The discriminant is then $D = 4c(x)$.

We write $c(x)$ as a series:

$$c(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

Since C contains p, the constant coefficient c_0 is zero. If $c_1 \neq 0$, p is a smooth point with tangent line $\widetilde{L}$, and D has a simple zero. If p is a node, $c_0 = c_1 = 0$ and $c_2 \neq 0$. Then D has a double zero. If p is a cusp, $c_0 = c_1 = c_2 = 0$, and $c_3 \neq 0$. Then D has a triple zero at p.

(ii) In this case, the polynomial $\widetilde{f}(y) = f(0,y)$ will have a triple zero at $y = 0$. Proceding as above, we may factor: $f = gh$ where g and h are polynomials in y whose coefficients are analytic funcicton so x, $g(x,y) = y^3 + a(x)y^2 + b(x)y + c(x)$, and $g(0,y) = y^3$. We eliminate the quadratic coefficient a by substituting $y - \frac{1}{3a}$ for y. With $g = y^3 + by + c$ in the new coordinates, the discriminant $\mathrm{Discr}_y(g)$ is $4b^3 + 27c^2$ (1.7.16). We write $c(x) = c_0 + c_1x + \cdots$ and $b(x) = b_0 + b_1x + \cdots$. Since p is a point of C with tangent line $\{y=0\}$, $c_0 = 0$ and $c_1 \neq 0$. Since the intersection multiplicity of C with the line $\{y=0\}$ at p is three, $b_0 = 0$. The discriminant $4b^3 + 27c^2$ has a zero of order two. □

Now let $f(x,y,z)$ be a homogeneous polynomial with no multiple factors, and let C be the (possibly reducible) plane curve $\{f = 0\}$. We project to $X = \mathbb{P}^1$ from a point q that is not on C. Let L_{pq} denote the line through a point $p = (x_0, y_0, 0)$ and q, the set of points (x_0, y_0, z_0), and let $\widetilde{p} = (x_0, y_0)$. Suppose that all intersections of C with L_{pq} except at q are transversal.

discrim-vanishing

1.9.12. Corollary. (i) *With notation as above:*

a) *If p is a smooth point of C with tangent line L_{pq}, the discriminant $\mathrm{Discr}_z(f)$ has a simple zero at $\widetilde{p}$.*

b) *If p is a node of C, $\mathrm{Discr}_z(f)$ has a double zero at $\widetilde{p}$.*

c) *If p is a cusp, $\mathrm{Discr}_z(f)$ has a double zero at $\widetilde{p}$.*

(ii) *If p is an ordinary flex point of C, $\mathrm{Discr}_z(f)$ has a double zero at $z=0$.* □

generic-cond

In cases **(i a,b,c)**, the hypotheses are satisfied when the center of projection q is in general position. To be precise about what is required of the generic point q in those cases, we ask that q not lie on any of these lines:

(1.9.13)

flex tangent lines and bitangent lines,
lines that contain more than one singular point,
special lines through singular points,
tangent lines that contain a singular point.

This is a list of finitely many lines that q must avoid.

transres

1.9.14. Corollary. *Let $C : \{g = 0\}$ and $D : \{h = 0\}$ be plane curves that intersect transversally at a point $p = (z_0, y_0, z_0)$. With coordinates in general position, $\mathrm{Res}_z(g,h)$ has a simple zero at (x_0, y_0).*

proof. Proposition 1.9.12 **(i b)** applies to the product gh, whose zero locus is the union $C \cup D$. It shows that the discriminant $\mathrm{Discr}_z(gh)$ has a double zero at p. We also have the formula

$$\mathrm{Discr}_z(gh) = \mathrm{Discr}_z(g)\,\mathrm{Discr}_z(h)\,\mathrm{Res}(g,h)^2$$

(1.7.21) with $f = gh$. When coordinates are in general position, $\mathrm{Discr}_z(g)$ and $\mathrm{Discr}_z(h)$ will not be zero at p. Then $\mathrm{Res}_z(g,h)$ has a simple zero at p. □

1.10 Bézout's Theorem

bezoutthm

Bézout's Theorem counts intersections of plane curves. We state it here in a form that is ambiguous because it contains a term "multiplicity" that hasn't yet been defined.

1.10.1. Bézout's Theorem. Let C and D be distinct curves of degrees m and n, respectively. When intersections are counted with an appropriate multiplicity, the number of intersections is mn. Moreover, the multiplicity at a transversal intersection is 1. bezoutone

As before, C and D intersect transversally at p if they are smooth at p and their tangent lines there are distinct.

1.10.2. Proposition. *Bézout's Theorem is true when one of the curves is a line.* *bezoutline*

See Corollary 1.3.10. The multiplicity of intersection of a curve and a line is the one that was defined there. □

The proof in the general case requires some algebra that we would rather defer. It will be given later (Theorem 7.8.1), but we will use the theorem in the rest of this chapter.

It is possible to determine the intersections by counting the zeros of the resultant with respect to one of the variables. To do this, one chooses coordinates x, y, z, so that neither C nor D contains the point $(0, 0, 1)$. One writes their defining polynomials f and g as polynomials in z with coefficients in $\mathbb{C}[x, y]$. The resultant R with respect to z will be a homogeneous polynomial in x, y, of degree mn. It will have mn zeros in $\mathbb{P}^1_{x,y}$, counted with multiplicity. Let $\widetilde{p} = (x_0, y_0)$ be a zero of R. Then $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$, which are polynomials in z, have a common root $z = z_0$, and then $p = (x_0, y_0, z_0)$ will be a point of $C \cap D$. It is a fact that the multiplicity of the zero of the resultant R at the image $\widetilde{p}$ is the (as yet undefined) intersection multiplicity of C and D at p. Unfortunately, this won't be obvious, even when the multiplicity is defined. However, one can prove the next proposition using this approach.

1.10.3. Proposition. *Let C and D be distinct plane curves of degrees m and n, respectively.* *nocom-*
(i) *C and D have at least one intersection, and the number of intersections is at most mn.* *monfactor*
(ii) *If all intersections are transversal, the number of intersections is precisely mn.*

It isn't obvious that two curves in the projective plane intersect. If two curves in the affine plane have no intersection, If they are parallel lines, for example, their closures in the projective plane meet on the line at infinity.

1.10.4. Lemma. *Let f and g be homogeneous polynomials in x, y, z of degrees m and n, respectively, and suppose that the point $(0, 0, 1)$ isn't a zero of f or g. If the resultant $\mathrm{Res}_z(f, g)$ with respect to z is identically zero, then f and g have a common factor.* *resnotzero*

proof. Let F denote the field of rational functions $\mathbb{C}(x, y)$. If the resultant is zero, f and g have a common factor in $F[z]$. There will be polynomials p and q in $F[z]$, of degrees at most $n-1$ and $m-1$ in z, respectively, such that $pf = qg$ (1.7.3). We may clear denominators, so we may assume that the coefficients of p and q are in $\mathbb{C}[x, y]$. This doesn't change their degree in z. Then $pf = qg$ is an equation in $\mathbb{C}[x, y, z]$, and p isn't divisible by g. Since $\mathbb{C}[x, y, z]$ is a unique factorization domain, f and g have a common factor. □

proof of Proposition 1.10.3. **(i)** Let C and D be distinct curves, defined by irreducible homogeneous polynomials f and g. Proposition 1.3.12 shows that there are finitely many intersections. We project to $\mathbb{P}^1$ from a point q that doesn't lie on any of the finitely many lines through pairs of intersection points. Then a line through q passes through at most one intersection, and the zeros of the resultant $\mathrm{Res}_z(f, g)$ that correspond to the intersection points will be distinct. The resultant has degree mn (1.7.10). It has at least one zero, and at most mn of them. Therefore C and D have at least one and at most mn intersections.

(ii) Every zero of the resultant will be the image of an intersection of C and D. To show that there are mn intersections if all intersections are transversal, it suffices to show that the resultant has simple zeros. This is Corollary 1.9.14. □

1.10.5. Corollary. *If the zero locus X of a homogeneous polynomial $f(x, y, z)$ is smooth, then f is irreducible, and therefore X is a smooth curve.* *smoothirred*

proof. Suppose that $f = gh$, and let p be a point of intersection of the loci $\{g = 0\}$ and $\{h = 0\}$. Proposition 1.10.3 shows that such a point exists. All partial derivatives of f vanish at p, so p is a singular point of the locus $f = 0$ (1.4.7). □

1.10.6. Proposition. **(i)** *Let d be an integer ≥ 3. A smooth plane curve of degree d has at least one flex point, and the number of flex points is at most $3d(d-2)$.* *numberof-flexes*
(ii) *If all flex points are ordinary, the number of flex points is $3d(d-2)$.*

Thus smooth curves of degrees $2, 3, 4, 5, ...$ have at most $0, 9, 24, 45, ...$ flex points, respectively.

proof. **(i)** Let C be the smooth curve defind by a homogeneous polynomial f of degree d. Let H be the Hessian matrix of f, let $detH = h_1^{e_1} \cdots h_k^{e_k}$ be the factorization of the determinant into irreducible polynomials h_i, and let Z_i be the locus of zeros of h_i. The *Hessian divisor* is defined to be the combination $D = e_1 Z_1 + \cdots + e_k Z_k$.

The flex points of C are its intersections with its Hessian divisor D (1.4.16). The entries of the 3×3 Hessian matrix H, the second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$, are homogeneous polynomials of degree $d-2$. So the Hessian determinant is homogeneous, of degree $3(d-2)$. Propositions 1.4.17 and 1.10.3 tell us that there are at most $3d(d-2)$ intersections.

(ii) A flex point is ordinary if the multiplicity of intersection of the curve and its tangent line is 3 (**1.4.8**). Bézout's Theorem asserts that the number of flex points is $3d(d-2)$ if the intersections of C with its Hessian divisor D are transversal, and therefore have multiplicity 1. So the next lemma completes the proof.

transversalH **1.10.7. Lemma.** *A curve $C : \{f = 0\}$ intersects its Hessian divisor D transversally at a point p if and only p is an ordinary flex point of C.*

proof. We prove the lemma by computation. I don't know a conceptual proof.

Let D be the Hessian divisor $\{\det H = 0\}$. The Hessian determinant $\det H$ vanishes at a smooth point p of C if and only if p is a flex point (1.4.16).

Assume that p is a flex point, let L be the tangent line to C at p, and let $\overline{h}$ denote the restriction of the determinant $\det H$ to L. The Hessian divisor D will be transversal to C at p if and only if it is transversal to L there, which will be true if and only if $\overline{h}$ has a zero of order 1.

We adjust coordinates x, y, z so that p is the point $(0, 0, 1)$ and L is the line $\{y = 0\}$, and we set $z = 1$ to work in the affine space $\mathbb{A}^2_{x,y}$. Because p is a flex point, the coefficients of the monomials $1, x$ and x^2 in the polynomial $f(x, y, 1)$ are zero. So

$$f(x, y, 1) = ay + bxy + cy^2 + dx^3 + ex^2y + \cdots$$

To restrict to L, we set $y = 0$, keeping $z = 1$: $f(x, 0, 1) = dx^3 + O(4)$, where $O(k)$ stands for a polynomial all of whose terms have degree $\geq k$.

To compute the determinant $\det H$, we put the variable z back. If f has degree n, then

$$f(x, y, z) = ayz^{n-1} + bxyz^{n-2} + cy^2z^{n-2} + dx^3z^{n-3} + ex^2yz^{n-3} + \cdots$$

We set $y = 0$ and $z = 1$ in the second order partial derivatives. With $v = 6dx$ and $w = (n-1)a + (n-2)bx$,

$f_{xx}(x, 0, 1) = 6dx + O(2) = v + O(2)$,

$f_{xz}(x, 0, 1) = 0 + O(2)$,

$f_{yz}(x, 0, 1) = (n-1)a + (n-2)bx + O(2) = w + O(2)$,

$f_{zz}(x, 0, 1) = 0 + O(2)$,

We won't need f_{xy} or f_{yy}. The Hessian matrix at $y = 0, z = 1$ has the form

$$H(x,0,1) = \begin{pmatrix} v & * & 0 \\ * & * & w \\ 0 & w & 0 \end{pmatrix} + O(2) \tag{1.10.8}$$

Because of the zeros, the entries marked with $*$ don't affect the determinant of $H(x, 0, 1)$. It is

$$\overline{h} = -vw^2 + O(2) = -6d(n-1)^2a^2x + O(2)$$

and it has a zero of order 1 at $x = 0$ if and only if a and d aren't zero there. Since C is smooth at p and since the coefficient of x in f is zero, the coefficient of y, which is a, can't be zero. Thus the curve C and its Hessian divisor D intersect transversally, if and only if d isn't zero. This is true if and only if p is an ordinary flex point. □

nineflexes **1.10.9. Corollary.** *A smooth cubic curve contains exactly 9 flex points.*

proof. Let C be a smooth cubic curve. The Hessian divisor D of C also has degree 3, so Bézout's Theorem predicts at most 9 intersections of D with C. To derive the corollary, we show that D intersects C transversally, and to do this, we show that D intersects the tangent line L to C at p transversally. According to Lemma 1.10.7, a nontransversal intersection of D and L would correspond to a point at which C has a flex that isn't ordinary, and at such a point, the intersection multiplicity of C and L would be greater than 3. This is impossible when the curve is a cubic. □

(1.10.10) singularities of the dual curve singdual

Let C be a plane curve. As before, an *ordinary flex point* is a smooth point p such that the intersection multiplicity of the curve and its tangent line L at p is equal to 3. A bitangent, a line L that is tangent to C at distinct points p and p', is an *ordinary bitangent* if neither p nor p' is a flex point. A tangent line L at a smooth point p of C is an *ordinary tangent* if p isn't a flex point and L isn't a bitangent.

The tangent line L at a point p will have other intersections with C. Most often, those other intersections will be transversal unless L is a bitangent, in which case it will be tangent to C at another point. However, it may also happen that one of the other intersections of L with C is a singular point of C. Or, L may be a *tritangent*, tangent to C at three points. Let's call such occurences *accidents*.

1.10.11. Definition. A plane curve C is *ordinary* if it is smooth, all of its bitangents and flex points are ordinary, and if there are no accidents. ordcurve

1.10.12. Lemma. *A generic curve C is ordinary.* *genisord*

We verify this by counting constants (see (1.1.3)). The reasoning is fairly convincing, though not completely precise.

There are three ways in which a curve C might fail to be ordinary:

- C may be singular.
- C may have a flex point that isn't ordinary.
- A bitangent to C may be a flex tangent or a tritangent.

The curve will be ordinary if none of these occurs.

Let the coordinates in the plane be x, y, z. The homogeneous polynomials of degree d form a vector space whose dimension is equal to the number N of monomials $x^i y^j z^k$ of degree $i+j+k = d$. Let f be a homogeneous polynomial of degree d, and let $f = \prod f_i^{e_i}$ be its factorization into irreducible polynomials. If Z_i denotes the zero locus of f_i, the divisor associated to f is $\sum e_i Z_i$. The divisors of degree d are parametrized by points of a projective space of dimension $n = N - 1$, and curves correspond to points in a subset of that space.

singular points. We look at the point $p_0 = (0,0,1)$, and we set $z = 1$. If p_0 is a singular point of a curve C defined by a polynomial f, the coefficients of $1, x, y$ in the polynomial $f(x, y, 1)$ will be zero. This is three conditions. So the curves that are singular at p_0 are parametrized by a linear subspace of dimension $n - 3$ in the projective space of dimension n, and the same will be true when p_0 is replaced by any other point of $\mathbb{P}^2$. The points of $\mathbb{P}^2$ depend on only 2 parameters. Therefore, in the space of divisors, the singular curves form a subset of dimension at most $n - 1$. Most curves are smooth.

flex points. Let's look at curves that have a four-fold tangency with the line $L : \{y = 0\}$ at p_0. Setting $z = 1$ as before, we see that the coefficients of $1, y, y^2, y^3$ in f must be zero. This is four conditions. The lines through p_0 depend on one parameter, and the points of $\mathbb{P}^2$ depend on two parameters, giving us three parameters to vary. We can't get all curves this way. Most curves have no four-fold tangencies, and therefore they have only ordinary flexes.

bitangents. To be tangent to the line $L : \{y = 0\}$ at the point p_0, the coefficients of 1 and y in f must be zero. This is two conditions. Then to be tangent to L at three given points p_0, p_1, p_2 imposes 6 conditions. A set of three points of L depends on three parameters, and a line depends on two parameters, giving us 5 parameters in all. Most curves don't have a tritangent. Similar reasoning takes care of bitangents in which one tangency is a flex. □

1.10.13. Proposition. *Let p be a point of an ordinary curve C, and let L be the tangent line at p.* *dualcusp*
If L is an ordinary tangent at p, then L^* is a smooth point of C^*.
If L is a bitangent, then L^* is a node of C^*.
If p is a flex, then L^* is a cusp of C^*.

proof. We refer to the map $C \xrightarrow{t} C^*$ from C to the dual curve (**1.6.3**). Because C is smooth, t is defined s at all points of C.

We dehomogenize the defining polynomial f by setting $z = 1$, and choose affine coordinates, so that p is the point $(x, y, z) = (0, 0, 1)$, the tangent line L at p is the line $\{y = 0\}$. Then L^* is the point $(u, v, w) = (0, 1, 0)$. Let $\widetilde{f}(x, y) = f(x, y, 1)$. We solve $\widetilde{f} = 0$ for $y = y(x)$ as an analytic function of x, as before. The tangent line L_1 to C at a nearby point $p_1 = (x_1, y_1)$ has the equation (1.6.11), and L_1^* is the point $(u, v, w) = (-y_1', 1, y_1'x_1 - y_1)$ of $\mathbb{P}^*$ (1.6.12). Since there are no accidents, this path traces out all points of C^* near to L^* (Corollary 1.6.18).

If L is an ordinary tangent line, $y(x)$ will have a zero of order 2 at $x = 0$. Then $u = -y'$ will have a simple zero. So the path $(-y', 1, y'x - y)$ is smooth at $x = 0$, and therefore C^* is smooth at the origin.

If L is an ordinary bitangent, tangent to C at two points p and p', the reasoning given for an ordinary tangent shows that the images in C^* of small neighborhoods of p and p' in C will be smooth at L^*. Their tangent lines p^* and ${p'}^*$ will be distinct, so p is a node.

Suppose that p is an ordinary flex point. Then, in the power series $y(x) = c_0 + c_1x + c_2x^2 + \cdots$, the coefficients c_0, c_1, c_2 are zero and since the flex is ordinary, $c_3 \neq 0$. We may assume that $c_3 = 1$ and that $y(x) = x^3 + \cdots$. Then, in the local equation $(u, v, w) = (-y', 1, y'x - y)$ for the dual curve, $u = -3x^2 + \cdots$ and $w = 2x^3 + \cdots$. Proposition 1.8.11 tells us that the singularity at the origin is a cusp. □

1.11 The Plücker Formulas

plucker

The *Plücker formulas* compute the number of flexes and bitangents of an ordinary plane curve. The fact that there is a sfle formula for bitangents is particularly interesting. The bitangents aren't very easy to count directly.

plform

1.11.1. Theorem: Plücker Formulas. *Let C be an ordinary curve of degree d at least two, and let C^* be its dual curve. Let f and b denote the numbers of flex points and bitangents of C, and let d^*, δ^* and κ^* denote the degree, the numbers of nodes, and the number of cusps of C^*, respectively. Then:*

(i) *The dual curve C^* has no flexes or bitangents. Its singularities are nodes or cusps.*

(ii) $d^* = d^2 - d, \quad f = \kappa^* = 3d(d-2),$ *and* $\quad b = \delta^* = \frac{1}{2}d(d-2)(d^2-9).$

proof. **(i)** A bitangent or a flex on C^* would produce a singularity on the bidual C^{**}, which is the smooth curve C.

(ii) The degree d^* was computed in Corollary 1.8.26. Bézout's Theorem counts the flex points: $f = 3d(d-2)$(1.10.6). The facts that $\kappa^* = f$ and $\delta^* = b$ are in Proposition 1.10.13. Thus $\kappa^* = 3d(d-2)$.

When we project C^* to $\mathbb{P}^1$ from a generic point s of $\mathbb{P}^*$. The number of branch points that correspond to tangent lines through s at smooth points of C^* is the degree d of the bidual C (1.8.26).

Next, let $F(u, v, w)$ be the defining polynomial for C^*. The discriminant $\mathrm{Discr}_w(F)$ has degree $d^{*2} - d^*$. Corollary 1.9.12 describes the order of vanishing of the discriminant at the images of the d tangent lines through s, the δ nodes of C^*, and the κ cusps of C^*. It tells us that $d^{*2} - d^* = d + 2\delta^* + 3\kappa^*$. Substituting the known values $d^* = d^2 - d$, and $\kappa^* = 3d(d-2)$ into this formula gives us $(d^2 - d)^2 - (d^2 - d) = d + 2\delta^* + 9d(d-2)$, or

$$2\delta^* \;=\; d^4 - 2d^3 - 9d^2 + 18d \;=\; d(d-2)(d^2-9)$$

□

some-plucker-formulas

1.11.2. Examples.

(i) All curves of degree 2 and all smooth curves of degree 3 are ordinary.

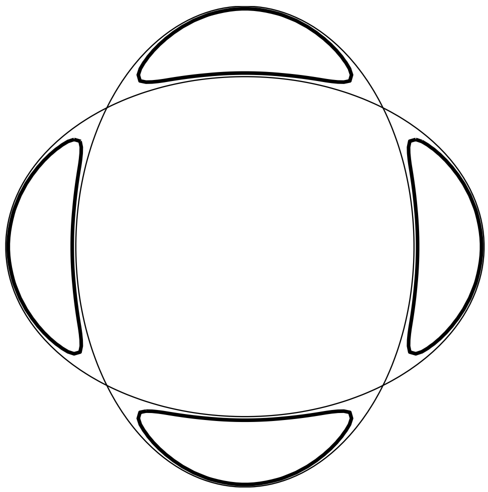
(ii) A curve of degree 2 has no flexes and no bitangents. Its dual curve has degree 2.

(iii) A smooth curve of degree 3 has 9 flexes and no bitangents. Its dual curve has degree 6.

(iv) An ordinary curve C of degree 4 has 24 flexes and 28 bitangents. Its dual curve has degree 12. □

We will make use of the fact that a quartic curve has 28 bitangents in Chapter 4 (see (4.6.31)). The Plücker Formulas are rarely used for curves of larger degree, but the fact that there is such a formula is interesting.

1.11.3. goober13



A Quartic Curve whose 28 **Bitangents are Real** (real locus)

To obtain this quartic, we added a small constant ϵ to the product of the quadratic equations of the two ellipses that are shown. The equation of the quartic is $(2x^2+y^2-1)(x^2+2y^2-1)+\epsilon=0$.

1.12 Exercises

chaponeex xinfpoints **1.12.1.** Prove that a plane curve contains infinitely many points.

xsin **1.12.2.** Prove that the path $x(t) = t,\ y(t) = \sin t$ doesn't lie on any plane algebraic curve in $\mathbb{A}^2$.

xtwovarirred **1.12.3.** Using counting constants, prove that most (nonhomogeneous) polynomials in two or more variables are irreducible.

xstaysirreducible **1.12.4.** Let $f(x, y, z)$ be a homogeneous polynomial not divisible by z. Prove that f is irreducible if and only if $f(x, y, 1)$ is irreducible.

xcoordtriangle **1.12.5.** **(i)** Describe the points that lie in the interior of the coordinate triangle in Figure1.2.11.
(ii) What can be deduced about the equation of the conic that is depicted in the figure?

xaffconictwo **1.12.6.** Prove that all affine conics can be put into one of the forms 1.1.6 by linear changes of variable, translations, and scalar multiplication.

xdiagform **1.12.7.** **(i)** Classify conics in $\mathbb{P}^2$ by writing an irreducible quadratic polynomial in three variables in the form X^tAX where A is symmetric, and diagonalizing this quadratic form.
(ii) Quadrics in projective space $\mathbb{P}^n$ are zero sets of irreducible homogeneous quadratic polynomials in $x_0, ..., x_n$. Classify quadrics in $\mathbb{P}^3$.

xlociequal **1.12.8.** Let f and g be irreducible homogeneous polynomials in x, y, z. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ are equal, then $g = cf$.

xcubicsing **1.12.9.** Without using Bézout's Theorem, prove that a plane cubic curve can have at most one singular point.

xtanconic **1.12.10.** Let C be the plane projective curve defined by the equation $x_0x_1 + x_1x_2 + x_2x_0 = 0$, and let p be the point $(-1, 2, 2)$. What is the equation of the tangent line to C at p?

xeqforcubic **1.12.11.** Let C be a smooth cubic curve in $\mathbb{P}^2$, and let p be a flex point of C. Choose coordinates so that p is the point $(0, 1, 0)$ and the tangent line to C at p is the line $\{z = 0\}$.
(i) Show that the coefficients of x^2y, xy^2, and y^3 in the defining polynomial f of C are zero.
(ii) Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form $y^2z + x^3 + axz^2 + bz^3$, and that $x^3 + ax + b$ will be a polynomial with distinct roots.
(iii) Show that one of the coefficients a or b can be eliminated, and therefore that smooth cubic curves in $\mathbb{P}^2$ depend on just one parameter.

xhessian **1.12.12.** Using Euler's formula together with row and column operations, show that the Hessian determinant is equal to $a \det H'$, where

$$H' = \begin{pmatrix} cf & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}, \quad a = \left(\tfrac{d-1}{x_0}\right)^2, \quad \text{and} \quad c = \tfrac{d}{d-1}$$

xhessianZero **1.12.13.** **(i)** Verify that the vanishing of the Hessian determinant isn't affected by a change of coordinates.
(ii) Prove that a smooth point of a curve is a flex point if and only if the Hessian determinant is zero, in this way: Given a smooth point p of X, choose coordinates so that $p = (0, 0, 1)$ and the tangent line ℓ is the line $\{x_1 = 0\}$. Then compute the Hessian.

xsymmfnindep **1.12.14.** Prove that the elementary symmetric functions $s_1 = x_1 + \cdots + x_n,\ \dots,\ s_n = x_1 \cdots x_n$ are algebaically independent.

xelementtransc **1.12.15.** Let K be a field extension of a field F, and let α be an element of K that is transcendental over F. Prove that every element of the field $F(\alpha)$ that isn't in F is transcendental over F.

xtdadds **1.12.16.** Let $td(K/F)$ denote the transcendence degree of a field extension K/F. Prove that, if $L \supset K \supset F$ are fields, then $td(L/F) = td(L/K) + td(L/F)$.

1.12.17. Let $f(x_0, x_1, x_2)$ be a homogeneous polynomial of degree d, let $f_i = \frac{\partial f}{\partial x_i}$, and let C be the plane curve $\{f = 0\}$. Use the following method to prove that the image in the dual plane of the set of smooth points of C is contained in a curve C^*: Let $N_r(k)$ be the dimension of the space of polynomials of degree $\leq k$ in r variables. Determine $N_r(k)$ for $r = 3$ and $r = 4$. Show that $N_4(k) > N_3(kd)$ if k is sufficiently large. Conclude that there is a nonzero polynomial $G(x_0, x_1, x_2)$ such that $G(f_0, f_1, f_2) = 0$. xdualis-curve

(This method doesn't give a good bound for the degree of C^*. One reason may be that f and its derivatives are related by Euler's Formula. It is tempting try using Euler's Formula to help compute the equation of C^*, but I haven't succeeded in getting anywhere that way.)

1.12.18. Let C be a smooth cubic curve in the plane $\mathbb{P}^2$, and let q be a generic point of $\mathbb{P}^2$. How many lines through q are tangent lines to C? xtangentq

1.12.19. Let X and Y be the surfaces in $\mathbb{A}^3_{x,y,z}$ defined by the equations $z^3 = x^2$ and $yz^2 + z + y = 0$, respectively. The intersection $C = X \cap Y$ is a curve. Determine the equation of its projection to the x, y-plane. xprojcurve

1.12.20. Complete the proof of Proposition 1.7.11 by computing the resultant of the polynomials x^m and $x^n - 1$. xercres-formula

1.12.21. Let f, g, and h be polynomials. Prove that xresfgh

(i) $\operatorname{Res}(f, gh) = \operatorname{Res}(f, g) \operatorname{Res}(f, h)$.

(ii) If the degree of gh is less than or equal to the degree of f, then $\operatorname{Res}(f, g) = \operatorname{Res}(f + gh, g)$.

1.12.22. With notation as in 1.7.3, suppose that a_0 and b_0 are not zero, and let α_i and β_j be the roots of $f(x, 1)$ and $g(x, 1)$, respectively. Show that $\operatorname{Res}(f, g) = a_0^n b_0^m \prod(\alpha_i - \beta_j)$. xres

1.12.23. Prove that a generic line meets a plane projective curve of degree d in d distinct points. gener-alline

1.12.24. Let $f = x^2 + xz + yz$ and $g = x^2 + y^2$. Compute the resultant $\operatorname{Res}_x(f, g)$ with respect to the variable x. xrest-wopol

1.12.25. Compute $\prod_{i \neq j}(\zeta^i - \zeta^j)$ when $\zeta = e^{2\pi i/n}$. xsignindisc

1.12.26. If $F(x) = \prod(x - \alpha_i)$, then $\operatorname{Discr}(F) = \pm \prod_{i<j}(\alpha_i - \alpha_j)^2$. Determine the sign. xsign

1.12.27. Let $f = a_0x^m + a_1x^{m-1} + \cdots a_m$ and $g = b_0x^n + b_1x^{n-1} + \cdots b_n$, and let $R = \operatorname{Res}(f, g)$ be the resultant of these polynomials. Prove that etdegres

(i) R is a polynomial that is homogeneous in each of the sets of variables a and b, and determine its degree.

(ii) If one assigns weighted degree i to the coefficients a_i and b_i, then R is homogeneous, of weighted degree mn.

1.12.28. Let coordinates in $\mathbb{A}^4$ be x, y, z, w, let Y be the variety defined by $z^2 = x^2 - y^2$ and $w(z - x) = 1$, and let π denote projection from Y to (x, y)-space. Describe the fibres and the image of π. xamap

1.12.29. Let p be a cusp of the curve C defined by a homogeneous polynomial f. Prove that there is just one line L through p such that the restriction of f to L has as zero of order > 2 at p, and that the order of zero for that line is precisely 3. xcusptan

1.12.30. Describe the intersection of the node $xy = 0$ at the origin with the unit 3-sphere in $\mathbb{A}^2$. geometry-ofnode

1.12.31. Prove that the Fermat curve $C : \{x^d + y^d + z^d = 0\}$ is connected by studying its projection to $\mathbb{P}^1$ from the point $(0, 0, 1)$. fermat-conn

1.12.32. Let $p(t, x) = x^3 + x^2 + t$. Then $p(0, x) = x^2(x + 1)$. Since x^2 and $x + 1$ are relatively prime, Hensel's Lemma predicts that p factors: $p = fg$, where g and g are polynomials in x whose coefficients are analytic functions in t, and f is monic, $f(0, x) = x^2$, and $g(0, x) = x + 1$. Determine this factorization up to degree 3 in t. Do the same for the polynomial $tx^4 + x^3 + x^2 + t$. xhensel

1.12.33. Let $f(t, y) = ty^2 - 4y + t$.

(i) Solve $f = 0$ for y by the quadratic formula, and sketch the real locus $f = 0$ in the t, y plane. xxhensellemma

(ii) What does Hensel's Lemma tell us about f?

(iii) Factor f, modulo t^4.

1.12.34. Factor $f(t,x) = x^3 + 2tx^2 + t^2x + x + t$, modulo t^2.

xxhensellemmatwo

xthreepts **1.12.35.** **(i)** Show that there is a conic C that passes through any five points of $\mathbb{P}^2$.
(ii) Use **(i)** to prove that a plane curve X of degree 4 can have at most three singular points.

xintconic **1.12.36.** By parametrizing a conic C, show that C meets a plane curve X of degree d and distinct from C in $2d$ points, when counted with multiplicity.

cuspcurvedual **1.12.37.** Using a generic projection to $\mathbb{P}^1$, determine the degree of the dual C^* of
(i) a plane cubic curve C with a cusp.
(ii) a plane curve C of degree 4 with three nodes.

xdualnode **1.12.38.** Let C be a cubic curve with a node. Determine the degree of the dual curve C^*, and the numbes of flexes, bitangents, nodes, and cusps of C and of C^*.

xcuspstandard **1.12.39.** Prove that every cusp (1.8.7) is analytically equivalent with the standard cusp.

1.12.40. Prove that a plane curve is connected.

xdefsing **1.12.41.** This is about the order of vanishing of the discriminant. With notation as in (**1.9.10**): If one perturbs the equation of C, the line L that meets C at p will be replaced by a finite set of nearby tangent lines. Choose particular examples for C in parts **(i b,c),(ii)** of (1.9.11) and compare the number of nearby tangents with the order of vanishing of the discriminant.

xsingsofcurve **1.12.42.** Analyze the singularities of the plane curve $x^3y^2 - x^3z^2 + y^3z^2 = 0$.

xthreecusps **1.12.43.** Exhibit an irreducible homogeneous polynomial $f(x,y,z)$ of degree 4 whose locus of zeros is a curve with three cusps.

notonline **1.12.44.** Let $f(x,y,z)$ be an irreducible homogeneous polynomial of degree > 1. Prove that the locus $f = 0$ in $\mathbb{P}^2$ contains three points that do not lie on a line.

Chapter 2 AFFINE ALGEBRAIC GEOMETRY

affine

The next chapters are about varieties of arbitrary dimension. We will use some of the terminology, such as discriminant and transcendence degree, that was introduced in Chapter 1, but many of the results in Chapter 1 won't be used again until we come back to curves in Chapter 8.

We begin with a review of some basic facts about rings and modules, omitting proofs. Give the next section a quick read, but don't spend too much time on it. You can refer back as needed, and look up information on the concepts that aren't familiar.

2.1 Rings and Modules

ringreview

By the word 'ring', we mean *commutative ring*: $ab = ba$, unless the contrary is stated explicitly. A commutative ring has two associative and commutative operations, addition and multiplication, that are related by the distributive law. It contains additive and multiplicative identity elements denoted by 0 and 1, respectively and it is a group with the operation of addition,

As before, a *domain* is a ring that has no zero divisors and isn't the zero ring. An *algebra* is a ring that contains the field $\mathbb{C}$ of complex numbers as a subring.

A set of elements $\alpha = \{\alpha_1, ..., \alpha_n\}$ *generates* an algebra A if every element of A can be expressed, usually not uniquely, as a polynomial in $\alpha_1, ..., \alpha_n$, with complex coefficients. Another way to state this is that the set α generates A if the homomorphism $\mathbb{C}[x_1, ..., x_n] \xrightarrow{\tau} A$ that evaluates a polynomial at $x = \alpha$ is surjective. If α generates A, then A will be isomorphic to the quotient $\mathbb{C}[x]/I$ of the polynomial algebra $\mathbb{C}[x]$, where I is the kernel of τ. A *finite-type algebra* is an algebra that can be generated by a finite set of elements.

We usually regard an R-module M as a *left module*, writing the scalar product of an element m of M by an element a of R as am. However, it is sometimes convenient to view M as a right module, writing ma instead of am. So we define $ma = am$. This is permissible when the ring is commutative.

A homomorphism of modules $M \to N$ over a ring R may also be called an *R-linear map*. When we say that a map is linear without mentioning a ring, we mean a $\mathbb{C}$-linear map, a homomorphism of vector spaces.

The term 'generate' is used in a second way, for modules and ideals. A set $(m_1, ..., m_k)$ of elements of an R-module M *generates* M if every element of M can be obtained as a combination $r_1m_1 + \cdots + r_km_k$ with coefficients r_i in R, or that the homomorphism from the free R-module R^k to M that sends a vector $(r_1, ..., r_k)$ to the combination $r_1m_1 + \cdots + r_km_k$ is surjective. A set $(m_1, ..., m_k)$ that generates M is an *R-basis* if every element of M is a combination in a unique way, or if $r_1m_1 + \cdots + r_km_k = 0$ only when $r_1 = \cdots = r_k = 0$. A module M that has a basis of order k is a *free R-module*, of rank k.

A set of elements $(\alpha_1,...,\alpha_k)$ generates an ideal I of a ring R it generates I as sR-module, if every element of I can be written as a combination $r_1\alpha_1+\cdots+r_k\alpha_k$, with r_i in R.

A *finite* module M is one that is spanned, or generated, by some finite set of elements. A ideal I of a ring R is *finitely generated* if it is a finite R-module.

It is important not to confuse the concept of a finite module with that of a finite-type algebra. An R-module M is a finite module if every element of M can be written as a *(linear) combination* $r_1m_1+\cdots+r_km_k$ of some finite set $\{m_1,...,m_k\}$ of elements of M, with coefficients in the ring R. A finite-type algebra A is an algebra in which every element can be written as a *polynomial* $f(\alpha_1,...,\alpha_k)$ in some finite set $\{\alpha_1,...,\alpha_k\}$ of elements of A, with complex coefficients.

If I and J are ideals of a ring R, the *product ideal*, which is denoted by IJ, is the ideal whose elements are finite sums of products $\sum a_ib_i$, with $a_i\in I$ and $b_i\in J$. The product ideal is usually different from the product *set* whose elements are products ab. The product set may not be an ideal.

The *power* I^k of I is the product of k copies of I — the ideal generated by products of k elements of I. The intersection $I\cap J$ of two ideals is an ideal, and

(2.1.1) $$(I\cap J)^2\subset IJ\subset I\cap J$$ intersect-product

An ideal M of a ring R is a *maximal ideal* if M isn't the unit ideal, $M<R$, and if there is no ideal I with $M<I<R$. An ideal M is a maximal ideal if and only if the quotient ring R/M is a field.

An ideal P of a R is a *prime ideal* if the quotient R/P is a domain. A maximal ideal is a prime ideal.

2.1.2. Lemma. *Let P be an ideal of a ring R, not the unit ideal. The following conditions are equivalent.* *defprime*

(i) *P is a prime ideal.*

(ii) *If a and b are elements of R, and if the product ab is in P, then $a\in P$ or $b\in P$.*

(iii) *If A and B are ideals of R, and if the product ideal AB is contained in P, then $A\subset P$ or $B\subset P$.* $\square$

The following equivalent version of **(iii)** is sometimes convenient:

(iii') *If A and B are ideals that contain P, and if the product ideal AB is contained in P, then $A=P$ or $B=P$.* $\square$

2.1.3. Lemma. *Let $A\xrightarrow{\varphi}B$ be a ring homomorphism. The inverse image of a prime ideal of B is a prime ideal of A.* $\square$ *invim-prime*

2.1.4. the Mapping Property of quotients. *mapprop*

(i) *Let K be an ideal of a ring R, let $R\xrightarrow{\tau}\overline{R}$ denote the canonical map from R to the quotient ring $\overline{R}=R/K$, and let S be another ring. Ring homomorphisms $\overline{R}\xrightarrow{\overline{\varphi}}S$ correspond bijectively to ring homomorphisms $R\xrightarrow{\varphi}S$ whose kernels contain K, the correspondence being $\varphi=\overline{\varphi}\circ\tau$:*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\tau"'] & S \arrow[d, equal] \\ \overline{R} \arrow[r, "\overline{\varphi}"] & S \end{tikzcd}$$

If $\ker\,\varphi=I,\ then\ \ker\,\overline{\varphi}=I/K$.

(ii) *Let M and N be modules over a ring R, let L be a submodule of M, and let $M\xrightarrow{\tau}\overline{M}$ denote the canonical map from M to the quotient module $\overline{M}=M/L$. Homomorphisms of modules $\overline{M}\xrightarrow{\overline{\varphi}}N$ correspond bijectively to homomorphisms $M\xrightarrow{\varphi}N$ whose kernels contain L, the correspondence being $\varphi=\overline{\varphi}\circ\tau$. If* $\ker\,\varphi=L,\ then\ \ker\,\overline{\varphi}=L/L$. $\square$

The word *canonical* that appears here is used often, to mean a construction that is the natural one in the given context. Exactly what this means is usually left unspecified.

(**2.1.5**) **commutative diagrams** commdiag

In the diagram displayed above, the maps $\overline{\varphi}\tau$ and φ from R to S are equal. This is referred to by saying that the diagram is *commutative*. A commutative diagram is one in which every map that can be obtained by composing its arrows depends only on the domain and range of that map. In these notes, almost all diagrams of maps are commutative. We won't mention commutativity most of the time. □

corrthm **2.1.6. Correspondence Theorem.**

(i) *Let $R \xrightarrow{\varphi} S$ be a surjective ring homomorphism with kernel K. (For instance, φ might be the canonical map from R to the quotient ring R/K. In any case, S will be isomorphic to R/K.) There is a bijective correspondence*

$$\{\textit{ideals of } R \textit{ that contain } K\} \quad \longleftrightarrow \quad \{\textit{ideals of } S\}$$

This correspondence associates an ideal I of R that contains K with its image $\varphi(I)$ in S and it associates an ideal J of S with its inverse image $\varphi^{-1}(J)$ in R.

If an ideal I of R that contains K corresponds to the ideal J of S, then φ induces an isomorphism of quotient rings $R/I \to S/J$. If one of the ideals, I or J, is prime or maximal, they both are.

(ii) *Let R be a ring, and let $M \xrightarrow{\varphi} N$ be a surjective homomorphism of R-modules with kernel L. There is a bijective correspondence*

$$\{\textit{submodules of } M \textit{ that contain } L\} \quad \longleftrightarrow \quad \{\textit{submodules of } N\}$$

This correspondence associates a submodule V of M that contains L with its image $\varphi(V)$ in N and it associates a submodule W of N with its inverse image $\varphi^{-1}(W)$ in M. □

comax **2.1.7. Chinese Remainder Theorem.** *Let $I_1, ..., I_k$ be comaximal ideals of a ring R.*

(i) *The product ideal $I_1 \cdots I_k$ is equal to the intersection $I_1 \cap \cdots \cap I_k$.*

(ii) *The map $R \longrightarrow R/I_1 \times \cdots \times R/I_k$ that sends an element a of R to the vector of its residues in R/I_ν is a surjective homomorphism. Its kernel is $I_1 \cap \cdots \cap I_k$, or $I_1 \cdots I_k$.*

(iii) *Let M be an R-module, and let $M_\nu = M/I_\nu$. The canonical homomorphism $M \to M_1 \times \cdots \times M_k$ is surjective.* □

Ideals $I_1, ..., I_k$ of a ring R are said to be *comaximal* if the sum of any two of them is the unit ideal.

quotof-prod **2.1.8. Proposition.** *Let R be a product of rings, $R = R_1 \times \cdots \times R_k$, let I be an ideal of R. There are ideals I_j of R_j such that $I = I_1 \times \cdots \times I_k$ and $R/I = R_1/I_1 \times \cdots \times R_k/I_k$.* □

noethr **(2.1.9) Noetherian rings**

A ring R is *noetherian* if all of its ideals are finitely generated. The ring $\mathbb{Z}$ of integers is noetherian. Fields are notherian. If I is an ideal of a noetherian ring R, the quotient ring R/I is noetherian.

basisthm **2.1.10. Hilbert Basis Theorem.** *Let R be a noetherian ring. The ring $R[x_1, ..., x_n]$ of polynomials with coefficients in R is noetherian.* □

Thus $\mathbb{Z}[x_1, ..., x_n]$ and $F[x_1, ..., x_n]$, F a field, are noetherian rings.

qnoeth **2.1.11. Corollary.** *Every finite-type algebra is noetherian.* □

ascchcond **(2.1.12) the ascending chain condition**

The condition that a ring R be noetherian can be rewritten in several ways that we review here.

Our convention is that, if X' and X are sets, the notation $X' \subset X$ means that X' is a subset of X, while $X' < X$ means that X' is a subset that is distinct from X. A *proper subset* X' of a set X is a nonempty subset distict from X, a set such that $\emptyset < X' < X$.

A sequence $X_1, X_2, ...$, finite or infinite, of subsets of a set Z forms an *increasing chain* if $X_n \subset X_{n+1}$ for all n, equality $X_n = X_{n+1}$ being permitted. If $X_n < X_{n+1}$ for all n, the chain is *strictly increasing*.

When $\mathcal{S}$ is a set whose elements are subsets of a set Z, we may refer to an element of $\mathcal{S}$ as a *member* of $\mathcal{S}$ to avoid confusion with the elements of Z. A member of $\mathcal{S}$ is a subset of Z. So the words 'member' and 'element' are synonymous.

A member M of $\mathcal{S}$ is a *maximal member* if it isn't properly contained in another member — if there is no member M' of $\mathcal{S}$ such that such that $M < M'$. For example, the set of proper subsets of a set of five elements contains five maximal members, the subsets of order four. The set of finite subsets of the set of integers contains no maximal member.

A maximal ideal of a ring R is a maximal member of the set of ideals of R that are different from the unit ideal.

2.1.13. Proposition. *The following conditions on a ring R are equivalent:* noether-conds

(i) *Every ideal of R is finitely generated.*

(ii) *The* ***ascending chain condition****: Every strictly increasing chain $I_1 < I_2 < \cdots$ of ideals of R is finite.*

(iii) *Every nonempty set of ideals of R contains a maximal member.* □

It is customary, though ungrammatical, to say that a ring has the ascending chain condition if it has no infinite, strictly increasing sequence of ideals.

The next corollaries follow from the ascending chain condition, though the conclusions are true whether or not R is noetherian.

2.1.14. Corollary. *Let R be a noetherian ring.* idealin-maximal

(i) *Every ideal of R except the unit ideal is contained in a maximal ideal.*

(ii) *A nonzero ring R contains at least one maximal ideal.*

(iii) *An element of R that isn't in any maximal ideal is a unit — an invertible element of R.* □

2.1.15. Corollary. *Let $s_1, ..., s_k$ be elements that generate the unit ideal of a ring R. For any positive integer n, the powers $s_1^n, ..., s_k^n$ generate the unit ideal.* □ powers-generate

2.1.16. Proposition. *Let R be a noetherian ring, and let M be a finite R-module.* noetheri-anmodule

(i) *Every submodule of M is a finite module.*

(ii) *The set of submodules of M satisfies the ascending chain condition.*

(iii) *Every nonempty set of submodules of M contains a maximal member.* □

(**2.1.17**) exact sequences exactseq

An *exact sequence*

$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

of modules over a ring R is a sequence of homomorphisms, finite or infinite, such that for all k, the image of d^{k-1} is equal to the kernel of d^k. For instance, a sequence $0 \to V \xrightarrow{d} V'$ is exact if d is injective, and a sequence $V \xrightarrow{d} V' \to 0$ is exact, if d is surjective.

A *short exact sequence* is an exact sequence of the form

$$0 \to V_0 \xrightarrow{a} V_1 \xrightarrow{b} V_2 \to 0 \tag{2.1.18}$$

To say that this sequence is exact means that the map a is injective, and that b induces an isomorphism from the quotient module V_1/aV_0 to V_2. sexseq

The short exact sequence (2.1) *splits* if there is a map $V_1 \xleftarrow{s} V_2$ such that bs is the identity on V_2. If the sequence splits, V_1 will be isomorphic to the direct sum $V_1 \oplus V_2$.

Let $V \xrightarrow{d} V'$ be a homomorphism of R-modules, and let W be the image of d. The *cokernel* of d is the module $C = V'/W \quad (= V'/dV)$. The homomorphism d embeds into an exact sequence

$$0 \to K \to V \xrightarrow{d} V' \to C \to 0 \tag{2.1.19}$$

kercok-erseq

where K and C are the kernel and cokernel of d, respectively.

A module homomorphism $V' \xrightarrow{f} M$ with cokernel C induces a homomorphism $C \to M$ if and only if the composed homomorphism fd is zero. This follows from the mapping property Corollary 2.1.4**(ii)**.

Let V be a finite-dimensional $\mathbb{C}$-module V (a vector space). The *dual module* V^* is the module of linear maps (homomorphisms of $\mathbb{C}$-modules) $V \to \mathbb{C}$. When $V \xrightarrow{d} V'$ is a homomorphism of $\mathbb{C}$-modules, there is a canonical dual homomorphism $V^* \xleftarrow{d^*} V'^*$. The dual of the sequence (2.1.19) is an exact sequence

$$0 \leftarrow K^* \leftarrow V^* \xleftarrow{d^*} V'^* \leftarrow C^* \leftarrow 0$$

So the dual of the kernel K is the cokernel of d^*, which is K^*, and the dual of the cokernel C is the kernel C^* of d^*. This is the reason for the term "cokernel".

snake **2.1.20. Proposition.** *(functorial property of the kernel and cokernel) Suppose given a diagram of R-modules*

$$\begin{tikzcd}
 & V \arrow[r, "u"] \arrow[d, "f"'] & V' \arrow[r] \arrow[d, "f'"'] & V'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & W \arrow[r] & W' \arrow[r, "v"'] & W'' &
\end{tikzcd}$$

whose rows are exact sequences. Let K, K', K'' and C, C', C'' denote the kernels and cokernels of f, f', and f'', respectively.

(i) (kernel is left exact) *The kernels form an exact sequence $K \to K' \to K''$. If u is injective, the sequence $0 \to K \to K' \to K''$ is exact.*

(ii) (cokernel is right exact) *The cokernels form an exact sequence $C \to C' \to C''$. If v is surjective, the sequence $C \to C' \to C'' \to 0$ is exact.*

(iii) (Snake Lemma) *There is a canonical homomorphism $K'' \xrightarrow{d} C$ that combines with the sequences above to form an exact sequence*

$$K \to K' \to K'' \xrightarrow{d} C \to C' \to C''.$$

If u is injective and/or v is surjective, the sequence remains exact with zeros at the appropriate ends. □

present-module (2.1.21) presenting a module

Let R be a ring. A *presentation* of an R-module M is an exact sequence of modules of the form

$$R^\ell \to R^k \to M \to 0$$

The map $R^\ell \to R^k$ will be given by an $\ell \times k$ *R-matrix*, a matrix with entries in R, and that matrix determines the module M up to isomorphism as the kernel of that map.

Every finite module over a noetherian ring R has a presentation. To obtain a presentation, one chooses a finite set $m = (m_1, ..., m_k)$ of generators for the finite module M, so that multiplication by m defines a surjective map $R^k \to M$. Let N be the kernel of that map. Because R is noetherian, N is a finite module. Next, one chooses a finits set of generators of N, which gives us a surjective map $R^\ell \to N$. Composition of that map with the inclusion $N \subset R^\ell$ produces an exact sequence $R^\ell \to R^k \to M \to 0$.

sumprod (2.1.22) direct sum and direct product

Let M and N be modules over a ring R. The *product module* $M \times N$ is the product set, whose elements are pairs (m, n), with m in M and n in N. The laws of composition are the same as the laws for vectors: $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ and $r(m, n) = (rm, rn)$. There are homomorphisms $M \xrightarrow{i_1} M \times N$ and $M \times N \xrightarrow{\pi_1} M$, defined by $i_1(m) = (m, 0)$ and $\pi_1(m, n) = m$, and similarly, homomorphisms $N \xrightarrow{i_2} M \times N$ and $M \times N \xrightarrow{\pi_2} N$. So i_1 and i_2 are inclusions, and π_1 and π_2 are projections.

The product module is characterized by this mapping property:

• Let T be an R-module. Homomorphisms $T \xrightarrow{\varphi} M \times N$ correspond bijectively to pairs of homomorphisms $T \xrightarrow{\alpha} M$ and $T \xrightarrow{\beta} N$. The homomorphism φ that corresponds to the pair α, β is $\varphi(m,n) = (\alpha m, \beta n)$, and when φ is given, the homomorphisms to M are $\alpha = \pi_1\varphi$ and $\beta = \pi_2\varphi$.

There is another product, the *tensor product* module $M \otimes_R N$ which is defined below.

The product module $M \times N$ is isomorphic to the *direct sum* $M \oplus N$. Elements of $M \oplus N$ can be written either as $m + n$, or with product notation, as (m, n).

The direct sum $M \oplus N$ is characterized by this mapping property:

• Let S be an R-module. Homomorphisms $M \oplus N \xrightarrow{\psi} S$ correspond bijectively to pairs u, v of homomorphisms $M \xrightarrow{u} S$ and $N \xrightarrow{v} S$. The homomorphism ψ that corresponds to the pair u, v is $\psi(m,n) = um + vn$, and when ψ is given, $u = \psi i_1$ and $v = \psi i_2$.

We use the direct product and direct sum notations interchangeably, but we note that the direct sum of an infinite set of modules isn't the same as the product.

(2.1.23) localization localiz

This is a preliminary dicsussion of an important construction. We will come back to it in Section 2.7.

Let s be a nonzero element of a domain A. The ring $A[s^{-1}]$, obtained by adjoining an inverse of s to A is called a *localization* of A. If $A[z]$ denotes the ring of polynomials in one variable z, with coefficients in A, the localization is isomorphic to the quotient $A[z]/(sz-1)$ of $A[z]$ modulo the principal ideal generated by $sz - 1$. The residue of z becomes the inverse of s. We will often denote this localization by A_s. If A is a finite-type domain, so is A_s.

(2.1.24) localizing a module locmod

Let A be a domain, and let M be an A-module. Let's regard M as a right module here. A *torsion element* of M is an element that is annihilated by some nonzero element s of A: $ms = 0$. A nonzero element m such that $ms = 0$ is an s-*torsion* element.

The set of all torsion elements of M is the *torsion submodule* of M, and a module whose torsion submodule is zero is *torsion-free*.

Let s be a nonzero element of a domain A. The *localization* M_s of an A-module M is defined in the natural way, as the A_s-module whose elements are equivalence classes of fractions $m/s^r = ms^{-r}$, with m in M and $r \geq 0$. An alternate notation for the localization M_s is $M[s^{-1}]$.

The only complication comes from the fact that M may contain s-torsion elements. If $ms = 0$, then m must map to zero in M_s, because in M_s, we will have $mss^{-1} = m$. To define M_s, one must to modify the equivalence relation, as follows: Two fractions $m_1 s^{-r_1}$ and $m_2 s^{-r_2}$ are defined to be equivalent if $m_1 s^{r_2+n} = m_2 s^{r_1+n}$ when n is sufficiently large. This takes care of torsion, and M_s becomes an A_s-module. There is a homomorphism $M \to M_s$ that sends an element m to the fraction $m/1$. If M is an s-torsion module, then $M_s = 0$.

In this definition of localization, it isn't necessary to assume that $s \neq 0$. But if $s = 0$, then $M_s = 0$ for every module M.

(2.1.25) tensor products tensprod

Let U and V be modules over a ring R. The *tensor product* $U \otimes_R V$ is an R-module that is generated by elements $u \otimes v$ called *tensors*, one for each u in U and each v in V. The elements of the tensor product are combinations $\sum_1^k r_i(u_i \otimes v_i)$ of tensors with coefficients in R.

The defining relations among the tensors are the *bilinear relations*:

bilinrels (2.1.26)
$$(u_1{+}u_2)\otimes v = u_1\otimes v{+}u_2\otimes v\,,\quad u\otimes(v_1{+}v_2) = u\otimes v_1{+}u\otimes v_2$$

and
$$r(u\otimes v) = (ru)\otimes v = u\otimes(rv)$$

for all u in U, v in V, and r in R. The symbol $\otimes$ is used as a reminder that tensors are to be manipulated using these relations.

One can absorb a coefficient from R into either one of the factors of a tensor, so every element of $U\otimes_R V$ can be written as a finite sum $\sum u_i\otimes v_i$ with u_i in U and v_i in V.

colxrow **2.1.27. Examples. (i)** If U is the space of m dimensional (complex) column vectors, and V is the space of n-dimensional row vectors, then $U\otimes_{\mathbb{C}} V$ identifies naturally with the space of $m\times n$-matrices.
(ii) If U and V are free R-modules of ranks m and n, with bases $\{u_1,...,u_m\}$ and $\{v_1,...,v_n\}$, respectively, the tensor product $U\otimes_R V$ is a free R-module of rank mn, with basis $\{u_i\otimes v_j\}$. In contrast, the product module $U\times V$ is a free module of rank $m+n$, with basis $\{(u_i,0)\}\cup\{(0,v_j)\}$.

There is an obvious map of sets

bilin (2.1.28)
$$U\times V \xrightarrow{\beta} U\otimes_R V$$

from the product *set* to the tensor product, that sends a pair (u,v) to the tensor $u\otimes v$. This map isn't a homomorphism of R-modules. The defining relations (2.1.26) show that β is R-*bilinear*, not R-linear.

The next corollary follows from the defining relations of the tensor product.

tensorbilin **2.1.29. Corollary.** *Let U,V, and W be R-modules. Homomorphisms of R-modules $U\otimes_R V \to W$ correspond bijectively to R-bilinear maps $U\times V\to W$.* □

Thus the map $U\times V \xrightarrow{\beta} U\otimes_R V$ is a *universal* bilinear map. Any R-bilinear map $U\times V \xrightarrow{f} W$ to a module W can be obtained from a module homomorphism $U\otimes_R V \xrightarrow{\widetilde{f}} W$ by composition with the bilinear map β: $U\times V \xrightarrow{\beta} U\otimes_R V \xrightarrow{\widetilde{f}} W$.

canonisom **2.1.30. Proposition.** *There are canonical isomorphisms*

- $U\otimes_R R\approx U$, *defined by* $u\otimes r \leftrightsquigarrow ur$
- $(U\oplus U')\otimes_R V\approx (U\otimes_R V)\oplus(U'\otimes_R V)$, *defined by* $(u_1+u_2)\otimes v \leftrightsquigarrow u_1\otimes v+u_2\otimes v$
- $U\otimes_R V\approx V\otimes_R U$, *defined by* $u\otimes v \leftrightsquigarrow v\otimes u$
- $(U\otimes_R V)\otimes_R W\approx U\otimes_R(V\otimes_R W)$, *defined by* $(u\otimes v)\otimes w \leftrightsquigarrow u\otimes(v\otimes w)$ □

rexacttensor **2.1.31. Proposition.** *tensor product is right exact Let $U \xrightarrow{f} U' \xrightarrow{g} U''\to 0$ be an exact sequence of R-modules. For any R-module V, the sequence*

$$U\otimes_R V \xrightarrow{f\otimes 1} U'\otimes_R V \xrightarrow{g\otimes 1} U''\otimes_R V\to 0$$

in which $[f\otimes 1](u\otimes v) = f(u)\otimes v$, is exact. □

Tensor product isn't left exact. For example, if $R=\mathbb{C}[x]$, then $R/xR\approx\mathbb{C}$. There is an exact sequence $0\to R\xrightarrow{x} R\to\mathbb{C}\to 0$. When we tensor with $\mathbb{C}$ we get a sequence $0\to\mathbb{C}\to\mathbb{C}\to\mathbb{C}\to 0$, in which the first map $\mathbb{C}\to\mathbb{C}$ is the zero map.

proof of Proposition 2.1.31. We suppose that an exact sequence of R-modules $U\xrightarrow{f} U'\xrightarrow{g} U''\to 0$ and another R-module V are given. We are to prove that the sequence $U\otimes_R V\xrightarrow{f\otimes 1} U'\otimes_R V\xrightarrow{g\otimes 1} U''\otimes_R V\to 0$ is exact. It is evident that the composition $(g\otimes 1)(f\otimes 1)$ is zero, and that $g\otimes 1$ is surjective. We must prove that $U''\otimes_R V$ is isomorphic to the cokernel of $f\otimes 1$.

Let C be the cokernel of $f\otimes 1$. The mapping property (2.1.4)**(ii)** gives us a canonical map $C\xrightarrow{\varphi} U''\otimes_R V$ that we want to show is an isomorphism. To show this, we construct the inverse of φ. We choose an element

v of V, and form a diagram of R-modules

$$\begin{tikzcd} U\times v \arrow[r] \arrow[d, "\beta_v"'] & U'\times v \arrow[r] \arrow[d, "\beta'_v"'] & U''\times v \arrow[r] \arrow[d, "\gamma_v"] & 0 \\ U\otimes_R V \arrow[r, "f\otimes 1"] & U'\otimes_R V \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

in which $U\times v$ denotes the module of pairs (u, v) with $u \in U$. It is isomorphic to U.

The rows in the diagram are exact sequences of modules. The vertical arrows β_v and β'_v are obtained by restriction from the canonical bilinear maps (2.1.28). They are R-linear because v is held constant. The map γ_v is determined by the definition of the cokernel, because the composition of the maps in the top row of the diagram is zero. Putting the maps γ_v together for all v in V gives us a bilinear map $U\times V \to C$. That bilinear map induces a linear map $U \otimes_R V \to C$, the inverse of φ. $\square$

2.1.32. Corollary. *Let U and V be modules over a domain R and let s be a nonzero element of R. Let R_s be the localization of RV.* *locistensor*
(i) *The localization $U_s$is isomorphic to $U\otimes_R(R_s)$.*
(ii) *Tensor product is compatible with localization: $U_s\otimes_{R_s}V_s \approx (U\otimes_R V)_s$*

proof. **(ii)** The composition of the canonical maps $U\times V \to U_s\times V_s \to U_s \otimes_{R_s} V_s$ is R-bilinear. It defines an R-linear map $U\otimes_R V \to U_s\otimes_{R_s} V_s$. Since s is inverible in $U_s\otimes_{R_s} V_s$, this map extends to an R_s-linear map $(U\otimes_R V)_s \to U_s\otimes_{R_s} V_s$. Next, we define an R_s-bilinear map $U_s\times V_s \to (U\otimes_R V)_s$ by mapping a pair (us^{-m}, vs^{-n}) to $(u\otimes v)s^{-m+n}$. This bilinear map induces the inverse map $U_s\otimes_{R_s}V_s \to (U\otimes_R V)_s$. $\square$

(2.1.33) extension of scalars extendscalars

Let $A \xrightarrow{\rho} B$ be a ring homomorphism. Extension of scalars is an operation that constructs an B-module from an A-module.

Let's write scalar multiplication on the right. So M will be a right A-module. Then $M\otimes_A B$ becomes a right B-module, scalar multiplication by $b \in B$ being defined by $(m\otimes b')b = m\otimes(b'b)$. This gives the functor

$$A\!-\!modules \xrightarrow{\otimes B} B\!-\!modules$$

called the *extension of scalars* from A to B.

(2.1.34) restriction of scalars resscal

If $A \xrightarrow{\rho} B$ is a ring homomorphism, a (left) B-module M can be made into an A-module by *restriction of scalars*. Scalar multiplication by an element a of A is defined by the formula

$$am = \rho(a)m \tag{2.1.35}$$ restrscal

It is customary to denote a module and the one obtained by restriction of scalars by the same symbol. But when it seems advisable, one can denote a B-module M and the A-module obtained from M by restriction of scalars by ${}_BM$ and ${}_AM$, respectively. The additive groups of ${}_BM$ and ${}_AM$ are the same.

For example, a module over the prime field $\mathbb{F}_p$ beomes a $\mathbb{Z}$-module by restriction of scalars. If $\overline{a}$ denotes the residue of an integer a in $\mathbb{F}_p$ and V is an $\mathbb{F}_p$-module, scalar multiplication in ${}_{\mathbb{Z}}V$ is defined in the obvious way, by $av = \overline{a}v$.

2.1.36. Lemma. *(extension and restriction of scalars are adjoint operators)* *xtscaltens*
Let $A \xrightarrow{\rho} B$ be a ring homomorphism, let M be an A-module, and let N be an B-module. Homomorphisms $M \xrightarrow{\varphi} {}_AN$ of A-modules correspond bijectively to homomorphisms of B-modules $M\otimes_A B \xrightarrow{\psi} {}_BN$. $\square$

This concludes our review of rings and modules.

2.2 The Zariski Topology

zartop

Affine algebraic geometry is a study of subsets of affine space that can be defined by systems of polynomial equations. Those subsets are the closed sets in the Zariski topology on $\mathbb{A}^n$, the *Zariski closed* sets. A *Zariski open* set is a set whose *complement*, the set of points not in U, is Zariski closed.

Let $f_1, ..., f_k$ be polynomials in $x_1, ..., x_n$. The set of points of $\mathbb{A}^n$ that solve the system of equations

fequat (2.2.1)
$$f_1 = 0\,,\ \ldots,\ f_k = 0$$

the *locus of zeros* of f, may be denoted by $V(f_1, ..., f_k)$ or by $V(f)$. Thus $V(f)$ is a Zariski closed set.

We use analogous notation for infinite sets. If $\mathcal{F}$ is any set of polynomials, $V(\mathcal{F})$ denotes the set of points of affine space at which all elements of $\mathcal{F}$ are zero. In particular, if I is an ideal of the polynomial ring, $V(I)$ denotes the set of points at which all elements of I vanish.

As before, the ideal I of $\mathbb{C}[x_1, ..., x_n]$ *generated* by polynomials $f_1, ..., f_k$ is the set of combinations $r_1f_1+\cdots+r_kf_k$ with polynomial coefficients r_i. Some notations for this ideal are $(f_1, ..., f_k)$ and (f). All elements of I vanish on the zero set $V(f)$, so $V(f) = V(I)$. The Zariski closed subsets of $\mathbb{A}^n$ can also be described as the sets $V(I)$, where I is an ideal.

An ideal isn't determined by its zero locus. For one thing, all powers f^k of a polynomial f have the same zeros as f.

IinJ **2.2.2. Lemma.** *Let I and J be ideals of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$.*
(i) *If $I \subset J$, then $V(I) \supset V(J)$.*
(ii) *$V(I^k) = V(I)$.*
(iii) *$V(I \cap J) = V(IJ) = V(I) \cup V(J)$.*
(iv) *If I_ν are ideals, then $V(\sum I_\nu)$ is the intersection $\bigcap V(I_\nu)$.*

proof. **(iii)** Recall that $(I \cap J)^2 \subset IJ \subset I \cap J$. Then **(i)** and **(ii)** show that $V(I \cap J) = V(IJ)$. Because I and J contain IJ, $V(IJ) \supset V(I) \cup V(J)$. To prove that $V(IJ) \subset V(I) \cup V(J)$, we note that $V(IJ)$ is the locus of common zeros of the products fg with f in I and g in J. Suppose that a point p is a common zero: $f(p)g(p) = 0$ for all f in I and all g in J. If there is an element f in I such that $f(p) \neq 0$, we must have $g(p) = 0$ for every g in J, and then p is a point of $V(J)$. If $f(p) = 0$ for all f in I, then p is a point of $V(I)$. In either case, p is a point of $V(I) \cup V(J)$. □

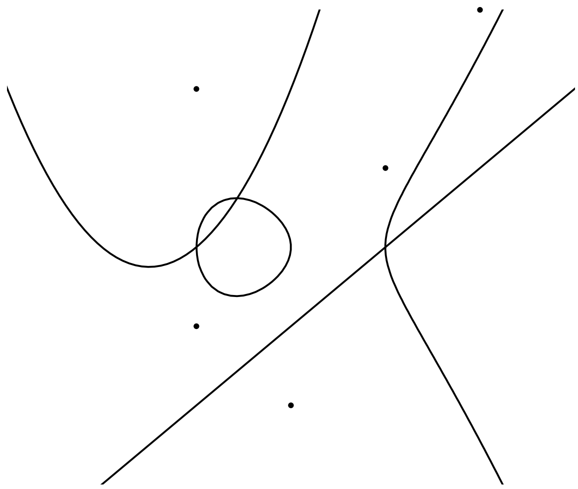
ztop **2.2.3.** To verify that the Zariski closed sets are the closed sets of a topology, one must show that

- the empty set and the whole space are Zariski closed,
- the intersection $\bigcap C_\nu$ of an arbitrary family of Zariski closed sets is Zariski closed, and
- the union $C \cup D$ of two Zariski closed sets is Zariski closed.

The empty set and the whole space are the zero sets of the elements 1 and 0, respectively. The other conditions follow from Lemma 2.2.2. □

ztopdimone **2.2.4. Example.** The proper Zariski closed subsets of the affine line, or of a plane affine curve, are the nonempty finite sets. The proper Zariski closed subsets of the affine plane are finite unions of points and curves. We omit the proofs of these facts. The corresponding facts for loci in the projective line and the projective plane have been noted before. (See (**1.3.4**) and (1.3.15).) □

2.2.5.



A Zariski closed subset of the affine plane (real locus)

A subset of a topological space X becomes a topological space with its *induced topology*. The closed (or open) subsets of a subset S in the induced topology are intersections $S \cap Y$, where Y is closed (or open) in X. When we speak of a subset S as a *subspace* of X, we mean that S is given the induced topology.

The topology induced on a subset S from the Zariski topology on $\mathbb{A}^n$ will be called the *Zariski topology* on S too. A subset of S is closed in its Zariski topology if it has the form $S \cap Z$ for some Zariski closed subset Z of $\mathbb{A}^n$. If S is a Zariski closed subset of $\mathbb{A}^n$, a closed subset of S can also be described as a closed subset of $\mathbb{A}^n$ that is contained in S.

2.2.6. Lemma. *Let $\{X^i\}$ be a covering of a topological space X by open sets. A subset V of X is open if and only if $V \cap X^i$ is open in X^i for every i, and V is closed if and only if $V \cap X^i$ is closed in X^i for every i. In particular, if $\{\mathbb{U}^i\}$ is the standard open covering of $\mathbb{P}^n$, a subset V of $\mathbb{P}^n$ is open (or closed) if and only if $V \cap \mathbb{U}^i$ is open (or closed) in $\mathbb{U}^i$ for every i.* $\square$ *closedinaff*

When two topologies T and T' on a set X are given, T' is said to be *coarser* than T if every closed set in T' is closed in T i.e., if T' contains fewer closed sets (or fewer open sets) than T, and T' is *finer* than T if it contains more closed sets (or more open sets) than T. The Zariski topology is coarser than the classical topology, and the next proposition shows that it is much coarser.

2.2.7. Proposition. *Every nonempty Zariski open subset of $\mathbb{A}^n$ is dense and path connected in the classical topology.* *opendense*

proof. The (complex) *line* L through distinct points p and q of $\mathbb{A}^n$ is a Zariski closed subset of $\mathbb{A}^n$, whose points can be written as $p + t(q - p)$, with t in $\mathbb{C}$. It corresponds bijectively to the affine t-line $\mathbb{A}^1$, and the Zariski closed subsets of L correspond to Zariski closed subsets of $\mathbb{A}^1$. They are the finite subsets, and L itself.

Let U be a nonempty Zariski open subset of $\mathbb{A}^n$, and let C be the Zariski closed complement of U. To show that U is dense in the classical topology, we choose distinct points p and q of $\mathbb{A}^n$, with p in U. If L is the line through p and q, $C \cap L$ will be a Zariski closed subset of L, a finite set that doesn't contain p. The complement of this finite set in L is $U \cap L$. In the classical topology, the closure of $U \cap L$ will be the whole line L. The closure of U contains the closure of $U \cap L$, which is L. So it contains q, and since q was arbitrary, the closure of U is $\mathbb{A}^n$.

Next, let L be the line through two points p and q of U. As before, $C \cap L$ will be a finite set of points. In the classical topology, L is a complex plane. The points p and q can be joined by a path in that plane that avoids a finite set. $\square$

Thus the Zariski topology is very different from the classical topology (**1.3.17**), but it is very useful in algebraic geometry. We will use the classical topology from time to time, but the Zariski topology will appear more often. Because of this, we refer to a Zariski closed subset simply as a *closed set*. Similarly, by an *open set* we mean a Zariski open set. We will mention the adjective "Zariski" only for emphasis.

irrclosed **(2.2.8) irreducible closed sets**

The fact that the polynomial algebra is a noetherian ring has an important consequence for the Zariski topology that we discuss here.

A topological space X has the *descending chain condition* on closed subsets there is no infinite, strictly descending chain $C_1 > C_2 > \cdots$ of closed subsets of X. (See (2.1.13).) The descending chain condition on closed subsets is equivalent with the *ascending chain condition* on open sets.

A *noetherian space* is a topological space that has the descending chain condition on closed sets. In a noetherian space, every nonempty family $\mathcal{S}$ of closed subsets has a minimal member, one that doesn't contain any other member of $\mathcal{S}$, and every nonempty family of open sets has a maximal member. (See (**2.1.12**).)

noethq-comp **2.2.9. Lemma.** *A noetherian topological space is quasicompact: Every open covering has a finite subcovering.* □

deschain **2.2.10. Proposition.** *With its Zariski topology, $\mathbb{A}^n$ is a noetherian space.*

proof. Suppose that a strictly descending chain $C_1 > C_2 > \cdots$ of closed subsets of $\mathbb{A}^n$ is given. Let I_j be the ideal of elements of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ that are identically zero on C_j. Then $C_j = V(I_j)$. The fact that $C_j > C_{j+1}$ implies that $I_j < I_{j+1}$. The ideals I_j form a strictly increasing chain. Since $\mathbb{C}[x_1, ..., x_n]$ is noetherian, that chain is finite. Therefore the chain C_j is finite. □

de-firrspace **2.2.11. Definition.** A topological space X is *irreducible* if it isn't the union of two proper closed subsets.

Another way to say that a topological space X is irreducible is this:

ir-redspacetwo **2.2.12.** *If C and D are closed subsets of an irreducible toplogical space X, and if $X = C \cup D$, then $X = C$ or $X = D$.*

The concept of irreducibility is useful primarily for noetherian spaces. The only irreducible subsets of a Hausdorff space are its points. So, in the classical topology, the only irreducible subsets of affine space are points.

Irreducibility may seem analogous to connectedness. A topological space is *connected* if it isn't the union $C \cup D$ of two proper *disjoint* closed subsets. However, the condition that a space be irreducible is much more restrictive because, in Definition 2.2.11, the closed sets C and D aren't required to be disjoint. In the Zariski topology on the affine plane, lines are irreducible closed sets. The union of two intersecting lines is connected, but not irreducible.

ir-redlemma **2.2.13. Lemma.** *The following conditions on a topological space X are equivalent.*

- *X is irreducible.*
- *The intersection $U \cap V$ of nonempty open subsets is nonempty.*
- *Every nonempty open subset U of X is dense — its closure is X.* □

The *closure* of a subset U of a topological space X, is the smallest closed subset of X that contains U. The closure exists because it is the intersection of all closed subsets that contain S.

closureinters **2.2.14. Lemma.** *Let Y be a subspace of a topological space X, let S be a subset of Y, and let C be the closure of S in X. The closure of S in Y is $C \cap Y$.*

proof. Let $\overline{S}$ be the closure of S in Y. It is is the intersection of the closed subsets of Y that contain S. A subset W is closed in Y if and only if $W = V \cap Y$ for some closed subset V of X, and if W contains S, so does V. The intersection of those subsets V is C. Then $\overline{S} = \bigcap W = \bigcap (V \cap Y) = \big(\bigcap V\big) \cap Y = C \cap Y$. □

quasicompact **2.2.15. Lemma. (i)** *The closure $\overline{Z}$ of a subspace Z of a topological space X is irreducible if and only if Z is irreducible.*

(ii) *A nonempty open subspace W of an irreducible space X is irreducible.*

(iii) *Let $Y \to X$ be a continuous map of topological spaces. The image of an irreducible subset D of Y is an irreducible subset of X.*

proof. **(i)** Let Z be an irreducible subset of X, and suppose that its closure $\overline{Z}$ is the union $\overline{C}\cup\overline{D}$ of two closed sets $\overline{C}$ and $\overline{D}$. Then Z is the union of the sets $C=\overline{C}\cap Z$ and $D=\overline{D}\cap Z$, and they are closed in Z. Therefore Z is one of those two sets, say $Z=C$. Then $Z\subset\overline{C}$, and since $\overline{C}$ is closed, $\overline{Z}\subset\overline{C}$. Because $\overline{C}\subset\overline{Z}$ as well, $\overline{C}=\overline{Z}$. Conversely, suppose that the closure $\overline{Z}$ of a subset Z of X is irreducible, and that Z is a union $C\cup D$ of closed subsets. Then $\overline{Z}=\overline{C}\cup\overline{D}$, and therefore $\overline{Z}=\overline{C}$ or $\overline{Z}=\overline{D}$. If $\overline{Z}=\overline{C}$, then $Z=\overline{C}\cap Z=C$ (2.2.14). So Z is irreducible.

(ii) See (2.2.13)). The closure of W is the irreducible space X.

(iii) Let D be an irreducible subspace of Y, and suppose that its image C is the union $C_1\cup C_2$ of closed subsets of C. The inverse image D_i of C_i is closed in D, and $D=D_1\cup D_2$. Therefore either $D_1=D$ or $D_2=D$. Say that $D_1=D$. Then the map $D\to C$ is surjective, and so is the map $D_1\to C_1$. Therefore $C_1=C$. □

2.2.16. Theorem. *In a noetherian topological space, every closed subset is the union of finitely many irreducible closed sets.* *unionirred*

proof. If a closed subset C_0 of a topological space X isn't a union of finitely many irreducible closed sets, then it isn't irreducible, so it is a union $C_1\cup D_1$, where C_1 and D_1 are proper closed subsets of C_0, and therefore closed subsets of X. Since C_0 isn't a finite union of irreducible closed sets, C_1 and D_1 cannot both be finite unions of irreducible closed sets. Say that C_1 isn't such a union. We have the beginning $C_0>C_1$ of a chain of closed subsets. We repeat the argument, replacing C_0 by C_1, and we continue in this way, to construct an infinite, strictly descending chain $C_0>C_1>C_2>\cdots$. So X isn't a noetherian space. □

2.2.17. Definition. An *affine variety* is an irreducible closed subset of affine space $\mathbb{A}^n$. defadffvar

Theorem 2.2.16 tells us that every closed subset of $\mathbb{A}^n$ is a finite union of affine varieties. Since an affine variety is irreducible, it is connected in the Zariski topology. An affine variety is also connected in the classical topology, but this isn't easy to prove. We may not get to the proof.

(2.2.18) noetherian induction noethind

In a noetherian space Z one can use *noetherian induction* in proofs. Suppose that a statement Σ is to be proved for every closed subvariety X of Z. It suffices to prove Σ for X under the assumption that it is true for every closed subvariety that is a proper subset of X. Or, to prove a statement Σ for every closed subset X of Z, it suffices to prove it for X under the assumption that Σ is true for every proper closed subset of X.

The justification of noetherian induction is similar to the justification of complete induction. Let $\mathcal{S}$ be the family of closed subvarieties for which Σ is false. If $\mathcal{S}$ isn't empty, it will contain a minimal member X. Then Σ will be true for every proper closed subvariety of X, etc.

(2.2.19) the coordinate algebra of a variety coordinateagebra

Let I be an ideal of R. The *radical* of I of is the set of elements α of R such that some power α^r is in I. It is an ideal that contains I. The radical will be denoted by $\operatorname{rad} I$:

(2.2.20) $$\operatorname{rad} I=\{\alpha\in R\,|\,\alpha^r\in I \text{ for some } r>0\}$$ raddef

An ideal that is equal to its radical is a *radical ideal*. A prime ideal is a radical ideal.

2.2.21. Lemma. **(i)** *An ideal I of a noetherian ring R contains a power of its radical.* *radpower*
(ii) *If I is an ideal of the polynomial ring $\mathbb{C}[x]$, then $V(I)=V(\operatorname{rad} I)$.*

proof. **(i)** Since R is noetherian, $\operatorname{rad} I$ is generated by a finite set of elements $\alpha=\{\alpha_1,...,\alpha_k\}$, and for large r, α_i^r is in I. We can use the same large integer r for every i. A monomial $\beta=\alpha_1^{e_1}\cdots\alpha_k^{e_k}$ of sufficiently large degree n in α will be divisible α_i^r for at least one i, and therefore it will be in I. The monomials of degree n generate $(\operatorname{rad} I)^n$, so $(\operatorname{rad} I)^n\subset I$. □

Consequently, if I and J are ideals and if $\operatorname{rad} I = \operatorname{rad} J$, then $V(I) = V(J)$. The converse of this statement is also true: If $V(I) = V(J)$, then $\operatorname{rad} I = \operatorname{rad} J$. This is a consequence of the *Strong Nullstellensatz* that is proved below (see (2.4.9)).

Because $(I \cap J)^2 \subset IJ \subset I \cap J$,

radIJ (2.2.22)
$$\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J)$$

Also, $\operatorname{rad}(I \cap J) = (\operatorname{rad} I) \cap (\operatorname{rad} J)$. Therefore $V(\operatorname{rad}(I \cap J)) = V(I) \cup V(J)$.

Recall that $V(P)$ denotes the set of points of affine space at which all elements of P vanish.

irredprime **2.2.23. Proposition.** *The affine varieties in $\mathbb{A}^n$ are the sets $V(P)$, where P is a prime ideal of the polynomial algebra $\mathbb{C}[x] = \mathbb{C}[x_1, ..., x_n]$. If P is a radical ideal of $\mathbb{C}[x]$, then $V(P)$ is an affine variety if and only if P is a prime ideal.*

We will use Proposition 2.2.23 in the next section, where we give a few examples of varieties, but we defer the proof to Section 2.5, where the proposition will be proved in a more general form. (See Proposition 2.5.13).)

defco-ordalg **2.2.24. Definition.** Let P be a prime ideal of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, and let V be the affine variety $V(P)$ in $\mathbb{A}^n$. The *coordinate algebra* of V is the quotient algebra $A = \mathbb{C}[x]/P$.

Geometric properties of the variety are reflected in algebraic properties of its coordinate algebra and vice versa. In a primitive sense, one can regard the geometry of an affine variety V as given by closed subsets and incidence relations — the inclusion of one closed set into another, as when a point lies on a line. A finer study of the geometry takes into account other things, tangency, for instance, but it is reasonable to begin by studying incidences $C' \subset C$ among closed subvarieties. Such incidences translate into inclusions $P' \supset P$ in the opposite direction among prime ideals.

2.3 Some affine varieties

somevarieties This section contains a few simple examples of varieties.

ptvar **2.3.1.** A *point* $p = (a_1, \dots, a_n)$ of affine space $\mathbb{A}^n$ is irreducible, so it is a variety. It is the set of solutions of the n equations $x_i - a_i = 0$, $i = 1, \dots, n$. The polynomials $x_i - a_i$ generate a maximal ideal in the polynomial algebra $\mathbb{C}[x]$, and a maximal ideal is a prime ideal. We will denote the maximal ideal that corresponds to the point p by $\mathfrak{m}_p$. It is the kernel of the substitution homomorphism $\pi_p : \mathbb{C}[x] \to \mathbb{C}$ that evaluates a polynomial $g(x_1, ..., x_n)$ at p: $\pi_p(g) = g(a_1, ..., a_n) = g(p)$.

The coordinate algebra of the point p is the quotient $\mathbb{C}[x]/\mathfrak{m}_p$. It is called the *residue field* at p, and it will be denoted by $k(p)$. The residue field $k(p)$ is isomorphic to the image of π_p, the field of complex numbers, but it is a particular quotient of the polynomial ring.

varinline **2.3.2.** The varieties in the affine line $\mathbb{A}^1$ are its points and the whole line $\mathbb{A}^1$. The varieties in the affine plane $\mathbb{A}^2$ are points, plane affine curves, and the whole plane.

This is true because the varieties correspond to the prime ideals of the polynomial ring. The prime ideals of $\mathbb{C}[x_1, x_2]$ are the maximal ideals, the principal ideals generated by irreducible polynomials, and the zero ideal. The proof is an exercise.

hsurf **2.3.3.** The set X of solutions of a single irreducible polynomial equation $f_1(x_1, ..., x_n) = 0$ in $\mathbb{A}^n$ is a variety called an *affine hypersurface*.

A hypersurface in the affine plane $\mathbb{A}^2$ is a plane affine curve. The *special linear group SL_2*, the group of complex 2×2 matrices with determinant 1, is a hypersurface in $\mathbb{A}^4$. It is the locus of zeros of the irreducible polynomial $x_{11}x_{22} - x_{12}x_{21} - 1$.

The reason that an affine hypersurface is a variety is that an irreducible element of a unique factorization domain is a prime element, and a prime element generates a prime ideal. The polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is a unique factorization domain.

apc **2.3.4.** A *line* in the plane, the locus of a linear equation $ax + by = c$, is a plane affine curve. Its coordinate algebra, which is $\mathbb{C}[x, y]/(ax + by - c)$, is isomorphic to a polynomial ring in one variable. Every line is isomorphic to the affine line $\mathbb{A}^1$.

2.3.5. Let $p = (a_1, \ldots, a_n)$ and $q = (b_1, \ldots, b_n)$ be distinct points of $\mathbb{A}^n$. The *point pair* (p, q) isn't irreducible, so it isn't a variety. It is the closed set defined by the system of n^2 equations $(x_i - a_i)(x_j - b_j) = 0$, $1 \le i, j \le n$, and the ideal I generated by the polynomials $(x_i - a_i)(x_j - b_j)$ isn't a prime ideal. The next corollary, which follows from the Chinese Remainder Theorem 2.1.7, describes that ideal: pointpr

2.3.6. Corollary. *The ideal I of polynomials that vanish on a point pair p, q is the product $\mathfrak{m}_p \mathfrak{m}_q$ of the maximal ideals at those points, and the quotient algebra $\mathbb{C}[x]/I$ is isomorphic to the product algebra $\mathbb{C} \times \mathbb{C}$.* *pointpair*
□

2.4 Hilbert's Nullstellensatz

The Hilbert Nullstellesatz establishes the fundamental relation between affine algebraic geometry and algebra. It identifies the points of an affine variety with maximal ideals. null

2.4.1. Nullstellensatz (version 1). *Let $\mathbb{C}[x]$ be the polynomial algebra in the variables $x_1, \ldots, x_n$. There are bijective correspondences between the following sets:* *nullone*

- *points p of the affine space $\mathbb{A}^n$,*
- *algebra homomorphisms $\pi_p : \mathbb{C}[x] \to \mathbb{C}$,*
- *maximal ideals $\mathfrak{m}_p$ of $\mathbb{C}[x]$.*

The homomorphism π_p evaluates a polynomial at a point p of $\mathbb{A}^n$. If $p = (a_1, ..., a_n)$, then $\pi_p(g) = g(p) = g(a_1, ..., a_n)$. The maximal ideal $\mathfrak{m}_p$ is the kernel of π_p. It is the ideal generated by the linear polynomials $x_1 - a_1, \ldots, x_n - a_n$. □

It is obvious that every algebra homomorphism $\mathbb{C}[x] \to \mathbb{C}$ is surjective, so its kernel is a maximal ideal. It isn't obvious that every maximal ideal of $\mathbb{C}[x]$ is the kernel of such a homomorphism. The proof can be found manywhere.[1]

The Nullstellensatz gives a way to describe the set $V(I)$ of zeros of an ideal I in affine space in terms of maximal ideals. The points of $V(I)$ are those at which all elements of I vanish — the points p such that I is contained in $\mathfrak{m}_p$.

(2.4.2) $$V(I) = \{p \in \mathbb{A}^n \mid I \subset \mathfrak{m}_p\}$$ maxide-alscheme

2.4.3. Proposition. *Let I be an ideal of the polynomial ring $\mathbb{C}[x]$. If the zero locus $V(I)$ is empty, then I is the unit ideal.* *Vempty*

proof. Every ideal I except the unit ideal is contained in a maximal ideal. □

2.4.4. Nullstellensatz (version 2). *Let A be a finite-type algebra. There are bijective correspondences between the following sets:* *nulltwo*

- *algebra homomorphisms $\overline{\pi} : A \to \mathbb{C}$,*
- *maximal ideals $\overline{\mathfrak{m}}$ of A.*

The maximal ideal $\overline{\mathfrak{m}}$ that corresponds to a homomorphism $\overline{\pi}$ is the kernel o, $\overline{\pi}$.

If A is presented as a quotient of a polynomial ring, say $A \approx \mathbb{C}[x_1, ..., x_n]/I$, then these sets also correspond bijectively to points of the set $V(I)$ of zeros of I in $\mathbb{A}^n$.

The symbol $\approx$ stands for an isomorphism.

As before, a finite-type algebra is an algebra that can be generated by a finite set of elements. A *presentation* of a finite-type algebra A is an isomorphism of A with a quotient $\mathbb{C}[x_1, ..., x_n]]/I$ of a polynomial ring. (This isn't the same as a presentation of a module (**2.1.21**).)

proof of version 2 *of the Nullstellensatz.* We choose a presentation of A as a quotient of a polynomial ring, to identify A with a quotient $\mathbb{C}[x]/I$. The Correspondence Theorem tells us that maximal ideals of A correspond to maximal ideals of $\mathbb{C}[x]$ that contain I. Those maximal ideals correspond to points of $V(I)$.

[1] While writing a paper, the mathematician Nagata decided that the English language needed this unusual word. Then he found it in a dictionary.

Let τ denote the canonical homomorphism $\mathbb{C}[x] \to A$.

polyring-toA (2.4.5)

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\pi"] \arrow[d, "\tau"'] & \mathbb{C} \arrow[d, equal] \\ A \arrow[r, "\overline{\pi}"] & \mathbb{C} \end{tikzcd}$$

The Mapping Property 2.1.4, applied to τ, tells us that homomorphisms $A \xrightarrow{\overline{\pi}} \mathbb{C}$ correspond to homomorphisms $\mathbb{C}[x] \xrightarrow{\pi} \mathbb{C}$ whose kernels contain I. Those homomorphisms also correspond to points of $V(I)$. $\square$

strongnull **2.4.6. Strong Nullstellensatz.** *Let I be an ideal of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$, and let V denote its locus of zeros in affine space: $V = V(I)$. If a polynomial $g(x)$ vanishes at every point of V, then I contains a power of g.*

proof. This is Rainich's beautiful proof. Let $g(x)$ be a polynomial that is identically zero on V. We are to show that I contains a power of g. The zero polynomial is in I, so we may assume that g isn't zero.

The Hilbert Basis Theorem tells us that I is a finitely generated ideal. Let $f = (f_1, \ldots, f_k)$ be a set of generators. We introduce a new variable y. In the $n{+}1$–dimensional affine space with coordinates $(x_1, \ldots, x_n, y)$, let W be the locus of solutions of the $k{+}1$ equations

fgy (2.4.7)
$$f_1(x) = 0, \ldots, f_k(x) = 0 \quad \text{and} \quad g(x)y - 1 = 0$$

Suppose that we have a solution of the equations $f(x) = 0$, say $(x_1, ..., x_n) = (a_1, ..., a_n)$. Then a is a point of V, and our hypothesis tells us that $g(a) = 0$ too. There can be no b such that $g(a)b = 1$. So there is no point $(a_1, ..., a_n, b)$ that solves the equations (2.4.7): The locus W is empty. Proposition 2.4.3 tells us that the polynomials $f_1, ..., f_k, gy - 1$ generate the unit ideal of $\mathbb{C}[x_1, ..., x_n, y]$. There are polynomials $p_1(x, y), \ldots, p_k(x, y)$ and $q(x, y)$ such that

rabinowitz (2.4.8)
$$p_1 f_1 + \cdots + p_k f_k + q(gy - 1) = 1$$

The ring $R = \mathbb{C}[x, y]/(gy - 1)$ can be described as the one obtained by adjoining an inverse of g to the polynomial ring $\mathbb{C}[x]$. The residue of y becomes the inverse. Since g isn't zero, $\mathbb{C}[x]$ is a subring of R. In R, $gy - 1 = 0$, and the equation (2.4.8) becomes $p_1 f_1 + \cdots + p_k f_k = 1$. When we multiply both sides of this equation by a large power g^N of g, we can use the equation $gy = 1$, which is true in R, to eliminate all occurences of y in the polynomials $p_i(x, y)$. Let $h_i(x)$ denote the polynomial in x that is obtained by eliminating y from $g^N p_i$. Then

$$h_1(x) f_1(x) + \cdots + h_k(x) f_k(x) \;=\; g^N(x)$$

is a polynomial equation that is true in R and in its subring $\mathbb{C}[x]$. Since $f_1, ..., f_k$ are in I, this equation shows that g^N is in I. $\square$

VIsupVJ **2.4.9. Corollary.** *Let $\mathbb{C}[x]$ denote the polynomial ring in the variables $x_1, ..., x_n$.*

(i) *Let P be a prime ideal of $\mathbb{C}[x]$, and let $V = V(P)$ be the variety of zeros of P in $\mathbb{A}^n$. If a polynomial g vanishes at every point of V, then g is an element of P.*

(ii) *Let f be an irreducible polynomial in $\mathbb{C}[x]$. If a polynomial g vanishes at every point of $V(f)$, then f divides g.*

(iii) *Let I and J be ideals of $\mathbb{C}[x]$. Then $V(I) \supset V(J)$ if and only if $\operatorname{rad} I \subset \operatorname{rad} J$, and $V(I) > V(J)$ if and only if $\operatorname{rad} I > \operatorname{rad} J$ (see* **(2.2.20)***).* $\square$

2.4.10. Examples.

strongnullex

(i) Let I be the ideal of the polynomial algebra $\mathbb{C}[x,y]$ generated by y^5 and $y^2 - x^3$. In the affine plane, the origin $(0,0)$, is the only common zero of these polynomials, and the polynomial x also vanishes at the origin. The Strong Nullstellensatz predicts that I contains a power of x. This is verified by the following equation:

$$yy^5 - (y^4 + y^2x^3 + x^6)(y^2 - x^3) \ = \ x^9$$

(ii) We may regard pairs A, B of $n{\times}n$ matrices as points of an affine space $\mathbb{A}$ of dimension $2n^2$, with coordinates a_{ij}, b_{ij}, $1 \le i,j \le n$. The pairs of *commuting matrices* ($AB = BA$) form a closed subset of $\mathbb{A}$, the locus of common zeros of the n^2 polynomials c_{ij} that compute the entries of the matrix $AB - BA$:

(2.4.11) $$c_{ij}(a,b) = \sum_\nu a_{i\nu}b_{\nu j} - b_{i\nu}a_{\nu j}$$ comm-mateq

If I is the ideal of the polynomial algebra $\mathbb{C}[a,b]$ generated by the set of polynomials $\{c_{ij}\}$, then $V(I)$ is the set of pairs of commuting complex matrices. The Strong Nullstellensatz asserts that, if a polynomial $g(a,b)$ vanishes on every pair of commuting matrices, some power of g is in I. Is g itself in I? It is a famous conjecture that I is a prime ideal. If so, g would be in I. Proving the conjecture would establish your reputation as a mathematician, but I don't recommend spending very much time on it right now. $\square$

2.5 The Spectrum

When a finite-type domain A is presented as a quotient of a polynomial ring $\mathbb{C}[x]/P$, where P is a prime ideal, A becomes the coordinate algebra of the variety $V(P)$ in affine space. The points of $V(P)$ correspond to maximal ideals of A and also to homomorphisms $A \to \mathbb{C}$. spectru-malg

The Nullstellensatz allows us to associate a set of points to a finite-type domain A without reference to a presentation. We can do this because the maximal ideals of A and the homomorphisms $A \to \mathbb{C}$ don't depend on a presentation. We replace the variety $V(P)$ by an abstract set of points, the *spectrum* of A, that we denote by $\operatorname{Spec} A$ and call an *affine variety*. We put one point p into the spectrum for every maximal ideal of A, and then we turn around and denote the maximal ideal that corresponds to a point p by $\overline{\mathfrak{m}}_p$. The Nullstellensatz tells us that p also corresponds to a homomorphism $A \to \mathbb{C}$ whose kernel is $\overline{\mathfrak{m}}_p$. We denote that homomorphism by $\overline{\pi}_p$. In analogy with (2.2.24), A is called the *coordinate algebra* of the affine variety $\operatorname{Spec} A$. To work with $\operatorname{Spec} A$, we may interpret its points as maximal ideals or as homomorphisms to $\mathbb{C}$, whichever is convenient.

When defined in this way, the variety $\operatorname{Spec} A$ isn't embedded into any affine space, but because A is a finite-type domain, it can be presented as a quotient $\mathbb{C}[x]/P$, where P is a prime ideal. When this is done, points of $\operatorname{Spec} A$ correspond to points of the subset $V(P)$ in $\mathbb{A}^n$.

Even when the coordinate ring A of an affine variety X is presented as $\mathbb{C}[x]/P$, we will often denote X by $\operatorname{Spec} A$ rather than by $V(P)$.

2.5.1. Note. In modern terminology, the word "spectrum" is usually used to denote the set of prime ideals of a ring. This becomes important when one studies rings that aren't finite-type algebras. When working with finite-type domains, there are enough maximal ideals. The other prime ideals aren't needed to fill out $\operatorname{Spec} A$, so we don't include them. $\square$ primespec

Let $X = \operatorname{Spec} A$. An element α of A defines a (complex-valued) function on X that we denote by the same letter α. The definition of the function α is as follows: A point p of X corresponds to a homomorphism $A \xrightarrow{\overline{\pi}_p} \mathbb{C}$. By definition The *value* $\alpha(p)$ of the function α at p is $\overline{\pi}_p(\alpha)$:

(2.5.2) $$\alpha(p) \overset{def}{=} \overline{\pi}_p(\alpha)$$ defalphap

Thus the kernel of $\overline{\pi}_p$, which is $\overline{\mathfrak{m}}_p$, is the set of elements α of the coordinate algebra A at which the value of α is 0:

$$\overline{\mathfrak{m}}_p = \{\alpha \in A \,|\, \alpha(p) = 0\}$$

The functions defined in this way by the elements of A are called the *regular functions* on X. (See Proposition 2.6.2 below.)

When A is a polynomial algebra $\mathbb{C}[x_1,...,x_n]$, the function defined by a polynomal $g(x)$ is simply the usual polynomial function, because π_p is defined by evaluating a polynomial at p: $g(p) = \pi_p(g)$ (2.3.1).

gpequal-salphap **2.5.3. Lemma.** *Let A be a quotient $\mathbb{C}[x]/P$ of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, modulo a prime ideal P, so that* $\operatorname{Spec} A$ *identifies with the closed subset $V(P)$ of $\mathbb{A}^n$. Then a point p of* $\operatorname{Spec} A$ *becomes a point of $\mathbb{A}^n$: $p = (a_1, ..., a_n)$. When an element α of A is represented by a polynomial $g(x)$, the value of α at p can be obtained by evaluating g: $\alpha(p) = g(p) = g(a_1, ..., a_n)$.*

So the value $\alpha(p)$ at a point p of $\operatorname{Spec} A$ can be obtained by evaluating a suitable polynomial g. However, unless P is the zero ideal, that polynomial won't be unique.

proof of Lemma 2.5.3. The point p of $\operatorname{Spec} A$ gives us a diagram (2.4.5), with $\pi = \pi_p$ and $\overline{\pi} = \overline{\pi}_p$, and where τ is the canonical map $\mathbb{C}[x] \to A$. Then $\alpha = \tau(g)$, and

redefinefn (2.5.4)
$$g(p) \overset{def}{=} \pi_p(g) = \overline{\pi}_p\tau(g) = \overline{\pi}_p(\alpha) \overset{def}{=} \alpha(p). \qquad \square$$

regfndetelt **2.5.5. Lemma.** *The regular functions determined by distinct elements α and β of A are distinct. In particular, the only element α of A that is zero at all points of* $\operatorname{Spec} A$ *is the zero element.*

proof. We replace α by $\alpha - \beta$. Then what is to be shown is that, if the function determined by an element α is the zero function, then α is the zero element.

We present A as $\mathbb{C}[x]/P$, $x = x_1, ..., x_n$, where P is a prime ideal. Let X be the locus of zeros of P in $\mathbb{A}^n$. Corollary 2.4.9 **(i)** tells us that P is the ideal of all elements that are zero on X. Let $g(x)$ be a polynomial that represents α. If p is a point of X, and if $\alpha(p) = 0$, then $g(p) = 0$. So if α is the zero function, then g is in P, and therefore $\alpha = 0$. $\square$

ztoponvar

(2.5.6) the Zariski topology on an affine variety

Let $X = \operatorname{Spec} A$ be an affine variety with coordinate algebra A. An ideal $\overline{J}$ of A defines a locus in X, a *closed subset*, that we denote by $V(\overline{J})$, using the same notation as for loci in affine space. The points of $V(\overline{J})$ are the points of X at which all elements of $\overline{J}$ vanish. This is analogous to (2.4.2):

locusin-spec (2.5.7)
$$V(\overline{J}) = \{p \in \operatorname{Spec} A \,|\, \overline{J} \subset \overline{\mathfrak{m}}_p\}$$

zerolocus-inX **2.5.8. Lemma.** *Let A be a finite-type domain that is presented as $A = \mathbb{C}[x]/P$. An ideal $\overline{J}$ of A corresponds to an ideal J of $\mathbb{C}[x]$ that contains P, and $\overline{J} = J/P$. Let $V(J)$ denote the zero locus of J in $\mathbb{A}^n$. When we regard* $\operatorname{Spec} A$ *as a subvariety of $\mathbb{A}^n$, the loci $V(\overline{J})$ in* $\operatorname{Spec} A$ *and $V(J)$ in $\mathbb{A}^n$ will be equal.* $\square$

empty **2.5.9. Proposition.** *Let $\overline{J}$ be an ideal of a finite-type domain A. The zero set $V(\overline{J})$ in $X =$* $\operatorname{Spec} A$ *is empty if and only if $\overline{J}$ is the unit ideal of A. If X is empty, then A is the zero ring.*

proof. The only ideal that isn't contained in a maximal ideal is the unit ideal. $\square$

nobar **2.5.10. Note.** We have put bars on the symbols $\overline{\mathfrak{m}}$, $\overline{\pi}$, and $\overline{J}$ in this section up to now, in order to distinguish ideals of A from ideals of $\mathbb{C}[x]$ and homomorphisms $A \to \mathbb{C}$ from homomorphisms $\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$. From now on we will put bars over the letters only when there is a danger of confusion. Most of the time, we will drop the bars, and write $\mathfrak{m}$, π, and J instead of $\overline{\mathfrak{m}}$, $\overline{\pi}$, and $\overline{J}$. $\square$

inter-sprimes **2.5.11. Proposition.** *Let I be an ideal of noetherian ring R. The radical of I is the intersection of the prime ideals of R that contain I.*

proof. Let x be an element of $\operatorname{rad} I$. Some power x^k is in I. If P is a prime ideal that contains I, then $x^k \in P$, and since P is a prime ideal, $x \in P$. So $\operatorname{rad} I \subset P$. Conversely, let x be an element not in $\operatorname{rad} I$. So no power of x is in I. We show that there is a prime ideal that contains I but not x. Let $\mathcal{S}$ be the set of ideals that contain I, but don't contain any power of x. The ideal I is one such ideal, so $\mathcal{S}$ isn't empty. Since R is noetherian, $\mathcal{S}$ contains a maximal member P (**2.1.12**). We show that P is a prime ideal by showing that, if two ideals A and B are strictly larger than P, their product AB isn't contained in P (2.1.2)**(iii')**. Since P is a maximal member of $\mathcal{S}$, A and B aren't in $\mathcal{S}$. They contain I and they contain powers of x, say $x^k \in A$ and $x^\ell \in B$. Then $x^{k+\ell}$ is in AB but not in P. Therefore $AB \not\subset P$. $\square$

The properties of closed sets in affine space that are given in Lemmas 2.2.2 and 2.2.21 are true for closed subsets of an affine variety. In particular, $V(J) = V(\operatorname{rad} J)$, and $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.

2.5.12. Corollary. *Let I and J be ideals of a finite-type domain A, and let $X = \operatorname{Spec} A$. Then $V(I) \supset V(J)$ if and only if $\operatorname{rad} I \subset \operatorname{rad} J$.* *subsofspecA*

This follows from the case of a polynomial ring, Corollary 2.4.9 **(iii)**, and Lemma 2.5.8. □

The next proposition includes Proposition 2.2.23 as a special case.

2.5.13. Proposition. *Let A be a finite-type domain, let $X = \operatorname{Spec} A$, and let P be a radical ideal of A. The closed set $V(P)$ of zeros of P is irreducible if and only if P is a prime ideal.* *proofirredprime*

proof. Let $Y = V(P)$, and let C and D be closed subsets of X such that $Y = C \cup D$. Say $C = V(I)$ and $D = V(J)$. We may suppose that I and J are radical ideals. Then the inclusion $C \subset Y$ implies that $I \supset P$, and similarly, $J \supset P$ (2.5.12). Because $Y = C \cup D$, we also have $V(P) = V(I) \cup V(J) = V(IJ)$. Therefore $\operatorname{rad}(IJ) = P$. If P is a prime ideal, then $P = I$ or $P = J$, and therefore $C = Y$ or $D = Y$. Then Y is irreducible. Conversely, suppose that P isn't a prime ideal. So there are ideals I, J strictly larger than the radical ideal P, such that $IJ \subset P$. In this case, Y will be the union of the two proper closed subsets $V(I)$ and $V(J)$ (2.5.12), so Y isn't irreducible. □

(2.5.14) the nilradical thenilradical

The *nilradical* of a ring is the set of its nilpotent elements. It is the radical of the zero ideal. If a ring R is noetherian, its nilradical N will be *nilpotent*: some power of N will be the zero ideal (Lemma 2.2.21). The nilradical of a domain is the zero ideal.

The next corollary follows from Proposition 2.5.11.

2.5.15. Corollary. *The nilradical of a noetherian ring R is the intersection of the prime ideals of R.* □ *intersectprimes*

Note. The conclusion of this corollary is true whether or not R is noetherian.

2.5.16. Corollary.

(i) *Let A be a finite-type algebra. An element of A that is in every maximal ideal of A is nilpotent.* *strongnullA*

(ii) *Let A be a finite-type domain. The intersection of the maximal ideals of A is the zero ideal.*

proof. **(i)** Say that A is presented as $\mathbb{C}[x_1, ..., x_n]/I$. Maximal ideals of A correspond to the maximal ideals of $\mathbb{C}[x]$ that contain I, and to points of the closed subset $V(I)$ of $\mathbb{A}^n$. Let α be the element of A that is represented by a polynomial $g(x)$ in $\mathbb{C}[x]$. Then α is in every maximal ideal of A if and only if $g = 0$ at all points of $V(I)$. If so, the Strong Nullstellensatz asserts that some power g^N is in I. Then $\alpha^N = 0$. □

2.5.17. Corollary. *An element α of a finite-type domain A is determined by the function that α defines on $\operatorname{Spec} A$.* *fndetermineselt*

proof. It is enough to show that an element α that defines the zero function is the zero element. Such an element α is in every maximal ideal (2.5.9), so it is nilpotent, and since A is a domain, $\alpha = 0$. □

2.6 Morphisms of Affine Varieties

morphism

Morphisms are the maps between varieties that are allowed. Morphisms between affine varieties, as will be defined in this section, correspond to algebra homomorphisms in the opposite direction between their coordinate algebras. Morphisms of projective varieties will be defined in the next chapter.

(2.6.1) regular functions regfnone

The *function field* K of an affine variety $X = \operatorname{Spec} A$ is the field of fractions of A. A *rational function* on X is a nonzero element of the function field. A rational function f is *regular* at a point p of X if it can be written as a fraction $f = a/s$ with $s(p) \neq 0$, and f is regular on a subset U of X if it is regular at every point of U.

In (2.5.2), we have seen that an element of the coordinate algebra A defines a function on X. The value $a(p)$ of a function a at a point p is $\pi_p(a)$, where π_p is the homomorphism $A \to \mathbb{C}$ that corresponds to p. A rational function $f = a/s$ is an element of A_s. It defines a function on the open subset X_s of X, with $f(p) = a(p)/s(p)$.

deloca **2.6.2. Proposition.** *The regular functions on an affine variety $X = \operatorname{Spec} A$ are the elements of the coordinate algebra A.*

proof. Let f be a rational function that is regular on X. So for every point p of X, f can be written as a fraction a/s such that $s(p) \neq 0$. The localization $X_s = \operatorname{Spec} A_s$ contains p, and f is an element of A_s. Because X is quasicompact, a finite set of such localizations, say $X_{s_1}, \dots, X_{s_k}$, will cover X. Then $s_1, ..., s_k$ have no common zeros on X, so they generate the unit ideal of A. Since f is in A_{s_i}, we can write $f = s_i^{-n} b_i$, or $s_i^n f = b_i$, with b_i in A, and we can use the same exponent n for each i. Since the elements s_i generate the unit ideal of A, so do the powers s_i^n. Writing $1 = \sum s_i^n c_i$ with c_i in A, $\; f = \sum s_i^n c_i f = \sum c_i b_i$. So f is an element of A.

This reasoning, in which one writes the identity element as a sum, occurs often. □

morphtwo **(2.6.3) morphisms**

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties, and let $A \xrightarrow{\varphi} B$ be an algebra homomorphism. A point q of Y corresponds to an algebra homomorphism $B \xrightarrow{\pi_q} \mathbb{C}$. When we compose π_q with φ, we obtain a homomorphism $A \xrightarrow{\pi_q\varphi} \mathbb{C}$. By definition, points p of $\operatorname{Spec} A$ correspond to homomorphisms $A \xrightarrow{\pi_p} \mathbb{C}$. So there is a unique point p of $X = \operatorname{Spec} A$ such that $\pi_q\varphi = \pi_p$.

defmorphaff **2.6.4. Definition.** Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. A *morphism* $Y \xrightarrow{u} X$ is a map that is defined, as above, by an algebra homomorphism $A \xrightarrow{\varphi} B$. If q is a point of Y, then uq is the point p of X such that $\pi_p = \pi_q\varphi$:

piqphi (2.6.5)
$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi_p"'] & B \arrow[d, "\pi_q"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} X & Y \arrow[l, "u"'] \\ p \arrow[u] & q \arrow[l] \arrow[u] \end{tikzcd}$$

So $p = uq$ means that $\pi_q\varphi = \pi_p$.

betaqalphap **2.6.6. Lemma.** *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $Y \xrightarrow{u} X$ be the morphism defined by a homomorphism $A \xrightarrow{\varphi} B$. Also, let q be a point of Y, and let $p = uq$ be its image in X.*
(i) *If α is an element of A and $\beta = \varphi(\alpha)$, then $\beta(q) = \alpha(p)$.*
(ii) *Let $\mathfrak{m}_p$ and $\mathfrak{m}_q$ be the maximal ideals of A and B at p and q, respectively. Then $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$.*

proof. **(i)** $\beta(q) = \pi_q(\beta) = \pi_q(\varphi\alpha) = \pi_p(\alpha) = \alpha(p)$.
(ii) $\alpha(p) = 0$ if and only if $[\varphi\alpha](q) = 0$. □

Thus the homomorphism φ is determined by the morphism u, and vice-versa. But just as a map $A \to B$ needn't be a homomorphism, a map $Y \to X$ needn't be a morphism.

brackets **Notation.** Parentheses tend to accumulate, and this can make expressions hard to read. When we want to denote the value of a complicated function such as $\varphi(\alpha)$ on an object q we may, for clarity, drop some parentheses and enclose the functor in square brackets, writing $[\varphi\alpha](q)$ instead of $(\varphi(\alpha))(q)$. When a square bracket is used this way, there is no logical difference between it and a parenthesis. □

A morphism $Y \xrightarrow{u} X$ is an *isomorphism* if and only if it is bijective, and its inverse function is a morphism. This will be true if and only if $A \xrightarrow{\varphi} B$ is an isomorphism of algebras.

2.6.7. Proposition. **(i)** *The morphism $Y \xrightarrow{u} X$ defined by a homomorphism $A \xrightarrow{\varphi} B$ is an isomorphism if and only if φ is an isomorphism.* uisomphiisom

(ii) *The morphism $X \xrightarrow{u} X$ defined by a homomorphism $A \xrightarrow{\varphi} A$ is the identity if and only if φ is the identity.* $\square$

The definition of a morphism can be confusing because the direction of the arrow is reversed. It will become clearer as we expand the discussion, but the reversal of arrows will remain a potential source of confusion.

morphisms to affine space.

A morphism $Y \xrightarrow{u} \mathbb{A}^1$ from a variety $Y = \operatorname{Spec} B$ to the affine line $\operatorname{Spec} \mathbb{C}[x]$ is defined by an algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} B$, and such a homomorphism substitutes an element β of B for x. The corresponding morphism u sends a point q of Y to the point $x = \beta(q)$ of the x-line.

For example, let Y be the space of 2×2 matrices, $Y = \operatorname{Spec} \mathbb{C}[y_{ij}]$, where y_{ij} are variable matrix entries, $1 \le i, j \le 2$. The determinant defines a morphism $Y \to \mathbb{A}^1$ that sends a matrix to its determinant. The corresponding algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} \mathbb{C}[y_{ij}]$ substitutes $y_{11}y_{22} - y_{12}y_{21}$ for x. It sends a polynomial $f(x)$ to $f(y_{11}y_{22} - y_{12}y_{21})$.

A morphism in the other direction, from the affine line $\mathbb{A}^1$ to a variety Y may be called a (complex) polynomial path in Y. When Y is the space of matrices, a morphism $\mathbb{A}^1 \to Y$ corresponds to a homomorphism $\mathbb{C}[y_{ij}] \to \mathbb{C}[x]$. It substitutes a polynomial in x for each variable y_{ij}.

A morphism from an affine variety $Y = \operatorname{Spec} B$ to affine space $\mathbb{A}^n$ is defined by a homomorphism $\mathbb{C}[x_1, ..., x_n] \xrightarrow{\Phi} B$, which substitutes elements β_i of B for x_i: $\Phi(f(x)) = f(\beta)$. (We use an upper case Φ here, keeping φ in reserve.) The corresponding morphism $Y \xrightarrow{u} \mathbb{A}^n$ sends a point q of Y to the point $(\beta_1(q), ..., \beta_n(q))$ of $\mathbb{A}^n$.

morphisms to affine varieties.

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties. Say that we have chosen a presentation $A = \mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$ of A, so that X becomes the closed subvariety $V(f)$ of affine space $\mathbb{A}^m$. There is no need to choose a presentation of B. A natural way to define a morphism from a variety Y to X is as a morphism $Y \xrightarrow{u} \mathbb{A}^m$ to affine space, whose image is contained in X. We check that this agrees with Definition 2.6.4.

A morphism $Y \xrightarrow{u} \mathbb{A}^m$ corresponds to a homomorphism $\mathbb{C}[x_1, ..., x_m] \xrightarrow{\Phi} B$, determined by a set $(\beta_1, ..., \beta_m)$ of elements of B, with the rule that $\Phi(x_i) = \beta_i$. Since X is the locus of zeros of the polynomials f, the image of Y will be contained in X if and only if $f_i(\beta_1(q), ..., \beta_m(q)) = 0$ for every point q of Y and every i, i.e., if and only if $f_i(\beta)$ is in every maximal ideal of B, in which case $f_i(\beta) = 0$ (2.5.16)**(ii)**. A better way to say this is: The image of Y is contained in X if and only if $\beta = (\beta_1, ..., \beta_m)$ solves the equations $f(x) = 0$. And, if β is a solution, the homomorphism Φ defines a homomorphism $A \xrightarrow{\varphi} B$.

$$\begin{array}{ccc} \mathbb{C}[x] & \xrightarrow{\Phi} & B \\ \downarrow & & \| \\ A & \xrightarrow{\varphi} & B \end{array}$$

This is an elementary, but important, principle:

- *Homomorphisms from the algebra $A = \mathbb{C}[x]/(f)$ to an algebra B correspond to solutions of the equations $f = 0$ in B.*

2.6.8. Corollary. *Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Spec} B$ be affine varieties, and suppose that $A = \mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$. There are bijective correspondences between the following sets:* morphandhomom

- *algebra homomorphisms $A \to B$,*
- *morphisms $Y \to X$,*
- *morphisms $Y \to \mathbb{A}^m$ whose images are contained in X,*
- *solutions of the equations $f_i(x) = 0$ in B, $i = 1, ..., k$.* $\square$

The second and third sets refer to an embedding of the variety X into affine space, but the first one does not. It shows that a morphism depends only on the varieties X and Y, not on their embeddings.

maptocusp **2.6.9. Example.** Let $B = \mathbb{C}[x]$ be the polynomial ring in one variable, and let A be the coordinate algebra $\mathbb{C}[u, v]/(v^2 - u^3)$ of a cubic curve with a cusp. A homomorphism $A \to B$ is determined by a solution of the equation $v^2 = u^3$ in $\mathbb{C}[x]$. The solutions have the form $u = g^2$, $v = g^3$ with g in $\mathbb{C}[x]$. For instance, $u = x^2$ and $v = x^3$ is a solution. $\square$

We note a few more facts about morphisms here. Their geometry will be analyzed further in Chapters 4 and 5.

phisurj **2.6.10. Proposition.** *Let $Y \xrightarrow{u} X$ be the morphism of affine varieties that corresponds to a homomorphism of finite-type domains $A \xrightarrow{\varphi} B$.*

(i) *Suppose that $B = A/P$, where P is a prime ideal of A, and that φ is the canonical homomorphism $A \to A/P$. Then u is the inclusion of the variety of zeros $Y = V(P)$ of P into X.*

(ii) *The homomorphism φ is surjective if and only if u maps Y isomorphically to a closed subvariety of X.*

(iii) *Let $Z \xrightarrow{v} Y$ be another morphism, that corresponds to a homomorphism $B \xrightarrow{\psi} R$ of finite-type domains, the composed map $Z \xrightarrow{uv} X$ corresponds to the composed homomorphism $A \xrightarrow{\psi\varphi} R$.* $\square$

It can be useful to phrase the definition of the morphism $Y \xrightarrow{u} X$ that corresponds to a homomorphism $A \xrightarrow{\varphi} B$ in terms of maximal ideals. Let $\mathfrak{m}_q$ be the maximal ideal of B at a point q of Y. The inverse image of $\mathfrak{m}_q$ in A is the kernel of the composed homomorphism $A \xrightarrow{\varphi} B \xrightarrow{\pi_q} \mathbb{C}$, so it is a maximal ideal of A: $\varphi^{-1}\mathfrak{m}_q = \mathfrak{m}_p$, for some p in X. That point p is the image of q: If $p = uq$, then $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$.

The fibre over a point p of the morphism $Y \xrightarrow{u} X$ defined by a homomorphism $A \xrightarrow{\varphi} B$ is described as follows: let $\mathfrak{m}_p$ be the maximal ideal at a point p of X, and let J be the extended ideal $\mathfrak{m}_p B$, the ideal generated by the image of $\mathfrak{m}_p$ in B. Its elements are finite sums $\sum \varphi(z_i)b_i$ with z_i in $\mathfrak{m}_p$ and b_i in B. (See (**2.7.5**) below.) If q is is a point of Y, then $uq = p$ if and only if $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$. This will be true if and only $J \subset \mathfrak{m}_q$.

cusp-normx **2.6.11. Example.** *(blowing up the plane)*

Let W and X be planes with coordinates (x, w) and (x, y), respectively. The affine blowup morphism $W \xrightarrow{\pi} X$ was described before (**1.8.5**). It is defined by the substitution $\pi(x, w) = (x, xw)$, and if corresponds to the algebra homomorphism $\mathbb{C}[x, y] \xrightarrow{\varphi} \mathbb{C}[x, w]$ defined by $\varphi(x) = x$ and $\varphi(y) = xw$. To be specific, the image of a point $q : (x, w) = (a, c)$ of W is the point $p : (x, y) = (a, ac)$ of X.

As was explained in (**1.8.5**), the blowup π is bijective at points (x, y) at which $x \neq 0$. The fibre of Z over a point of Y of the form $(0, y)$ is empty unless $y = 0$, and the fibre over the origin $(0, 0)$ in Y is the w-axis, the line $x = 0$ in the plane W. $\square$

mcont **2.6.12. Proposition.** *A morphism $Y \xrightarrow{u} X$ of affine varieties is a continuous map in the Zariski topology and also in the classical topology.*

proof. the Zariski topology: Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, so that u corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. A closed subset C of X will be the zero locus of a set $\alpha = \{\alpha_1, ..., \alpha_k\}$ of elements of A. Let $\beta_i = \varphi\alpha_i$. The inverse image $u^{-1}C$ is the set of points q such that $p = uq$ is in C, i.e., such that $\alpha_i(uq) = \beta_i(q) = 0$ (2.6.6). So $u^{-1}C$ is the zero locus in Y of the elements $\beta_i = \varphi(\alpha_i)$. It is a closed set.

the classical topology: We use the fact that polynomials are continuous functions. First, a morphism of affine spaces $\mathbb{A}^n_y \xrightarrow{U} \mathbb{A}^m_x$ is defined by an algebra homomorphism $\mathbb{C}[x_1, ..., x_m] \xrightarrow{\Phi} \mathbb{C}[y_1, ..., y_n]$, and that homomorphism is determined by the polynomials $h_1(y), ..., h_m(y)$ that are the images of the variables $x_1, ..., x_m$. The morphism U sends the point $(y_1, ..., y_n)$ of $\mathbb{A}^n$ to the point $(h_1(y), ..., h_m(y))$ of $\mathbb{A}^m$. It is continuous because polynomials are continuous functions.

Next, say that a morphism $Y \xrightarrow{u} X$ is defined by a homomorphism $A \xrightarrow{\varphi} B$ of algebras that are presented as $A = \mathbb{C}[x]/I$ and $B = \mathbb{C}[y]/J$. We form a diagram of homomorphisms and the associated diagram of morphisms:

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & \mathbb{C}[y] \arrow[d, "\beta"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^n_x & \mathbb{A}^m_y \arrow[l, "U"'] \\ X \arrow[u] & Y \arrow[l, "u"'] \arrow[u] \end{tikzcd}$$

Here the map α sends $x_1, ..., x_n$ to $\alpha_1, ..., \alpha_n$, and β sends y_i to $\beta_i = \varphi(\alpha_i)$. Then Φ is obtained by choosing elements h_i of $\mathbb{C}[y]$, such that $\beta(h_i) = \beta_i$.

In the diagram on the right, U is a continuous map, and the vertical arrows are the embeddings of X and Y into their affine spaces. Since the topologies on X and Y are induced from their embeddings into affine spaces, u is continuous. $\square$

Thus every morphism of affine varieties can be obtained by restriction from a morphism of affine spaces. However, in the diagram above, the morphism U depends on the choice of the polynomials h_i and on the presentations of A and B. It isn't unique.

2.7 Localization

In these notes, the word "localization" refers to the process of adjoining inverses to an algebra, and to the effect of that process on the spectrum. boldloc

Let s be a nonzero element of a domain A. As before (**2.1.23**), the ring $A_s = A[s^{-1}]$ obtained by adjoining an inverse of s to A is called a *localization* of A. If X denotes the variety $\operatorname{Spec} A$, X_s will denote the variety $\operatorname{Spec} A_s$. It will be called a localization of X too.

2.7.1. Proposition. *The localization $X_s = \operatorname{Spec} A_s$ is homeomorphic to the open subspace of X of points at which the function defined by s isn't zero.* *topology-onlocal-ization*

proof. Let p be a point of X, let $A \xrightarrow{\pi_p} \mathbb{C}$ be the corresponding homomorphism. If s isn't zero at p, say $s(p) = c \neq 0$, then π_p extends uniquely to a homomorphism $A_s \to \mathbb{C}$ that sends s^{-1} to c^{-1}. This gives us a unique point of X_s that corresponds to p. If $c = 0$, then π_p doesn't extend to A_s.

A closed subset C of X will be the set of zeros of the elements $a_1, ..., a_k$ of A. Then $C \cap X_s$ will be the set of zeros of those same elements in X_s. It will be closed in X_s. Conversely, let D be a closed subset of X_s, say the zero set in X_s of some elements $\beta_1, ..., \beta_k$, where $\beta_i = b_i s^{-n_i}$ with b_i in A. Since s^{-1} doesn't vanish on X_s, the elements b_i and β_i have the same zeros in X_s. If C is the zero set of $b_1, ..., b_k$ in X, then $C \cap X_s = D$. $\square$

Thus we may identify a localization X_s with the open subset of X of points at which the value of s isn't zero. Then the effect of adjoining the inverse is to throw out the points of X at which s vanishes. For example, the spectrum of the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ becomes the complement of the origin in the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$.

Most varieties contain open sets that aren't localizations. The complement X' of the origin in the affine plane $X = \operatorname{Spec} \mathbb{C}[x_1, x_2]$ is a simple example. Every polynomial that vanishes at the origin vanishes on an affine curve, which has points distinct from the origin. Its inverse doesn't define a function on X'. So X' isn't a localization of X. This is rather obvious, but in other situations, it is often hard to tell whether or not a given open set is a localization.

Localizations are important for two reasons:

2.7.2. basistop

- The relation between an algebra A and a localization A_s is easy to understand.
- The localizations X_s of an affine variety X form a basis for the Zariski topology on X.

A *basis* for the topology on a topological space X is a family $\mathcal{B}$ of open sets with this property: Every open subset of X is a union of open sets that are members of $\mathcal{B}$.

To show that the localizations X_s of an affine variety X form a basis for the topology on X, we must show that every open subset U of $X = \operatorname{Spec} A$ can be covered by sets of the form X_s. Let C be the complement

$X-U$ of U in X. Then C is closed, so it is the set of common zeros of some nonzero elements $s_1,...,s_k$ of A. The zero set $V(s_i)$ of s_i is the complement of the locus X_{s_i} in X, C is the intersection of the sets $V(s_i)$, and U is the union of the sets X_{s_i}. □

coverby-locs **2.7.3. Corollary.** *Let $X = \operatorname{Spec} A$ be an affine variety.*

(i) *Let $s_1,...,s_k$ be elements of A. If the localizations $X_{s_1},...,X_{s_k}$ cover X, then $s_1,...,s_k$ generate the unit ideal of A.*

(ii) *If $\{U_\nu\}$ is an open covering of X, a covering by open sets, there are elements $s_1,...,s_k$ of A such that each X_{s_i} is contained in one of the open sets U_ν, and the localizations $X_{s_1},...,X_{s_k}$ cover X.* □

locloc **2.7.4. Lemma.** *Let $X = \operatorname{Spec} A$ be an affine variety.*

(i) *If if A_s and A_t are localizations of A, and if $A_s \supset A_t$, then A_s is a localization of A_t. Or, if X_s and X_t are localizations of X, and if $X_s \subset X_t$, then X_s is a localization of X_t.*

(ii) *If u is an element of a localization A_s of A, then $(A_s)_u$ is also a localization of A.* □

extcontr **(2.7.5) extension and contraction of ideals**

Let $A \subset B$ be the inclusion of a ring A as a subring of a ring B. The *extension* of an ideal I of A is the ideal IB of B generated by I. Its elements are finite sums $\sum_i z_ib_i$ with z_i in I and b_i in B. The *contraction* of an ideal J of B is the intersection $J \cap A$. It is an ideal of A.

When A_s is a localization of A and I is an ideal of A, the elements of the extended ideal IA_s are fractions of the form zs^{-k}, with z in I. We denote this extended ideal by I_s.

xtcontr-prop **2.7.6. Lemma.** *Let s be a nonzero element of a domain A.*

(i) *Let J be an ideal of the localization A_s and let $I = J\cap A$. Then $J = I_s$. Every ideal of A_s is the extension of an ideal of A.*

(ii) *Let P be a prime ideal of A. If s isn't in P, the extended ideal P_s is a prime ideal of A_s. If I is any ideal of A that contains I, the extended ideal I_s is the unit ideal.* □

multsys **(2.7.7) multiplicative systems**

To work with the inverses of finitely many nonzero elements, one may simply adjoin the inverse of their product. For working with an infinite set of inverses, the concept of a multiplicative system is useful. A *multiplicative system* S in a domain A is a subset of A that consists of nonzero elements, is closed under multiplication, and contains 1. If S is a multiplicative system, the ring of *S-fractions* AS^{-1} is the ring obtained by adjoining inverses of all elements of S. Its elements are equivalence classes of fractions as^{-1} with a in A and s in S, the equivalence relation and the laws of composition being the usual ones for fractions. The ring AS^{-1} will be called a *localization* too. When necessary to avoid confusion, the ring obtained by inverting a single nonzero element s may be called a *simple localization*.

inverseex-amples **2.7.8. Examples.** **(i)** The set consisting of the powers of a nonzero element s of a domain A is a multiplicative system. Its ring of fractions is the simple localization A_s.

(ii) The set S of all nonzero elements of a domain A is a multiplicative system. Its ring of fractions is the field of fractions of A.

(iii) An ideal P of a domain A is a prime ideal if and only if its complement, the set of elements of A *not* in P, is a multiplicative system. □

extendide-altoloc **2.7.9. Proposition.** *Let S be a multiplicative system in a domain A, and let A' be the localization AS^{-1}.*

(i) *Let I be an ideal of A. The extended ideal IA' is the set IS^{-1} whose elements are classes of fractions xs^{-1}, with x in I and s in S. The extended ideal is the unit ideal if and only if I contains an element of S.*

(ii) *Let J be an ideal of the localization A' and let $I = J \cap A$. Then $IA' = J$.*

(iii) *If P is a prime ideal of A and if $P \cap S$ is empty, the extended ideal $P' = PA'$ is a prime ideal of A', and the contraction $P' \cap A$ is equal to P. If $P \cap S$ isn't empty, the extended ideal is the unit ideal. Thus prime ideals of AS^{-1} correspond bijectively to prime ideals of A that don't meet S.* □

2.7.10. Corollary. *Every localization AS^{-1} of a noetherian domain A is a noetherian domain.* $\square$ *locfintype*

2.7.11. When S is a multiplicative system in a domain A, the *localization* MS^{-1} of an A-module M is defined in a way analogous to the one used for simple localizations: It is the AS^{-1}-module whose elements are equivalence classes of fractions ms^{-1} with m in M and s in S. To take care of torsion, two fractions $m_1s_1^{-1}$ and $m_2s_2^{-1}$ are defined to be equivalent if there is a nonzero element s in S such that $m_1s_2s = m_2s_1s$. Then $ms_1^{-1} = 0$ if and only if $ms = 0$ for some nonzero s in S. As with simple localizations, there will be a homomorphism $M \to MS^{-1}$ that sends an element m to the fraction $m/1$. locmodtwo

2.7.12. Proposition. *Let S be a multiplicative system in a domain A.* *localexact*

(i) *Localization is an exact functor: A homomorphism $M \xrightarrow{\varphi} N$ of A-modules induces a homomorphism $MS^{-1} \xrightarrow{\varphi'} NS^{-1}$ of AS^{-1}-modules. If $M \xrightarrow{\varphi} N \xrightarrow{\psi} P$ is an exact sequence of A-modules, the localized sequence $MS^{-1} \xrightarrow{\varphi'} NS^{-1} \xrightarrow{\psi'} PS^{-1}$ is exact.*
(ii) *Let M be an A-module and let N be an AS^{-1}-module. When N is made into an A-module by restriction of scalars, homomorphisms of A-modules $M \to_A N$ correspond bijectively to homomorphisms of AS^{-1}-modules $MS^{-1} \to N$.*
(iii) *If multiplication by s is an injective map $M \to M$ for every s in S, then $M \subset MS^{-1}$. If multiplication by every s is a bijective map $M \to M$, then $M \approx MS^{-1}$.* $\square$

(2.7.13) a general principle importprinc

An elementary principle for working with fractions is that any finite sequence of computations in a localization AS^{-1} will involve finitely many denominators, and can therefore be done in a simple localization A_s, where s is a common denominator for the fractions that occur.

2.8 Finite Group Actions

grpi

Let G be a finite group of automorphisms of a finite-type domain B. An *invariant element* β of B is an element such that $\sigma\beta = \beta$ for every element σ of G. For example, for all b in B, the product and the sum

(2.8.1) $$\prod_{\sigma\in G} \sigma b \quad , \quad \sum_{\sigma\in G} \sigma b$$ invarelts

are invariant elements. The invariant elements form a subalgebra of B that is often denoted by B^G. Theorem 2.8.5 below asserts that B^G is a finite-type domain, and that points of the variety $\operatorname{Spec} B^G$ correspond bijectively to G-orbits in the variety $\operatorname{Spec} B$.

2.8.2. Examples. actonplaneex
(i) The symmetric group $G = S_n$ operates on the polynomial algebra $R = \mathbb{C}[y_1, ..., y_n]$ by permuting the variables. The Symmetric Functions Theorem asserts that the elementary symmetric functions

$$s_1(y) = \sum_i y_i\,, \quad s_2(y) = \sum_{i<j} y_iy_j \;\;, \ldots, \quad s_n(y) = y_1y_2\cdots y_n$$

generate the algebra R^G of invariant polynomials. Moreover, $s_1, ..., s_n$ are algebraically independent, so R^G is the polynomial algebra $\mathbb{C}[s_1, ..., s_n]$. The inclusion of R^G into R gives us a morphism $Y \to S$, from affine y-space $Y = \mathbb{A}^n_y$ to affine s-space $S = \mathbb{A}^n_s$. The operation of G on R defines an operation on Y. We use the same symbol s_j to denote the symmetric function $s_j(y)$ and the coordinate variable in the affine space S. If $c_1, ..., c_n$ are scalars, one can evaluate the variables $s_1, ..., s_n$ at $y = c$, to obtain a point $c = (c_1, ..., c_n)$ of S. The points $a = (a_1, ..., a_n)$ of Y with image c in S are those such that $s_j(a) = c_j$, and $a_1, ..., a_n$ are the roots of the polynomial $y^n - c_1y^{n-1} + \cdots \pm c_n$. The roots form a G-orbit, so the set of G-orbits in Y maps bijectively to S.

(ii) Let σ be the automorphism of the polynomial ring $B = \mathbb{C}[y_1, y_2]$ defined by $\sigma y_1 = \zeta y_1$ and $\sigma y_2 = \zeta^{-1}y_2$, where $\zeta = e^{2\pi i/n}$. Let G be the cyclic group of order n generated by σ, and let A denote the algebra B^G

of invariant elements. A monomial $m = y_1^i y_2^j$ is invariant if and only if n divides $i-j$, and an invariant polynomial is a linear combination of invariant monomials. You will be able to show that the three monomials

zetainvar (2.8.3)
$$u_1 = y_1^n\ ,\ u_2 = y_2^n\ ,\ \text{and } w = y_1y_2$$

generate the algebra A of invariants. Let's use the same symbols u_1, u_2, w to denote variables in a polynomial ring $\mathbb{C}[u_1, u_2, w]$. Let J be the kernel of the canonical homomorphism $\mathbb{C}[u_1, u_2, w] \xrightarrow{\tau} A$ that sends u_1, u_2 and w to y_1^n, y_2^n and y_1y_2, respectively.

slaction **2.8.4. Lemma.** *With notation as above, the kernel of τ is the principal ideal of $\mathbb{C}[u_1, u_2, w]$, generated by the polynomial $f = w^n - u_1u_2$. Thus $A \approx \mathbb{C}[u_1, u_2, w]/(w^n - u_1u_2)$.*

proof. First, f is in J. Let $g(u_1, u_2, w)$ be any element of J. So $g(y_1^n, y_2^n, y_1y_2) = 0$. We divide g by f, considered as a monic polynomial in w, say $g = fq + r$, where the remainder $r(u_1, u_2, w)$ has degree $< n$ in w. The remainder will be in J too: $r(y_1^n, y_2^n, y_1y_2) = 0$. We write r as a polynomial in w: $r = r_0(u_1, u_2) + r_1(u_1, u_2)w + \cdots + r_{n-1}(u_1, u_2)w^{n-1}$. When we substitute y_1^n, y_2^n, y_1y_2, the term $r_i(u_1, u_2)w^i$ becomes $r_i(y_1^n, y_2^n)(y_1y_2)^i$. The degree in y_1 of every monomial that appears there will be congruent to i modulo n, and the same is true for the degree in y_2. Since $r(y_1^n, y_2^n, y_1y_2) = 0$, and since the indices i are distinct, $r_i(y_1^n, y_2^n)$ must be zero for every i. This implies that $r_i(u_1, u_2) = 0$ for every i. So $r = 0$, which means that f divides g. □

We go back to the operation of the cyclic group G on $B = \mathbb{C}[y_1, y_2]$ and the algebra of invariants A. Let Y denote the affine plane $\operatorname{Spec} B$, and let $X = \operatorname{Spec} A$. The group G operates on Y, and except for the origin, which is a fixed point, the orbit of a point (y_1, y_2) consists of the n points $(\zeta^i y_1, \zeta^{-i} y_2)$, $i = 0, \ldots, n-1$. To show that G-orbits in Y correspond bijectively to points of X, we fix complex numbers u_1, u_2, w with $w^n = u_1u_2$, and look for solutions of the equations (2.8.3). When $u_1 \neq 0$, the equation $u_1 = y_1^n$ has n solutions for y_1, and when a soluion is given, y_2 is determined by the equation $y_1y_2 = w$. So the fibre has order n. Similarly, there are n points in the fibre if $u_2 \neq 0$. If $u_1 = u_2 = 0$, then $y_1 = y_2 = w = 0$, and the fibre contains just one point. In all cases, the fibres are the G-orbits. □

groupoper-one **2.8.5. Theorem.** *Let G be a finite group of automorphisms of a finite-type domain B, and let A denote the algebra B^G of invariant elements. Let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.*
(i) *A is a finite-type domain and B is a finite A-module.*
(ii) *G operates by automorphisms on Y.*
(iii) *The morphism $Y \to X$ defined by the inclusion $A \subset B$ is surjective. Its fibres are the G-orbits of points of Y.*

When a group G operates on a set Y, one often denotes the set of G-orbits of Y by Y/G, which is read as 'Y mod G'. With that notation, part **(iii)** of the theorem asserts that there is a bijective map

$$Y/G \to X$$

proof of 2.8.5 **(i)**: *The invariant algebra $A = B^G$ is a finite-type algebra, and B is a finite A-module.*

This is an interesting indirect proof. To show that A is a finite-type algebra, one constructs a finite-type subalgebra R of A such that B is a submodule of a finite R-module.

Let $\{z_1, \ldots, z_k\}$ be the G-orbit of an element z_1 of B. The orbit is the set of roots of the polynomial

spoly
$$f(t) \ = \ (t - z_1) \cdots (t - z_k) \ = \ t^k - s_1t^{k-1} + \cdots \pm s_k$$

Its coefficients $s_i(z)$ are the elementary symmetric functions in $\{z_1, ..., z_k\}$. Let R_1 denote the algebra generated by those symmetric functions. Because the symmetric functions are invariant, $R_1 \subset A$. Using the equation $f(z_1) = 0$, we can write any power of z_1 as a polynomial in z_1 of degree less than k, with coefficients in R_1.

We choose a finite set of generators $\{y_1, \ldots, y_r\}$ for the algebra B. If the order of the orbit of y_j is k_j, then y_j will be the root of a monic polynomial f_j of degree k_j with coefficients in A. Let R denote the finite-type algebra generated by all of the coefficients of all of the polynomials $f_1, ..., f_r$. For every $j = 1, ..., r$, we can write any power of y_j as a polynomial in y_j with coefficients in R, and of degree less than k_j. Using such expressions, we can write every monomial in $y_1, ..., y_r$ as a polynomial with coefficients in R, whose degree in the variable y_j is less than k_j. Since $y_1, ..., y_r$ generate B, we can write every element of B as such a polynomial. Then the finite set of monomials $y_1^{e_1} \cdots y_r^{e_r}$ with $e_j < k_j$ spans B as an R-module. Therefore B is a finite R-module.

The algebra A of invariants is a subalgebra of B that contains R. Since R is a finite-type algebra, it is noetherian. When regarded as an R-module, A is a submodule of the finite R-module B. Therefore A is also a finite R-module. When we put a finite set of algebra generators for R together with a finite set of R-module generators for A, we obtain a finite set of algebra generators for A, so A is a finite-type algebra. And, since B is a finite R-module, it is also a finite module over the larger ring A.

proof of 2.8.5 **(ii)**: *The group G operates on Y.*

A group element σ is a homomorphism $B \xrightarrow{\sigma} B$. It defines a morphism $Y \xleftarrow{u_\sigma} Y$, as in Definition 2.6.4. Since σ is an invertible homomorphism, i.e., an automorphism of B, u_σ is an automorphism of Y. Thus G operates on Y. However, there is a point that should be mentioned.

We write the operation of G on B on the left as usual, so that a group element σ maps an element β of B to σb. Then if σ and τ are two group elements, the product $\sigma\tau$ acts as *first do τ, then σ*: $(\sigma\tau)\beta = \sigma(\tau\beta)$.

(2.8.6) $$B \xrightarrow{\tau} B \xrightarrow{\sigma} B$$ BBB

We substitute $u = u_\sigma$ into Definition 2.6.4: If q is a point of Y, the morphism $Y \xleftarrow{u_\sigma} Y$ sends q to the point p such that $\pi_p = \pi_q\sigma$. It seems permissible to drop the symbol u, and to write the morphism simply as $Y \xleftarrow{\sigma} Y$. But since arrows are reversed when going from homomorphisms of algebras to morphisms of their spectra (2.6.5), the maps displayed in (2.8.6) above, give us morphisms

(2.8.7) $$Y \xleftarrow{\tau} Y \xleftarrow{\sigma} Y$$ YYY

On $Y = \operatorname{Spec} B$, the product $\sigma\tau$ acts as *first do σ, then τ*. This is a problem, but we can get around it by putting the symbol σ on the right when it operates on Y, so that σ sends a point q to $q\sigma$. Then if q is a point of Y, we will have $q(\sigma\tau) = (q\sigma)\tau$, as required of the operation.

- *If G operates on the left on B, then it operates on the right on* $\operatorname{Spec} B$.

This is important only when one wants to compose morphisms. In Definition 2.6.4, we followed custom and wrote the morphism u that corresponds to an algebra homomorphism φ on the left. We will continue to write morphisms on the left where possible, but not here.

Let β be an element of B and let q be a point of Y. The value of the function $\sigma\beta$ at a point q is the same as the value of β at the point $q\sigma$ (2.6.6):

(2.8.8) $$[\sigma\beta](q) = \beta(q\sigma)$$ $\square$ bsigmay

proof of 2.8.5 **(iii)**: *The fibres of the morphism $Y \to X$ are the G-orbits in Y.*

We go back to the subalgebra $A = B^G$. For σ in G, we have a diagram of algebra homomorphisms and the corresponding diagram of morphisms of varieties

(2.8.9) $$\begin{tikzcd} B \arrow[r, "\sigma"] & B & & Y & Y \arrow[l, "\sigma"'] \\ A \arrow[u] \arrow[r, equal] & A \arrow[u] & & X \arrow[from=u] \arrow[r, equal] & X \arrow[from=u] \end{tikzcd}$$ actonB

The diagram of morphisms shows that points of Y that are in a G-orbit have the same image in X, and therefore that the set of G-orbits in Y, which we may denote by Y/G, maps to X. We show that the map $Y/G \to X$ is bijective.

2.8.10. Lemma. **(i)** *Let $p_1, \ldots, p_k$ be distinct points of affine space $\mathbb{A}^n$, and let $c_1, \ldots, c_k$ be complex numbers. There is a polynomial $f(x_1, \ldots, x_n)$ such that $f(p_i) = c_i$ for $i = 1, \ldots, n$.* *arbvals*

(ii) *Let B be a finite-type algebra, let $q_1, \ldots, q_k$ be distinct points of* $\operatorname{Spec} B$, *and let $c_1, \ldots, c_k$ be distinct complex numbers. There is an element β in B such that $\beta(q_i) = c_i$ for $i = 1, \ldots, k$.* $\square$

injectivity of the map $Y/G \to X$: Let O_1 and O_2 be distinct G-orbits in Y. Lemma 2.8.10 tells us that there is an element β in B whose value is 0 at every point of O_1, and 1 at every point of O_2. Since G permutes the orbits, $\sigma\beta$ will also be 0 at points of O_1 and 1 at points of O_2. Then the product $\gamma = \prod_\sigma \sigma\beta$ will be 0 at points of O_1 and 1 at points of O_2, and the product γ is invariant. If p_i denotes the image in X of the orbit O_i, the maximal ideal $\mathfrak{m}_{p_i}$ of A is the intersection $A \cap \mathfrak{m}_q$, where q is any point in the orbit O_i. Therefore γ is in the maximal ideal $\mathfrak{m}_{p_1}$, but not in $\mathfrak{m}_{p_2}$. The images of the two orbits are distinct.

surjectivity of the map $Y/G \to X$: It suffices to show that the map $Y \to X$ is surjective.

extideal **2.8.11. Lemma.** *If I is an ideal of the invariant algebra A, and if the extended ideal IB is the unit ideal of B, then I is the unit ideal of A.*

As before, the extended ideal IB is the ideal of B generated by I.

Let's assume the lemma for the moment, and use it to prove surjectivity of the map $Y \to X$. Let p be a point of X. The lemma tells us that the extended ideal $\mathfrak{m}_p B$ isn't the unit ideal. So it is contained in a maximal ideal $\mathfrak{m}_q$ of B, where q is a point of Y. Then $\mathfrak{m}_p \subset (\mathfrak{m}_p B) \cap A \ \subset \mathfrak{m}_q \cap A$. The contraction $\mathfrak{m}_q \cap A$ is an ideal of A, and it isn't the unit ideal because it doesn't contain 1, which isn't in $\mathfrak{m}_q$. Since $\mathfrak{m}_p$ is contained in $\mathfrak{m}_q \cap A$ and $\mathfrak{m}_p$ is a maximal ideal, $\mathfrak{m}_p = \mathfrak{m}_q \cap A$. This means that q maps to p in X. □

proof of the lemma. If $IB = B$, there will be an equation $\sum_i z_i b_i = 1$, with z_i in I and b_i in B. The sums $\alpha_i = \sum_\sigma \sigma b_i$ are invariant, so they are elements of A, and the elements z_i are invariant because they are in A. Therefore $\sum_\sigma \sigma(z_i b_i) = z_i \sum_\sigma \sigma b_i = z_i \alpha_i$ is in I. Then

$$\sum_\sigma 1 = \sum_\sigma \sigma(1) = \sum_{\sigma,i} \sigma(z_i b_i) = \sum_i z_i \alpha_i$$

The right side is in I, and the left side is the order of the group which, because A contains the complex numbers, is an invertible element of A. So I is the unit ideal. □

2.9 Exercises

2.9.1. Prove that relatively prime polynomials in F, G two variables x, y, not necessarily homogeneous, have finitely many common zeros in $\mathbb{A}^2$. chaptwoex xfinmanyzeros

2.9.2. Prove that if A, B are finite-type domains, defining $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2)$ makes the tensor product $A \otimes B$ into a finite-type domain. xtensftyp

2.9.3. Prove that if a noetherian ring contains just one prime ideal, then that ideal is nilpotent. xidealnilp

2.9.4. Prove that an algebra A that is a complex vector space of dimension d contains at most d maximal ideals. xatmostmax

2.9.5. Let T denote the ring $\mathbb{C}[\epsilon]$, with $\epsilon^2 = 0$. If A is the coordinate ring of an affine variety X, an (infinitesimal) tangent vector to X is, by definition, given by an algebra homomorphism $\varphi : A \to T$. xtanvect

(i) Show that such a homomorphism can be written in the form $\varphi(a) = f(a) + d(a)\epsilon$, where f and d are functions $A \to \mathbb{C}$. Show that f is an algebra homomorphism, and that d is an *f-derivation*, a linear map that satisfies the identity $d(ab) = f(a)d(b) + d(a)f(b)$.

(ii) Let $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_r)$. Show that the tangent vectors to $X = \operatorname{Spec} A$ are defined by the equations $\nabla f_i(p)x = 0$. In other words, the tangent vectos are the vectors that are the vectors that are orthogonal to the gradients.

2.9.6. Let $i = (i_1, ..., i_n)$ be a set of non-negative integers, and let $x^{(i)}$ denote the monomial $x_1^{i_1} \cdots x_n^{i_n}$. A *formal power series* is a sum $\sum a_{(i)} x^{(i)}$, where $a_{(i)}$ are arbitrary complex numbers. There is no condition of convergence. Prove that the set of formal power series forms a ring $\mathbb{C}[[x_1, ..., x_n]]$, and that an element whose constant term is nonzero is invertible. exinvertseries

2.9.7. Prove that that the varieties in the affine plane $\mathbb{A}^2$ are points, curves, and the affine plane $\mathbb{A}^2$ itself. xvarinplane

2.9.8. Classify algebras that are complex vector spaces of dimensions two or three. xdimthree

2.9.9. Derive version 1 of the Nullstellensatz from the Strong Nulletellensatz. xstrongimpliesN

2.9.10. Find generators for the ideal of $\mathbb{C}[x, y]$ of polynomials that vanish at the three points $(0, 0), (0, 1), (1, 0)$. xptsideal

2.9.11. Let A be a noetherian ring. Prove that a radical ideal I of A is the intersection of finitely many prime ideals. xintprime

2.9.12. Let C and D be closed subsets of an affine variety $X = \operatorname{Spec} A$. Suppose that no component of D is contained in C. Prove that there is a regular function f that vanishes on C and isn't identically zero on any component of D. xnotvanish

2.9.13. A minimal prime ideal is an ideal that doesn't properly contain any other prime ideal. Prove that a nonzero, finite-type algebra A (not necessarily a domain) contains at least one and only finitely many minimal prime ideals. Try to find a proof that doesn't require much work. xminprime

2.9.14. Let K be a field and let R be the polynomial ring $K[x_1, ..., x_n]$, with $n > 0$. Prove that the field of fractions of R is not a finitely generated K-algebra. franotfg

2.9.15. Prove that the algebra $A = \mathbb{C}[x, y]/(x^2 + y^2 + 1)$ is isomorphic to the Laurent Polynomial Ring $\mathbb{C}[t, t^{-1}]$, but that $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ is not isomorphic to $\mathbb{R}[t, t^{-1}]$. xcircle

2.9.16. Let B be a finite type domain, and let p and q be points of the affine variety $Y = \operatorname{Spec} B$. Let A be the set of elements $f \in B$ such that $f(p) = f(q)$. Prove xgluepts

(i) A is a finite type domain.

(ii) B is a finite A-module.

(iii) Let $\varphi :\ \operatorname{Spec} B \to \operatorname{Spec} A$ be the morphism obtained from the inclusion $A \subset B$. Show that $\varphi(p) = \varphi(q)$, and that φ is bijective everywhere else.

2.9.17. The equation $y^2 = x^3$ defines a plane curve X with a cusp at the origin, the spectrum of the algebra $A = \mathbb{C}[x, y]/(y^2 - x^3)$. There is a homomorphism $A \xrightarrow{\varphi} \mathbb{C}[t]$, such that $\varphi(x) = t^2$ and $\varphi(y) = t^3$, and the associated morphism $\mathbb{A}^1_t \xrightarrow{u} X$ sends a point t of $\mathbb{A}^1$ to the point $(x, y) = (t^2, t^3)$ of X. Prove that u is a homeomorphism in the Zariski topology and in the classical topology. xcusp xspellout-morph **2.9.18.** Explain what a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ means in terms of polynomials, when $A = \mathbb{C}[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ and $B = \mathbb{C}[y_1, \ldots, y_n]/(g_1, \ldots, g_k)$.

xadjoin-frac **2.9.19.** Let $A = \mathbb{C}[x_1, ..., x_2]$, and let $B = A[\alpha]$, where α is an element of the fraction field $\mathbb{C}(x)$ of A. Describe the fibres of the morphism $Y = \operatorname{Spec} B \to \operatorname{Spec} A = X$.

xparam-curve **2.9.20.** Let X be the plane curve $y^2 = x(x-1)^2$, let $A = \mathbb{C}[x, y]/(y^2 - x(x-1)^2)$ be its coordinate algebra, and let x, y denote the residues of those elements in A too.

(i) Points of the curve can be parametrized by a variable t. Use the lines $y = t(x-1)$ to determine such a parametrization.

(ii) Let $B = \mathbb{C}[t]$ and let T be the affine line $\operatorname{Spec} \mathbb{C}[t]$. The parametrization gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.

grplawtwo **2.9.21.** Let X be the affine line $\operatorname{Spec} \mathbb{C}[x]$. When we view $\operatorname{Spec} \mathbb{C}[x_1, x_2]$ as the product $X \times X$, a homomorphism $\mathbb{C}[x] \to \mathbb{C}[x_1, x_2]$ defines a law of composition on X, a morphism $X \times X \to X$. Determine the homomorphisms that are group laws on X with the point $x = 0$ as the identity.

xzetaxy **2.9.22.** The cyclic group $G = \langle \sigma \rangle$ of order n operates on the polynomial algebra $A = \mathbb{C}[x, y]$ by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, where $\zeta = e^{2\pi i/n}$.

(i) Describe the invariant ring A^G by exhibiting generators and defining relations.

(ii) Prove that the there is a $2 \times n$ matrix whose 2×2-minors are defining relations for A^G.

(iii) Prove that the morphism $\operatorname{Spec} A = \mathbb{A}^2 \to \operatorname{Spec} B$ defined by the inclusion $B \subset A$ is surjective, and that its fibres are the G-orbits. Don't use Theorem 2.8.5.

remainirrd **2.9.23.** Let A be a finite-type domain, and let f be an irreducible element of $\mathbb{C}[x_1, ..., x_n]$ of positive degree. Prove that f is an irreducible element of $A[x_1, ..., x_n]$.

2.9.24. try to make exercise: missing points in C^* are L^*, where L is a special line.

Chapter 3 PROJECTIVE ALGEBRAIC GEOMETRY

projgeom

As before, points of projective space $\mathbb{P}^n$ are equivalence classes of nonzero vectors $(x_0, ..., x_n)$, the equivalence relation being that, for any nonzero complex number λ

xlambdax (3.0.1)
$$(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n).$$

Projective varieties are irreducible closed subsets of projective space.

Though affine varieties are important, most of algebraic geometry concerns projective varieties. It isn't completely clear why this is so, but one property of projective space gives a hint of its importance: With its classical topology, projective space is *compact*. Therefore a projective variety is compact.

A topological space is compact if:

It is a *Hausdorff space*: Distinct points p, q of X have disjoint open neighborhoods, and
it is *quasicompact*: If X is covered by a family $\{U^i\}$ of open sets, then a finite subfamily covers X.

By the way, when we say that the sets $\{U^i\}$ *cover* a topological space X, we mean that X is the union $\bigcup U^i$. We don't allow U^i to contain elements that aren't in X, though that would be a customary usage in English.

In the classical topology, affine space $\mathbb{A}^n$ isn't quasicompact, and therefore it isn't compact. The *Heine-Borel Theorem* asserts that a subset of $\mathbb{A}^n$ is compact in the classical topology if and only if it is closed and bounded.

We show that $\mathbb{P}^n$ is compact, assuming that the Hausdorff property has been verified. The $2n+1$-dimensional sphere $\mathbb{S}$ of unit length vectors in $\mathbb{A}^{n+1}$ is a bounded set, and because it is the zero locus of the equation $\overline{x}_0 x_0 + \cdots + \overline{x}_n x_n = 1$, it is closed. The Heine-Borel Theorem tells us that $\mathbb{S}$ is compact, and the map $\mathbb{S} \to \mathbb{P}^n$ that sends a vector $(x_0, ..., x_n)$ to the point of projective space with that coordinate vector is continuous and surjective. The next lemma of topology shows that $\mathbb{P}^n$ is compact.

image-compact **3.0.2. Lemma.** *Let $Y \xrightarrow{f} X$ be a continuous map. Suppose that Y is compact and that X is a Hausdorff space — that it has the Hausdorff property. Then the image $f(Y)$ is a closed, compact subset of X.* □

3.1 Projective Varieties

pvariety

A subset of $\mathbb{P}^n$ is *Zariski closed* if it is the set of common zeros of a family of *homogeneous* polynomials $f_1, ..., f_k$ in the coordinate variables $x_0, ..., x_n$, or if it is the set of zeros of the ideal $\mathcal{I}$ generated by such a family. As was explained in (1.3.1), $f(\lambda x) = 0$ for all λ if and only if all of the homogeneous parts of f vanish at x.

The Zariski closed sets are the closed sets in the *Zariski topology* on $\mathbb{P}^n$. We usually refer to Zariski closed sets simply as *closed sets*.

Because the polynomial ring $\mathbb{C}[x_0, ..., x_n]$ is noetherian, projective space $\mathbb{P}^n$ is a noetherian space: Every strictly increasing family of ideals of $\mathbb{C}[x]$ is finite, and every strictly decreasing family of closed subsets of $\mathbb{P}^n$ is finite. Therefore every closed subset of $\mathbb{P}^n$ is a finite union of irreducible closed sets (2.2.16).

3.1.1. Definition. A *projective variety* is an irreducible closed subset of a projective space $\mathbb{P}^n$. defprojvar

We will want to know when two projective varieties are isomorphic. This will be explained in Section 3.5, when morphisms are defined.

The Zariski topology on a projective variety X is induced from the topology on the projective space that contains it (**??**). Since a projective variety X is closed in $\mathbb{P}^n$, a subset of X is closed in X if and only if it is closed in $\mathbb{P}^n$.

3.1.2. Lemma. *The one-point sets in projective space are closed.* *point-closed*

proof. Let p be the point $(a_0, ..., a_n)$. The first guess might be that the one-point set $\{p\}$ is defined by the equations $x_i = a_i$, but the polynomials $x_i - a_i$ aren't homogeneous in x. This is reflected in the fact that, for any $\lambda \neq 0$, the vector $(\lambda a_0, ..., \lambda a_n)$ represents the same point, but it doesn't satisfy those equations. The equations that define the set $\{p\}$ are

(3.1.3) $$a_i x_j = a_j x_i,$$ define-point

for $i, j = 0, ..., n$, which imply that the ratios a_i/a_j and x_i/x_j are equal. $\square$

3.1.4. Lemma. *The proper closed subsets of the projective line are its nonempty finite subsets, and the proper closed subsets of the projective plane are finite unions of points and curves.* $\square$ *closedin-line*

The rest of this section contains a few examples of projective varieties.

(3.1.5) linear subspaces linsubsp

If W is a subspace of dimension $r+1$ of the vector space $\mathbb{C}^{n+1}$, the points of $\mathbb{P}^n$ that are represented by the nonzero vectors in W form a *linear subspace* L of $\mathbb{P}^n$, of dimension r. If $(w_0, ..., w_r)$ is a basis of W, the linear subspace L corresponds bijectively to a projective space of dimension r, by

$$c_0 w_0 + \cdots + c_r w_r \quad \longleftrightarrow \quad (c_0, ..., c_r)$$

For example, the set of points $(x_0, ..., x_r, 0, ..., 0)$ is a linear subspace of dimension r. A *line* is a linear subspace of dimension 1. $\square$

(3.1.6) a quadric surface quadric-surface

A *quadric* in projective three-space $\mathbb{P}^3$ is the locus of zeros of an irreducible homogeneous quadratic polynomial in four variables.

We describe a bijective map from the product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines to a quadric in $\mathbb{P}^3$. Let coordinates in the two copies of $\mathbb{P}^1$ be (x_0, x_1) and (y_0, y_1), respectively, and let the four coordinates in $\mathbb{P}^3$ be z_{ij}, with $0 \le i, j \le 1$. The map is defined by $z_{ij} = x_i y_j$. Its image is the quadric Q whose equation is

(3.1.7) $$z_{00} z_{11} = z_{01} z_{10}$$ ponepone

To check that the map $\mathbb{P}^1 \times \mathbb{P}^1 \to Q$ defined by the euation $z_{ij} = x_i y_j$ is bijective, we choose a point w of Q. One of its coordinates, say z_{00}, will be nonzero. Then if (x, y) is a point of $\mathbb{P}^1 \times \mathbb{P}^1$ whose image is w, so that $z_{ij} = x_i y_j$, the coordinates x_0 and y_0 must be nonzero too. When we normalize z_{00}, x_0, and y_0 to 1, the equation of the quadric becomes $z_{11} = z_{01} z_{10}$. This equation has a unique solution for x_1 and y_1 such that $z_{ij} = x_i y_j$, namely $x_1 = z_{10}$ and $y_1 = z_{01}$.

The quadric whose equation is (3.1.7) contains two families of *lines*, the images of the subsets $x \times \mathbb{P}^1$ and $\mathbb{P}^1 \times y$ of $\mathbb{P} \times \mathbb{P}$.

The equation (3.1.7) can be diagonalized by substituting $z_{00} = s+t$, $z_{11} = s - t$, $z_{01} = u+v$, $z_{10} = u - v$. This changes the equation (3.1.7) to $s^2 - t^2 = u^2 - v^2$. When we look at the affine open set $\{u = 1\}$, the equation becomes $s^2 + v^2 - t^2 = 1$. The real locus of this equation is a one-sheeted hyerboloid in $\mathbb{R}^3$, and the two families of complex lines in the quadric correspond to the familiar rulings of that hyperboloid by real lines.

###insert figure###

hsurftwo **(3.1.8) hypersurfaces**

A *hypersurface* in projective space $\mathbb{P}^n$ is the locus of zeros of an irreducible homogeneous polynomial $f(x_0, ..., x_n)$. Its *degree* is the degree of the polynomial f. Plane projective curves and quadric surfaces are hypersurfaces.

segreemb **(3.1.9) the Segre embedding of a product**

The product $\mathbb{P}^m_x \times \mathbb{P}^n_y$ of projective spaces can be embedded by its *Segre embedding* into a projective space $\mathbb{P}^N_z$ that has coordinates z_{ij}, with $i = 0, ..., m$ and $j = 0, ..., n$. So $N = (m+1)(n+1) - 1$. The Segre embedding is defined by

segreco-ords (3.1.10)
$$z_{ij} = x_i y_j$$

We call the coordinates z_{ij} the *Segre variables*. The map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ that was described in (**3.1.6**) is the simplest case of a Segre embedding.

segreeq **3.1.11. Proposition.** *The Segre embedding maps the product $\mathbb{P}^m \times \mathbb{P}^n$ bijectively to the locus S of the **Segre equations***

segree-quations (3.1.12)
$$z_{ij} z_{k\ell} - z_{i\ell} z_{kj} = 0$$

The proof is analogous to the one given in (**3.1.6**). □

The Segre embedding is important because it makes the product of projective spaces into a projective variety, the closed subvariety of $\mathbb{P}^N$ defined by the Segre equations. However, to show that the product is a variety, we need to show that the locus of the Segre equations is irreducible, and this isn't obvious. We defer the proof to Section 3.3 below. (See Proposition 3.3.4.)

verone-seemb **(3.1.13) the Veronese embedding of projective space**

Let the coordinates in $\mathbb{P}^n$ be x_i, and let those in $\mathbb{P}^N$ be v_{ij}, with $0 \le i \le j \le n$. So $N = \binom{n+2}{2} - 1$. The *Veronese embedding* is the map $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^N$ defined by $v_{ij} = x_i x_j$. The Veronese embedding resembles the Segre embedding, but in the Segre embedding, there are distinct sets of coordinates x and y, and $i \le j$ isn't required.

The proof of the next proposition is similar to the proof of (3.1.11), once one has untangled the inequalities.

veroneq **3.1.14. Proposition.** *The Veronese embedding f maps $\mathbb{P}^n$ bijectively to the locus X in $\mathbb{P}^N$ of the equations*

$$v_{ij} v_{k\ell} = v_{i\ell} v_{kj} \quad \text{for} \quad 0 \le i \le k \le j \le \ell \le n$$
□

For example, the Veronese embedding maps $\mathbb{P}^1$ bijectively to the conic $v_{00} v_{11} = v_{01}^2$ in $\mathbb{P}^2$.

(3.1.15) the twisted cubic twistcubic

There are higher order Veronese embeddings. They are defined by evaluating the monomials of some degree $d > 2$. The first example is the embedding of $\mathbb{P}^1$ by the cubic monomials in two variables, which maps $\mathbb{P}^1$ to $\mathbb{P}^3$. Let the coordinates in $\mathbb{P}^3$ be $v_0, ..., v_3$. The cubic Veronese embedding is defined by

$$v_0 = x_0^3, \quad v_1 = x_0^2 x_1, \quad v_2 = x_0 x_1^2, \quad v_3 = x_1^3$$

Its image, the locus $(v_0, v_1, v_2, v_3) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$, is called a *twisted cubic* in $\mathbb{P}^3$. It is the set of common zeros of three polynomials:

(3.1.16) $$v_0 v_2 - v_1^2 \ , \ \ v_1 v_2 - v_0 v_3 \ , \ \ v_1 v_3 - v_2^2$$ twcubic

the 2×2 minors of the 2×3 matrix

(3.1.17) $$\begin{pmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \end{pmatrix}$$ twothree-matrix

A 2×3 matrix has rank ≤ 1 if and only if its 2×2 minors are zero. So a point (v_0, v_1, v_2, v_3) lies on the twisted cubic if (3.1.17) has rank one, which means that the vectors (v_0, v_1, v_2) and (v_1, v_2, v_3), if both are nonzero, represent the same point of $\mathbb{P}^2$.

Setting $x_0 = 1$ and $x_1 = t$, the twisted cubic becomes the locus of points $(1, t, t^2, t^3)$. There is one point on the twisted cubic at which $x_0 = 0$, the point $(0, 0, 0, 1)$. □

3.2 Homogeneous Ideals

Let R denote the polynomial algebra $\mathbb{C}[x_0, ..., x_n]$. homogen

3.2.1. Lemma. *Let $\mathcal{I}$ be an ideal of R. The following conditions are equivalent.* *homogideal*

(i) *$\mathcal{I}$ can be generated by homogeneous polynomials.*

(ii) *A polynomial is in $\mathcal{I}$ if and only if its homogeneous parts are in $\mathcal{I}$.* □

An ideal $\mathcal{I}$ that satisfies these conditions is a *homogeneous ideal*.

3.2.2. Corollary. *Let S be a subset of projective space $\mathbb{P}^n$. The set of elements of R that vanish at all points of S is a homogeneous ideal.* *idealishom*

This follows from Lemma 1.3.2. □

3.2.3. Lemma. *The radical of a homogeneous ideal is homogeneous.* *radicalhomogeneous*

proof. Let $\mathcal{I}$ be a homogeneous ideal, and let f be an element of its radical $\operatorname{rad}\mathcal{I}$. So for some r, f^r is in $\mathcal{I}$. When f is written as the sum $f_0 + \cdots + f_d$ of its homogeneous parts, the highest degree part of f^r is $(f_d)^r$. Since $\mathcal{I}$ is homogeneous, $(f_d)^r$ is in $\mathcal{I}$ and f_d is in $\operatorname{rad}\mathcal{I}$. Then $f_0 + \cdots + f_{d-1}$ is also in $\operatorname{rad}\mathcal{I}$. By induction on d, all of the homogeneous parts $f_0, ..., f_d$ are in $\operatorname{rad}\mathcal{I}$. □

3.2.4. The locus of zeros of a set f of homogeneous polynomials in $\mathbb{P}^n$ may be denoted by $V(f)$, and the locus of zeros of a homogeneous ideal $\mathcal{I}$ may be denoted by $V(\mathcal{I})$. We use the same notation as for closed subsets of affine space. defVXI

The complement of the origin in the affine space $\mathbb{A}^{n+1}$ is mapped to the projective space $\mathbb{P}^n$ by sending a vector $(x_0, ..., x_n)$ to the point of $\mathbb{P}^n$ defined by that vector. A homogeneous ideal $\mathcal{I}$ has a zero locus W in affine space and V in projective space. Unless $\mathcal{I}$ is the unit ideal, the origin $x = 0$ will be a point of W. The complement of the origin in W will map surjectively to V.

If W contains a point x other than the origin, then every point of the one-dimensional subspace of $\mathbb{A}^{n+1}$ spanned by x is in W, because a homogeneous polynomial f vanishes at x if and only if it vanishes at λx. An affine variety that is the union of lines through the origin is called an *affine cone*. If the locus W contains a point x other than the origin, it is an affine cone.

The loci $\{x_0^2 + x_1^2 - x_2^2 = 0\}$ and $\{x_0x_1^3 + x_1^2x_2^2 + x_0^3x_2 = 0\}$ are affine cones in $\mathbb{A}^3$.

Note. The real locus $x_0^2 + x_1^2 - x_2^2 = 0$ in $\mathbb{R}^3$ decomposes into two parts when the origin is removed. Because of this, it is sometimes called a "double cone". However, the complex locus doesn't decompose.

irrel **(3.2.5) the irrelevant ideal**

In the polynomial algebra $R = \mathbb{C}[x_0, ..., x_n]$, the maximal ideal $\mathcal{M} = (x_0, ..., x_n)$ generated by the variables is called the *irrelevant ideal* because its zero locus in projective space is empty.

nozeros **3.2.6. Proposition.** *The zero locus $V(\mathcal{I})$ in $\mathbb{P}^n$ of a homogeneous ideal $\mathcal{I}$ of R is empty if and only if $\mathcal{I}$ contains a power of the irrelevant ideal $\mathcal{M}$.*

Another way to say this is: The zero locus of a homogeneous ideal $\mathcal{I}$ is empty if and only if either $\mathcal{I}$ is the unit ideal R, or its radical is the irrelevant ideal.

proof of Proposition 3.2.6. Let V be the zero locus of $\mathcal{I}$ in $\mathbb{P}^n$. If $\mathcal{I}$ contains a power of $\mathcal{M}$, it contains a power of each variable. Powers of the variables have no common zeros in projective space, so V is empty.

Suppose that V is empty, and let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$. Since the complement of the origin in W maps to the empty locus V, it is empty. The origin is the only point that might be in W. If W is the one point space consisting of the origin, then $\operatorname{rad}\mathcal{I} = \mathcal{M}$. If W is empty, $\mathcal{I}$ is the unit ideal. □

homstrnull **3.2.7. Strong Nullstellensatz, projective version.** *Let g be a nonconstant homogeneous polynomial in $x_0, ..., x_n$, and let $\mathcal{I}$ be a homogeneous ideal of $\mathbb{C}[x]$, not the unit ideal. If g vanishes at every point of the zero locus $V(\mathcal{I})$ in $\mathbb{P}^n$, then $\mathcal{I}$ contains a power of g.*

proof. Let W be the locus of zeros of $\mathcal{I}$ in the affine space with coordinates $x_0, ..., x_n$. A homogeneous polynomial g that vanishes on $V(\mathcal{I})$ vanishes at every point of W different from the origin, and if g isn't a constant, it vanishes at the origin too. So the affine Strong Nullstellensatz 2.4.6 applies. If a nonconstant homogeneous polynomial g vanishes on W, then $\mathcal{I}$ contains a power of g. □

hstrongnull-cor **3.2.8. Corollary.** **(i)** *Let f and g be homogeneous polynomials. If f is irreducible and if $V(f) \subset V(g)$, then f divides g.*
(ii) *Let $\mathcal{I}$ and $\mathcal{J}$ be homogeneous radical ideals, neither of which is the unit ideal. If $V(\mathcal{I}) = V(\mathcal{J})$, then $\mathcal{I} = \mathcal{J}$.*

proof. **(ii)** Suppose that $V(f) = V(g)$. Let g be a homogeneous element of $\mathcal{J}$ that vanishes on $V(\mathcal{J})$ and therefore on $V(\mathcal{I})$. Since $\mathcal{I}$ is a radical ideal, the Strong Nullstellensatz tells us that $\mathcal{I}$ contains g. This shows that $\mathcal{J} \subset \mathcal{I}$. Similarly, $\mathcal{I} \subset \mathcal{J}$. □

homprime **3.2.9. Lemma.** *Let $\mathcal{P}$ be a homogeneous ideal in the polynomial algebra $\mathbb{C}[x_0, ..., x_n]$, not the unit ideal. The following conditions are equivalent:*
(i) *$\mathcal{P}$ is a prime ideal.*
(ii) *If f and g are homogeneous polynomials, and if $fg \in \mathcal{P}$, then $f \in \mathcal{P}$ or $g \in \mathcal{P}$.*
(iii) *If $\mathcal{A}$ and $\mathcal{B}$ are homogeneous ideals, and if $\mathcal{AB} \subset \mathcal{P}$, then $\mathcal{A} \subset \mathcal{P}$ or $\mathcal{B} \subset \mathcal{P}$. Or, if $\mathcal{A}$ and $\mathcal{B}$ are homogeneous ideals that contain $\mathcal{P}$, and if $\mathcal{AB} \subset \mathcal{P}$, then $\mathcal{A} = \mathcal{P}$ or $\mathcal{B} = \mathcal{P}$.*

In other words, a homogeneous ideal is a prime ideal if the usual conditions for a prime ideal are satisfied when the polynomials or ideals are homogeneous.

proof of the lemma. When the word homogeneous is omitted, **(ii)** and **(iii)** become the definition of a prime ideal. So **(i)** implies **(ii)** and **(iii)**. The fact that **(iii)** $\Rightarrow$ **(ii)** is proved by considering the principal ideals generated by f and g.

(ii) ⇒ (i) Suppose that a homogeneous ideal $\mathcal{P}$ satisfies condition **(ii)**, and that the product fg of two polynomials, not necessarily homogeneous, is in $\mathcal{P}$. If f has degree d and g has degree e, the highest degree part of fg is the product of the homogeneous parts f_d and g_e. Since $\mathcal{P}$ is a homogeneous ideal that contains fg, it contains $f_d g_e$. Therefore one of the factors, say f_d, is in $\mathcal{P}$. Let $h = f - f_d$. Then $hg = fg - f_d g$ is in $\mathcal{P}$, and it has lower degree than fg. By induction on the degree of fg, h or g is in $\mathcal{P}$, and if h is in $\mathcal{P}$, so is f. □

3.2.10. Proposition. *Let V be the zero locus in $\mathbb{P}^n$ of a homogeneous radical ideal $\mathcal{I}$ that isn't the irrelevant ideal or the unit ideal. Then V is a projective variety (an irreducible closed subset of $\mathbb{P}^n$) if and only if $\mathcal{I}$ is a prime ideal. Thus a subset V of $\mathbb{P}^n$ is a projective variety if and only if it is the zero locus of a homogeneous prime ideal other than the irrelevant ideal.* *closedirred*

proof. The closed set V isn't empty, so the locus W of zeros of the radical ideal $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ contains points other than the origin. Let W' be the complement of the origin in W. Then W' maps surjectively to V. If V is irreducible, then W' is irreducible and therefore W is irreducible (2.2.15) **(ii)**. Proposition 2.2.23 tells us that $\mathcal{I}$ is a prime ideal.

Conversely, suppose that $\mathcal{I}$ isn't a prime ideal. Then there are homogeneous ideals $\mathcal{A} > \mathcal{I}$ and $\mathcal{B} > \mathcal{I}$, such that $\mathcal{AB} \subset \mathcal{I}$. Since $\mathcal{I}$ is a radical ideal, $\operatorname{rad}(\mathcal{AB}) \subset \mathcal{I}$, and since $\operatorname{rad}\mathcal{A}\operatorname{rad}\mathcal{B} \subset \operatorname{rad}(\mathcal{AB})$, $\operatorname{rad}\mathcal{A}\operatorname{rad}\mathcal{B} \subset \mathcal{I}$. Therefore we may suppose that $\mathcal{A}$ and $\mathcal{B}$ are radical ideals. If α is an element of $\mathcal{A}$ that isn't in $\mathcal{I}$, the Strong Nullstellensatz asserts that α doesn't vanish on $V(\mathcal{I})$. So $V(\mathcal{A}) < V(\mathcal{I})$ and similarly, $V(\mathcal{B}) < V(\mathcal{I})$. But $V(\mathcal{A}) \cup V(\mathcal{B}) = V(\mathcal{AB}) \supset V(\mathcal{I})$. Then $V(\mathcal{I})$ isn't an irreducible space. □

(3.2.11) quasiprojective varieties var-quasiproj

We may somwtimes want to study a nonempty open subset of a projective variety in addition to the projective variety itself. We call such an open subset a *variety* too. The topology on a variety is induced from the topology on projective space. It will be an irreducible topological space (Lemma 2.2.15). However, most of the the varieties we encounter will be either affine or projective.

For example, the complement of a point in a projective variety is a variety.

We denote the open subspace $\{x_i \neq 0\}$ of $\mathbb{P}^n$ by $\mathbb{U}^i$, as we did for subsets of $\mathbb{P}^2$. The points of $\mathbb{U}^0$ can be written as $(1, u_1, ..., u_n)$, with $u_i = x_i/x_0$. This subspace is an affine space of dimension n that we refer to as a *standard open subset* of projective space.

An affine variety $X = \operatorname{Spec} A$ may be embedded as a closed subvariety into the standard open set $\mathbb{U}^0$. It becomes an open subset of its closure in $\mathbb{P}^n$, which is a projective variety (Lemma 2.2.15). So it is a variety. And of course, a projective variety is a variety.

Elsewhere, what we call a variety is called a *quasiprojective variety*. We drop the adjective 'quasiprojective'. There are abstract varieties that aren't quasiprojective, varieties that cannot be embedded into any projective space. But such varieties aren't very important and we won't study them. In fact, it is hard enough to find examples that we won't try to give one here. So for us, the adjective 'quasiprojective' is superfluous as well as ugly.

3.2.12. Lemma. *The topology on the affine open subset $\mathbb{U}^0$ of $\mathbb{P}^n$ induced from the Zariski topology on $\mathbb{P}^n$ is same as the Zariski topology that is obtained by viewing $\mathbb{U}^0$ as the affine space $\operatorname{Spec}\mathbb{C}[u_1, ..., u_n]$, $u_i = x_i/x_0$.* *topon-standaff*

This follows from the fact that a homogeneous polynomial $f(x_0, ..., x_n)$ and its dehomogeniztion $F(u_1, ..., u_n) = f(1, u_1, ..., u_n)$ have the same zeros on $\mathbb{U}^0$. □

3.3 Product Varieties

The properties of products of varieties are intuitively plausible, but one must be careful because the Zariski topology on a product isn't the product topology. prodvar

The *product topology* on the product $X \times Y$ of topological spaces is the coarsest topology such that the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are continuous. If C and D are closed subsets of X and Y respectively, then $C \times D$ is a closed subset of $X \times Y$ in the product topology, and every closed set in the product topology is a finite union of such subsets. The product topology is much coarser than the Zariski topology. For example, the proper Zariski closed subsets of $\mathbb{P}^1$ are the nonempty finite subsets. In the product topology, the proper closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ are finite unions of sets of the form $p \times \mathbb{P}^1$, $\mathbb{P}^1 \times q$, and $p \times q$ ('vertical' lines, 'horizontal' lines, and points). Most Zariski closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$, the diagonal $\Delta = \{(p,p) \mid p \in \mathbb{P}^1\}$ for instance, aren't of this form.

zartopprod **(3.3.1) the Zariski topology on** $\mathbb{P}^m \times \mathbb{P}^n$

As has been mentioned, the product of projective spaces $\mathbb{P}^m \times \mathbb{P}^n$ can be embedded into a projective space $\mathbb{P}^N$ by the Segre map, which identifies the product as a closed subset of $\mathbb{P}^N$, with $N = mn+m+n$. It is the locus of the Segre equations $z_{ij}z_{k\ell} = z_{i\ell}z_{kj}$, Since $\mathbb{P}^m \times \mathbb{P}^n$, with its Segre embedding, becomes a closed subset of $\mathbb{P}^N$, we don't really need a separate definition of its Zariski topology. Its closed subsets are the zero sets of families of homogeneous polynomials in the Segre variables z_{ij}, families that include the Segre equations. However, it is important to know that the Segre embedding maps the product $\mathbb{P}^m \times \mathbb{P}^n$ to an *irreducible* closed subset of $\mathbb{P}^N$, so that the product becomes a projective variety. This will be proved below, in Corollary 3.3.5.

One can describe the closed subsets of $\mathbb{P}^m \times \mathbb{P}^n$ directly, in terms of *bihomogeneous* polynomials. A polynomial $f(x,y)$ in $x = (x_0,...,x_m)$ and $y = (y_0,...,y_n)$ is bihomogeneous if it is homogeneous in the variables x and homogeneous in the variables y. For example, $x_0^2y_0 + x_0x_1y_1$ is a bihomogeneous polynomial, of degree 2 in x and degree 1 in y.

The bihomogeneous part of bidegree i,j of a polynomial $f(x,y)$ is the sum of terms whose degrees in x and y are i and j, respectively. Because (x,y) and $(\lambda x, \mu y)$ represent the same point of $\mathbb{P}^m \times \mathbb{P}^n$ for all nonzero scalars λ and μ, we want to know that $f(x,y) = 0$ if and only if $f(\lambda x, \mu y) = 0$ for all nonzero λ and μ. This will be true if and only if all of the bihomogeneous parts of f are zero. (See (1.3.2).)

closedin-pxp **3.3.2. Proposition. (i)** *Let Z be a subset of $\mathbb{P}^m \times \mathbb{P}^n$. The Segre image of Z is closed if and only if Z is the locus of zeros of a family of bihomogeneous polynomials.*

(ii) *If X and Y are closed subsets of $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively, then $X \times Y$ is a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$.*

(iii) *The projection maps $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\pi_1} \mathbb{P}^m$ and $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\pi_2} \mathbb{P}^n$ are continuous.*

(iv) *For all x in $\mathbb{P}^m$ the fibre $x \times \mathbb{P}^n$ is homeomorphic to $\mathbb{P}^n$, and for all y in $\mathbb{P}^n$, the fibre $\mathbb{P}^m \times y$ is homeomorphic to $\mathbb{P}^m$.*

proof. **(i)** Let Π denote the Segre image of $\mathbb{P}^m \times \mathbb{P}^n$, and let $f(z)$ be a homogeneous polynomial in the Segre variables z_{ij}. When we substitute $z_{ij} = x_iy_j$ into f, we obtain a bihomogeneous polynomial $\widetilde{f}(x,y)$ whose degree in x and in y is the same as the degree of f. The inverse image of the zero set of f in Π is the zero set of $\widetilde{f}$ in $\mathbb{P}^m \times \mathbb{P}^n$. Therefore the inverse image of a closed subset of Π is the zero set of a family of bihomogeneous polynomials in $\mathbb{P}^m \times \mathbb{P}^n$.

Conversely, let $\widetilde{g}(x,y)$ be a bihomogeneous polynomial, say of degrees r in x and degree s in y. If $r = s$, we may collect variables that appear in $\widetilde{g}$ in pairs x_iy_j and replace each pair x_iy_j by z_{ij}. We will obtain a homogeneous polynomial g in z such that $g(z) = \widetilde{g}(x,y)$ when $z_{ij} = x_iy_j$. The zero set of g in Π is the image of the zero set of $\widetilde{g}$ in $\mathbb{P}^m \times \mathbb{P}^n$.

Suppose that $r \geq s$, and let $k = r-s$. Because the variables y cannot all be zero at any point of $\mathbb{P}^n$, the equation $g = 0$ on $\mathbb{P}^m \times \mathbb{P}^n$ is equivalent with the system of equations $gy_0^k = gy_1^k = \cdots = gy_n^k = 0$. The polynomials gy_i^k are bihomogeneous, of same degree in x as in y. This puts us back in the first case.

(ii) A homogeneous polynomial $f(x)$ is a bihomogeneous polynomial of degree zero in y, and a homogeneous polynomial $g(y)$ as a bihomogeneous polynomial of degree zero in x. So $X \times Y$, which is a locus of the form $f(x) = g(y) = 0$ in $\mathbb{P}^m \times \mathbb{P}^n$, is closed in $\mathbb{P}^m \times \mathbb{P}^n$.

(iii) For the projection π_1, we must show that if X is a closed subset of $\mathbb{P}^m$, its inverse image $X \times \mathbb{P}^n$ is closed. This is a special case $Y = \mathbb{P}^n$ of **(ii)**.

(iv) It will be best to denote the chosen point of $\mathbb{P}^m$ by a symbol other than x here. We'll denote it by $\underline{x}$. Part **(i)** tells us that the bijective map $\underline{x} \times \mathbb{P}^n \to \mathbb{P}^n$ is continuous. To show that the inverse map is continuous, we must show that a closed subset Z of $\underline{x} \times \mathbb{P}^n$ is the inverse image of a closed subset of $\mathbb{P}^n$. Say that Z is the zero locus of a set of bihomogeneous polynomials $f(x,y)$. The polynomials $\overline{f}(y) = f(\underline{x}, y)$ are homogeneous in y, and the inverse image of their zero locus is Z. $\square$

3.3.3. Corollary. *Let X and Y be projective varieties, and let Π denote the product $X\times Y$, regarded as a closed subspace of $\mathbb{P}^m\times\mathbb{P}^n$.* *projcont*

- *The projections $\Pi\to X$ and $\Pi\to Y$ are continuous.*
- *For all x in X and all y in Y, the fibres $x\times Y$ and $X\times y$ are homeomorphic to Y and X, respectively.* $\square$

3.3.4. Proposition. *Suppose that a topology is given on the product $X\times Y=\Pi$ of irreducible topological spaces X and Y, and that it has these properties:* *Pirred*

- *The projections $\Pi\xrightarrow{\pi_1}X$ and $\Pi\xrightarrow{\pi_2}Y$ are continuous.*
- *For all x in X and all y in Y, the fibres $x\times Y$ and $X\times y$ are homeomorphic to Y and X, respectively.*

Then Π is an irreducible topological space.

The first condition tells us that the topology on $X\times Y$ is at least as fine as the product topology, and the second one tells us that the topology isn't too fine. (We don't want to give Π the discrete topology.)

Some notation for use in the proof of the proposition: Let x be a point of X. If W is a subset of $X\times Y$, we denote the intersection $W\cap(x\times Y)$ by ${}_xW$. Similarly, if y is a point of Y, we denote $W\cap(X\times y)$ by W_y. By analogy with the x,y-plane, we call ${}_xW$ and W_y a *vertical slice* and a *horizontal slice*, of W, respectively.

proof of Proposition 3.3.4. We prove irreducibility by showing that the intersection of two nonempty open subsets W and W' of $X\times Y$ isn't empty (2.2.13).

We show first that the image $U=\pi_2W$ of an open subset W of $X\times Y$ via projection to Y is an open subset of Y. We are given that, for every x, the fibre $x\times Y$ is homeomorphic to Y. Since W is open in $X\times Y$, the vertical slice ${}_xW$ is open in $x\times Y$. Its image $\pi_2({}_xW)$ is open in the homeomorphic space Y. Since W is the union of the sets ${}_xW$, U is the union of the open sets $\pi_2({}_xW)$. So U is open.

Now let W and W' be nonempty open subsets of $X\times Y$, and let U and U' be their images via projection to Y. So U and U' are nonempty open subsets of Y. Since Y is irreducible, $U\cap U'$ isn't empty. Let y be a point of $U\cap U'$. Since $U=\pi_2W$ and y is a point of U, the horizontal slice W_y, which is an open subset of the fibre $X\times y$, isn't empty. Similarly, W'_y isn't empty. Since $X\times y$ is homeomorphic to the irreducible space X, it is irreducible. So $W_y\cap W'_y$ isn't empty. Therefore $W\cap W'$ isn't empty, as was to be shown. $\square$

3.3.5. Corollary. *The product $X\times Y$ of projective varieties X and Y is a projective variety.* $\square$ *prodirred*

(3.3.6) products of affine varieties

Let $X=\operatorname{Spec}A$ and $Y=\operatorname{Spec}B$ be affine varieties. Say that X is embedded as a closed subvariety of $\mathbb{A}^m$, so that $A=\mathbb{C}[x_1,...,x_m]/P$ for some prime ideal P, and that Y is embedded similarly into $\mathbb{A}^n$, $B=\mathbb{C}[y_1,...,y_n]/Q$ for some prime ideal Q. Then in affine x,y-space $\mathbb{A}^{m+n}$, $X\times Y$ is the locus of the equations $f(x)=0$ and $g(y)=0$, with f in P and g in Q. Proposition 3.3.4 shows that $X\times Y$ is irreducible, so it is a variety. Let P' be the ideal of $\mathbb{C}[x,y]$ generated by the elements of P. It consists of sums of products of elements of P with polynomials in x,y. Let Q' be defined in the analogous way.

3.3.7. Proposition. *The elements of the ideal $I=P'+Q'$ are the polynomials that vanish on the variety $X\times Y$. Therefore I is a prime ideal.* *fgprime*

The fact that $X\times Y$ is a variety tells us only that the radical of I is a prime ideal.

proof of Proposition 3.3.7. Let $R=\mathbb{C}[x,y]/I$. The projection $X\times Y\to X$ is surjective. Therefore the map $A\to R$ is injective, and similarly, $B\to R$ is injective. We identify A and B with their images in R.

Any polynomial $f(x,y)$ can the written, in many ways, as a sum, each of whose terms is a product of a polynomial in x with a polynomial in y: $f(x,y)=\sum a_i(x)b_i(y)$. Therefore any element ρ of R can be written as a finite sum of products

$$\rho=\sum_{i=1}^k a_ib_i \tag{3.3.8}$$

with a_i in A and b_i in B. To show that 0 is the only element of R that vanishes identically on $X\times Y$, we show that a sum ρ of k products a_ib_i that vanishes on $X\times Y$ can also be written as a sum of $k-1$ products.

Say that $\rho = \sum_1^k a_i b_i$. If $a_k = 0$, then ρ is the sum $\sum_{i=1}^{k-1} a_i b_i$ of $k-1$ products. If $a_k \neq 0$, the function on X defined by a_k isn't identically zero. We choose a point $\underline{x}$ of X such that $a_k(\underline{x}) \neq 0$. Let $\overline{a}_i = a_i(\underline{x})$ and $\overline{\rho}(y) = \rho(\underline{x}, y)$. Then $\overline{\rho}(y) = \sum_{i=1}^k \overline{a}_i b_i$ is an element of B. Since ρ vanishes on $X \times Y$, $\overline{\rho}$ vanishes on $Y = \operatorname{Spec} B$. Therefore $\overline{\rho} = 0$. Let $c_i = \overline{a}_i / \overline{a}_k$. Then $b_k = -\sum_{i=1}^{k-1} c_i b_i$. Substituting for b_k into ρ and collecting coefficients of $b_1, ..., b_{k-1}$ gives us an expression for ρ as a sum of $k-1$ terms. When $k = 1$, $b_1 = 0$, and therefore $\rho = 0$. □

3.4 Rational Functions

ratfnsec

fnfld **(3.4.1) the function field**

Let X be a projective variety, say a closed subvariety of $\mathbb{P}^n$, and let $\mathbb{U}^i : \{x_i \neq 0\}$ be one of the standard open subsets of $\mathbb{P}^n$. The intersection $X^i = X \cap \mathbb{U}^i$, if it isn't empty, will be a closed subvariety of the affine space $\mathbb{U}^i$ and a dense open subset of X. It will be an affine variety, and its localizations will also be affine varieties. (The intersection $X \cap \mathbb{U}^i$ is empty when X is contained in the hyperplane $\{x_i = 0\}$.) Let's call the nonempty sets $X^i = X \cap \mathbb{U}^i$ the *standard open subsets* of X.

enufaff **3.4.2. Lemma.** *The localizations of the standard open sets sets $X^i = X \cap \mathbb{U}^i$ are affine varieties, and they form a basis for the topology on X.*

This follows from (2.7.2). □

There are affine open sets that aren't localizations of these standard open sets, but we don't yet have a definition of affine varieties. Rather than defining affine open sets here, we postpone discussion to Section 3.6.

Let X be a closed subvariety of $\mathbb{P}^n$, and let $x_0, ..., x_n$ be coordinates in $\mathbb{P}^n$. For each $i = 0, ..., n$, let $X^i = X \cap \mathbb{U}^i$. We omit the indices for which X^i is empty. Then X^i will be affine, and the intersection $X^{ij} = X^i \cap X^j$ will be a localization, both of X^i and of X^j. The coordinate algebra A_i of X^i is generated by the images of the elements $u_{ij} = x_j / x_i$ in A_i, and if we denote those images by u_{ij} too, then $X^{ij} = \operatorname{Spec} A_{ij}$, where $A_{ij} = A_i[u_{ij}^{-1}] = A_j[u_{ji}^{-1}]$.

deffnfld **3.4.3. Definition.** The *function field* K of a projective variety X is the function field of any one of the standard open subsets X^i (**3.4.1**), and the *function field* of an open subvariety X' of a projective variety X is the function field of X. All open subvarieties of variety have the same function field.

For example, let x_0, x_1, x_2 be coordinates in $\mathbb{P}^2$. To describe the function field of $\mathbb{P}^2$, we can use the standard open set $\mathbb{U}^0$, which is an affine plane $\operatorname{Spec} \mathbb{C}[u_1, u_2]$ with $u_i = x_i / x_0$. The function field of $\mathbb{P}^2$ is the field of rational functions: $K = \mathbb{C}(u_1, u_2)$. We must use u_1, u_2 as coordinates here. It wouldn't be good to normalize x_0 to 1 and use coordinates x_1, x_2, because we may want to change to another standard open set such as $\mathbb{U}^1$. The coordinates in $\mathbb{U}^1$ are $v_0 = x_0/x_1$ and $v_2 = x_2/x_1$, and the function field K is also the field of rational functions $\mathbb{C}(v_0, v_2)$. The two fields $\mathbb{C}(u_1, u_2)$ and $\mathbb{C}(v_0, v_2)$ are the same.

A *rational function* on a variety X is an element of its function field. If a point p of X lies in a standard open set $X^i = \operatorname{Spec} A_i$, a rational function α is *regular* at p if it can be written as a fraction a/s of elements of A_i with $s(p) \neq 0$. If so, its *value* at p is $\alpha(p) = a(p)/s(p)$. If X' is an open subvariety of a projective variety X, a rational function on X' is *regular* at a point p of X' if it is a regular function on X at p.

When we regard an affine variety $X = \operatorname{Spec} A$ as a closed subvariety of $\mathbb{U}^0$, its function field will be the field of fractions of A, and Proposition 2.6.2 shows that the regular functions on the affine variety $\operatorname{Spec} A$ are the elements of A.

Thus a rational function on a projective variety X will define a function on a nonempty open subset of X. It can be evaluated at some points of X, not at all points.

regindep **3.4.4. Lemma. (i)** *Let p be a point of a projective variety X. The regularity of a rational function at p doesn't depend on the choice of a standard open set that contains p.*

(ii) *A rational function that is regular on a nonempty open subset X' is determined by the function it defines on X'.*

Part **(ii)** follows from Corollary 2.5.17. □

(3.4.5) points with values in a field ptvlfld

Let K be a field that contains the complex numbers. A *point* of projective space $\mathbb{P}^n$ *with values in* K is an equivalence class of nonzero vectors $\alpha = (\alpha_0, ..., \alpha_n)$ with α_i in K, the equivalence relation being analogous to the one for ordinary points: $\alpha \sim \alpha'$ if $\alpha' = \lambda\alpha$ for some λ in K. If X is the subvariety of $\mathbb{P}^n$ defined by a homogeneous prime ideal $\mathcal{P}$ of $\mathbb{C}[x]$, a *point α of X with values in K* is a point of $\mathbb{P}^n$ with values in K such that $f(\alpha) = 0$ for all f in $\mathcal{P}$.

Let K be the function field of a projective variety X. If X is embedded into $\mathbb{P}^n$, the embedding defines a point of X with values in K. To get this point, we choose a standard affine open set $\mathbb{U}^i$ of $\mathbb{P}^m$ such that $X^i = X \cap \mathbb{U}^i$ isn't empty, say $i = 0$. Then X^0 will be affine, $X^0 = \operatorname{Spec} A_0$. The embedding of X^0 into the affine space $\mathbb{U}^0$ is defined by a homomorphism $\mathbb{C}[u_1, ..., u_n] \to A_0$, with $u_i = x_i/x_0$. If α_i denotes the image of u_i in A_0, for $i = 1, ..., n$ and $\alpha_0 = 1$, then $(\alpha_0, ..., \alpha_n)$ is the point of $\mathbb{P}^n$ with values in K defined by the projective embedding of X.

3.4.6. Note. *(the function field of a product)* The function field of the product $X \times Y$ of varieties isn't generated by the function fields K_X and K_Y of X and Y. For example, let $X = \operatorname{Spec} \mathbb{C}[x]$ and $Y = \operatorname{Spec} \mathbb{C}[y]$ (one x and one y). Then $X \times Y = \operatorname{Spec} \mathbb{C}[x, y]$. The function field of $X \times Y$ is the field of rational functions $\mathbb{C}(x, y)$ in two variables. The algebra generated by the fraction fields $\mathbb{C}(x)$ and $\mathbb{C}(y)$ consists of the rational functions $p(x, y)/q(x, y)$ in which $q(x, y)$ is a product of a polynomial in x and a polynomial in y. Most rational functions, $1/(x + y)$ for instance, aren't of that type. □ ffprod

(3.4.7) interlude: rational functions on projective space ratfn-spspace

Let R denote the polynomial ring $\mathbb{C}[x_0, ..., x_n]$. A *homogeneous fraction* is a fraction of homogeneous polynomials in $x_0, ..., x_n$. The *degree* of a homogeneous fraction is the difference of degrees: $\deg g/h = \deg g - \deg h$.

A homogeneous fraction f is *regular* at a point p of $\mathbb{P}^n$ if, when it is written as a fraction g/h of relatively prime homogeneous polynomials, the denominator h isn't zero at p, and f is *regular on a subset U* if it is regular at every point of U.

A homogeneous fraction f of degree $d \neq 0$ won't define a function anywhere on projective space, beause $f(\lambda x) = \lambda^d f(x)$. In particular, a nonconstant homogeneous polynomial g of won't define a function, though it makes sense to say that such a polynomial vanishes at a point of $\mathbb{P}^n$.

On the other hand, a homogeneous fraction $f = g/h$ of degree zero, so that g and h have the same degree r, then f does define a function wherever h isn't zero, because $g(\lambda x)/h(\lambda x) = \lambda^r g(x)/\lambda^r h(x) = g(x)/h(x)$.

3.4.8. Lemma. (i) *Let h be a homogeneous polynomial of positive degree d, and let V be the open subset of $\mathbb{P}^n$ of points at which h isn't zero. The rational functions that are regular on V are those of the form g/h^k, where $k \geq 0$ and g is a homogeneous polynomial of degree dk.* *ho-mogfractsfn-fld*

(ii) *The only rational functions that are regular at every point of $\mathbb{P}^n$ are the constant functions.*

For example, the homogeneous polynomials that don't vanish at any point of the standard open set $\mathbb{U}^0$ are the scalar multiples of powers of x_0. So the rational functions that are regular on $\mathbb{U}^0$ are those of the form g/x_0^k, with g homogeneous of degree k. This agrees with the fact that the coordinate algebra of $\mathbb{U}^0$ is the polynomial ring $\mathbb{C}[u_1, ..., u_n]$, $u_i = x_i/x_0$, because $g(x_0, ..., x_m)/x_0^k = g(u_1, ..., u_n)$.

proof of Lemma 3.4.8. **(i)** Let α be a regular function on the open set V, say $\alpha = g_1/h_1$, where g_1 and h_1 are relatively prime homogeneous polynomials. Then h_1 doesn't vanish on V, so its zero locus in $\mathbb{P}^n$ is contained in the zero locus of h. According to the Strong Nullstellensatz, h_1 divides a power of h. Say that $h^k = fh_1$. Then $g_1/h_1 = fg_1/fh_1 = fg_1/h^k$.

(ii) If a rational function f is regular at every point of $\mathbb{P}^n$, then it is regular on $\mathbb{U}^0$. It will have the form g/x_0^k, where g is a homogeneous polynomial of degree k not divisible by x_0. Since f is also regular on $\mathbb{U}^1$, it will have the form h/x_1^ℓ, where h is homogeneous and not divisible by x_1. Then $gx_1^\ell = hx_0^k$. Since x_0 doesn't divide g, $\;k = 0$. Therefore g is a constant. $\square$

It is also true that the only rational functions that are regular at every point of a projective variety are the constants. The proof of this will be given later (Corollary 8.2.9). When studying projective varieties, the constant functions are useless, so one has to look at at regular functions on open subsets. Affine varieties appear in projective algebraic geometry, as open subsets on which there are enough regular functions.

3.5 Morphisms

sectmorph As with affine varieties, morphisms are the allowed maps between varieties. Some morphisms, such as the projection from a product $X \times Y$ to X, are sufficiently obvious that they don't require much discussion, but many morphisms aren't obvious.

Let X and Y be subvarieties of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. A morphism $Y \to X$, as defined below, will be determined by a morphism from Y to $\mathbb{P}^m$ whose image is contained in X. However, in most cases, such a morphism won't be the restriction of a morphism from $\mathbb{P}^n$ to $\mathbb{P}^m$. This is an important point: It is usually impossible to define f using polynomials in the coordinate variables of $\mathbb{P}^n$.

veronex **3.5.1. Example.** Let the coordinates in $\mathbb{P}^2$ be y_0, y_1, y_2. The Veronese map from the projective line $\mathbb{P}^1$ to $\mathbb{P}^2$, defined by $(x_0, x_1) \rightsquigarrow (x_0^2, x_0x_1, x_1^2)$, is an obvious morphism. Its image is the conic C in $\mathbb{P}^2$ defined by the polynomial $\{y_0y_2 - y_1^2\}$. The Veronese map defines a bijective morphism $\mathbb{P}^1 \xrightarrow{f} C$, whose inverse function π sends a point (y_0, y_1, y_2) of C with $y_0 \neq 0$ to the point $(x_0, x_1) = (y_1, y_2)$, and it sends the remaining point, which is $(0, 0, 1)$, to $(0, 1)$. Though π is a morphism $C \to \mathbb{P}^1$, there is no way to extend it to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. In fact, the only morphisms from $\mathbb{P}^2$ to $\mathbb{P}^1$ are the constant morphisms, whose images are points. $\square$

It is convenient, though somewhat artificial, to use points with values in a field to define morphisms.

morphtoP **(3.5.2) morphisms to projective space**

In this section, it will be helpful to have a separate notation for the point with values in a field K determined by a nonzero vector $\alpha = (\alpha_0, ..., \alpha_n)$, with entries in K. We'll denote that point by $\underline{\alpha}$. So if α and α' are points with values in K, then $\underline{\alpha} = \underline{\alpha}'$ if $\alpha' = \lambda\alpha$ for some nonzero λ in K. We'll drop this notation later.

Let Y be a variety with function field K. A morphism from Y to projective space $\mathbb{P}^n$ will be defined by a point of $\mathbb{P}^n$ with values in K. The fact that points of projective space are equivalence classes of vectors, not the vectors themselves, will be useful.

Let $\alpha = (\alpha_0, ..., \alpha_n)$ be a vector with entries in the function field K of a variety Y. We try to use the point $\underline{\alpha}$ to define a morphism from Y to projective space $\mathbb{P}^n$. To define the image $\underline{\alpha}(q)$ of a point q of Y (an ordinary point), we look for a vector $\alpha' = (\alpha_0', ..., \alpha_n')$, such that $\underline{\alpha}' = \underline{\alpha}$, i.e., $\alpha' = \lambda\alpha$, with $\lambda \in K$, and such that the rational functions α_i' are regular at q and not all zero there. Such a vector may exist or not. If α' exists, we define

defmorphP (3.5.3)
$$\underline{\alpha}(q) = (\alpha_0'(q), ..., \alpha_n'(q)) \quad \big(= \underline{\alpha}'(q)\big)$$

We call $\underline{\alpha}$ a *good point* if such a vector α' exists for every point q of Y.

twoconds **3.5.4. Lemma.** *A point $\underline{\alpha}$ of $\mathbb{P}^n$ with values in the function field K of a variety Y is a good point if either one of the two following conditions holds for every point q of Y:*

- *There is an element λ of K such that the rational functions $\alpha_i' = \lambda\alpha_i$, $i = 0, ..., n$, are regular and not all zero at q, for $i = 0, ..., n$.*
- *There is an index j such that $\alpha_j \neq 0$, and the rational functions α_i/α_j are regular at q, for $i = 0, ..., n$.*

proof. The first condition simply restates the definition. We show that it is equivalent with the second one. Suppose that α_i/α_j is regular at q for every i. Let $\lambda = \alpha_j^{-1}$, and let $\alpha_i' = \lambda\alpha_i = \alpha_i/\alpha_j$. The rational functions α_i' are regular at q, and they aren't all zero there because $\alpha_j' = 1$. Conversely, suppose that for some nonzero λ in K, $\;\alpha_i' = \lambda\alpha_i$ are all regular at q and that α_j' isn't zero there. Then ${\alpha_j'}^{-1}$ is a regular function at q, so the rational functions α_i'/α_j', which are equal to α_i/α_j, are regular at q for all i. $\square$

3.5.5. Lemma. *Let $\underline{\alpha}$ be a good point with values in the function field K of a variety Y. The image $\underline{\alpha}(q)$ in $\mathbb{P}^n$ of a point q of Y is independent of the choice of the vector that represents $\underline{\alpha}$.* *doesntdepend*

This follows from Lemma 3.5.4 because the second condition doesn't involve λ. □

3.5.6. Definition. Let Y be a variety with function field K. A *morphism* from Y to projective space $\mathbb{P}^n$ is a map that can be defined, as in (3.5.3), by a good point $\underline{\alpha}$ with values in K. defmorphtoP

We may denote the morphism defined by a good point $\underline{\alpha}$ by the same symbol $\underline{\alpha}$.

3.5.7. Proposition. *Let α be a vector with values in the function field K of a variety Y, and suppose that $\underline{\alpha}$ is a good point, that defines a morphism $Y \to \mathbb{P}^n$. Suppose that the inverse image of the standard open set $\mathbb{U}^0$ in $\mathbb{P}^n$ is nonempty. Then $\alpha_0 \neq 0$, and the inverse image of $\mathbb{U}^0$ the set of points $q \in Y$ at which the functions α_i/α_0 are regular, for $j = 1, ..., n$.* *maptoui*

proof. If α_0 were zero, $\underline{\alpha}$ would map Y to the hyperplane $\{x_0 = 0\}$. So $\alpha_0 \neq 0$. Let q be a point of Y. Since α is a good point, there is a λ such that $\alpha'_i = \lambda\alpha_i$ are all regular at q and not all zero, and then $\underline{\alpha}(q) = (\alpha'_0(q), ..., \alpha'_n(q))$. The image will be in $\mathbb{U}^0$ if $\alpha'_0(q) \neq 0$. If so, we let $\alpha'' = {\alpha'_0}^{-1}\alpha' = \alpha_0^{-1}\alpha$. Then α''_i are all regular at q and $\alpha''_0(q) = 1$. □

3.5.8. Examples. identmap

(i) The identity map $\mathbb{P}^1 \to \mathbb{P}^1$. Let $X = \mathbb{P}^1$, and let (x_0, x_1) be coordinates in X. The function field of X is the field $K = \mathbb{C}(t)$ of rational functions in the variable $t = x_1/x_0$. The identity map $X \to X$ is defined by the point $\alpha = (1, t)$ with values in K. For every point p of X except the point $(0, 1)$, $\underline{\alpha}(p) = \alpha(p) = (1, t(p))$. For the point $q = (0, 1)$, we let $\alpha' = t^{-1}\alpha = (t^{-1}, 1)$. Then $\underline{\alpha}(q) = \alpha'(q) = (x_0(q)/x_1(q), 1) = (0, 1)$. So $\underline{\alpha}$ is a good point.

(ii) We go back to Example 3.5.1, in which C is the conic $y_0y_2 = y_1^2$ and f is the morphism $\mathbb{P}^1 \to C$ defined by $f(x_0, x_1) = (x_0^2, x_0x_1, x_1^2)$. The inverse morphism π can be described as the projection from C to the line $L_0 : \{y_0 = 0\}$, $\pi(y_0, y_1, y_2) = (y_1, y_2)$. This formula for π is undefined at the point $q = (1, 0, 0)$, though the map extends to the whole conic C. Let's write this projection using a point with values in the function field K of C. The standard affine open set $\{y_0 \neq 0\}$ of $\mathbb{P}^2$ is the polynomial algebra $\mathbb{C}[u_1, u_2]$, with $u_1 = y_1/y_0$ and $u_2 = y_2/y_0$. Denoting the restriction of the function u_i to $C^0 = C \cap \mathbb{U}^0$ by u_i too, the restricted functions are related by the equation $u_2 = u_1^2$ that is obtained by dehomogenizing f. The function field K is $\mathbb{C}(u_1)$.

The projection π is defined by the point $\alpha = (u_1, u_1^2)$ with values in K: $\pi(y_0, y_1, y_2) = \pi(1, u_1, u_2) = (u_1, u_1^2)$. Lemma 3.5.4 tells us that α is a good point if and only if one of the two vectors $\alpha' = (1, u_1)$ or $\alpha'' = (u_1^{-1}, 1)$ is regular at every (ordinary) point p of C. Since $u_1 = y_1/y_0$, α' is regular at all points at which $y_0 \neq 0$. This leaves just one point $p = (0, 0, 1)$ to consider. Noting that $u_1^{-1} = y_0/y_1 = y_1/y_2$, we see that α'' is regular there. So α is a good point. □

(3.5.9) morphisms to projective varieties morphtoV

3.5.10. Definition. Let Y be a variety, and let X be a subvariety of a projective space $\mathbb{P}^m$. A morphism of varieties $Y \xrightarrow{\alpha} X$ is the restriction of a morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ whose image is contained in X. defmorphtoX

3.5.11. Lemma. *Let X be a projective variety that is the locus of zeros of a family f of homogeneous polynomials. A morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ defines a morphism $Y \to X$ if and only if $f(\alpha) = 0$.* *ptfvalKfzero*

proof. Let $f(x_0, ..., x_m)$ be a homogeneous polynomial of degree d, with zero locus X in $\mathbb{P}^m$. We show that the image of a morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ is contained in X if and only if $f(\alpha) = 0$. Whether or not X is a variety is irrelevant. Suppose that $f(\alpha) = 0$, and let q be a point of Y. Since $\underline{\alpha}$ is a good point, the ratios $\alpha'_j = \alpha_j/\alpha_i$ are regular at q for some i, and $\underline{\alpha}(q) = \underline{\alpha}'(q)$. Then $f(\alpha') = \alpha_i^{-d}f(\alpha) = 0$. Therefore q is a point of X.

Conversely, suppose that $f(\alpha) \neq 0$. Let Y' be the open subset of Y of points at which all nonzero α_i are invertible regular functions. Then $f(\alpha') = \alpha_i^d f(\alpha)$ will be a nonzero rational function on Y'. It will be nonzero at some points q. □

We remark that a morphism $Y \xrightarrow{\alpha} X$ won't restrict to a map of function fields $K_X \to K_Y$ unless the image of Y is dense in X.

morphcont **3.5.12. Proposition.** *A morphism of varieties $Y \xrightarrow{\alpha} X$ is a continuous map in the Zariski topology, and a continuous map in the classical topology.*

proof. Since the topologies on a projective variety X are induced from those on projective space $\mathbb{P}^m$, we may suppose that $X = \mathbb{P}^m$. Let $\mathbb{U}^i$ be a standard open subset of X whose inverse image in Y isn't empty, and let Y' be a localization of a standard open subset of that inverse image. The restriction $Y' \to \mathbb{U}^i$ of the morphism $\underline{\alpha}$ is continuous in either topology because it is a morphism of affine varieties (3.5.7). Since Y is covered by open sets such as Y', $\underline{\alpha}$ is continuous. $\square$

firstprop-morph **3.5.13. Lemma.**

(i) *The inclusion of an open or a closed subvariety Y into a variety X is a morphism.*

(ii) *Let $Y \xrightarrow{f} X$ be a map whose image lies in an open or a closed subvariety Z of X. Then f is a morphism if and only if its restriction $Y \to Z$ is a morphism.*

(iii) *A composition of morhisms $Z \xrightarrow{\beta} Y \xrightarrow{\alpha} X$ is a morphism.*

(i) *Let $\{Y^i\}$ be an open covering of a variety Y, and let $Y^i \xrightarrow{f^i} X$ be morphisms. If the restrictions of f^i and f^j to the intersections $Y^i \cap Y^j$ are equal for all i, j, there is a unique morphism f whose restriction to Y^i is f^i.*

We omit the proofs of **(i) - (iii)**. Part **(iv)** is true because the points with values in K that define the morphisms f^i will be equal. $\square$

mapprop-prod **(3.5.14) the mapping property of a product**

The product $X{\times}Y$ of sets X and Y can be characterized by this property: Maps from a set T to the product $X{\times}Y$ correspond bijectively to pairs of maps $T \xrightarrow{f} X$ and $T \xrightarrow{g} Y$. The map $T \xrightarrow{h} X{\times}Y$ that corresponds to a pair of maps f, g sends a point t to the point pair $(f(t), g(t))$. So $h = (f, g)$. If $T \xrightarrow{h} X \times Y$ is a map to the product, the corresponding maps to X and Y are the compositions with the projections $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$: $f = \pi_1 h$ and $g = \pi_2 h$.

The analogous statements are true for morphisms of varieties:

mapprop-var **3.5.15. Proposition.** *Let X and Y be varieties, and let $X \times Y$ be the product variety.*

(i) *The projections $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$ are morphisms.*

(ii) *Morphisms from a variety T to the product variety $X \times Y$ correspond bijectively to pairs of morphisms $T \to X$ and $T \to Y$, the correspondence being the same as for maps of sets.*

(iii) *If $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$ are morphisms of varieties, the product map $X{\times}Y \xrightarrow{f\times g} U{\times}V$, which is defined by $[f \times g](x, y) = (f(x), g(y))$, is a morphism.*

proof. Perhaps it suffices to exhibit the points with values in the function fields that define the morphisms.

(i) The function field of $X \times Y$ contains the function field K_X of X. The point with values in K that defines the projection π_1 is the point with values in K_X defined by the embedding of X into projective space.

(ii) Let $z_{ij} = x_i y_j$ be the Segre coordinates for $X \times Y$, and let $x = \alpha$ and $y = \beta$ be the points with values in the function field K_T of T that define the morphisms $T \to X$ and $T \to Y$. The point with values in K_T the defines the map $T \to X \times Y$ is $z_{ij} = \alpha_i \beta_j$.

(iii) Let the coordinates in U, V, and $U{\times}V$ be u_i and v_j and $w_{ij} = u_i v_j$, respectively. Say that the morphism f is defined by the point α with values in K_X, and that g is defined by the point β with values in K_Y. The function field $K_{X\times Y}$ contains K_X and K_Y. Then $w_{ij} = \alpha_i \beta_j$ defines the product morphism $X \times Y \to U \times V$. $\square$

isomor-phisms **(3.5.16) isomorphisms**

An *isomorphism* of varieties is a bijective morphism $Y \xrightarrow{u} X$ whose inverse function is also a morphism. Isomorphisms are important because they allow us to identify different incarnations of what might be called the "same" variety.

3.5.17. Example. The projective line $\mathbb{P}^1$, a conic in $\mathbb{P}^2$, and a twisted cubic in $\mathbb{P}^3$ are isomorphic. Let Y denote the projective line with coordinates y_0, y_1. The function field K of Y is the field of rational functions in $t = y_1/y_0$. The degree 3 Veronese map $Y \longrightarrow \mathbb{P}^3$ (**3.1.15**) defines an isomorphism from Y to its image X, a twisted cubic. It is defined by the vector $\alpha = (1, t, t^2, t^3)$ of $\mathbb{P}^3$ with values in K, and $\alpha' = (t^{-3}, t^{-2}, t^{-1}, 1)$ defines the same point. twistcubicinverse

The twisted cubic X is the locus of zeros of the equations $v_0v_2 = v_1^2$, $v_2v_1 = v_0v_3$, $v_1v_3 = v_2^2$. To identify the function field of X, we put $v_0 = 1$, obtaining relations $v_2 = v_1^2, v_3 = v_1^3$. The function field is the field $F = \mathbb{C}(v_1)$. The point of $Y = \mathbb{P}^1$ with values in F that defines the inverse $X \to Y$ of the morphism $\underline{\alpha}$ is defined by the point $\beta = (1, v_1)$. □

3.5.18. Lemma. *Let $Y \xrightarrow{f} X$ be a morphism of varieties, let $\{X^i\}$ and $\{Y^i\}$ be open coverings of X and Y, respectively, such that the image of Y^i in X is contained in X^i. If the restrictions $Y^i \xrightarrow{f^i} X^i$ of f are isomorphisms, then f is an isomorphism.* *propisom*

proof. Let g^i denote the inverse of the morphism f^i. Then $g^i = g^j$ on $X^i \cap X^j$, because $f^i = f^j$ on $Y^i \cap Y^j$. By (3.5.13) **(iv)**, there is a unique morphism $X \xrightarrow{g} Y$ whose restriction to Y^i is g^i. That morphism is the inverse of f. □

(3.5.19) the diagonal diagonal

Let X be a variety. The *diagonal* X_Δ is the set of points (p, p) in the product variety $X \times X$. It is a subset of the product that is closed in the Zariski topology, but not in the product topology.

3.5.20. Proposition. *Let X be a variety. The diagonal X_Δ is a closed subvariety of the product $X \times X$, and it is isomorphic to X.* *diagclosed*

proof. Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$ that contains X, and let $x_0, ..., x_n$ and $y_0, ..., y_n$ be coordinates in the two factors of $\mathbb{P} \times \mathbb{P}$. The diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$ is the closed subvariety defined by the bilinear equations $x_iy_j = x_jy_i$, or in the Segre variables, by the equations $z_{ij} = z_{ji}$, which show that the ratios x_i/x_j and y_i/y_j are equal.

Next, let X be the closed subvariety of $\mathbb{P}$ defined by a system of homogeneous equations $f(x) = 0$. The diagonal X_Δ can be identified as the intersection of the product $X \times X$ with the diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$, so it is a closed subvariety of $X \times X$. As a closed subvariety of $\mathbb{P} \times \mathbb{P}$, the diagonal X_Δ is defined by the equations

$$x_iy_j = x_jy_i \quad \text{and} \quad f(x) = 0 \tag{3.5.21}$$

Xdelta

The morphisms $X \xrightarrow{(id,id)} X_\Delta \xrightarrow{\pi_1} X$ show that X_Δ is isomorphic to X. □

It is interesting to compare Proposition 3.5.20 with the Hausdorff condition for a topological space. The proof of the next lemma is often assigned as an exercise in topology.

3.5.22. Lemma. *A topological space X is a Hausdorff space if and only if, when $X \times X$ is given the product topology, the diagonal X_Δ becomes a closed subset of $X \times X$.* □ *hausdorffdiagonal*

Though a variety X, with its Zariski topology, isn't a Hausdorff space unless it is a point, Lemma 3.5.22 doesn't contradict Proposition 3.5.20 because the Zariski topology on $X \times X$ is finer than the product topology.

(3.5.23) the graph of a morphism ggraph

Let $Y \xrightarrow{f} X$ be a morphism of varieties. The *graph* Γ_f of f is the subset of $Y \times X$ of pairs (q, p) such that $f(q) = p$.

3.5.24. Proposition. *The graph Γ_f of a morphism $Y \xrightarrow{f} X$ is a closed subvariety of $Y \times X$, isomorphic to Y.* *graph* *proof.* We form a diagram of morphisms

graphdiagram (3.5.25)

$$\begin{tikzcd} \Gamma_f \arrow[r] \arrow[d, "v"'] & Y\times X \arrow[d, "f\times id"] \\ X_\Delta \arrow[r] & X\times X \end{tikzcd}$$

where v sends a point (q,p) of Γ_f to the point $(f(q),p)=(p,p)$ of the diagonal X_Δ. The graph is the inverse image of X_Δ in $Y\times X$. Since X_Δ is closed in $X\times X$, Γ_f is closed in $Y\times X$.

Let π_1 denote the projection from $Y\times X$ to Y. The composition of the morphisms $Y \xrightarrow{(id,f)} Y\times X \xrightarrow{\pi_1} Y$ is the identity map on Y, and the image of the map (id,f) is the graph Γ_f. The two maps $Y\to\Gamma_f$ and $\Gamma_f\to Y$ are inverses, so Γ_f is isomorphic to Y. $\square$

defprojection **(3.5.26) projection**

The map

projectiontwo (3.5.27)
$$\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^{n-1}$$

that drops the last coordinate of a point: $\pi(x_0,...,x_n)=(x_0,...,x_{n-1})$ is called a *projection*. It is defined at all points of $\mathbb{P}^n$ except at the *center of projection*, the point $q=(0,...,0,1)$, . So π is a morphism from the complement $U=\mathbb{P}^n-\{q\}$ to $\mathbb{P}^{n-1}$:

$$U \xrightarrow{\pi} \mathbb{P}^n$$

The points of U are the ones that can be written in the form $(x_0,...,x_{n-1},1)$

Let the coordinates in $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ be $x=x_0,...,x_n$ and $y=y_0,...,y_{n-1}$, respectively. The fibre $\pi^{-1}(y)$ over a point $(y_0,...,y_{n-1})$ is the set of points $(x_0,...,x_n)$ such that $(x_0,...,x_{n-1})=\lambda(y_0,...,y_{n-1})$, while x_n is arbitrary. It is the line in $\mathbb{P}^n$ through the points $(y_1,...,y_{n-1},0)$ and $q=(0,...,0,1)$, with the center of projection q omitted.

In Segre coordinates, the graph of π in $U\times\mathbb{P}^{n-1}_y$ is the locus Γ of solutions of the equations $z_{ij}=z_{ji}$ for $0\le i,j\le n-1$, which imply that the vectors $(x_0,...,x_{n-1})$ and $(y_0,...,y_{n-1})$ are proportional.

projgrah **3.5.28. Proposition.** *In $\mathbb{P}^n_x\times\mathbb{P}^{n-1}_y$, the locus W of the equations $x_iy_j=x_jy_i$, or $z_{ij}=z_{ji}$, with $0\le i,j\le n-1$ is the closure of the graph Γ of π.*

proof. At points x distinct from q, the solutions of these equations are the points of Γ, and all remaining points of $\mathbb{P}^n\times\mathbb{P}^{n-1}$, points of the form (q,y) are also solutions. So the locus W, a closed set, is contained in the union $\Gamma\cup(q\times\mathbb{P}^{n-1})$. To show that W is equal to that union, we show that a homogeneous polynomial $g(w)$ that vanishes on Γ, vanishes at all points of $q\times\mathbb{P}^{n-1}$. Given $(y_0,...,y_{n-1})$ in $\mathbb{P}^{n-1}$, let $x_t=(ty_0,...,ty_{n-1},1)$. For all $t\neq 0$, the point (x_t,y) is in Γ and therefore $g(x_t,y)=0$. Since g is a continuous function, $g(x_t,y)$ approaches $g(q,y)$ as $t\to 0$. So $g(q,y)=0$. $\square$

We denote the closure W of Γ by $\overline{\Gamma}$ now. The projection $\overline{\Gamma}\to\mathbb{P}^n_x$ that sends a point (x,y) to x is bijective except when $x=q$, and the fibre over q, which is $q\times\mathbb{P}^{n-1}$, is a projective space of dimension $n-1$. Because the point q of $\mathbb{P}^n$ is replaced by a projective space in $\overline{\Gamma}$, the map $\overline{\Gamma}\to\mathbb{P}^n_x$ is called a *blowup* of the point q. This is a projective blowup.

3.6 Affine Varieties

affineV We have used the term 'affine variety' in several contexts: An irreducible closed subset of affine space $\mathbb{A}^n_x$ is an affine variety. The spectrum $\operatorname{Spec} A$ of a finite type domain A is an affine variety. A closed subvariety in $\mathbb{A}^n$ becomes a variety in $\mathbb{P}^n$ when the ambient affine space $\mathbb{A}^n$ is identified with the standard open subset $\mathbb{U}^0$. We combine these definitions now, in a rather obvious way: An *affine variety* X is a variety that is isomorphic to a variety of the form $\operatorname{Spec} A$.

If X is an affine variety with coordinate algebra A, the function field K of X will be the field of fractions of A, and as Proposition 2.6.2 shows, the regular functions on X are the elements of A. So A and $\operatorname{Spec} A$ are determined uniquely by X. The isomorphism $\operatorname{Spec} A \to X$ is also determined uniquely. It seems permissible to identify X with $\operatorname{Spec} A$, when A is the coordinate algebra of an affine variety X.

(**3.6.1**) **affine open sets** affopens

Now that we have a definition of an affine variety, we can make the next definition. Though obvious, it is important: An *affine open subset* of a variety X is an open subvariety that is an affine variety. From now on, this will be the definition. A nonempty open subset V of X is an affine open subset if and only if

(a) the algebra R of regular functions on V is a finite-type domain, so that $\operatorname{Spec} R$ is defined, and

(b) V is isomorphic to $\operatorname{Spec} R$.

Since the localizations of the standard open sets are affine, the affine open subsets form a basis for the topology on X.

3.6.2. Lemma. *Let U and V be affine open subsets of an affine variety X.* *locloctwo*

(i) *If U is a localization of X and V is a localization of U, then V is a localization of X.*

(ii) *If $V \subset U$ and V is a localization of X, then V is a localization of U.*

(iii) *Let p be a point of $U \cap V$. There is an open set Z containing p that is a localization of U and also a localization of V.* □

3.6.3. Lemma. *Let $X = \operatorname{Spec} A$ be an affine variety, and let R be the algebra of regular functions on an arbitrary variety Y. An algebra homomorphism $A \to R$ defines a morphism $Y \xrightarrow{f} X$.* *Xmaps*

proof. **(i)** The point here is that Y isn't assumed to be affine. The algebra R consists of the regular functions on Y. Other than that, we don't know much about R.

Let $\{Y^i\}$ be a covering of Y by affine open sets, and let R_i be the coordinate algebra of Y^i. A rational function that is regular on Y is regular on Y^i, so $R \subset R_i$. The composition of the homomorphism $A \to R \subset R_i$ define morphisms $Y^i = \operatorname{Spec} R_i \xrightarrow{f^i} \operatorname{Spec} A$ for each i, and it is true that $f^i = f^j$ on the affine variety $Y^i \cap Y^j$. Lemma 3.5.13 **(iv)** shows that there is a unique morphism $Y \xrightarrow{f} \operatorname{Spec} A$ that restricts to f^i on Y^i. □

3.6.4. Corollary. *Let $Y \xrightarrow{f} X$ be a morphism of varieties, let q be a point of X, and let $p = f(q)$. If a rational function g on X is regular at p, its pullback $g \circ f$ is a regular function on Y at q.* *pullback-reg*

proof. We choose an affine open neighborhood U of p in X on which g is a regular function and an affine open neighborhood V of q in Y that is contained in the inverse image $f^{-1}U$. The morphism f restricts to a morphism $V \to U$ that we denote by the same letter f. Let A and B be the coordinate algebras of U and V, respectively. The morphism $V \xrightarrow{f} U$ corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. On U, the function g is an element of A, and $g \circ f = \varphi(g)$. □

3.6.5. Proposition. *The complement of a hypersurface is an affine open subvariety of $\mathbb{P}^n$.* *comphy-per*

proof. Let H be the hypersurface defined by an irreducible homogeneous polynomial f of degree d, and let Y be the complement of H in $\mathbb{P}^n$. Let R be the algebra of regular functions on Y. The elements of R are the homogeneous fractions of degree zero, of the form g/f^k (**3.4.7**). The fractions m/f, where m are the monomials of degree d, generate R. Since there are finitely many monomials of degree d, R is a finite-type domain. Lemma 3.6.3 gives us a morphism $Y \xrightarrow{u} X = \operatorname{Spec} R$. We show that u is an isomorphism.

Let A be the algebra of regular functions on the standard affine open set $\mathbb{U}^0$ of $\mathbb{P}^n$. The intersection $Y^0 = Y \cap \mathbb{U}^0$ is a localization of $\mathbb{U}^0$, the spectrum of $A[s^{-1}]$, where $s = f/x_0^d$. Let $t = s^{-1} = x_0^d/f$. This is an element of R.

3.6.6. Lemma. *The localizations $A[s^{-1}]$ and $R[t^{-1}]$ are equal.* *Yzero*

proof of the lemma. The generators m/f of R, m a monomial of degree d, can be written as products $s^{-1}(m/x_0^d)$. Since m/x_0^d is in A, the generators are in A_s. So $R \subset A_s$, and since $t^{-1} = s$ is in A,

$R_t \subset A_s$. Next, the fractions x_i/x_0 generate A, and x_i/x_0 can be written as $t^{-1}(m/f)$, with $m = x_i x_0^{d-1}$, so they are in R_t. Then $A \subset R_t$ and since $s^{-1} = t$ is in R, $A_s \subset R_t$. □

We go back to the proof of the Proposition 3.6.5. According Lemma 3.6.6, the morphism $Y \xrightarrow{u} X$ restricts to an isomorphism $Y^0 \to X^0 = \operatorname{Spec} A[s^{-1}]$, and the index 0 can be replaced by any $i = 0, ..., n$. The next lemma, together with Lemma 3.5.18 shows that u is an isomorphism. □

moverfzero **3.6.7. Lemma.** *The open sets X^i cover X.*

proof. Suppose that a point p of X isn't contained in any of the subsets X^i. Let $s_i = f/x_i^d$ and let $t_i = s_i^{-1}$. Then $t_i(p) = 0$ for all i. If m is any monomial of degree d, m^d will divisible by x_i^d for some i and then $(m/f)^d$ will be divisible by t_i. So if $t_i(p) = 0$, then $[m/f](p) = 0$ for every monomial of degree d, and therefore $[f/f](p) = 0$. Since $f/f = 1$, this is a contradiction. □

intersectaffine **3.6.8. Theorem.** *Let U and V be affine open subvarieties of a variety X. The intersection $U \cap V$ is an affine open subvariety. If, say $U \approx \operatorname{Spec} A$ and $V \approx \operatorname{Spec} B$, the coordinate algebra of $U \cap V$ is generated by the two algebras A and B.*

proof. Let $[A, B]$ denote the subalgebra generated by two subalgebras A and B of the function field K of X. The elements of $[A, B]$ are finite sums of products $\sum a_i b_i$ with a_i in A and b_i in B. If $A = \mathbb{C}[a_1, ..., a_r]$, and $B = \mathbb{C}[b_1, ..., b_s]$, the algebra $[A, B]$ is generated by the set $\{a_i\} \cup \{b_j\}$. It is a finite-type algebra.

The algebras A and B that appear in the statement of the theorem are subalgebras of the function field K of X. Let $R = [A, B]$ and let $W = \operatorname{Spec} R$. To prove the theorem, we show that W is isomorphic to $U \cap V$. The varieties U, V, W, and X have the same function field K, and the inclusions of coordinate algebras $A \to R$ and $B \to R$ give us morphisms $W \to U$ and $W \to V$. We also have inclusions $U \subset X$ and $V \subset X$, and X is a subvariety of a projective space $\mathbb{P}^n$. Restricting the projective embedding of X gives us embeddings of U and V and it gives us a morphism from W to $\mathbb{P}^n$. All of these morphisms to $\mathbb{P}^n$ will be defined by the same good point α with values in K, the point that defines the projective embedding of X. Let's denote the morphisms to $\mathbb{P}^n$ by $\underline{\alpha}_X, \underline{\alpha}_U, \underline{\alpha}_V$ and $\underline{\alpha}_W$. The morphism $\underline{\alpha}_W$ can be obtained as the composition of the morphisms $W \to U \subset X \subset \mathbb{P}^n$, and also as the analogous composition, in which V replaces U. Therefore the image of W in $\mathbb{P}^n$ is contained in $U \cap V$. Thus $\underline{\alpha}_W$ restricts to a morphism $W \xrightarrow{\epsilon} U \cap V$. We show that ϵ is an isomorphism.

Let p be a point of $U \cap V$. We choose an affine open subset U_s of $U \cap V$ that is a localization of U and that contains p (3.6.2). The coordinate algebra of U_s will be A_s, with s in A, and B will be a subalgebra of A_s. Then

$$R_s = [A, B]_s = [A_s, B] = A_s$$

So ϵ maps the localization $W_s = \operatorname{Spec} R_s$ of W isomorphically to the open subset $U_s = \operatorname{Spec} A_s$ of $U \cap V$. Since we can cover $U \cap V$ by open sets such as U_s, Lemma 3.5.13 **(ii)** shows that ϵ is an isomorphism. □

3.7 Lines in Three-Space

linesinp

The *Grassmanian* $\mathbf{G}(m, n)$ is a variety whose points correspond to subspaces of dimension m of the vector space $\mathbb{C}^n$, or to linear subspaces of dimension $m-1$ of $\mathbb{P}^{n-1}$. One says that $\mathbf{G}(m, n)$ *parametrizes* those subspaces. For example, the Grassmanian $\mathbf{G}(1, n+1)$ is the projective space $\mathbb{P}^n$. The points of $\mathbb{P}^n$ parametrize one-dimensional subspaces of $\mathbb{C}^{n+1}$.

The Grassmanian $\mathbf{G}(2, 4)$ parametrizes two-dimensional subspaces of $\mathbb{C}^4$, or lines in $\mathbb{P}^3$. We denote that Grassmanian by $\mathbb{G}$, and we describe it in this section. The point of $\mathbb{G}$ that corresponds to a line ℓ in $\mathbb{P}^3$ will be denoted by $[\ell]$.

One can get some insight into the structure of $\mathbb{G}$ using row reduction. Let $V = \mathbb{C}^4$, let u_1, u_2 be a basis of a two-dimensional subspace U of V, and let M be the 2×4 matrix whose rows are u_1, u_2. The rows of the matrix M' obtained from M by row reduction span the same space U, and the row-reduced matrix M' is uniquely determined by U. Provided that the left hand 2×2 submatrix of M is invertible, M' will have the form

rowreduced (3.7.1)
$$M' = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

The Grassmanian $\mathbb{G}$ contains, as an open subset, a four-dimensional affine space whose coordinates are the variable entries of M'.

In any 2×4 matrix M with independent rows, some pair of columns will be independent, and the corresponding 2×2 submatrix will be invertible.. That pair of columns can be used in place of the first two in a row reduction. So $\mathbb{G}$ is covered by six four-dimensional affine spaces that we denote by $\mathbb{W}^{ij}$, $1\le i<j\le 4$, $\mathbb{W}^{ij}$ being the space of 2×4 matrices such that column i is $(1,0)^t$ and column j is $(0,1)^t$.

The fact that $\mathbb{P}^4$ and $\mathbb{G}$ are both covered by affine spaces of dimension four might lead one to guess that they are similar. They are quite different.

(3.7.2) the exterior algebra extalgone

Let V be a complex vector space. The *exterior algebra* $\bigwedge V$ ('wedge V') is a noncommutative algebra — an algebra whose multiplication law isn't commutative. It is generated by the elements of V, with the relations

(3.7.3) $$vw=-wv \quad \text{for all } v,w \text{ in } V.$$ extalg

3.7.4. Lemma. *The condition (3.7.3) is equivalent with: $vv=0$ for all v in V.* *vvzero*

proof. To get $vv=0$ from (3.7.3), one sets $w=v$. Suppose that $vv=0$ for all v in V. Then vv, ww, and $(v+w)(v+w)$ are all zero. Since $(v+w)(v+w)=vv+vw+wv+ww$, it follows that $vw+wv=0$. □

To familiarize yourself with computation in $\bigwedge V$, verify that $v_3v_2v_1=-v_1v_2v_3$ and that $v_4v_3v_2v_1=v_1v_2v_3v_4$.

Let $\bigwedge^r V$ denote the subspace of $\bigwedge V$ spanned by products of length r of elements of V. The exterior algebra $\bigwedge V$ is the direct sum of the subspaces $\bigwedge^r V$. An algebra A that is a direct sum of subspaces A^i, and such that multiplication maps $A^i\times A^j$ to A^{i+j}, is called a *graded algebra*. The exterior algebra is a *noncommutative graded algebra*.

3.7.5. Proposition. *If $(v_1,...,v_n)$ is a basis for V, the products $v_{i_1}\cdots v_{i_r}$ of length r, with increasing indices $i_1<i_2<\cdots<i_r$, form a basis for $\bigwedge^r V$.* *dependentproduct*

The proof of this proposition is at the end of the section.

3.7.6. Corollary. *Let $v_1,...,v_r$ be elements of V. In $\bigwedge^r V$, the product $v_1\cdots v_r$ is zero if and only if the elements are dependent.* □ *wedgezero*

For the rest of the section, we let V be a vector space of dimension four, with basis $(v_1,...,v_4)$. Proposition 3.7.5 tells us that

(3.7.7) extbasistwo

$\bigwedge^0 V=\mathbb{C}$ is a space of dimension 1, with basis $\{1\}$
$\bigwedge^1 V=V$ is a space of dimension 4, with basis $\{v_1,v_2,v_3,v_4\}$
$\bigwedge^2 V$ is a space of dimension 6, with basis $\{v_iv_j\,|\,i<j\}=\{v_1v_2,v_1v_3,v_1v_4,v_2v_3,v_2v_4,v_3v_4\}$
$\bigwedge^3 V$ is a space of dimension 4, with basis $\{v_iv_jv_k\,|\,i<j<k\}=\{v_1v_2v_3,v_1v_2v_4,v_1v_3v_4,v_2v_3v_4\}$
$\bigwedge^4 V$ is a space of dimension 1, with basis $\{v_1v_2v_3v_4\}$
$\bigwedge^q V=0$ when $q>4$.

The elements of $\bigwedge^2 V$ are combinations

(3.7.8) $$w=\sum_{i<j}a_{ij}v_iv_j$$ wedgetwo

We regard $\bigwedge^2 V$ as an affine space of dimension 6, identifying the combination w with the vector $(a_{12},a_{13},a_{14},a_{23},a_{24},a_{34})$, and we use the same symbol w to denote the corresponding element of the projective space $\mathbb{P}^5$.

defdecomp **3.7.9. Definition.** An element of $\bigwedge^2 V$ is *decomposable* if it is the product of two elements of V.

describedecomp **3.7.10. Proposition.** *The decomposable elements w of $\bigwedge^2 V$ are those such that $ww = 0$, and the relation $ww = 0$ is equivalent with the following equation in its coefficients a_{ij}:*

eqgrass (3.7.11)
$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$$

proof. If w is decomposable, say $w = u_1u_2$ with u_i in V, then $w^2 = u_1u_2u_1u_2 = -u_1^2u_2^2$ is zero because $u_1^2 = 0$. For the converse, we compute w^2 with $w = \sum_{i<j} a_{ij}v_iv_j$. The result is

$$ww = 2\big(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\big)v_1v_2v_3v_4$$

To show that w is decomposable if $w^2 = 0$, it seems simplest to factor w explictly. Since the assertion is trivial when $w = 0$, we may suppose that some coefficient of w is nonzero. Say that $a_{12} \neq 0$. Then if $w^2 = 0$,

factorw (3.7.12)
$$w \;=\; \frac{1}{a_{12}}\big(a_{12}v_2 + a_{13}v_3 + a_{14}v_4\big)\big(-a_{12}v_1 + a_{23}v_3 + a_{24}v_4\big)$$

The computation for another pair of indices is similar. $\square$

decomptwo **3.7.13. Corollary.** **(i)** *Let w be a nonzero decomposable element of $\bigwedge^2 V$, say $w = u_1u_2$, with u_i in V. Then (u_1, u_2) is a basis for a two-dimensional subspace of V.*

(ii) *Let (u_1, u_2) and (u_1', u_2') be bases for two subspaces U and U' of V, and let $w = u_1u_2$ and $w' = u_1'u_2'$. Then $U = U'$, if and only if w and w' differ by a scalar factor — if and only if they represent the same point of $\mathbb{P}^5$.*

(iii) *The Grassmanian $\mathbb{G}$ corresponds bijectively to the quadric Q in $\mathbb{P}^5$ whose equation is (3.7.11). If U is a two-dimensional subspace of V with basis (u_1, u_2), the point of $\mathbb{G}$ that represents U is sent to the point $w = u_1u_2$ of Q.*

Thus the Grassmanian $\mathbb{G}$ can be represented as the quadric (3.7.11) in $\mathbb{P}^5$.

proof. **(i)** If an element w of $\bigwedge^2 V$ is decomposable, say $w = u_1u_2$, and if w isn't zero, then u_1 and u_2 must be independent (3.7.6). They span a two-dimensional subspace.

(ii) Suppose that $U' = U$. Then, when we write the second basis in terms of the first one, say $(u_1', u_2') = (au_1 + bu_2, cu_1 + du_2)$, the product w' becomes the scalar multiple $(ad-bc)w$ of w, and $ad-bc \neq 0$.

If $U' \neq U$, then at least three of the vectors u_1, u_2, u_1', u_2' will be independent. Say that u_1, u_2, u_1' are independent. Then, according to Corollary 3.7.6, the product $u_1u_2u_1'$ isn't zero. Since $u_1'u_2'u_1' = 0$, $u_1'u_2'$ cannot be a scalar multiple of u_1u_2.

(iii) This follows from **(i)** and **(ii)**. $\square$

For the rest of this section, we will use the concept of the algebraic dimension of a variety X. This dimension can be defined as the length d of the longest chain $C_0 > C_1 > \cdots > C_d$ of closed subvarieties of X. We refer to the algebraic dimension simply as the *dimension*, and we use some of its properties informally here, deferring proofs to the discussion of dimension in the next chapter.

The topological dimension of X, its dimension in the classical topology, is always twice the algebraic dimension. Because the Grassmanian $\mathbb{G}$ is covered by affine spaces of dimension 4, its algebraic dimension is 4 and its topological dimension is 8.

pinlclosed **3.7.14. Proposition.** *Let $\mathbb{P}^3$ be the projective space associated to a four dimensional vector space V. In the product $\mathbb{P}^3 \times \mathbb{G}$, the locus Γ of pairs $p,[\ell]$ such that p lies on ℓ is a closed subset of dimension 5.*

proof. Let ℓ be the line in $\mathbb{P}^3$ that corresponds to a subspace U with basis (u_1, u_2), let $w = u_1u_2$, and let p be the point represented by a vector x in V. Then $p \in \ell$ means $x \in U$, which is true if and only if (x, u_1, u_2) is a dependent set — if and only if $xw = 0$ (3.7.5). An element w of $\bigwedge^2 V$ is decomposable when $w^2 = 0$. So Γ is the closed subset of points (x, w) of $\mathbb{P}^3_x \times \mathbb{P}^5_w$ defined by the bihomogeneous equations $xw = 0$ and $w^2 = 0$. $\square$

When we project Γ to $\mathbb{G}$, the fibre over a point $[\ell]$ of $\mathbb{G}$ is the set of pairs $p, [\ell]$ such that $p \in \ell$. The projection maps that fibre bijectively to the line ℓ. Thus Γ can be viewed as a family of lines, parametrized by $\mathbb{G}$. Its dimension is $\dim \ell + \dim \mathbb{G} = 1 + 4 = 5$.

(3.7.15) lines on a surface linesina-surface

When one is given a surface S in $\mathbb{P}^3$, one may ask: Does S contain a line? One surface that contains lines is the quadric Q in $\mathbb{P}^3$ whose equation is $z_{01}z_{10} = z_{00}z_{11}$, the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3_w$ (**3.1.6**). It contains two families of lines, the lines that correspond to the two "rulings" $p \times \mathbb{P}^1$ and $\mathbb{P}^1 \times q$ of $\mathbb{P}^1 \times \mathbb{P}^1$. There are surfaces of arbitrary degree that contain lines, but a generic surface of degree four or more won't contain any line.

We use coordinates x_i with $i = 1, 2, 3, 4$ for $\mathbb{P}^3$ here. There are $N = \binom{d+3}{3}$ monomials of degree d in four variables, so homogeneous polynomials of degree d are parametrized by an affine space of dimension N, and surfaces of degree d in $\mathbb{P}^3$ by a projective space of dimension $n = N-1$. Let $\mathbb{S}$ denote that projective space, let $[S]$ denote the point of $\mathbb{S}$ that corresponds to a surface S, and let f be the irreducible polynomial whose zero locus is S. The coordinates of $[S]$ are the coefficients of f. Speaking infomally, we say that a point of $\mathbb{S}$ *is* a surface of degree d in $\mathbb{P}^3$. (When f is reducible, its zero locus isn't a variety. Let's not worry about this.)

Consider the line ℓ_0 defined by $x_3 = x_4 = 0$. Its points are those of the form $(x_1, x_2, 0, 0)$, and a surface $S : \{f = 0\}$ will contain ℓ_0 if and only if $f(x_1, x_2, 0, 0) = 0$ for all x_1, x_2. Substituting $x_3 = x_4 = 0$ into f leaves us with a polynomial in two variables:

(3.7.16) $$f(x_1, x_2, 0, 0) = c_0x_1^d + c_1x_1^{d-1}x_2 + \cdots + c_dx_2^d$$ scon-tainslzero

where c_i are some of the coefficients of the polynomial f. If $f(x_1, x_2, 0, 0)$ is identically zero, all of those coefficients will be zero. So the surfaces that contain ℓ_0 correspond to the points of the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by the equations $c_0 = \cdots = c_d = 0$. Its dimension is $n-d-1$. This is a satisfactory answer to the question of which surfaces contain ℓ_0, and we can use it to make a guess about lines in a generic surface of degree d.

3.7.17. Lemma. *In the product variety $\mathbb{G}\times\mathbb{S}$, the set Σ of pairs $[\ell]$,$[S]$ such that ℓ is a line, S is a surface of degree d, and $\ell \subset S$, is a closed set.* *scon-tainslclosed*

proof. Let $\mathbb{W}^{ij}$, $1 \le i < j \le 4$ denote the six affine spaces that cover the Grassmanian, as at the beginning of this section. It suffices to show that the intersection $\Sigma^{ij} = \Sigma \cap (\mathbb{W}^{ij}\times\mathbb{S})$ is closed in $\mathbb{W}^{ij}\times\mathbb{S}$ for all i, j (2.2.6). We inspect the case $i, j = 1, 2$.

A line ℓ such that $[\ell]$ is in $\mathbb{W}^{12}$ corresponds to a subspace of $\mathbb{C}^2$ with basis (u_1, u_2) of the form $u_1 = (1, 0, a_2, a_3)$, $u_2 = (0, 1, b_2, b_3)$, and the coordinates of the points of ℓ are combinations $ru_1 + su_2$ of u_1, u_2. Let $f(x_1, x_2, x_3, x_4)$ be the polynomial that defines a surface S of degree d. The line ℓ is contained in S if and only if $f(r, s, ra_2+sb_2, ra_3+sb_3) = \widetilde{f}(r, s)$ is zero for all r and s, and $\widetilde{f}(r, s)$ is a homogeneous polynomial of degree d in r, s. If we write $\widetilde{f}(r, s) = z_0r^d + z_1r^{d-1}s + \cdots + z_ds^d$, the coefficients z_ν will be polynomials in a_i, b_i and in the coefficients of f. The locus $z_0 = \cdots = z_d = 0$ is the closed subset Σ^{12} of $\mathbb{W}^{12} \times \mathbb{S}$ that represents surfaces containing a line. $\square$

The set of surfaces that contain our special line ℓ_0 corresponds to the linear space $\mathbb{L}_0$ of $\mathbb{S}$ of dimension $n-d-1$, and ℓ_0 can be carried to any other line ℓ by a linear map $\mathbb{P}^3 \to \mathbb{P}^3$. So the sufaces that contain another line ℓ also form a linear subspace of $\mathbb{S}$ of dimension $n-d-1$. Those subspaces are the fibres of Σ over $\mathbb{G}$. The dimension of the Grassmanian $\mathbb{G}$ is 4. Therefore the dimension of Σ is

$$\dim \Sigma \;=\; \dim \mathbb{L}_0 + \dim \mathbb{G} \;=\; (n-d-1) + 4$$

Since $\mathbb{S}$ has dimension n,

(3.7.18) $$\dim \Sigma \;=\; \dim \mathbb{S} - d + 3$$ dimspace-lines

When we project the product $\mathbb{G}\times\mathbb{S}$ and its subvariety Σ to $\mathbb{S}$, the fibre of Σ over a point $[S]$ is the set of pairs $[\ell]$,$[S]$ such that ℓ is contained in S — the set of lines in S.

lines-lowdeg **3.7.19.** When the degree d of the surfaces we are studying is 1, $\dim \Sigma = \dim \mathbb{S}+2$. Every fibre of Σ over $\mathbb{S}$ will have dimension at least 2. In fact, every fibre has dimension equal to 2. Surfaces of degree 1 are planes, and the lines in a plane form a two-dimensional family.

When $d = 2$, $\dim \Sigma = \dim \mathbb{S}+1$. We can expect that most fibres of Σ over $\mathbb{S}$ will have dimension 1. This is true: A smooth quadric contains two one-dimensional families of lines. (All smooth quadrics are equivalent with the quadric (3.1.7).) But if a quadratic polynomial $f(x_1, x_2, x_3, x_4)$ is the product of linear polynomials, its locus of zeros will be a union of planes. It will contain two-dimensional families of lines. Some fibres have dimension 2.

When $d \geq 4$, $\dim \Sigma < \dim \mathbb{S}$. The projection $\Sigma \to \mathbb{S}$ cannot be surjective. Most surfaces of degree 4 or more contain no lines.

The most interesting case is that the degree d is 3. In this case, $\dim \Sigma = \dim \mathbb{S}$. Most fibres will have dimension zero. They will be finite sets. In fact, a generic cubic surface contains 27 lines. We have to wait to see why the number is precisely 27 (see Theorem 4.6.27).

Our conclusions are intuitively plausible, but to be sure about them, we need to study dimension carefully. We do this in the next chapters.

prdep-prrod **(3.7.20) proofofProposition3.7.5.**

Let $v = (v_1, ..., v_n)$ be a basis of a vector space V. The proposition asserts that the products $v_{i_1} \cdots v_{i_r}$ of length r with increasing indices $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$. To prove this, we need to be more precise about the definition of the exterior algebra $\bigwedge V$.

tensalg **3.7.21.** We start with the algebra $T(V)$ of noncommutative polynomials in the basis v, which is also called the *tensor algebra* on V. The part $T^r(V)$ of $T(V)$ of degree r has, as basis, the n^r noncommutative monomials of degree r, the products $v_{i_1} \cdots v_{i_r}$ of length r of elements of the basis. Its dimension is n^r. For example, when $n{=}2$, the eight-dimensional space $T^3(V)$ has basis $(x_1^3,\ x_1^2x_2,\ x_1x_2x_1,\ x_2x_1^2,\ x_1x_2^2,\ x_2x_1x_2,\ x_2^2x_1,\ x_2^3)$.

The exterior algebra $\bigwedge V$ is the quotient of $T(V)$ obtained by forcing the relations $vw+wv = 0$ (3.7.3). Using the distributive law, one sees that the relations $v_iv_j+v_jv_i = 0$, $1 \leq i, j \leq n$, are sufficient to define this quotient.

We can multiply the relations $v_iv_j+v_jv_i$ on left and right by noncommutative monomials $p(v)$ and $q(v)$ in $v_1, ..., v_n$. When we do this with all pairs p, q of monomials whose degrees sum to $r-2$, the noncommutative polynomials

vivj
$$p(v)(v_iv_j+v_jv_i)q(v) \tag{3.7.22}$$

span the kernel of the linear map $T^r(V) \to \bigwedge^r V$. So in $\bigwedge^r V$, $p(v)(v_iv_j)q(v) = -p(v)(v_jv_i)q(v)$. Using these relations, any product $v_{i_1} \cdots v_{i_r}$ in $\bigwedge^r V$ is, up to sign, equal to a product in which the indices i_ν are in increasing order. Thus the products with indices in increasing order span $\bigwedge^r V$, and because $v_iv_i = 0$, such a product will be zero unless the indices are strictly increasing.

We go to the proof now. Let $v = (v_1, ..., v_n)$ be a basis for V. We show first that the product $w = v_1 \cdots v_n$ of the basis elements in increasing order is a basis of the space $\bigwedge^n V$. We have shown that w spans $\bigwedge^n V$, and it remains to show that $w \neq 0$, or that $\bigwedge^n V \neq 0$.

Let's use multi-index notation, writing $(i) = (i_1, ..., i_r)$, and $v_{(i)} = v_{i_1} \cdots v_{i_r}$. We define a surjective linear map $T^n(V) \xrightarrow{\varphi} \mathbb{C}$. The products $v_{(i)} = (v_{i_1} \cdots v_{i_n})$ of length n form a basis of $T^n(V)$. If there is no repetition among the indices $i_1, ..., i_n$, then (i) will be a permutation of the indices $1, ..., n$. In that case, we set $\varphi(v_{(i)}) = \varphi(v_{i_1} \cdots v_{i_n}) = sign(i)$. If there is a repetition, we set $\varphi(v_{(i)}) = 0$.

Let p and q be noncommutative monomials whose degrees sum to $n-2$. If the product $p(v_iv_j)q$ has no repeated index, the indices in $p(v_iv_j)q$ and in $p(v_jv_i)q$ will be permutations of $1, ..., n$, and those permutations will have opposite signs. So $p(v_iv_j + v_jv_i)q$ will be in the kernel of φ. Since these elements span the space of relations that define $\bigwedge^n V$ as a quotient of $T^n(V)$, the surjective map $T^n(V) \xrightarrow{\varphi} \mathbb{C}$ defines a surjective map $\bigwedge^n V \to \mathbb{C}$. Therefore $\bigwedge^n V \neq 0$.

To prove (3.7.5), we must show that for $r \leq n$, the products $v_{i_1} \cdots v_{i_r}$ with $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$, and we have seen that those products span $\bigwedge^r V$. We must show that they are independent. Suppose that a combination $z = \sum c_{(i)} v_{(i)}$ is zero, the sum being over the sets $\{i_1, ..., i_r\}$ of strictly increasing indices. We choose a particular set $(j_1, ..., j_r)$ of n strictly increasing indices, and we let $(k) = (k_1, ..., k_{n-r})$ be the set of indices that don't occur in (j), listed in arbitrary order. Then all terms in the sum $zv_{(k)} = \sum c_{(i)} v_{(i)} v_{(k)}$ will be zero except the term with $(i) = (j)$. On the other hand, since $z = 0$, $zv_{(k)} = 0$. Therefore $c_{(j)} v_{(j)} v_{(k)} = 0$, and since $v_{(j)} v_{(k)}$ differs by sign from $v_1 \cdots v_n$, it isn't zero. It follows that $c_{(j)} = 0$. This is true for all (j), so $z = 0$. $\square$

3.8 Exercises

closurePP chapthreeex **3.8.1.** Let X be the affine surface in $\mathbb{A}^3$ defined by the equation $x_1^3 + x_1x_2x_3 + x_1x_3 + x_2^2 + x_3 = 0$, and let $\overline{X}$ be its closure in $\mathbb{P}^3$. Describe the intersection of $\overline{X}$ with the plane at infinity in $\mathbb{P}^3$.

xcubiceq **3.8.2.** Let C be a cubic curve, the locus of a homogeneous cubic polynomial $f(x, y, z)$ in $\mathbb{P}^2$. Suppose that $(0, 0, 1)$ and $(0, 1, 0)$ are flex points of C, that the tangent line to C at $(0, 0, 1)$ is the line $\{y = 0\}$, and the tangent line at $(0, 1, 0)$ is the line $\{z = 0\}$. What are the possible polynomials f? Disregard the question of whether f is irreducible.

xtendtoinf **3.8.3.** Let Y and Z be the zero sets in $\mathbb{P}$ of relatively prime homogeneous polynomials g and h of the same degree r. Prove that the rational function $\alpha = g/h$ will tend to infinity as one approaches a point of Z that isn't also a point of Y and that, at intersections of Y and Z, α is indeterminate in the sense that the limit depends on the path.

xf-zoneirred **3.8.4.** Let f be a homogeneous polynomial in x, y, z, not divisible by z. Prove that f is irreducible if and only if $f(x, y, 1)$ is irreducible.

homog-primecon-verse **3.8.5.** Let $\mathcal{P}$ be a homogeneous ideal in $\mathbb{C}[x_0, ..., x_n]$, and suppose that its dehomogenization P is a prime ideal. Is $\mathcal{P}$ a prime ideal?

xcondpt **3.8.6.** Let U be the open complement of a closed subset Z in a projective variety X in $\mathbb{P}^n$. Say that X is the set of solutions of the homogeneous polynomial equations $f = 0$ and that Z is the set of solutions of some equations $g = 0$. What conditions must a point p of $\mathbb{P}^n$ satisfy in order to be a point of U?

xideali-nAxP **3.8.7.** Describe the ideals that define closed subsets of $\mathbb{A}^m \times \mathbb{P}^n$.

xlocusin-PxP **3.8.8.** With coordinates x_0, x_1, x_2 in the plane $\mathbb{P}$ and s_0, s_1, s_2 in the dual plane $\mathbb{P}^*$, let C be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where f is an irreducible homogeneous polynomial in x. Let Γ be the locus of pairs (x, s) of $\mathbb{P} \times \mathbb{P}^*$ such that the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$ is the tangent line to C at x. Prove that Γ is a Zariski closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

xfn-bounded **3.8.9.** Let U be a nonempty open subset of $\mathbb{P}^n$. Prove that if a rational function is bounded on U, it is a constant.

xmapcusp **3.8.10.** Let Y be the cusp curve $\operatorname{Spec} B$, where $B = \mathbb{C}[x, y]/(y^2 - x^3)$. This algebra embeds as subring into $\mathbb{C}[t]$, by $x = t^2$. $y = t^3$. Show that the two vectors $v_0 = (x-1, y-1)$ and $v_1 = (t+1, t^2+t+1)$ define the same point of $\mathbb{P}^1$ with values in the fraction field K of B, and that they define morphisms from Y to $\mathbb{P}^1$ wherever the entries are regular functions on Y. Prove that the two morphisms they define piece together to give a morphism $Y \to \mathbb{P}^1$.

xdoes-ntextend **3.8.11.** Let C be a conic in $\mathbb{P}^2$, and let π be the projection to $\mathbb{P}^1$ from a point q of C. Prove that there is no way to extend this map to a morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$.

xverify-morph **3.8.12.** Verify that the following maps are morphisms of projective varieties:
(i) the projection from a product variety $X \times Y$ to X,
(ii) the inclusion of X into the product $X \times Y$ as the set $X \times y$ for a point y of Y,
(iii) the morphism of products $X \times Y \to X' \times Y$ when a morphism $X \to X'$ is given.

zautPone **3.8.13.** A pair f_0, f_1 of homogeneous polynomials in x_0, x_1 of the same degree d can be used to define a morphism $\mathbb{P}^1 \to \mathbb{P}^1$. At a point q of $\mathbb{P}^1$, the morphism evaluates $(1, f_1/f_0)$ or $(f_0/f_1, 1)$ at q.
(i) The *degree* of such a morphism is the number of points in a generic fibre. Determine the degree.
(ii) Describe the group of automorphisms of $\mathbb{P}^1$.

3. A pair (f_0, f_1) of relatively prime, homogeneous polynomials in x_0, x_1 of the same degree d defines a morphism $u : \mathbb{P}^1 \to \mathbb{P}^1$ that maps a point q to $(1, f_1(q)/f_0(q))$ if $f_0(q) \neq 0$ and to $(f_0(q)/f_1(q), 1)$ if $f_1(q) \neq 0$. By inspecting the inverse images of a few points, determine the maps that are injective, and use your result to describe the group of automorphisms of $\mathbb{P}^1$.

3.8.14. **(i)** What are the conditions that a triple of $f = (f_0, f_1, f_2)$ homogeneous polynomials in x_0, x_1, x_2 of the same degree d must satisfy in order to define a morphism $\mathbb{P}^2 \to \mathbb{P}^2$? xmorph-Ptwo

(ii) If f does define a morphism, what is its degree?

3.8.15. Let C be the plane projective curve $x^3 - y^2 z = 0$. cuspto-pone

(i) Show that the function field K of C is the field $\mathbb{C}(t)$ of rational functions in $t = y/x$.

(ii) Show that the point $(t^2 - 1, t^3 - 1)$ of $\mathbb{P}^1$ with values in K defines a morphism $C \to \mathbb{P}^1$.

3.8.16. Describe all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$. xPtwo-Pone

3.8.17. *blowing up a point in* $\mathbb{P}^2$. Consider the Veronese embedding of $\mathbb{P}^2_{xyz} \to \mathbb{P}^5_u$ by monomials of degree 2 defined by $(u_0, u_1, u_2, u_3, u_4, u_5) = (z^2, y^2, x^2, yz, xz, xy)$. If we drop the coordinate u_0, we obtain a map $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^4$: $\varphi(x, y, z) = (y^2, x^2, yz, xz, xy)$ that is defined at all points except the point $q = (0, 0, 1)$. Find defining equations for the closure of the image X. Prove that the inverse map $X \xrightarrow{\varphi^{-1}} \mathbb{P}^2$ is everywhere defined, that the fibre of φ^{-1} over q is a projective line, and that f is bijective everywhere else. xblowup

3.8.18. Show that the conic C in $\mathbb{P}^2$ defined by the polynomial $y_0^2 + y_1^2 + y_2^2 = 0$ and the twisted cubic V in $\mathbb{P}^3$, the zero locus of the polynomials $v_0 v_2 - v_1^2$, $v_0 v_3 - v_1 v_2$, $v_1 v_3 - v_2^2$ are isomorphic by exhibiting inverse morphisms between them. xconic-cubic

3.8.19. Let X be the affine plane with coordinates (x, y). Given a pair of polynomials $u(x, y), v(x, y)$ in x, y, one may try to define a morphism $f : X \to \mathbb{P}^1$ by $f(x, y) = (u, v)$. Under what circumstances is f a morphism? xmapplan-etoPone

3.8.20. Let x_0, x_1, x_2 be the coordinate variables in the projective plane X, and for $i = 1, 2$, let $u_i = x_i/x_0$. The function field K of X is the field of rational functions in the variables u_i. Let $f(u_1, u_2)$ and $g(u_1, u_2)$ be polynomials. Under what circumstances does the point $(1, f, g)$ with values in K define a morphism $X \to \mathbb{P}^2$? xmap-toPthree

3.8.21. Prove that every finite subset S of a projective variety X is contained in an affine open subset. easyk-leiman

3.8.22. Describe the affine open subsets of the projective plane $\mathbb{P}^2$. xaffinopen-plane

3.8.23. What is the dimension of the Grassmanian $\mathbf{G}(m, n)$? xdimgrass

3.8.24. According to (3.7.19), a generic quartic surface in $\mathbb{P}^3$ won't contain any lines. Will a generic quartic surface contain a plane conic? xconicin-quartic

3.8.25. Let V be a vector space of dimension 5, and let $\mathbb{G}$ denote the Grassmanian $\mathbf{G}(2, 5)$ of lines in $\mathbb{P}(V) = \mathbb{P}^4$. So $\mathbb{G}$ is a subvariety of the projective space $\mathbb{P}(W)$, $W = \bigwedge^2 V$, which has dimension 10. let D denote the subset of decomposable vectors in $\mathbb{P}(W)$. Prove that there is a bijective correspondence between two-dimensional subspaces of V and points of D, and that a vector w in $\bigwedge^2 V$ is decomposable if and only if $ww = 0$. Exhibit defining equations for $\mathbb{G}$ in the space $\mathbb{P}(W)$. xlinesp-five

3.8.26. *a flag variety*. Let $\mathbb{P} = \mathbb{P}^3$. The space of planes in $\mathbb{P}$ is the *dual projective space* $\mathbb{P}^*$. The variety F that parametrizes triples (p, ℓ, H) consisting of a point p, a line ℓ, and a plane H in $\mathbb{P}$, with $p \in \ell \subset H$, is called a *flag variety*. Exhibit defining equations for F in $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^{3*}$. The equations should be homogeneous in each of 3 sets of variables. xflag

3.8.27. Let Y be an affine variety. Prove that morphisms $Y \to \mathbb{P}^n$ whose images are in $\mathbb{U}^0$ correspond bijectively to morphisms of affine varieties $Y \to \mathbb{U}^0$, as defined in (2.6.4). xmorphaff

3.8.28. Determine all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$. xtwotoone

Chapter 4 INTEGRAL MORPHISMS

The concept of an algebraic integer was one of the essential ideas in the development of algebraic number theory in the 19th century. Then, largely through the work of Noether and Zariski, an analog was seen to be essential in algebraic geometry. We study that analog here.

4.1 The Nakayama Lemma

nakayama

eigen **(4.1.1) eigenvectors**

It won't be a surprise that eigenvectors are important, but the way that they are used to study modules may be less familiar.

Let P be an $n \times n$ matrix with entries in a ring A. The concept of an eigenvector for P makes sense when the entries of a vector are in an A-module. A column vector $v = (v_1, ..., v_n)^t$ with entries in an A-module M is an *eigenvector* of P with *eigenvalue* λ in A if $Pv = \lambda v$.

When the entries of a vector are in a module, it becomes hard to adapt the usual requirement that an eigenvector must be nonzero. So we drop it, though the zero vector tells us nothing.

eigenval **4.1.2. Lemma.** *Let P be a square matrix with entries in a ring A and let $p(t)$ be the characteristic polynomial* $\det(tI-P)$ *of P. If v is an eigenvector of P with eigenvalue λ, then $p(\lambda)v = 0$.*

The usual proof, in which one multiplies the equation $(\lambda I-P)v = 0$ by the cofactor matrix of $(\lambda I-P)$, carries over. $\square$

The next lemma is a cornerstone of the theory of modules. In it, JM denotes the set of (finite) sums $\sum_i a_i m_i$ with a_i in J and m_i in M.

nakaya-malem **4.1.3. Nakayama Lemma.** Let M be a finite module over a ring A, and let J be an ideal of A such that $M = JM$. There is an element z in J such that $m = zm$ for all m in M, i.e., such that $(1-z)M = 0$.

Since it is always true that $M \supset JM$, the hypothesis $M = JM$ could be replaced by $M \subset JM$.

proof of the Nakayama Lemma. Let $v_1, ..., v_n$ be generators for the finite A-module M. The equation $M = JM$ tells us that there are elements p_{ij} in J such that $v_i = \sum p_{ij} v_j$. We write this equation in matrix notation, as $v = Pv$, where v is the column vector $(v_1, ..., v_n)^t$ and P is the matrix $P = (p_{ij})$. Then v is an eigenvector of P with eigenvalue 1, and if $p(t)$ is the characteristic polynomial of P, then $p(1)v = 0$. Since the entries of P are in the ideal J, inspection of the determinant of $I - P$ shows that $p(1)$ has the form $1 - z$, with z in J. Then $(1-z)v_i = 0$ for all i. Since $v_1, ..., v_n$ generate M, $(1-z)M = 0$. □

With notation as in the Nakayama Lemma, let $s = 1-z$, so that $sM = 0$. The localized module M_s is the zero module.

4.1.4. Corollary. *Let I and J be ideals of a noetherian domain A.* *idealzero*
(i) *If $I = JI$, then either I is the zero ideal or J is the unit ideal.*
(ii) *Let B be a domain that contains A, and that is a finite A-module. If the extended ideal JB is the unit ideal of B, then J is the unit ideal of A.*

proof. **(i)** Since A is noetherian, I is a finite A-module. If $I = JI$, the Nakayama Lemma tells us that there is an element z of J such that $zx = x$ for all x in I. Suppose that I isn't the zero ideal. We choose a nonzero element x of I. Because A is a domain, we can cancel x from the equation $zx = x$, obtaining $z = 1$. Then 1 is in J, and J is the unit ideal.

(ii) The elements of the extended ideal JB are sums $\sum u_i b_i$ with u_i in J and b_i in B. Suppose that $B = JB$. Then there is an element z of J such that $b = zb$ for all b in B. Setting $b = 1$ shows that $z = 1$. So J is the unit ideal. □

4.1.5. Corollary. *Let x be an element of a noetherian domain A, not a unit, and let J be the principal ideal xA.* *xkdividesy*
(i) *The intersection $\bigcap J^n$ is the zero ideal.*
(ii) *If y is a nonzero element of A, the integers k such that x^k divides y in A are bounded.*
(iii) *For every $k > 0$, $J^k > J^{k+1}$.*

proof. Let $I = \bigcap J^n$ and let y be an element of I. Since $J^n = x^n A$, the elements of I are divisible by x^n for n. So for every n, there is an element a_n in A such that $y = a_n x^n$. Then $y/x = a_n x^{n-1}$, which is an element of J^{n-1}. Since this is true for every n, y/x is in I, and y is in JI. Here y can be any element of I, so $I = JI$. Since x isn't a unit, J isn't the unit ideal. Corollary 4.1.4**(i)** tells us that $I = 0$. This proves **(i)**, and **(ii)** follows. For **(iii)**, we note that if $J^k = J^{k+1}$, then, multiplying by J^{n-k}, we see that $J^n = J^{n+1}$ for every $n \geq k$. Therefore $J^k = \bigcap J^n = 0$. But since A is a domain and $x \neq 0$, $J^k = x^k A \neq 0$. □

4.2 Integral Extensions

int

An *extension* of a domain A is a domain B that contains A as a subring.

Let B be an extension of a domain A. An element β of B is *integral over A* if it is a root of a monic polynomial with coefficients in A. An extension B of A is an *integral extension* if all elements of B are integral over A.

4.2.1. Lemma. *Let $A \subset B$ be an extension of noetherian domains.* *aboutintegral*
(i) *An element b of B is integral over A if and only if the subring $A[b]$ of B generated by b is a finite A-module.*
(ii) *The set of elements of B that are integral over A is a subring of B.*
(iii) *If B is generated as A-algebra by finitely many integral elements, then B is a finite A-module.*
(iv) *Let $R \subset A \subset B$ be domains, and suppose that A is an integral extension of R. An element of B is integral over A if and only if it is integral over R. Therefore, if A is an integral extension of R and B is an integral extension of A, then B is an integral extension of R.* □

4.2.2. Corollary. *An extension $A \subset B$ of finite-type domains is an integral extension if and only if B is a finite A-module.* □ *integraliffinite*

Thus, if $f(x)$ is a monic irreducible polynomial with cofficients in A, and if $B = A[x]/(f)$, then every element of B will be integral over A.

4.2.3. Lemma. *Let $A \subset B$ be an extension of domains, with A noetherian, let I be a nonzero ideal of A, and let b be an element of B. If $bI \subset I$, then b is integral over A.* *betaintegral* *proof.* Because A is noetherian, I is finitely generated. Let $v = (v_1, ..., v_n)^t$ be a vector whose entries generate I. The hypothesis $bI \subset I$ allows us to write $bv_i = \sum p_{ij} v_j$ with p_{ij} in A, or in matrix notation, $bv = Pv$. So v is an eigenvector of P with eigenvalue b, and if $p(t)$ is the characteristic polynomial of P, then $p(b)v = 0$. Since I isn't zero, at least one v_i is nonzero, but $p(b)v_i = 0$. Since A is a domain, $p(b) = 0$. The characteristic polynomial is a monic polynomial with coefficients in A, so b is integral over A. □

deffinmorphaff **4.2.4. Definition.** Let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ be affine varieties. The morphism $Y \xrightarrow{u} X$ defined by an integral extension $A \subset B$ will be called an *integral morphism* of affine varieties.

Thus an integral morphism of affine varieties $Y \to X$ is a morphism whose associated algebra homomorphism $A \xrightarrow{\varphi} B$ is injective, and such that B is a finite A-module. The inclusion u of a proper closed subvariety Y into X isn't an integral morphism, though B is a finite A-module.

integralsurj **4.2.5. Proposition.** *An integral morphism $Y \xrightarrow{u} X$ of affine varieties is a surjective map.*

proof. Let $\mathfrak{m}_x$ be the maximal ideal at point x of X. Corollary 4.1.4 **(ii)** shows that the extended ideal $\mathfrak{m}_x B$ isn't the unit ideal of B, so $\mathfrak{m}_x B$ is contained in a maximal ideal $\mathfrak{m}_y$ of B, where y is a point of Y. Then $\mathfrak{m}_y \cap A$ contains $\mathfrak{m}_x$ and it isn't the unit ideal because it doesn't contain 1. So $\mathfrak{m}_y \cap A = \mathfrak{m}_x$, and x is the image uy. Therefore u is surjective. □

grpii **4.2.6. Example.** Let G be a finite group of automorphisms of a normal, finite-type domain B, let A be the algebra B^G of invariant elements of B. According to Theorem 2.8.5, A is a finite-type domain, B is a finite integral extension of A, and points of $X = \operatorname{Spec} A$ correspond to G-orbits of points of $Y = \operatorname{Spec} B$. □

The next example is helpful for an intuitive understanding of the geometric meaning of integrality.

integralovercx **4.2.7. Example.** Let f be an irreducible polynomial in $\mathbb{C}[x, y]$ (one x and one y), let $A = \mathbb{C}[x]$, and let $B = \mathbb{C}[x, y]/(f)$. So $X = \operatorname{Spec} A$ is an affine line and $Y = \operatorname{Spec} B$ is a plane affine curve. The canonical map $A \to B$ defines the morphism $Y \to X$ that is obtained by restricting the projection $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_x$ to Y.

We write f as a polynomial in y, whose coefficients are polynomials in x, say

fxy (4.2.8)
$$f(x, y) = a_0 y^n + a_1 y^{n-1} + \cdots + a_n$$

with $a_i = a_i(x)$. Let x_0 be a point of X. The fibre of the map $Y \to X$ over x_0 consists of the points (x_0, y_0) such that y_0 is a root of the one-variable polynomial $f(x_0, y)$.

Let $\delta(x)$ be the discriminant of $f(x, y)$, viewed as a polynomial in y. This discriminant isn't identically zero because f is irreducible (1.7.22). For all but finitely many values x_0 of x, both a_0 and δ will be nonzero. Then $f(x_0, y)$ will have n distinct roots, and the fibre of Y over x_0 will have order n.

When $f(x, y)$ is a monic polynomial in y, the morphism $Y \to X$ will be an integral morphism. If so, the leading term y^n of f will be the dominant term, when y is large. For x_1 near to a point x_0 of X, there will be a positive real number N such that

$$|y^n| > |a_1(x_1) y^{n-1} + \cdots + a_n(x_1)|$$

when $|y| > N$, and therefore $f(x_1, y) \neq 0$. So the roots y of $f(x_1, y)$ are bounded by N for all x_1 near to x_0.

On the other hand, when the leading coefficient $a_0(x)$ isn't a constant, B won't be integral over A. If x_0 is a root of $a_0(x)$, $f(x_0, y)$ will have degree less than n. What happens there is that, for points x_1 near to x_0, some roots of $f(x_1, y)$ are unbounded. In calculus, one says that the locus $f(x, y) = 0$ has a vertical asymptote at x_0.

To see this, we divide f by its leading coefficient. Let $g(x, y) = f(x, y)/a_0 = y^n + c_1 y^{n-1} + \cdots + c_n$ with $c_i(x) = a_i(x)/a_0(x)$. For any x at which $a_0(x)$ isn't zero, the roots of g are the same as those of f. However, let x_0 be a root of a_0. Because f is irreducible. At least one coefficient $a_j(x)$ doesn't have x_0 as a root. Then $c_j(x)$ is unbounded near x_0, and because the coefficient c_j is a symmetric function in the roots, the roots are not all bounded.

This is the general picture: The roots of a polynomial remain bounded near points at which the leading coefficient isn't zero, but some roots are unbounded near to a point at which the leading coefficient is zero. □

4.2.9. Noether Normalization Theorem. Let A be a finite-type domain over an infinite field k. There exist elements $y_1, \dots, y_n$ in A that are algebraically independent over k, such that A is a finite module over the polynomial subalgebra $R = k[y_1, \dots, y_n]$, i.e., such that A is an integral extension of R. noether-normal

When $k = \mathbb{C}$, the theorem can be stated by saying that every affine variety X admits an integral morphism to an affine space. (It is trivial that an affine variety admits a finite morphism to affine space, because its embedding into affine space is a finite morphism.)

The Noether Normalization Theorem remains true when A is a finite-type algebra over a finite field, though the proof given below needs to be modified.

4.2.10. Lemma. *Let k be an infinite field, and let $f(x)$ be a nonzero polynomial of degree d in $x_1, \dots, x_n$, with coefficients in k. After a suitable linear change of variable and scaling, f will be a monic polynomial in x_n.* *nonzero-coeff*

proof. Let f_d be the homogeneous part of f of maximal degree d. We regard f_d as a function $k^n \to k$. Since k is infinite, that function isn't identically zero. We choose coordinates $x_1, ..., x_n$ so that the point $q = (0, ..., 0, 1)$ isn't a zero of f_d. Then $f_d(0, ..., 0, x_n) = cx_n^d$, and the coefficient c, which is $f_d(0, ..., 0, 1)$, will be nonzero. We can multiply by c^{-1} to make f monic. $\square$

proof of the Noether Normalization Theorem. Say that the finite-type domain A is generated by elements $x_1, \dots, x_n$. If those elements are algebraically independent over k, A will be isomorphic to the polynomial algebra $\mathbb{C}[x]$. In that case we let $R = A$. If $x_1, ..., x_n$ aren't algebraically independent, they satisfy a polynomial relation $f(x) = 0$ of some positive degree d, with coefficients in k. The lemma tells us that, after a suitable change of variable and scaling, the coefficient of x_n^d in f will be 1. Then f will be a monic polynomial in x_n with coefficients in the subalgebra B of A generated by $x_1, \dots, x_{n-1}$. So x_n will be integral over B, and A will be a finite B-module. By induction on n, we may assume that B is a finite module over a polynomial subalgebra R. Then A will be a finite module over R too. $\square$

The next corollary is an example of the general principle, as has been noted before, that in any localization, a construction involving finitely many denominators can be done in a simple localization.

4.2.11. Corollary. *Let $A \subset B$ be finite-type domains. There is a nonzero element s in A such that B_s is a finite module over a polynomial subring $A_s[y_1, ..., y_r]$.* *localntheorem*

proof. Let S be the multiplicative system of nonzero elements of A, so that $K = AS^{-1}$ is the fraction field of A, and let $B_K = BS^{-1}$ be the ring obtained from B by adjoining inverses of all elements of S. Also, let $\beta = (\beta_1, ..., \beta_k)$ be a set of algebra generators for the finite-type algebra B. Then as K-algebra, B_K is generated by β. It is a finite-type K-algebra. The Noether Normalization Theorem tells us that B_K is a finite module over a polynomial subring $R_K = K[y_1, ..., y_r]$. So B_K is an integral extension of R_K. An element of B will be in B_K. Therefore it will be the root of a monic polynomial, say

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$$

where the coefficients $c_j(y)$ are elements of R_K. Each coefficient c_j is a combination of finitely many monomials in y, with coefficients in K. If $d \in A$ is a common denominator of those coefficients, $c_j(x)$ will have coefficients in $A_d[y]$. Since the generators $\beta_1, ..., \beta_k$ of B are integral over R_K, we may choose a single denominator s so that they are all integral over $A_s[y]$. The algebra B_s is generated over A_s by β, so B_s will be an integral extension of $A_s[y]$. $\square$

4.3 Normalization

finint

Let A be a domain with fraction field K. The *normalization* $A^\#$ of A is the set of elements of K that are integral over A. The normalization is a domain that contains A (4.2.1) **(ii)**.

A domain A is *normal* if it is equal to its normalization, and a *normal variety* X is a variety that has an *affine covering* $\{X^i = \operatorname{Spec} A_i\}$, a covering by affine open sets, in which the algebras A_i are normal domains.

To justify the definition of normal variety, we need to show that if an affine variety $X = \operatorname{Spec} A$ has an affine covering $\{X^i = \operatorname{Spec} A_i\}$, in which A_i are normal domains, then A is a normal domain. This follows from Lemma 4.3.4 **(iii)** below.

Our goal here is the next theorem, whose proof is at the end of the section.

normalfinite **4.3.1. Theorem.** *Let A be a finite-type domain with fraction field K of characteristic zero. The normalization $A^\#$ of A is a finite A-module and a finite-type domain.*

Thus there is an integral morphism $\operatorname{Spec} A^\# \to \operatorname{Spec} A$.

The theorem is true also when the characteristic of K isn't zero.

Asharp-modA **4.3.2. Corollary.** *With notation as above, there is a nonzero element s in A such that $sA^\# \subset A$.*

proof. We assume that the theorem has been proved. Since A and $A^\#$ have the same fraction field, every element α of $A^\#$ can be written as a fraction $\alpha = a/s$ with a, s in A, and then $s\alpha$ is in A. Since $A^\#$ is a finite A-module, one can find a nonzero element s in A such that $s\alpha$ is in A for all α in $A^\#$. Then $sA^\# \subset A$. $\square$

nodecurve **4.3.3. Example.** *(normalization of a nodal cubic curve)* The algebra $A = \mathbb{C}[u,v]/(v^2-u^3-u^2)$ can be embedded into the one-variable polynomial algebra $B = \mathbb{C}[x]$, by $u = x^2-1$ and $v = x^3-x$. The fraction fields of A and B are equal because $x = v/u$, and the equation $x^2 = u+1$ shows that x is integral over A. The algebra B is normal, so it is the normalization of A.

The plane curve $C = \operatorname{Spec} A$ has a node at the origin $p = (0,0)$, and $\operatorname{Spec} B$ is the affine line $\mathbb{A}^1$. The inclusion $A \subset B$ defines an integral morphism $\mathbb{A}^1 \to C$ whose fibre over p is the point pair $x = \pm 1$, and that is bijective at all other points. I think of C as the variety obtained by gluing the points $x = \pm 1$ of the affine line together.

In this example, the effect of normalization can be visualized geometrically. This is fairly unusual, because normalization is an algebraic process. Its effect on geometry may be subtle. $\square$

ufdnormal **4.3.4. Lemma.** **(i)** *A unique factorization domain is normal. In particular, a polynomial algebra over a field is normal.*

(ii) *A localization A_s of a normal domain A is normal.*

(iii) *Let $s_1, ..., s_k$ be nonzero elements of a domain A that generate the unit ideal. If the localizations A_{s_i} are normal for all i, then A is normal.*

proof. **(i)** Let A be a unique factorization domain, and let β be an element of its fraction field that is integral over A. Say that

eqntwo (4.3.5)
$$\beta^n + a_1\beta^{n-1} + \cdots + a_{n-1}\beta + a_n = 0$$

with a_i in A. We write $\beta = r/s$, where r and s are relatively prime elements of A. Multiplying by s^n gives us the equation

$$r^n = -s\,(a_1 r^{n-1} + \cdots + a_n s^{n-1})$$

This equation shows that if a prime element of A divides s, it also divides r. Since r and s are relatively prime, there is no such prime element. So s is a unit, and β is in A.

(ii) Let β be an element of the fraction field of A that is integral over A_s. There will be a polynomial relation of the form (4.3.5), where the coefficients a_i are elements of A_s. The element $\gamma = s^k\beta$ is a root of the polynomial

$$\gamma^n + (s^k a_1)\gamma^{n-1} + (s^{2k}a_2)\gamma^{n-2} + \cdots + +(s^{nk}a_n) = 0$$

Since a_i are in A_s, all coefficients in this polynomial will be in A when k is sufficiently large, and then γ will be integral over A. Since A is normal, γ will be in A, and $\beta = s^{-k}\gamma$ will be in A_s.

(iii) This proof follows a common pattern. Suppose that A_{s_i} is normal for every i. If an element β of K is integral over A, it will be in A_{s_i} for all i, and $s_i^n\beta$ will be an element of A, when n is large. We can use the same exponent n for all i. Since $s_1, ..., s_k$ generate the unit ideal, so do their powers $s_i^n, ..., s_k^n$. Say that $\sum r_i s_i^n = 1$, with r_i in A. Then $\beta = \sum r_i s_i^n \beta$ is in A. $\square$

We prove Theorem 4.3.1 in a slightly more general form. Let A be a finite type domain with fraction field K, and let L be a finite field extension of K. The *integral closure* of A in L is the set of all elements of L that are integral over A.

intclo **4.3.6. Theorem.** *Let A be a finite type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. The integral closure of A in L is a finite A-module.*

The proof that we give at the end of the section makes use of the characteristic zero hypothesis, though the theorem is true for a finite-type algebra over any field k.

4.3.7. Lemma. *Let A be a normal noetherian domain with fraction field K of characteristic zero, and let L be an algebraic field extension of K. An element β of L is integral over A if and only if the monic irreducible polynomial f for β over K has coefficients in A.* *about-tracetwo*

proof. If the monic polynomial f has coefficients in A, then β is integral over A. Suppose that β is integral over A. We may replace L by any field extension that contains β. So we may replace L by $K[\beta]$. Then L becomes a finite extension of K, which embeds into a Galois extension. So we may replace L by a Galois extension. Let G be its Galois group, and let $\{\beta_1, \ldots, \beta_r\}$ be the G-orbit of β, with $\beta = \beta_1$. Then the irreducible polynomial for β over K is

(4.3.8) $$f(x) = (x - \beta_1) \cdots (x - \beta_r)$$ orbitpoly

If β is integral over A, all elements of the orbit are integral over A. Therefore the coefficients of f, which are elemetary symmetric functions in the orbit, are integral over A, and since A is normal, they are in A. So f has coefficients in A. $\square$

4.3.9. Example. A nonconstant polynomial $f(x, y)$ in the polynomial ring $A = \mathbb{C}[x, y]$ is said to be *square-free* if it has no nonconstant square factors. dplanenormal

Let f be an irreducible, square-free polynomial, and let B denote the integral extension $\mathbb{C}[x, y, w]/(w^2 - f)$ of A. Let K and L be the fraction fields of A and B, respectively. Then L is a Galois extension of K. Its Galois group is generated by the automorphism σ of order 2 that is defined by $\sigma(w) = -w$. The elements of L have the form $\beta = a + bw$ with a, b in K, and $\sigma(\beta) = \beta' = a - bw$.

We show that B is the integral closure of A in L. Suppose that $\beta = a + bw$ is integral over A, with a, b in K. If $b = 0$, then $\beta = a$. This is an element of A and therefore it is in B. If $b \neq 0$, the irreducible polynomial for β will be

$$(x - \beta)(x - \beta') = x^2 - 2ax + (a^2 - b^2 f)$$

Because β is integral over A, $2a$ and $a^2 - b^2 f$ will be in A, and because the characteristic isn't 2, this is true if and only if a and $b^2 f$ are in A. We write $b = u/v$, with u, v relatively prime elements of A, so $b^2 f = u^2 f / v^2$. If v weren't a constant, then since u and v are relatively prime and f is square-free, v^2 couldn't be canceled from $u^2 f$. So $b^2 f$ wouldn't be in A. From $b^2 f$ in A we can conclude that v is a constant and that b is in A. Summing up, β is integral if and only if a and b are in A, which means that β is in B. $\square$

(4.3.10) trace deftrace

Let L be a finite field extension of a field K and let β be an element of K. When L is viewed as a K-vector space, multiplication by β becomes a K-linear operator $L \to L$. The *trace* of that operator will be denoted by $\mathrm{trace}(\beta)$. The trace is a K-linear map $L \to K$.

4.3.11. Lemma. *Let L/K be a field extension of degree n, let $K' = K[\beta]$ be the extension of K generated by an element β of L, and let $f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$ be the irreducible polynomial of β over K. Say that $[L\!:\!K'] = d$, so that $n = rd$. Then $\mathrm{trace}(\beta) = -da_1$. If β is an element of K, then $\mathrm{trace}(\beta) = n\beta$.* *about-traceone*

proof. The set $(1, \beta, \ldots, \beta^{r-1})$ is a K-basis for K'. On this basis, the matrix of multiplication by β has the form illustrated below for the case $r = 3$. Its trace is $-a_1$.

$$M = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix}.$$

Next, if $(u_1, \ldots, u_d)$ is a basis for L over K', the set $\{\beta^i u_j\}$, with $i = 0, \ldots, r-1$ and $j = 1, \ldots, d$, will be a basis for L over K. When this basis is listed in the order

$$(u_1, u_1\beta, ..., u_1\beta^{n-1}; u_2, u_2\beta, \ldots u_2\beta^{n-1};\ \ldots\ ; u_d, u_d\beta, ..., u_d\beta^{n-1}),$$

the matrix of multiplication by β will be made up of d blocks of the matrix M. $\square$

traceinA **4.3.12. Corollary.** *Let A be a normal domain with fraction field K and let L be a finite field extension of K. If an element β of L is integral over A, its trace is in A.*

This follows from Lemmas 4.3.7 and 4.3.11. □

formnondeg **4.3.13. Lemma.** *Let A be a normal noetherian domain with fraction field K* of characteristic zero, *and let L be a finite field extension of K. The form $L\times L \to K$ defined by $\langle\alpha,\beta\rangle = \mathrm{trace}(\alpha\beta)$ is K-bilinear, symmetric, and nondegenerate. If α and β are integral over A, then $\langle\alpha,\beta\rangle$ is an element of A.*

proof. The form is obviously symmetric, and it is K-bilinear because multiplication is K-bilinear and trace is K-linear. A form is nondegenerate if its nullspace is zero, which means that when α is a nonzero element, there is an element β such that $\langle\alpha,\beta\rangle \neq 0$. Given $\alpha \neq 0$, let $\beta = \alpha^{-1}$. Then $\langle\alpha,\beta\rangle = \mathrm{trace}(1)$, which, according to (4.3.11), is the degree $[L:K]$ of the field extension. It is here that the hypothesis on the characteristic of K enters: The degree is a nonzero element of K.

If α and β are integral over A, so is their product $\alpha\beta$ (4.2.1) **(ii)**. Corollary 4.3.12 shows that $\langle\alpha,\beta\rangle$ is an element of A. □

clearde-nom **4.3.14. Lemma.** *Let A be a domain with fraction field K, let L be a field extension of K, and let β be an element of L that is algebraic over K. If β is a root of a polynomial $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with a_i in A, then $\gamma = a_n\beta$ is integral over A.*

proof. One finds a monic polynomial with root γ by substituting $x = y/a_n$ into f and multiplying by a_n^{n-1}. □

normfin-proof (**4.3.15**) **proof of Theorem 4.3.1**

Let A be a finite-type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. We are to show that the integral closure of A in L is a finite A-module.

Step 1. We may assume that A is normal.

We use the Noether Normalization Theorem to write A as a finite module over a polynomial subalgebra $R = \mathbb{C}[y_1,\dots,y_d]$. Let F be the fraction field of R. Then K and L are finite extensions of F. An element of L will be integral over A if and only if it is integral over R (4.2.1) **(iv)**. So the integral closure of A in L is the same as the integral closure of R in L. We replace A by the normal algebra R and K by F.

Step 2. Bounding the integral extension.

We assume that A is normal. Let $(v_1,\dots,v_n)$ be a K-basis for L whose elements are integral over A. Such a basis exists because we can multiply any element of L by a nonzero element of K to make it integral (Lemma 4.3.14). Let

mapvector (4.3.16)
$$T : L \to K^n$$

be the map defined by $T(\beta) = \big(\langle v_1,\beta\rangle,\dots,\langle v_n,\beta\rangle\big)$, where $\langle\ ,\ \rangle$ is the bilinear form defined in Lemma 4.3.13. This map is K-linear. If $\langle v_i,\beta\rangle = 0$ for all i, then because $(v_1,\dots,v_n)$ is a basis for L, $\langle\gamma,\beta\rangle = 0$ for every γ in L, and since the form is nondegenerate, $\beta = 0$. Therefore T is injective.

Let B be the integral closure of A in L. We are to show that B is a finite A-moule. The basis elements v_i are in B, and if β is in B, then $v_i\beta$ will be in B too. Then $\langle v_i,\beta\rangle = \mathrm{trace}(v_i b)$ will be in A, and $T(\beta)$ will be in A^n (4.3.13). When we restrict T to B, we obtain an injective map $B \to A^n$ that we denote by T_0. Since T is K-linear, T_0 is A-linear. It is an injective homomorphism of A-modules that maps B isomorphically to its image, a submodule of A^n. Since A is noetherian, every submodule of the finite A-module A^n is a finite module. Therefore the image of T_0 is a finite A-module, and so is the isomorphic module B. □

4.4 Geometry of Integral Morphisms

prmint The main geometric properties of an integral morphism of affine varieties are summarized by the theorems in this section, which show that the geometry is as nice as could be expected.

Let $Y \to X$ be an integral morphism of affine varieties. We say that a closed subvariety D of Y *lies over* a closed subvariety C of X if C is the image of D.

Similarly, if $A \to B$ is an integral extension of finite-type domains, we say that a prime ideal Q of B *lies over* a prime ideal P of A if P is the contraction $Q \cap A$. For example, if $Y \to X$ is the morphism of affine varieties that corresponds to a homomorphism $A \to B$, and if a point y of Y has image x in X, then y lies over x, and the maximal ideal $\mathfrak{m}_y$ lies over the maximal ideal $\mathfrak{m}_x$.

4.4.1. Lemma. *Let $A \subset B$ be an integral extension of finite-type domains, and let J be an ideal of B. If J isn't the zero ideal of B, then its contraction $J \cap A$ isn't the zero ideal of A.* contrzero

proof. An element β of J will be a root of a monic polynomial with coefficients in A, say $\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_0$. If $a_0 = 0$, then since B is a domain, we can cancel β from this polynomial. So we may assume that $a_0 \neq 0$. The equation shows that a_0 is in J as well as in A. $\square$

4.4.2. Proposition. *Let $A \to B$ be an integral extension of finite-type domains, and let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$.* QoverP

(i) *Let P and Q be prime ideals of A and B, respectively, let C be the locus of zeros of P in X, and let D be the locus of zeros of Q in Y. Then Q lies over P if and only if D lies over C.*

(ii) *Let Q and Q' be prime ideals of B that lie over the same prime ideal P of A. If $Q \subset Q'$, then $Q = Q'$. Therefore, if D' and D are closed subvarieties of Y that lie over the same subvariety C of X and if $D' \subset D$, then $D' = D$.*

proof. **(i)** Let $\overline{A} = A/P$ and $\overline{B} = B/Q$. Then $D = \operatorname{Spec} \overline{B}$ and $C = \operatorname{Spec} \overline{A}$. Suppose that Q lies over P. So $P = Q \cap A$. Then the canonical map $\overline{A} \to \overline{B}$ will be injective, and $\overline{B}$ will be generated as $\overline{A}$-module by the residues of a set of generators of the finite A-module B. So $\overline{B}$ is an integral extension of $\overline{A}$, and the map from D to C is surjective (Proposition 4.2.5), which means that D lies over C. Conversely, if D lies over C, the morphism $D \to C$ is surjective. Then the canonical map $\overline{A} \to \overline{B}$ is injective, and this implies that $Q \cap A = P$.

(ii) Suppose that Q and Q' lie over P and that $Q \subset Q'$. With $\overline{A} = A/P$ and $\overline{B} = B/Q$ as before, let $\overline{Q}' = Q'/Q$. Because B is an integral extension of A, $\overline{B}$ is an integral extension of $\overline{A}$, and $\overline{Q}'$ is an ideal of $\overline{B}$. Since Q and Q' lie over P, $Q \cap A = Q' \cap A = P$. We show that $\overline{Q}' \cap \overline{A} = 0$. Let $\underline{x}$ be an element of $\overline{Q}' \cap \overline{A}$, and let $x \in Q'$ and $z \in A$ be elements whose residues in $\overline{Q}'$ are equal to $\underline{x}$. Then the residue of $x - z$ is zero, so $x - z$ is in Q and in Q'. Therefore z is in $Q' \cap A = P$, and $\underline{x} = 0$. So $\overline{Q}' \cap \overline{A} = 0$. Lemma 4.4.1 tells us that $\overline{Q}' = 0$. Therefore $Q' = Q$. $\square$

4.4.3. Theorem. *Let $Y \xrightarrow{u} X$ be an integral morphism of affine varieties.* closedim-

(i) *The fibres of u have bounded cardinality.* age

(ii) *The image of a closed subset of Y is a closed subset of X, and the image of a closed subvariety of Y is a closed subvariety of X.*

(iii) *The set of closed subvarieties of Y that lie over a closed subvariety C of X is finite and nonempty.*

proof. Let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, and let $A \subset B$ be the extension that corresponds to the integral morphism u.

(i) *(bounding the fibres)* Let $y_1, ..., y_r$ be points of Y in the fibre over a point x of X. For each i, the contraction of the maximal ideal $\mathfrak{m}_{y_i}$ of B at y_i is the maximal ideal $\mathfrak{m}_x$ of A at x. To bound the number r, we use the Chinese Remainder Theorem to show that B cannot be spanned as A-module by fewer than r elements.

Let k_i and k denote the residue fields $B/\mathfrak{m}_{y_i}$, and $A/\mathfrak{m}_x$, respectively, all of these fields being isomorphic to $\mathbb{C}$. Let $\overline{B} = k_1 \times \cdots \times k_r$. We form a diagram of algebra homomorphisms

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & \overline{B} \approx \mathbb{C}^r \\ \uparrow & & \uparrow \\ A & \longrightarrow & k \approx \mathbb{C} \end{array}$$

which we interpret as a diagram of A-modules. The minimal number of generators of the A-module $\overline{B}$ is equal to its dimension as k-module, which is r. The Chinese Remainder Theorem asserts that φ is surjective, so B cannot be spanned by fewer than r elements.

(ii) (*the image of a closed set is closed*) The image of an irreducible set via a continuous map is irreducible (2.2.15)(**iii**), so it suffices to show that the image of a closed subvariety is closed. Let D be the closed subvariety of Y that corresponds to a prime ideal Q of B, and let $P = Q \cap A$ be its contraction, which is a prime ideal of A. Let C be the variety of zeros of P in X. The coordinate algebras of the affine varieties D and C are $\overline{B} = B/Q$ and $\overline{A} = A/P$, respectively, and $\overline{B}$ is an integral extension of $\overline{A}$ because B is an integral extension of A (4.2.5). The map $D \to C$ is surjective. Therefore C is the image of D.

(iii) (*subvarieties that lie over a closed subvariety*) Let C be a closed subvariety of X. Its inverse image $Z = u^{-1}C$ is closed in Y. It is the union of finitely many irreducible closed sets, say $Z = D'_1 \cup \cdots \cup D'_k$. Part **(i)** tells us that the image C'_i of D'_i is a closed subvariety of X. Since u is surjective, $C = \bigcup C'_i$, and since C is irreducible, $C'_i = C$ for at least one i. For such an i, D'_i lies over C. Next, any subvariety D that lies over C will be contained in the inverse image Z, and therefore contained in D'_i for some i. Proposition 4.4.2 **(ii)** shows that $D = D'_i$. Therefore the varieties that lie over C are among the varieties D'_i. □

4.5 Dimension

dim Every variety has a dimension, and as is true for the dimension of a vector space, the dimension is important, though it is a very coarse measure. We give two definitions of dimension of a variety X here. However, the proof that they are equivalent requires work.

The first definition is that the dimension of a variety X is the transcendence degree of its function field. For now, we'll refer to this as the *t-dimension* of X.

intdim **4.5.1. Corollary.** *Let $Y \to X$ be an integral morphism of affine varieties. The t-dimensions of X and Y are equal.* □

The second definition of dimension is the combinatorial dimension, which is defined as follows: A *chain* of closed subvarieties of a variety X is a strictly decreasing sequence

chntwo (4.5.2)
$$C_0 > C_1 > C_2 > \cdots > C_k$$

of closed subvarieties. The *length* of this chain is defined to be k. The chain is *maximal* if it cannot be lengthened by inserting another closed subvariety, which means that $C_0 = X$, that there is no closed subvariety $\widetilde{C}$ with $C_i > \widetilde{C} > C_{i+1}$ for $i < k$, and that C_k is a point.

For example, $\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$, where $\mathbb{P}^i$ is the linear subspace of points $(x_0, ..., x_i, 0, ..., 0)$, is a maximal chain in projective space $X = \mathbb{P}^n$, and its length is n.

Theorem 4.5.6 below shows that all maximal chains of closed subvarieties have the same length. The *combinatorial dimension* of X is the length of a maximal chain. We'll refer to it as the *c-dimenson*. Theorem 4.5.6 also shows that the t-dimension and the c-dimension of a variety are equal. When we have proved that theorem, we will refer to the t-dimension and to the c-dimension simply as the *dimension*, and we will use the two definitions interchangeably.

In an affine variety $\operatorname{Spec} A$, a strictly decreasing chain (4.5.2) of closed subvarieties corresponds to a strictly increasing chain

chn (**4.5.3**)
$$P_0 < P_1 < P_2 < \cdots < P_k,$$

of prime ideals of A of length k, a *prime chain*. This prime chain is *maximal* if it cannot be lengthened by inserting another prime ideal, which means that P_0 is the zero ideal, that there is no prime ideal $\widetilde{P}$ with $P_i < \widetilde{P} < P_{i+1}$ for $i < k$, and that P_k is a maximal ideal. The *c-dimension* of a finite-type domain A is the length k of a maximal chain (**4.5.3**) of prime ideals. If $X = \operatorname{Spec} A$, then the c-dimensions of X and of A are equal.

The next theorem is the basic tool for studying dimension. Though the statement is intuitively plausible, its proof isn't easy. It is a subtle theorem. We have put the proof at the end of this section.

krullthm **4.5.4. Krull's Principal Ideal Theorem.** Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let V be the zero locus in X of a nonzero element α of A. Every irreducible component of V has t-dimension $d-1$.

4.5.5. Corollary. *Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let C be a component of the zero locus of a nonzero element α of A. Then among proper closed subvarieties, C maximal. There is no closed subvariety D such that $C < D < X$.* *compmax*

proof. Let $C < D < X$ be closed subvarieties of $X = \operatorname{Spec} A$. Some nonzero element β of A will vanish on D. Then D will be a subvariety of the zero locus of β, so by Krull's Theorem, its t-dimension will be at most $d-1$. Similarly, if $D = \operatorname{Spec} B$, some nonzero element of B will vanish on C, so the t-dimension of C will be at most $d-2$, and C isn't the zero locus of a nonzero element of A. □

4.5.6. Theorem. *Let X be a variety of t-dimension d. All chains of closed subvarieties of X have length at most d, and all maximal chains have length d. Therefore the c-dimension and the t-dimension of X are equal.* *dimtheorem*

proof. We do the case that X is affine. Induction allows us to assume that the theorem is true for an affine variety whose t-dimension is less than d. Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let $C_0 > C_1 > \cdots > C_k$ be a chain of closed subvarieties of X. We must show that $k \le d$ and that $k = d$ if the chain is maximal. We may insert closed subvarieties into the chain where possible, so we may assume that $C_0 = X$. Next, C_1, being a proper closed subset of X, is contained in the zero locus Z of a nonzero element α of A, and it will be contained in an irreducible component $\widetilde{C}$ of Z. If $\widetilde{C} > C_1$, we insert $\widetilde{C}$ into the chain, to reduce ourselves to the case that C_1 is a component of the zero locus of α. By Krull's Theorem, C_1 has t-dimension $d-1$. By Corollary 4.5.5 C_1 is a maximal proper closed subvariety, and induction applies to the chain $C_1 > \cdots > C_k$ of closed subvarieties of C_1. The length of that chain, which is $k-1$ is less than $d-1$, and it is equal to $d-1$ if the chain is maximal. Therefore the chain $\{C_i\}$ has length at most n, and it has length n if it is a maximal chain.

Theorem 4.5.6 for an arbitrary variety follows from the next lemma. □

4.5.7. Lemma. *Let X' be an open subvariety of a variety X. There is a bijective correspondence between chains $C_0 > \cdots > C_k$ of closed subvarieties of X such that $C_k \cap X' \neq \emptyset$ and chains $C'_0 > \cdots > C'_k$ of closed subvarieties of X'. Given the chain $\{C_i\}$ in X, the chain $\{C'_i\}$ in X' is defined by $C'_i = C_i \cap X'$. Given a chain C'_i in X', the corresponding chain in X consists of the closures C_i in X of the varieties C'_i.* *restrictchain*

proof. Suppose given a chain C_i and that $C_k \cap X' \neq \emptyset$. Then for every i, the intersection $C'_i = C_i \cap X'$ is a dense open subset of the irreducible closed set C_i (2.2.13). So the closure of C'_i is C_i, and since $C_i > C_{i+1}$, it is also true that $C'_i > C'_{i+1}$. Therefore $C'_0 > \cdots > C'_k$ is a chain of closed subsets of X'. Conversely, if $C'_0 > \cdots > C'_k$ is a chain in X', the closures in X form a chain in X. (See 2.2.14).) □

From now on, we use the word *dimension* to denote either the t-dimension or the c-dimension, and we denote the dimension of a variety by $\dim X$.

4.5.8. Examples. (i) The polynomial algebra $\mathbb{C}[x_0, \dots, x_n]$ in $n{+}1$ variables has dimension $n{+}1$. The chain of prime ideals dimpolyalg

(4.5.9) $$0 < (x_0) < (x_0, x_1) < \cdots < (x_0, \dots, x_n)$$ primechain

is a maximal prime chain. When the irrelevant ideal $(x_0, ..., x_n)$ is removed from this chain, it corresponds to a maximal chain

$$\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$$

of closed subvarieties of projective space $\mathbb{P}^n$, which has dimension n.

(ii) The maximal chains of closed subvarieties of $\mathbb{P}^2$ have the form $\mathbb{P}^2 > C > p$, where C is a plane curve and p is a point. □

If (4.5.2) is a maximal chain in X, then

(4.5.10) $$C_1 > C_2 > \cdots > C_k$$ chaini

will be a maximal chain in the variety C_1. So when X has dimension k, the dimension of C_1 will be $k-1$. Similarly, let $P_0 < P_1 < \cdots < P_k$ be a maximal chain of prime ideals in a finite-type domain A, let $\overline{A} = A/P_1$ and let $\overline{P}_j$ denote the image P_j/P_1 of P_j in $\overline{A}$, for $j \ge 1$. Then

$$\overline{0} = \overline{P}_1 < \overline{P}_2 < \cdots < \overline{P}_k$$

will be a maximal prime chain in $\overline{A}$, and therefore the dimension of the domain $\overline{A}$ will be $k-1$.

integraldimequal **4.5.11. Corollary.** *Let X be a variety.*
(i) *If X' is an open subvariety of a X, then* $\dim X' = \dim X$.
(ii) *If $Y \to X$ is an integral morphism of varieties, then* $\dim Y = \dim X$.
(iii) *If Y is a proper closed subvariety of X, then* $\dim Y < \dim X$. □

One more term: A closed subvariety C of a variety X has *codimension* 1 if $C < X$ and if $\dim C = \dim X - 1$. If so, there is no closed set $\widetilde{C}$ with $C < \widetilde{C} < X$. A prime ideal P of a noetherian domain has *codimension* 1 if it isn't the zero ideal, and if there is no prime ideal $\widetilde{P}$ with $(0) < \widetilde{P} < P$.

In the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$, the prime ideals of codimension 1 are the principal ideals generated by irreducible polynomials.

Krullproof **(4.5.12) proof of Krull's Theorem**

lemmaone **4.5.13. Lemma.** *Krull's Theorem is true when X is an affine space.*

proof. Here A is the polynomial ring $\mathbb{C}[x_1, ..., x_n]$. Let $\alpha_1 \cdots \alpha_k$ be the factorization of the polynomial α into irreducible polynomials, and let V_i be the zero locus of α_i. The irreducible factors generate prime ideals of A, so V_i are irreducible, and V is their union. We replace α by the irreducible factor α_1 whose zero locus is V_1, and we relabel that factor as $f(x)$. Then V becomes the zero locus $V(f)$, and the coordinate algebra of V is $\overline{A} = A/Af$. We may assume that f is monic in x_d (1.3.20). Then $\overline{A}$ is an integral extension of $\mathbb{C}[x_1, ..., x_{d-1}]$. Its t-dimension is $d-1$. □

lemmatwo **4.5.14. Lemma.** *To prove Krull's Theorem, it suffices to prove it when the coordinate ring A is normal and the zero locus of α is irreducible.*

proof. We are given an affine variety $X = \operatorname{Spec} A$ of t-dimension d, a nonzero element α of A, and an irreducible component C of the zero locus of α. We are to show that the t-dimension of C is $d - 1$.

Let $A^\#$ be the normalization of A and let $X^\# = \operatorname{Spec} A^\#$. There is an integral morphism $X^\# \to X$. The t-dimensions of $X^\#$ and X are the same. Let V' and V be the zero loci of α in $X^\#$ and in X, respectively. Then V' is the inverse image of V in $X^\#$. The map $V' \to V$ is surjective because the integral morphism $X^\# \to X$ is surjective.

Let $D_1, \cdots, D_k$ be the irreducible components of V', and let C_i be the image of D_i in X. The closed sets C_i are irreducible (4.4.3) **(ii)**, and their union is V. So at least one C_i is equal to the chosen component C. Let D be a component of V' whose image is C. The map $D \to C$ is also an integral morphism, so the t-dimensions of D and C are equal. We may therefore replace X and C by $X^\#$ and D, respectively. Hence we may assume that A is normal.

Next, suppose that the zero locus of α has the form $C \cup \Delta$, where C is the chosen irreducible component, and Δ is the union of the other components. We choose an element s of A that is identically zero on Δ but not identically zero on C. Inverting s eliminates all points of Δ, but $X_s \cap C = C_s$ won't be empty. If X is normal, so is X_s (4.3.4) **(ii)**. Since localization doesn't change t-dimension, we may replace X and C by X_s and C_s, respectively. □

We go to the proof of Krull's Theorem now. According to Lemma 4.5.14, we may assume that $X = \operatorname{Spec} A$ is a normal affine variety of dimension d, and that the zero locus of α is an irreducible closed set C. We are to prove that the t-dimension of C is $d - 1$.

We apply the Noether Normalization Theorem. Let $X \to S$ be an integral morphism to an affine space $S = \operatorname{Spec} R$ of dimension d, where R is a polynomial ring $\mathbb{C}[u_1, ..., u_d]$.

Let K and F be the function fields of X and S, respectively, and let $f(t)$ be the monic irreducible polynomial for α over F. The coefficients of f are in R. Let $\alpha_1, ..., \alpha_r$ be the roots of f in a splitting field L of f over K, with $\alpha_1 = \alpha$, let B be the integral closure of A in L, and let $Y = \operatorname{Spec} B$. Then Y is an integral extension of X and of S. We have morphisms

$$Y \xrightarrow{u} X \xrightarrow{v} S$$

Let w denote the composed morphism $Y \xrightarrow{vu} S$. The Galois group G of L/F operates on B and on Y. and S is the space Y/G of G-orbits.

The coefficients of $f(t) = (t-\alpha_1)\cdots(t-\alpha_r)$ are G-invariant. They are elements of R, and the constant term is the product $\alpha_1\cdots\alpha_r$. Let's denote that product by β.

4.5.15. Lemma. *Let Z be the zero locus of β in S. The morphism $X \xrightarrow{v} S$ maps C surjectively to Z.* *lemmathree*

$$\begin{tikzcd} & C \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

Assuming the lemma, Z will be irreducible because C is irreducible. Lemma 4.5.13 shows that the t-dimension of Z is $d-1$. Therefore the t-dimension of C is at least $d-1$, and since it is a closed subset of X, it is less than d. So the t-dimension of C is equal to $d-1$. $\square$

proof of Lemma 4.5.15 The element β of R defines functions on S, X, and Y, the functions on X and Y being obtained from the function on S by composition with the maps v and w, respectively. We denote all of those functions by β. If y is a point of Y, $x = uy$ and $s = wy$, then $\beta(y) = \beta(x) = \beta(s)$. Similarly, α defines functions on X and on Y that we denote by α: $\alpha(y) = \alpha(x)$.

Let x be a point of C. So $\alpha(x) = 0$. Since α divides β, $\beta(x) = 0$. If s is the image of x in S, then $\beta(s) = \beta(x)$, so $\beta(s) = 0$. This shows that s is a point of Z. Therefore Z contains the image of C.

For the other inclusion, let z be a point of Z. Then $\beta(z) = 0$. Let y be a point of Y such that $wy = z$. So $\beta(y) = 0$. The fibre of Y over z is the G-orbit of y, and since β is a function on S, it vanishes at every point of that orbit. Since $\beta = \alpha_1\cdots\alpha_k$, $\alpha_i(y) = 0$ for some i. Let σ be an element of G such that $\alpha_i = \sigma\alpha$. We recall that $[\sigma\alpha](y) = \alpha(y\sigma)$. So $\alpha(y\sigma) = 0$. We replace y by $y\sigma$. Then $\alpha(y) = 0$, and it is still true that $wy = z$. Let $x = uy$. Because $\alpha(y) = 0$, it is also true that $\alpha(x) = 0$. So x is a point of C. The image of x in S is $vx = wy = z$. Since z can be any point of Z, the map $C \to Z$ is surjective. $\square$

4.6 Chevalley's Finiteness Theorem

(4.6.1) finite morphisms finmorph prodagain

The concept of an integral morphism of affine varieties was defined in Section 4.2. A morphism $Y \xrightarrow{u} X$ of affine varieties $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ is a *finite morphism* if the homomorphism $A \xrightarrow{\varphi} B$ that corresponds to u makes B into a finite A-module. We extend these definitions to varieties that aren't necessarily affine here.

As has been noted, the difference between a finite morphism and an integral morphism of affine varieties is that for a finite morphism, the homomorphism φ needn't be injective. If u is a finite morphism and φ is injective, B will be an integral extension of A, and u will be an integral morphism.

By the *restriction* of a morphism $Y \xrightarrow{u} X$ to an open subset X' of X, we mean the induced morphism $Y' \to X'$, where Y' is the inverse image of X'.

4.6.2. Definition. A morphism of varieties $Y \xrightarrow{u} X$ is a *finite morphism* if X can be covered by affine open subsets X^i such that the restriction of u to each X^i is a finite morphism of affine varieties, as defined in (4.2.4). Similarly, a morphism u is an *integral morphism* if X can be covered by affine open sets X^i to which the restriction of u is an integral morphism of affine varieties. deffinmorph

4.6.3. Corollary. *An integral morphism is a finite morphism. The composition of finite morphisms is a finite morphism. The inclusion of a closed subvariety into a variety is a finite morphism.* $\square$ *finiteexamp*

When X is affine, Definitions 4.2.4 and 4.6.2 both apply. Proposition 4.6.5, which is below, shows that the two definitions are equivalent. Unfortunately, the proof is rather long. Such verifications are a cost of doing business with affine open sets.

4.6.4. Lemma. **(i)** *Let $A \xrightarrow{\varphi} B$ be a homomorphism of finite-type domains that makes B into a finite A-module, and let s be a nonzero element of A. Then B_s is a finite A_s-module.* *restfinmorph*
(ii) *Using Definition 4.6.2, the restriction of a finite (or an integral) morphism $Y \xrightarrow{u} X$ to an open subset of a variety X is a finite (or an integral) morphism.*

proof. **(i)** Here B_s denotes the localization of B as an A-module. This localization can also be obtained by localizing the algebra B with respect to the image $s' = \varphi(s)$, provided that s' isn't zero. If s' is zero, then s annihilates B, so $B_s = 0$. In either case, a set of elements that spans B as A-module will span B_s as A_s-module, so B_s is a finite A_s-module.

(ii) Say that X is covered by affine open sets to which the restriction of u is a finite morphism. The localizations of these open sets form a basis for the Zariski topology on X, so X' can be covered by such localizations. Part **(i)** shows that the restriction of u to X' is a finite morphism. □

onecoverfinite **4.6.5. Proposition.** *Let $Y \xrightarrow{u} X$ be a finite (or an integral) morphism, as defined in (4.6.2), and let X_1 be an affine open subset of X. The restriction of u to X_1 is a finite (or an integral) morphism of affine varieties, as defined in (4.2.4).*

The proof isn't difficult proof, but there are several things to check. We've put it at the end of the section.

Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$ with coordinates $y_0, ..., y_n$, and let X be a variety. For next theorem, we abbreviate the notation for a product of a variety V with X, writing

$$\widetilde{V} = V \times X$$

chevfin **4.6.6. Chevalley's Finiteness Theorem.** *Let X be a variety, let Y be a closed subvariety of the product $\widetilde{\mathbb{P}} = \mathbb{P} \times X$, and let π denote the projection $Y \to X$, respectively. If all fibres of π are finite sets, then π is a finite morphism.*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] & \widetilde{\mathbb{P}} \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

familyofvars **4.6.7. Example.** Let $A = \mathbb{C}[t]$, and let $X = \operatorname{Spec} A$. The zero locus of the polynomial $y_0^3 + y_1^3 + y_2^3 + ty_0y_1y_2 = 0$ in $\mathbb{P}^2 \times X$ can be regarded as a family of plane cubic curves, parametrized by X. □

projchevfin **4.6.8. Corollary.** *Let Y be a projective variety, and let $Y \xrightarrow{\pi} X$ be a morphism. If the fibres of π are finite sets, then π is a finite morphism. If Y is a projective curve, every nonconstant morphism $Y \xrightarrow{\pi} X$ is a finite morphism.*

proof. This follows from the theorem when one replaces Y by the graph of π in $\widetilde{Y} = Y \times X$. The graph is isomorphic to Y. If Y is a closed subvariety of $\mathbb{P}$, the graph will be a closed subvariety of $\widetilde{\mathbb{P}}$ (Proposition 3.5.24). When Y is a curve, the fibres of any nonconstant morphism $Y \to X$ will be finite sets. □

We need two lemmas for the proof of Chevalley's Theorem.

Let $y_0, ..., y_n$ be coordinates in $\mathbb{P} = \mathbb{P}^n$, and let $A[y_0, ..., y_n]$ be the algebra of polynomials in y with coefficients in A. In analogy for terminology used with complex polynomials, we say that a polynomial with coefficients in A is *homogeneous* if it is homogeneous as a polynomial in y, and that an ideal of $A[y]$ that can be generated by homogeneous polynomials is a *homogeneous ideal*.

hompolycoeffA **4.6.9. Lemma. (i)** *Let $X = \operatorname{Spec} A$ be an affine variety, and let Y be a nonempty subset of $\widetilde{\mathbb{P}} = \mathbb{P} \times X$. The ideal $\mathcal{I}$ of elements of $A[y]$ that vanish at every point of Y is a homogeneous radical ideal. If Y is a closed subvariety of $\widetilde{\mathbb{P}}$, then $\mathcal{I}$ is a prime ideal.*

(ii) *If the zero locus of a homogeneous ideal $\mathcal{I}$ of $A[y]$ is empty, then $\mathcal{I}$ contains a power of the irrelevant ideal $\mathcal{M} = (y_0, ..., y_n)$ of $A[y]$.*

proof. This is similar to Proposition 3.2.6 and 2.5.13. Let $y_0, ..., y_n$ be coordinates in $\mathbb{P}$, let $\mathbb{A}$ be the affine space of dimension $n+1$ with those coordinates, and let o bethe origin in $\mathbb{A}$. Let Z be the inverse image of Y in $\widetilde{\mathbb{A}} = \mathbb{A} \times X$, and let Z' be the complement of $\widetilde{o} = o \times X$ in Z. Because Y isn't empty, $\mathcal{I}$ is the ideal of all polynomials that vanish on Z', and also the ideal of all polynomials that vanish on Z. Proposition 2.5.13 shows that $\mathcal{I}$ is a prime ideal.

If the zero locus of $\mathcal{I}$ in $\mathbb{P} \times X$ is empty, the zero locus in $\widetilde{\mathbb{A}}$ will be contained in $\widetilde{o}$. The radical of $\mathcal{I}$ will contain the ideal of $\widetilde{o}$ in $A[y]$, which is the irrelevant ideal. □

4.6.10. Lemma. *Let A be a finite type domain, let I be an ideal of the polynomial algebra $A[u_1, ..., u_n]$, and let k be a positive integer. Suppose that, for each $i = 1, ..., n$, there is a polynomial $g_i(u_1, ..., u_n)$ of degree at most $k-1$, with coefficients in A, such that $u_i^k - g_i(u)$ is in I. Then $B = A[u]/I$ is a finite A-module.* *powersofyenuf*

proof. Let's denote the residue of u_i in B by the same symbol u_i. In B, we will have $u_i^k = g_i(u)$. Any monomial m of degree at least nk in $u_1, ..., u_n$ will be divisible by u_i^k for at least one i. Then in B, m is equal to a polynomial in $u_1, ..., u_n$ of degree less than d, with coefficients in A. It follows by induction that the monomials in $u_1, ..., u_n$ of degree at most $nk-1$ span B as an A-module. □

proof of Chevelley's Finiteness Theorem. This is Schelter's proof.

We are given a closed subvariety Y of $\widetilde{\mathbb{P}} = \mathbb{P} \times X$, with $\mathbb{P} = \mathbb{P}^n$, and the fibres of Y over X are finite sets. We are to prove that the projection $Y \to X$ is a finite morphism. By induction, we may assume that the theorem is true when $\mathbb{P}$ is a projective space of dimension $n-1$.

We may suppose that X is affine, say $X = \operatorname{Spec} A$ (see Definition 4.6.2).

Case 1. There is a hyperplane H in $\mathbb{P}$ such that Y is disjoint from $\widetilde{H} = H \times X$ in $\widetilde{\mathbb{P}}$.

This is the main case. We adjust coordinates $y_0, ..., y_n$ in $\mathbb{P}$ so that H is the hyperplane at infinity $\{y_0 = 0\}$. Because Y is disjoint from $\widetilde{H}$, it is a subset of the affine variety $\widetilde{\mathbb{U}}^0 = \mathbb{U}^0 \times X$, $\mathbb{U}^0$ being the standard open set $\{y_0 \neq 0\}$ in $\mathbb{P}$. Since Y is irreducible and closed in $\widetilde{\mathbb{P}}$, it is a closed subvariety of the affine variety $\widetilde{\mathbb{U}}^0$. So Y is affine.

Let $\mathcal{P}$ be the homogeneous prime ideal of $A[y]$ whose zero set in $\widetilde{\mathbb{P}}$ is Y. The ideal $\mathcal{Q}$ whose zero set is $\widetilde{H}$ is the principal ideal of $A[y]$generated by y_0. Let $\mathcal{I} = \mathcal{P} + \mathcal{Q}$. A homogeneous polynomial of degree k in $\mathcal{I}$ has the form $f(y) + y_0 g(y)$, where f is a homogeneous polynomial in $\mathcal{P}$ of degree k, and g is a homogeneous polynomial of degree $k-1$ in $A[y]$.

By hypothesis, $Y \cap \widetilde{H}$ is empty. Therefore the sum $\mathcal{I} = \mathcal{P} + \mathcal{Q}$ contains a power of the irrelevant ideal $\mathcal{M} = (y_0, ..., y_n)$ of $A[y]$. Say that $\mathcal{M}^k \subset \mathcal{I}$. Then y_i^k is in $\mathcal{I}$, for $i = 0, ..., n$. So there are polynomial equations

$$y_i^k = f_i(y) + y_0 g_i(y) \tag{4.6.11}$$ fplusg

with f_i in $\mathcal{P}$ homogeneous, of degree k and g_i in $A[y]$ homogeneous, of degree $k-1$.

Recall that Y is a closed subset of $\widetilde{\mathbb{U}}^0$. Its (nonhomogenous) ideal P in $A[u]$ can be obtained by dehomogenizing the ideal $\mathcal{P}$. We dehomogenize the equations (4.6.11). With $u_i = y_i/y_0$, let $F_i = f_i(1, u_1, ..., u_n)$ and $G_i = g_i(1, u_1, ..., u_n)$. Then $F_i = u_i^k - G_i$. The important points are that F_i is in the ideal P, and that the degree of G_i is at most $k-1$. Lemma 4.6.10 shows that $Y \to X$ is a finite morphism. This completes the proof of Case 1.

Case 2. the general case.

We have taken care of the case in which there exists a hyperplane H such that Y is disjoint from $\widetilde{H}$. The next lemma shows that we can cover the given variety X by open subsets to which this special case applies. Then Lemma 4.6.4 and Proposition 4.6.5 apply, to complete the proof. □

4.6.12. Lemma. *Let Y be a closed subvariety of $\mathbb{P}^n \times X$, and suppose that the projection $Y \xrightarrow{\pi} X$ has finite fibres. Suppose also that Chevalley's Theorem has been proved for closed subvarieties of $\mathbb{P}^{n-1} \times X$. For every point p of X, there is an open neighborhood X' of p in X, and there is a hyperplane H in $\mathbb{P}$ such that $Y' = \pi^{-1} X'$ is disjoint from $\widetilde{H}$.* *avoidhplane*

proof. Let p be a point of X, and let $\widetilde{q} = (\widetilde{q}_1, ..., \widetilde{q}_r)$ be the finite set of points of Y making up the fibre over p. We project $\widetilde{q}$ from $\mathbb{P} \times X$ to $\mathbb{P}$, obtaining a finite set $q = (q_1, ..., q_r)$ of points of $\mathbb{P}$, and we choose a hyperplane H in $\mathbb{P}$ that avoids this finite set. Then $\widetilde{H}$ avoids the fibre $\widetilde{q}$. Let Z denote the closed set $Y \cap \widetilde{H}$. Because the fibres of Y over X are finite, so are the fibres of Z over X. By hypothesis, Chevalley's Theorem is true for subvarieties of $\mathbb{P}^{n-1} \times X$, and $\widetilde{H}$ is isomorphic to $\mathbb{P}^{n-1} \times X$. It follows that, for every component Z' of Z, the morphism $Z' \to X$ is a finite morphism, and therefore its image is closed in X (Theorem 4.4.3). Thus the image of Z is a closed subset of X that doesn't contain p. Its complement is the required neighborhood of p.□

(4.6.13) proof of Proposition 4.6.5

proofcovfin

We'll do the case of an integral morphism. The case of a finite morphism is similar.

Step 1. Preliminaries.

We are given a morphism $Y \xrightarrow{u} X$, and we are given an affine covering $\{X^i\}$ of X, such that, for every i, the restriction u^i of u to X^i is an integral morphism of affine varieties. We are to show that the restriction of u to any affine open subset X_1 of X is an integral morphism of affine varieties.

The affine open set X_1 is covered by the affine open sets $X_1^i = X_1 \cap X^i$. For every i, the restriction u_1^i of u to X_1^i can also be obtained by restricting u^i. So u_1^i are integral morphisms (4.6.4) **(ii)**. We may replace X by X_1. Since the localizations of an affine variety form a basis for its Zariski topology, we see that what is to be proved is this:

A morphism $Y \xrightarrow{u} X$ is given in which $X = \operatorname{Spec} A$ is affine. There are elements $s_1, ..., s_k$ that generate the unit ideal of A such that, for every i, the inverse image Y^i of $X^i = X_{s_i}$ is affine, and its coordinate algebra B_i is an integral extension of the localized algebra $A_i = A_{s_i}$. We must show that Y is affine, and that its coordinate algebra B is an integral A-module.

Step 2. The algebra of regular functions on Y.

We assume that X is affine, $X = \operatorname{Spec} A$. Let B be the algebra of regular functions on Y. If Y is affine, B will be its coordinate algebra, and Y will be its spectrum. Here Y isn't assumed to be affine. By hypothesis, the inverse image Y^i of X^i is the spectrum of an integral A_i-algebra B_i. Then B and B_i are subalgebras of the function field of Y. Since the localizations X^i cover X, the affine varieties Y^i cover Y. A function is regular on Y if and only if it is regular on each Y^i, and therefore

$$B = \bigcap B_i$$

Step 3. The coordinate algebra B_j of Y^j is a localization of B.

Denoting the images in B of the elements s_i by the same symbols, we show that B_j is the localization $B[s_j^{-1}]$. The localization X^i is the set of points of X at which $s_i \neq 0$. The inverse image Y^i of X^i is the set of points of Y at which $s_i \neq 0$, and the affine variety $Y^j \cap Y^i$ is the set of points of Y^j at which $s_i \neq 0$. So the coordinate algebra of $Y^j \cap Y^i$ is the localization $B_j[s_i^{-1}]$. Then

$$B[s_j^{-1}] \overset{(1)}{=} \bigcap_i \big(B_i[s_j^{-1}]\big) \overset{(2)}{=} \bigcap_i B_j[s_i^{-1}] \overset{(3)}{=} B_j[s_j^{-1}] \overset{(4)}{=} B_j$$

The explanation of the numbered equalities is as follows:

(1) A rational function β is in $B_i[s_j^{-1}]$ if $s_j^n\beta$ is in B_i for large n, and we can use the same exponent n for all $i = 1, ..., r$. Then β is in $\bigcap_i \big(B_i[s_j^{-1}]\big)$ if and only if $s_j^n\beta$ is in $\bigcap_i B_i = B$. So β is in $\bigcap_i \big(B_i[s_j^{-1}]\big)$ if and only if it is in $B[s_j^{-1}]$.

(2) $B_i[s_j^{-1}] = B_j[s_i^{-1}]$ because $Y^j \cap Y^i = Y^i \cap Y^j$.

(3),(4) Since s_j is one of the elements s_i, $\bigcap_i B_j[s_i^{-1}] \subset B_j[s_j^{-1}]$. For all i, $B_j \subset B_j[s_i^{-1}]$. Moreover, s_j doesn't vanish on Y^j. It is a unit in B_j, and therefore $B_j[s_j^{-1}] = B_j$. Then $B_j \subset \bigcap_i B_j[s_i^{-1}] \subset B_j[s_j^{-1}] = B_j$.

Step 4. B is an integral extension of A.

With $A_i = A_{s_i}$ as before, we choose a finite set $(b_1, ..., b_n)$ of elements of B that generates the A_i-module B_i for every i. We can do this because we can span the finite A_i-module $B_i = B[s_i^{-1}]$ by finitely many elements of B, and there are finitely many algebras B_i. We show that the set $(b_1, ..., b_n)$ generates the A-module B.

Let x be an element of B. Then x is in B_i, so it is a combination of $(b_1, ..., b_n)$ with coefficients in A_i. For large k, $s_i^k x$ will be a combination of those elements with coefficients in A, say

$$s_i^k x = \sum_\nu a_{i,\nu} b_\nu$$

with $a_{i,\nu}$ in A. We can use the same exponent k for all i. The powers s_i^k generate the unit ideal. With $\sum r_i s_i^k = 1$,

$$x = \sum_i r_i s_i^k x = \sum_i r_i \sum_\nu a_{i,\nu} b_\nu$$

The right side is a combination of $b_1, ..., b_n$ with coefficients in A.

Step 5. Y is affine.

The algebra B of regular functions on Y is a finite-type domain because it is a finite module over the finite-type domain A. Let $\widetilde{Y} = \operatorname{Spec} B$. The fact that B is the algebra of regular functions on Y gives us a morphism $Y \xrightarrow{\epsilon} \widetilde{Y}$ (Corollary 3.6.3). Restricting to the open subset X^j of X gives us a morphism $Y^j \xrightarrow{\epsilon^j} \widetilde{Y}^j$ in which, since $B_j = B[s_j^{-1}]$, Y^j and $\widetilde{Y}^j$ are both equal to $\operatorname{Spec} B_j$. Therefore ϵ^j is an isomorphism. Corollary 3.5.13 **(ii)** shows that ϵ is an isomorphism. So Y is affine, and by *Step 4*, its coordinate algebra B is an integral A-module. $\square$

(4.6.14) affine double planes affdplanes

Let A be the polynomial algebra $\mathbb{C}[x, y]$ and let X be the affine plane $\operatorname{Spec} A$. An *affine double plane* is a locus of the form $w^2 = f(x, y)$ in the affine 3-space with coordinates w, x, y, where f is a square-free polynomial in x, y (see Example 4.3.9). The affine double plane is $Y = \operatorname{Spec} B$, where $B = \mathbb{C}[w, x, y]/(w^2 - f)$, and the inclusion $A \subset B$ gives us an integral morphism $Y \to X$.

Let w, x, y denote the variables and also their residues in B. As in Example 4.3.9, B is a normal domain of dimension two, and a free A-module with basis $(1, w)$. It has an automorphism σ of order 2, defined by $\sigma(a + bw) = a - bw$.

The fibres of Y over X are the σ-orbits in Y. If $f(x_0, y_0) \neq 0$, the fibre over the point x_0 of X consists of two points, and if $f(x_0, y_0) = 0$, it consists of one point. The reason that Y is called a *double plane* is that most points of the plane X are covered by two points of Y. The *branch locus* of the covering, which will be denoted by Δ, is the (possibly reducible) curve $\{f = 0\}$ in X. The fibres over the *branch points*, the points of Δ, are single points.

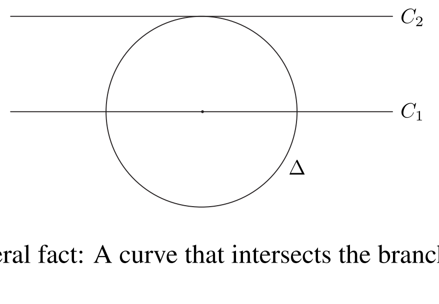
If a closed subvariety D of Y lies over a curve C in X, then $D' = D\sigma$ lies over C too. The curves D and D' may be equal or not. They will have dimension one, and we call them curves too. Let g be the polynomial whose zero locus in X is C. Krull's Theorem tells us that the components of the zero locus of g in Y have dimension one. If a point q of Y lies over a point p of C, then q and $q\sigma$ are the only points of Y lying over p. One of them will be in D, the other in D'. So the inverse image of C is $D \cup D'$. There are no isolated points in the inverse image, and there is no room for another curve.

Thus if $D = D'$, then D is the only curve lying over C. Otherwise, there will be two curves D and D' that lie over C. In that case, we say that C *splits* in Y.

A curve C in the plane X will be the zero set of a principal prime ideal P of the polynomial algebra A, and if D lies over C, it will be the zero set of a prime ideal Q of B that lies over P (4.4.2) **(i)**.

4.6.15. Example. Let $f(x, y) = x^2 + y^2 - 1$. The double plane $Y : \{w^2 = x^2 + y^2 - 1\}$ is an *affine quadric* in $\mathbb{A}^3$. Its branch locus Δ in the affine plane X is the curve $\{x^2 + y^2 = 1\}$. circleexample

The line $C_1 : \{y = 0\}$ in X meets the branch locus Δ transversally at the points $(x, y) = (\pm 1, 0)$, and when we set $y = 0$ in the equation for Y, we obtain $w^2 = x^2 - 1$. The polynomial $w^2 - x^2 + 1$ is irreducible, so y generates a prime ideal of B. On the other hand, the line $C_2 : \{y = 1\}$ is tangent to Δ at the point $(0, 1)$, and it splits. When we set $y = 1$ in the equation for Y, we obtain $w^2 = x^2$. The locus $\{w^2 = x^2\}$ is the union of the two lines $\{w = x\}$ and $\{w = -x\}$ that lie over C_1. The prime ideals of B that correspond to these lines aren't principal ideals.



$\square$

This example illustrates a general fact: A curve that intersects the branch locus transversally doesn't split. We explain this now.

localanal **(4.6.16) local analysis**

Suppose that a plane curve $C : \{g = 0\}$ and the branch locus $\Delta : \{f = 0\}$ of a double plane $w^2 = f$ meet at a point p. We adjust coordinates so that p becomes the origin $(0, 0)$, and we write

$$f(x, y) = \sum a_{ij} x^i y^j = a_{10}x + a_{01}y + a_{20}x^2 + \cdots$$

Since p is a point of Δ, the constant coefficient of f is zero. If the two linear coefficients aren't both zero, p will be a smooth point of Δ, and the tangent line to Δ at p will be the line $\{a_{10}x + a_{01}y = 0\}$. Similarly, writing $g(x, y) = \sum b_{ij} x^i y^j$, the tangent line to C, if p is a smooth point, is the line $\{b_{10}x + b_{01}y = 0\}$.

Let's suppose that the two tangent lines are defined and distinct, i.e., that Δ and C intersect transversally at p. We change coordinates once more, to make the tangent lines the coordinate axes. After adjusting by scalar factors, the polynomials f and g will have the form

$$f(x, y) = x + u(x, y) \quad \text{and} \quad g(x, y) = y + v(x, y),$$

where u and v are polynomials all of whose terms have degree at least 2.

Let $X_1 = \operatorname{Spec} \mathbb{C}[x_1, y_1]$ be another affine plane. The map $X_1 \to X$ defined by the substitution $x_1 = x + u(x, y)$, $y_1 = y + v(x, y)$ is invertible analytically near the origin, because the Jacobian matrix

jacob (4.6.17)
$$\left(\frac{\partial(x_1, y_1)}{\partial(x, y)} \right)_{(0,0)}$$

at the origin p is the identity matrix. When we make that substitution, Δ becomes the locus $\{x_1 = 0\}$ and C becomes the locus $\{y_1 = 0\}$. In this local analytic coordinate system, the equation $w^2 = f$ that defines the double plane becomes $w^2 = x_1$. When we restrict it to C by setting $y_1 = 0$, x_1 becomes a local coordinate function on C. The restriction of the equation remains $w^2 = x_1$. So the inverse image Z of C can't be split analytically. Therefore it doesn't split algebraically either.

splitnot-transversal **4.6.18. Corollary.** *A curve that intersects the branch locus transversally at some point doesn't split.* □

This isn't a complete analysis. When a curve C and the branch locus Δ are tangent at every point of intersection, C may split or not, and which possibility occurs cannot be decided locally in most cases. However, one case in which a local analysis suffices to decide splitting is that C is a line. Let t be a coordinate in a line L, so that $L \approx \operatorname{Spec} \mathbb{C}[t]$. The restriction of the polynomial f to L will give us a polynomial $\overline{f}(t)$ in t. A root of $\overline{f}$ corresponds to an intersection of L with Δ, and a multiple root corresponds to an intersection at which L and Δ are tangent, or at which Δ is singular. The line L will split if and only if the polynomial $w^2 - \overline{f}$ factors, i.e., if and only if $\overline{f}$ is a square in $\mathbb{C}[t]$. This will be true if and only if every root of $\overline{f}$ has even multiplicity — if and only if the intersection multiplicity of L and Δ at every intersection point is even.

projdplane **(4.6.19) projective double planes**

Let X be the projective plane $\mathbb{P}^2$, with coordinates x_0, x_1, x_2. A *projective double plane* is a locus of the form

wtdplane (4.6.20)
$$y^2 = f(x_0, x_1, x_2)$$

where f is a square-free, homogeneous polynomial of *even degree* $2d$. To regard (4.6.20) as a homogeneous equation, we must assign weight d to the variable y (see **1.7.9**). Then, since we have weighted variables, we must work in a *weighted projective space* $\mathbb{WP}$ with coordinates x_0, x_1, x_2, y, where x_i have weight 1 and y has weight d. A point of this weighted space is represented by a nonzero vector (x_0, x_1, x_2, y), with the equivalence relation that, for all nonzero λ, $(x_0, x_1, x_2, y) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda^d y)$. The points of the projective double plane Y are the points of $\mathbb{WP}$ that solve the equation (4.6.20).

The projection $\mathbb{WP} \to X$ that sends (x_0, x_1, x_2, y) to (x_0, x_1, x_2) is defined at all points except at $(0, 0, 0, 1)$. If (x, y) solves (4.6.20) and if $x = 0$, then $y = 0$ too. So $(0, 0, 0, 1)$ isn't a point of Y. The projection is defined at all points of Y. The fibre of the morphism $Y \to X$ over a point x consists of points (x, y) and $(x, -y)$, which will be equal if and only if x lies on the *branch locus* of the double plane, the (possibly reducible) plane curve $\Delta : \{f = 0\}$ in X. The map $\sigma : (x, y) \rightsquigarrow (x, -y)$ is an automorphism of Y, and points of X correspond bijectively to σ-orbits in Y.

Since the double plane Y is embedded into a weighted projective space, it isn't presented to us as a projective variety in the usual sense. However, it can be embedded into a projective space in the following way: The projective plane X can be embedded by a Veronese embedding of higher order, using as coordinates the monomials $m = (m_1, m_2, \ldots)$ of degree d in the variables x. This embeds X into a projective space $\mathbb{P}^N$ where $N = \binom{d+2}{2} - 1$. When we add a coordinate y of weight d, we obtain an embedding of the weighted projective space $\mathbb{WP}$ into $\mathbb{P}^{N+1}$, that sends the point (x, y) to (m, y). The double plane can be realized as a projective variety by this embedding.

When $Y \to X$ is a projective double plane, then, as with affine double planes, a curve C in X may split in Y or not. If C has a transversal intersection with the branch locus Δ, it will not split. On the other hand, if L is a line all of whose intersections with the branch locus Δ have even multiplicity, it will split.

4.6.21. Corollary. *Let Y be a generic quartic double plane – a double plane whose branch locus Δ is a generic quartic curve. The lines that split in Y are the bitangent lines to Δ.* $\square$ *bitansplit*

(4.6.22) homogenizing an affine double plane

ho-
mogdplane

To construct a projective double plane from an affine double plane, we write the affine double plane as

$$w^2 = F(u_1, u_2) \tag{4.6.23}$$

rela-
belaffd-
plane

for some nonhomogeneous polynomial F. We suppose that F has even degree $2d$, and we homogenize F, setting $u_i = x_i/x_0$. We multiply both sides of this equation by x_0^{2d} and set $y = x_0^d w$. This produces an equation of the form (4.6.20), where y has weight d and f is the homogenization of F.

If F has odd degree $2d - 1$, one needs to multiply F by x_0 in order to make the substitution $y = x_0^d w$ permissible. When one does this, the line at infinity $\{x_0 = 0\}$ becomes a part of the branch locus.

(4.6.24) cubic surfaces and quartic double planes

cubicisd-
plane

Let $\mathbb{P}^3$ be the ordinary projective 3-space with coordinates x_0, x_1, x_2, z of weight one, and let X be be the projective plane $\mathbb{P}^2$ with coordinates x_0, x_1, x_2. We consider the projection $\mathbb{P}^3 \xrightarrow{\pi} X$ that sends (x, z) to x. It is defined at all points except at the center of projection $q = (0, 0, 0, 1)$, and its fibres are the lines through q, with q omitted.

Let S be a cubic surface in $\mathbb{P}^3$, the locus of zeros of an irreducible homogeneous cubic polynomial $g(x, z)$, and suppose that q *is* a point of S. Then the coefficient of z^3 in g will be zero, so g will be quadratic in z: $g(x, z) = az^2 + bz + c$, where a, b, c are homogeneous polynomials in x, of degrees $1, 2, 3$, respectively. The defining equation g for S becomes

$$az^2 + bz + c = 0 \tag{4.6.25}$$

projequa-
tion

The discriminant $f(x) = b^2 - 4ac$ of g with respect to z is a homogeneous polynomial of degree 4 in x. Let Y be the projective double plane

$$y^2 = b^2 - 4ac \tag{4.6.26}$$

quarticd-
plane

in which the variable y is given weight 2.

The quadratic formula solves for z in terms of the chosen square root y of the disriminant, wherever $a \neq 0$:

$$z = \frac{-b + y}{2a} \quad \text{or} \quad y = 2az + b \tag{4.6.27}$$

quadrfor-
mula

The formula $y = 2az + b$ remains correct when $a = 0$, and it defines a map $S \to Y$. The inverse map $Y \to Z$ given by the quadratic formula (4.6.27) is defined wherever $a \neq 0$. So the cubic surface and the quartic double plane are isomorphic except above the line $\{a = 0\}$ in X.

quarticis-generic **4.6.28. Lemma.** *The discriminants of the cubic polynomials $az^2 + bz + c$ include every homogeneous quartic polynomial $f(x)$ whose divisor of zeros $\Delta : \{f = 0\}$ has at least one bitangent line. Therefore the discriminants of those polynomials form a dense subset of the space of quartic polynomials.*

proof. Let f be a quartic polynomial whose zero locus has a bitangent line ℓ_0. Then ℓ_0 splits in the double plane $y^2 = f$. Say that ℓ_0 is the zero set of a homogeneous linear polynomial $a(x)$. Then f is congruent to a square, modulo a. There is a homogeneous quadratic polynomial $b(x)$ such that $f \equiv b^2$, modulo a. Then $f = b^2 - 4ac$ for some homogeneous cubic polynomial $c(x)$. The cubic polynomial $g(x, z) = az^2 + bz + c$ has discriminant f.

Conversely, let $g(x, z) = az^2 + bz + c$ be given. The intersections of the line $\ell_0 : \{a = 0\}$ with the discriminant divisor $\Delta : \{b^2 - 4ac = 0\}$ are the solutions of the equations $a = 0$ and $b = 0$. Since the quadratic polynomial b appears as a square in the discriminant, the intersections of ℓ_0 and Δ have even multiplicity. So ℓ_0 will be a bitangent, provided that the locus $b = 0$ meets ℓ_0 two distinct points, and this will be true when g is generic. □

From now on, we suppose that S is a generic cubic surface. With a suitable change of coordinates any point of a generic surface can become the point q, so we may suppose that both S and q are generic. Then S contains only finitely many lines, and those lines won't contain q (3.7.19).

Let ℓ be a line in the plane X, say the locus of zeros of the linear equation $r_0x_0 + r_1x_1 + r_2x_2 = 0$. The same equation defines a plane H in $\mathbb{P}^3_{x,y}$ that contains q, and the intersection $S \cap H$ is a cubic curve C in the plane H. The curve C is the inverse image of ℓ in S.

cubicsplits **4.6.29. Lemma.** *Let S be a generic cubic surface. The lines L contained in S correspond bijectively to lines ℓ in X whose inverse images C are reducible cubic curves. If C is reducible, it will be the union $L \cup Q$ of a line and a conic.*

proof. A line L in S won't contain q. So its image in X will be a line, call it ℓ, and L will be a component of its inverse image. Therefore C will be reducible.

Let ℓ be a line in X. At least one irreducible component of its inverse image C will contain q, and there are no lines through q. So if the cubic C is reducible, it will be the union of a conic and a line L, and q will be a point of the conic. Then L will be one of the lines in S. □

Let ℓ_0 be the line $\{a = 0\}$. The points of Y that lie above ℓ_0 are the points (x, y) such that $a = 0$ and $y = \pm b$. Also, let H_0 denote the inverse image of ℓ_0 in $\mathbb{P}^3$, the plane $\{a = 0\}$, and let C_0 be the cubic curve $S \cap H_0$. The points of C_0 are the solutions in $\mathbb{P}^3$ of the equations $a = 0$ and $bz + c = 0$.

ae-qualzero **4.6.30. Lemma.** *The curve C_0 is irreducible.*

proof. We may adjust coordinates so that a becomes the linear polynomial x_0. When we restrict to H_0 by setting $x_0 = 0$ in the polynomial $bz+c$, we obtain a polynomial $\bar{b}z+\bar{c}$, where $\bar{b}$ and $\bar{c}$ are generic homogeneous polynomials in x_1, x_2 of degrees 2 and 3, respectively. Such a polynomial is irreducible, and C_0 is the locus $\bar{b}z + \bar{c} = 0$. □

twenty-seven **4.6.31. Theorem.** *A generic cubic surface S in $\mathbb{P}^3$ contains precisely 27 lines.*

This theorem follows from next lemma, which relates the 27 lines in the generic cubic surface S to the 28 bitangents of its generic quartic discriminant curve Δ.

linesplits **4.6.32. Lemma.** *Let S be a generic cubic surface $az^2 + bz + c = 0$, and suppose that coordinates are chosen so that $q = (0, 0, 0, 1)$ is a generic point of S. Let $\Delta : \{b^2 - 4ac = 0\}$ be the quartic discriminant curve, and let Y be the double plane $y^2 = b^2 - 4ac$.*

(i) *If a line L is contained in S, its image in X will be a bitangent to the quartic curve Δ. Distinct lines in S have distinct images in X.*

(ii) *The line $\ell_0 : \{a = 0\}$ is a bitangent. It isn't the image of a line in S.*

(iii) *Every bitangent ℓ except ℓ_0 is the image of a line in S.*

proof. Let L be a line in S, let ℓ be its image in X, and let C be the inverse image of ℓ in S. Lemma 4.6.29 tells us that C is the union of the line L and a conic. So L is the only line in S that has ℓ as its image. The quadratic formula (4.6.27) shows that, because the inverse image C of ℓ is reducible, ℓ splits in the double plane Y too, and therefore ℓ is a bitangent to δ. This proves **(i)**. Moreover, Lemma 4.6.30 shows that ℓ cannot be the line ℓ_0. This proves **(ii)**. If a bitangent ℓ is distinct from ℓ_0, the map $Y \to Z$ given by the quadratic formula is defined except at the finite set $\ell \cap \ell_0$. Since ℓ splits in Y, its inverse image C in S will be reducible, and one component of C is a line in S. This proves **(iii)**. $\square$

4.7 Exercises

chapfourex xdcc

4.7.1. A ring A is said to have the *descending chain condition* (dcc) if every strictly decreasing chain of ideals $I_1 > I_2 > \cdots$ is finite. Let A be a finite type $\mathbb{C}$-algebra. Prove

(i) A has dcc if and only if it is a finite dimensional complex vector space.

(ii) If A has dcc, then it has finitely many maximal ideals, and every prime ideal is maximal.

(iii) If a finite-type algebra A has finitely many maximal ideals, then it has dcc.

(iv) Suppose that A has dcc, let M be an arbitrary A-module, and let I denote the intersection of the maximal ideals of A. If $IM = M$, then $M = 0$. (This might be called the Stong Nakayama Lemma. The usual Nakayama Lemma requires that M be finitely generated.)

xfinmodfld

4.7.2. Let $A \subset B$ be noetherian domains and suppose that B is a finite A-module. Prove that A is a field if and only if B is a field.

xaltnull

4.7.3. Prove this alternate form of the Nullstellensatz: Let k be a field, and let B be a domain that is a finitely generated k-algebra. If B is a field, then $[B:k] < \infty$.

xinverseintegral

4.7.4. Let α be an element of a domain A, and let $\beta = \alpha^{-1}$. Prove that if β is integral over A, then it is an element of A.

xopenisom

4.7.5. Let X and Y be varieties with the same function field K. Show that there are nonempty open subsets X' of X and and Y' of Y that are isomorphic.

xAsintclosed

4.7.6. Let $A \subset B$ be finite type domains with fraction fields $K \subset L$ of characteristic zero, and let $Y \to X$ be the corresponding morphism of affine varieties. Prove the following:

(i) There is a nonzero element s in A such that A_s is integrally closed.

(ii) There is a nonzero element s in A such that B_s is a finite module over a polynomial ring $A_s[y_1, ..., y_d]$.

(iii) Suppose that L is a finite extension of K of degree d. There is a nonzero element $s \in A$ such that all fibres of the morphism $Y \to X$ consist of d points.

chainmaximalx

4.7.7. Verify directly that the prime chain **4.5.9** is maximal.

xpchainmax

4.7.8. Prove that $\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$ is a maximal chain of closed subsets of $\mathbb{P}^n$.

xorbits

4.7.9. Let G be a finite group of automorphisms of a normal, finite-type domain B, let A be the algebra of invariant elements of B, and let $Y \xrightarrow{u} X$ be the integral morphism of varieties corresponding to the inclusion $A \subset B$. Prove that there is a bijective correspondence between G-orbits of closed subvarieties of Y and closed subvarieties of X.

xatmostdeg

4.7.10. Let $A \subset B$ be an extension of finite-type domains such that B is a finite A-module, and let P be a prime ideal of A. Prove that the number of prime ideals of B that lie over P is at most equal to the degree $[L:K]$ of the field extension.

xnotidzero

4.7.11. Let $Y = \operatorname{Spec} B$ be an affine variety, let $D_1, \dots, D_n$ be distinct closed subvarieties of Y and let V be a closed subset of Y. Assume that V doesn't contain any of the sets D_j. Prove that there is an element β of B that vanishes on V, but isn't identically zero on any D_j.

dimdim

4.7.12. Let $Y \xrightarrow{u} X$ be a surjective morphism of affine varieties, and let K and L be the function fields of X and Y, respectively. Show that if $\dim Y = \dim X$, there is a nonempty open subset X' of X such that all fibres over points of X' have the same order n, and that $n = [L:K]$.

xchevthmdimone

4.7.13. Work out the proof of Chevalley's Theorem in the case that Y is a closed subset of $\mathbb{P}^1 \times X$ that doesn't meet the locus at infinity $\widetilde{H} = H \times X$. (In $\mathbb{P}^1$, H will be the point at infinity.) Do this in the following way: Say that $X = \operatorname{Spec} A$. Let $B_0 = A[u]$, $B_1 = A[v]$, and $B_{01} = A[u,v]$, where $u = y_1/y_0$ and $v = u^{-1} = y_0/y_1$. Then $\widetilde{U}^0 = U^0 \times X = \operatorname{Spec} B_0$, $\widetilde{U}^1 = \operatorname{Spec} B_1$, and $\widetilde{U}^{01} = \operatorname{Spec} B_{01}$. Let P_1 be the ideal of B_1 that defines $Y \cap \widetilde{U}^1$, and let P_0 be the analogous ideal of B_0. In B_1, the ideal of $\widetilde{H}$ is the principal ideal vB_1. Since $Y \cap \widetilde{H} = \emptyset$, $P_1 + vB_1$ is the unit ideal of B_1. Write out what this means. Then go over to the open set $\widetilde{U}^0$, and show that the residue of u in the coordinate algebra B_0/P_0 of Y is the root of a monic polynomial.

xmapcurvefin

4.7.14. Prove that a nonconstant morphism from a curve Y to $\mathbb{P}^1$ is a finite morphism without appealing to Chevalley's Theorem.

4.7.15. Let A be a finite type domain, $R = \mathbb{C}[t]$, $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} R$. Let $\varphi : A \to R$ be a homomorphism whose image is not $\mathbb{C}$, and let $\pi : Y \to X$ be the corresponding morphism. ximclosed

(i) Show that R is a finite A-module.

(ii) Show that the image of π is a closed subset of X.

4.7.16. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism. xPtwoPt-wofinite

4.7.17. Let $Y \xrightarrow{u} X$ be a finite morphism of curves, and let K and L be the function fields of X and Y, respectively, and suppose $[L : K] = n$. Prove that all fibres have order at most n, and all but finitely many fibres of Y over X have order equal to n. countfib

4.7.18. Prove that a variety of any dimension contains no isolated point. xnoisolpt

4.7.19. Let X be the subset obtained by deleting the origin from $\mathbb{A}^2$. Prove that there is no injective morphism from an affine variety Y to $\mathbb{A}^2$ whose image is X. xmiss-point

4.7.20. With reference to Example 4.6.15, show that the prime ideal that corresponds to the line $w = x$ is not a principal ideal. xnotprinc

4.7.21. Identify the double plane $y^2 = f(x)$ defined as in (**4.6.19**) by a quadratic polynomial f. xquadrd-plane

4.7.22. A *double line* is a locus $y^2 = f(x_0, x_1)$ analogous to a double plane (4.6.20), where f is a homogeneous polynomial of even degree $2d$ with distinct roots. Determine the genus of a double line. xdblcurve

4.7.23. Let $Y \to X$ be an affine double plane, and let D be a curve in Y whose image in X is a plane curve C. Say that C has degree d. Define $\deg D$ to be d if C splits and $2d$ if C remains prime or ramifies. Most curves C in X will intersect the branch locus transversally. Therefore they won't split. On the other hand, most curves D in Y will not be symmetric with respect to the automorphism σ of Y over X. Then there will be two curves $D, D\sigma$ lying over C, so C will split. Try to explain this curious point. xstrange

4.7.24. Let Y be a closed subvariety of projective space $\mathbb{P}^n$ with coordinates $y = (y_0, ..., y_n)$, let d be a positive integer, and let $w = (w_0, ..., w_k)$ be homogeneous polynomials in y of degree d with no common zeros on Y. Prove that sending a point q of Y to $(w_0(q), ..., w_k(q))$ defines a finite morphism $Y \xrightarrow{u} \mathbb{P}^k$. Consider the case that w_i are linear polynomials first. xfmorph

4.7.25. Let M be a module over a finite-type domain A, and let α be an element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s = \alpha - c$ is an injective map $M \xrightarrow{s} M$. xmultbyf-plusc

4.7.26. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism. Do this by showing that the fibres cannot have positive dimension. xpt-woptwo

Chapter 5 STRUCTURE OF VARIETIES IN THE ZARISKI TOPOLOGY

In this chapter, we will see how algebraic curves control the geometry of higher dimensional varieties.

5.1 Local Rings

localrings

A *local ring* is a noetherian ring that contains just one maximal ideal. We make a few comments about local rings here though we will be interested mainly in some special ones, the discrete valuation rings that are discussed below.

Let R be a local ring with maximal ideal M. An element of R that isn't in M isn't in any maximal ideal, so it is a unit. The quotient R/M is a field called the *residue field* of R. For us, the residue field will often be the field of complex numbers.

The Nakayama Lemma 4.1.3 has a useful version for local rings:

local-nakayama

5.1.1. Local Nakayama Lemma. *Let R be a local ring with maximal ideal M and residue field $k = R/M$. Let V be a finite R-module, and let $\overline{V} = V/MV$. If $\overline{V} = 0$, then $V = 0$.*

proof. If $\overline{V} = 0$, then $V = MV$. The usual Nakayama Lemma tells us that M contains an element z such that $1-z$ annihilates V. Then $1-z$ isn't in M, so it is a unit. A unit annihilates V, and therefore $V = 0$. $\square$

gen-modmsq

5.1.2. Corollary. *Let R be a local ring. A set $z_1, ..., z_k$ of elements generates M if the set of its residues generates M/M^2.* $\square$

A local domain R with maximal ideal M has *dimension one* if it contains only two prime ideals, (0) and M, and they are distinct. We describe the *normal* local domains of dimension one in this section. They are the discrete valuation rings that are defined below.

local

5.1.3. A note about the overused word local.

A property is true *locally* on a topological space X if every point p of X has an open neighborhood U such that the property is true on U.

In these notes, the words *localize* and *localization* refer to the process of adjoining inverses. The (simple) localizations of an affine variety $X = \operatorname{Spec} A$ form a basis for the topology on X. If some property is true locally on X, one can cover X by localizations on which the property is true. There will be elements $s_1, ..., s_k$ of A that generate the unit ideal, such that the property is true on each of the localizations X_{s_i}.

Let A be a noetherian domain. An A-module M is *locally free* if there are elements $s_1, ..., s_k$ that generate the unit ideal of A, such that M_{s_i} is a free A_{s_i}-module for each i. The free modules M_{s_i} will have equal rank (2.1.27. That rank is the *rank* of the locally free A-module M.

An ideal I of a domain A is *locally principal* if A contains elements s_i that generate the unit ideal, such that I_{s_i} is a principal ideal of A_{s_i} for every i. A locally principal ideal is a locally free module of rank one. $\square$

(5.1.4) Valuations dvr

Let K be a field. A *discrete valuation* v on K is a surjective homomorphism

(5.1.5) $$K^\times \xrightarrow{\mathrm{v}} \mathbb{Z}^+ \quad , \quad \mathrm{v}(ab) = \mathrm{v}(a) + \mathrm{v}(b)$$ dval

from the multiplicative group $K^\times$ of nonzero elements of K to the additive group $\mathbb{Z}^+$ of integers, such that, if a, b are elements of K, and if a, b and $a{+}b$ aren't zero, then

$$\mathrm{v}(a{+}b) \geq \min\{\mathrm{v}(a), \mathrm{v}(b)\}$$

The word "discrete" refers to the fact that $\mathbb{Z}^+$ is given the discrete topology. Other valuations exist. They are interesting, but less important, and we won't use them. To simplify terminology, we refer to a discrete valuation simply as a *valuation*.

Let r be a positive integer. If v is a valuation and if $\mathrm{v}(a) = r$, then r is the *order of zero of* a, and if $\mathrm{v}(a) = -r$, then r is the *order of pole of* a, with respect to the valuation.

The *valuation ring* R associated to a valuation v on a field K is the subring of K of elements with non-negative value, together with zero:

(5.1.6) $$R = \{a \in K^\times \,|\, \mathrm{v}(a) \geq 0\} \ \cup \ \{0\}$$ valnring

Valuation rings are usually called "discrete valuation rings", but we are dropping the word discrete.

5.1.7. Proposition. *Valuations of the field $\mathbb{C}(t)$ of rational functions in one variable correspond bijectively to points of the projective line. The valuation ring that corresponds to a point $p \neq \infty$ is the ring of rational functions that are regular at p..* *valsinCt*

beginning of the proof. Let a be a complex number. To define the valuation v that corresponds to the point $p: \{t = a\}$ of $\mathbb{P}^1$, we write a nonzero polynomial f as $(t-a)^k h$, where $t - a$ doesn't divide h, and we define, $\mathrm{v}(f) = k$. Then we define $\mathrm{v}(f/g) = \mathrm{v}(f) - \mathrm{v}(g)$. You will be able to check that, with this definition, v becomes a valuation whose valuation ring is the algebra of functions that are regular at p (**2.6.1**). This valuation ring is called the *local ring* of $\mathbb{P}^1$ at p (see (**5.1.10**) below). Its elements are rational functions in t whose denominators aren't divisible by $t - a$. The valuation that corresponds to the point of $\mathbb{P}^1$ at infinity is obtained by working with t^{-1} in place of t.

The proof that these are all of the valuations of $\mathbb{C}(t)$ will be given at the end of the section.

5.1.8. Proposition. *Let* v *be a valuation on a field K, and let x be a nonzero element of K with value* *idealsin-*
$\mathrm{v}(x) = 1$. *valring*

(i) *The valuation ring R of* v *is a normal local domain of dimension one. Its maximal ideal M is the principal ideal xR. The elements of M are the elements of K with positive value, together with zero:*

$$M = \{a \in K^\times \mid \mathrm{v}(a) > 0\} \cup \{0\}$$

(ii) *The units of R are the elements of $K^\times$ with value zero. Every nonzero element z of K has the form $z = x^k u$, where u is a unit and k, is an integer, not necessarily positive.*

(iii) *The proper R-submodules of K are the sets $x^k R$, where k is an integer. The set $x^k R$ consists of zero and the elements of $K^\times$ with value $\geq k$. The sets $x^k R$ with $k \geq 0$ are the nonzero ideals of R. They are the powers of the maximal ideal, and they are principal ideals.*

(iv) *There is no ring properly between R and K: If R' is a ring and if $R \subset R' \subset K$, then either $R = R'$ or $R' = K$.*

proof. We prove **(i)** last.

(ii) Since v is a homomorphism, $\mathrm{v}(u^{-1}) = -\mathrm{v}(u)$ for any nonzero u in K. Then u and u^{-1} are both in R, i.e., u is a unit of R, if and only if $\mathrm{v}(u)$ is zero. If z is a nonzero element of K with $\mathrm{v}(z) = k$, then $u = x^{-k}z$ has value zero, so u is a unit, and $z = x^k u$.

(iii) It follows from **(ii)** that x^kR consists of the elements of K of value at least k. Suppose that a nonzero R-submodule J of K contains an element z with value k. Then $z = ux^k$ and u is a unit, so J contains x^k. Therefore J contains x^kR. If $x^kR < J$, then J contains an element with value $< k$. So if k is the smallest integer such J contains an element with value k, then $J = x^kR$. If there is no minimum value among the elements of J, then J contains x^kR for every k, and $J = K$.

(iv) This follows from **(iii)**. The ring R' will be a nonzero R-submodule of K. If $R' < K$, then $R' = x^kR$ for some k, and if $R \subset R'$, then $k \le 0$. But x^kR isn't closed under multiplication when $k < 0$. So the only possibility is that $k = 0$ and $R = R'$.

(i) Part **(iii)** tells us that R is a principal ideal domain, so it is noetherian. Its maximal ideal is $M = xR$. It also follows from **(iii)** that M and $\{0\}$ are the only prime ideals of R. So R is a local ring of dimension 1. If the normalization of R were larger than R, then according to **(iv)**, it would be equal to K, and x^{-1} would be integral over R. There would be a polynomial relation $x^{-r} + a_1x^{-(r-1)} + \cdots + a_r = 0$ with a_i in R. When one multiplies this relation by x^r, one sees that 1 would be a multiple of x. Then x would be a unit, which it is not, because its value is 1. □

character-izedvr

5.1.9. Theorem.
(i) *A local domain whose maximal ideal is a nonzero principal ideal is a valuation ring.*
(ii) *A normal local domain of dimension* 1 *is a valuation ring.*

proof. **(i)** Let R be a local domain whose maximal ideal M is a nonzero principal ideal, say $M = xR$, with $x \neq 0$, and let y be a nonzero element of R. The integers k such that x^k divides y are bounded (4.1.5). Let x^k be the largest power that divides y. Then $y = ux^k$, where $k \ge 0$ and u is in R but not in M. So u is a unit. Every nonzero element z of the fraction field K of R will have the form $z = ux^r$ where u is a unit and r is an integer, possibly negative. This is shown by writing the numerator and denominator of a fraction in such a form.

The valuation whose valuation ring is R is defined by $\mathrm{v}(z) = r$ when $z = ux^r$ with u a unit, as above. Suppose that $z_i = u_ix^{r_i}$ for $i = 1, 2$, where u_i are units and $0 \le r_1 \le r_2$, then $z_1 + z_2 = \alpha x^{r_1}$ and $\alpha = u_1 + u_2x^{r_2-r_1}$ is an element of R. Therefore $\mathrm{v}(z_1 + z_2) \ge r_1 = \min\{\mathrm{v}(z_1), \mathrm{v}(z_2)\}$. We also have $\mathrm{v}(z_1z_2) = \mathrm{v}(z_1) + \mathrm{v}(z_2)$. Thus v is a surjective homomorphism. The requirements for a valuation are satisfied.

(ii) The fact that a valuation ring is a normal, one-dimensional local ring is Proposition 5.1.8 **(i)**. We show that a normal local domain R of dimension 1 is a valuation ring by showing that its maximal ideal M is a principal ideal. The proof is tricky.

Let z be a nonzero element of M. Because R is a local ring of dimension 1, M is the only prime ideal that contains z, so M is the radical of the principal ideal zR, and $M^r \subset zR$ if r is large (Proposition 2.5.11). Let r be the smallest integer such that $M^r \subset zR$. Then there is an element y in M^{r-1} but not in zR, such that $yM \subset zR$. We restate this by saying that $w = y/z$ isn't in R, but $wM \subset R$. Since M is an ideal, multiplication by an element of R carries wM to wM. So wM is an ideal. Since M is the maximal ideal of the local ring R, either $wM \subset M$, or $wM = R$. If $wM \subset M$, Corollary 4.1.4 **(iii)** shows that w is integral over R. This can't happen because R is normal and w isn't in R. Therefore $wM = R$ and $M = w^{-1}R$. This implies that w^{-1} is in R and that M is a principal ideal. □

local-ringatp

(5.1.10) the local ring at a point

Let $\mathfrak{m}$ be the maximal ideal at a point p of an affine variety $X = \operatorname{Spec} A$, and let S be the complement of $\mathfrak{m}$ in A, a multiplicative system (**2.7.7**). The prime ideals P of the localization AS^{-1} are the extensions of the prime ideals Q of A that are contained in $\mathfrak{m}$: $P = QS^{-1}$ (2.7.9). Since $\mathfrak{m}$ is a maximal ideal of A, $\mathfrak{m}S^{-1}$ is the unique maximal ideal of AS^{-1}, and AS^{-1} is a local ring. This ring is called the *local ring of A at p* . It is often denoted by A_p. Lemma 4.3.4 shows that, if A is a normal domain, then A_p is normal.

For example, let $X = \operatorname{Spec} A$ be the affine line, $A = \mathbb{C}[t]$, and let p be the point $t = 0$. The local ring A_p is the ring whose elements are fractions of polynomials $f(t)/g(t)$ with $g(0) \neq 0$.

The local ring at a point p of any variety, not necessarily affine, is the the local ring at p of an affine open neighborhood of p.

5.1.11. Corollary. *Let $X = \operatorname{Spec} A$ be an affine variety.* *intloc*

(i) *The coordinate algebra A is the intersection of the local rings A_p at the points of X.*

$$A = \bigcap_{p \in X} A_p$$ *intersect-codimonea-gain*

(ii) *The coordinate algebra A is normal if and only if all of its local rings A_p are normal.* □

5.1.12. Proposition. *Let M be a finite module over a finite-type domain A, and let p be a point of $\operatorname{Spec} A$. If the localized module M_p (2.7.11) is a free A_p-module, then there is an element s, not in $\mathfrak{m}_p$, such that M_s is a free A_s-module.* *generateMinloc*

This is an example of the general principle (**2.7.13**). □

5.1.13. Note. The notations A_s and A_p are traditional, though inconsistent. In the localization A_s, the element s is the one that is inverted, while in the local ring A_p, the elements of the maximal ideal $\mathfrak{m}_p$ are the ones that are *not* inverted. □ inconsist

Completion of the proof of Proposition 5.1.7. We show that every valuation v of the function field $\mathbb{C}(t)$ of $\mathbb{P}^1$ corresponds to a point of $\mathbb{P}^1$.

Let R be the valuation ring of v. If $\mathrm{v}(t) < 0$, we replace t by t^{-1}, so that $\mathrm{v}(t) \geq 0$. Then t is an element of R, and therefore $\mathbb{C}[t] \subset R$. The maximal ideal M of R isn't zero. It contains a nonzero fraction g/h of polynomials in t. The denominator h is in R, so M also contains the numerator g. Since M is a prime ideal, it contains a monic irreducible factor of g of the form $t - a$ for some complex number a. The local ring R_0 of $\mathbb{C}[t]$ at the point $t = a$ is a valuation ring that is obtained by inverting $t - c$ for all $c \neq a$. When $c \neq a$, the scalar $c - a$ isn't in M, so $t - c$ won't be in M. Since R is a local ring, $t - c$ will be a unit of R for all $c \neq a$. So R_0 is contained in R (5.1.7). There is no ring properly containing R_0 except K (5.1.8), so $R_0 = R$. □

5.2 Smooth Curves

A *curve* is a variety of dimension 1. The proper closed subsets of a curve are finite subsets. smaffcurve

A point p of a curve X is a *smooth point* if the local ring at p is a valuation ring. Otherwise, it is a *singular point*. A curve X is *smooth* if all of its points are smooth. If $X = \operatorname{Spec} A$ is a smooth affine curve, the prime ideals different from the zero ideal are maximal.

If v_p is the valuation associated to a smooth point p of a curve X and r is a positive integer, a rational function α on X has a zero of order $r > 0$ at p if $\mathrm{v}_p(\alpha) = r$, and it has a pole of order r at p if $\mathrm{v}_p(\alpha) = -r$.

5.2.1. Note. Suppose that an affine curve X is the spectrum of an algebra $A = \mathbb{C}[x_1, ..., x_n]/P$, and that $f_1, ..., f_k$ generate the prime ideal P. A better definition of a smooth point p is that the rank of the Jacobian matrix $J = \frac{\partial f_i}{\partial x_j}$, evaluated at p, is $n-1$. However, we will use the Jacobian matrix just once, at the end of this section. For us, the definition given above is more convenient. □ jacobian

5.2.2. Lemma. **(i)** *An affine curve X is smooth if and only if its coordinate algebra is a normal domain.* *smptsopen*

(ii) *A curve has finitely many singular points.*

(iii) *The normalization $X^\#$ of a curve X is a smooth curve, and the finite morphism $X^\# \to X$ becomes an isomorphism when singular points of X and their inverse images are deleted.*

proof. **(i)** This follows from Theorem 5.1.9 and Proposition 4.3.4.

(ii) The statement that a morphism is an isomorphism can be verified locally, so we may replace X by an affine open subset $\operatorname{Spec} A$. Let $A^\#$ be the normalization of A. There is a nonzero element s in A such that $sA^\# \subset A$ (Corollary 4.3.2). Then $A_s = A^\#_s$. So $\operatorname{Spec} A_s$, which is the complement of a finite set in $\operatorname{Spec} A$, is smooth.

(iii) This is rather obvious. □

node-curvetwo **5.2.3. Example.** We go back to Example 4.3.3 of a nodal cubic curve $C = \operatorname{Spec} A,\ A = \mathbb{C}[u,v]/(v^2–u^3–u^2)$ and its normalization $B = \mathbb{C}[x]$, the map $A \xrightarrow{\varphi} B$ being defined by $\varphi(u) = x^2 - 1$ and $\varphi(v) = x^3 - x$. The normalization $C^\# = \operatorname{Spec} B$ is the affine line. The curve C has a node at the origin p, and the fibre of $C^\#$ over p is the point pair $x = \pm 1$. Let's denote the points $x = 1$ and $x = -1$ by q_1 and q_2, respectively, and the polynomial $x^2 - 1$ by w. The open subset $U = C - \{p\}$ is the localization $\operatorname{Spec} A_u$. Its inverse image in $C^\#$ is the complement of the point pair q_1, q_2, which is the localization $W = \operatorname{Spec} B_w$. Since $\varphi(u) = w,\ \varphi$ extends to a map $A_u \to B_w$, and its inverse maps x to v/u. So W and U are isomorphic. $\square$

5.2.4. Proposition. *Let X be a smooth curve with function field K. Every point of $\mathbb{P}^n$ with values in K*
pointsofcurve *defines a morphism $X \to \mathbb{P}^n$.*

proof. A point $(\alpha_0, ..., \alpha_n)$ of $\mathbb{P}^n$ with values in K determines a morphism $X \to \mathbb{P}^n$ if it is a good point, which means that, for every (ordinary) point p of X, there is an index j such that the functions α_i/α_j are regular at p for $i = 0, ..., n$ (3.5.6). This will be true when j is chosen so that the order of zero of α_j at p is the minimal integer among the orders of zero of α_i of the elements α_i that aren't zero. $\square$

The next example shows that this proposition doesn't extend to varieties X of dimension greater than one.

nomaptop **5.2.5. Example.** Let X' be the complement of the origin in the affine plane $X = \operatorname{Spec} \mathbb{C}[x,y]$, and let $K = \mathbb{C}(x,y)$ be the function field of X. The vector (x,y) defines a good point of X' with values in K, and therefore a morphism $X' \to \mathbb{P}^1$. If (x,y) were a good point of X then, according to Proposition 3.5.4, at least one of the two rational functions x/y or y/x would be regular at the origin $q = (0,0)$. This isn't the case, so (x,y) isn't a good point of X. The morphism $X' \to \mathbb{P}^1$ doesn't extend to X. $\square$

ptsvals **5.2.6. Proposition.** *Let $X = \operatorname{Spec} A$ be a smooth affine curve with function field K.*
(i) *The local rings of X are the valuation rings of K that contain A.*
(ii) *The maximal ideals of A are locally principal.*

In fact, it follows from Proposition 5.2.9 below that every ideal of A is locally principal.

proof of the proposition. Since A is a normal domain of dimension one, its local rings are valuation rings that contain A (see Theorem 5.1.9 and Corollary 5.1.11). Let R be a valuation ring of K that contains A, let v be the associated valuation, and let M be the maximal ideal of R. The intersection $M \cap A$ is a prime ideal of A. Since A has dimension 1, the zero ideal is the only prime ideal of A that isn't a maximal ideal. We can multiply by an element of R to clear the denominator of an element of M, obtaining an element of A while staying in M. So $M \cap A$ isn't the zero ideal. It is the maximal ideal $\mathfrak{m}_p$ of A at a point p of X. The elements of A that aren't in $\mathfrak{m}_p$ aren't in M either. They are invertible in R. So the local ring A_p at p, which is a valuation ring, is contained in R, and is therefore equal to R (5.1.8) **(iii)**. Since M is a principal ideal, so is the maximal ideal of A_p, and $\mathfrak{m}_p$ is locally principal. $\square$

5.2.7. Proposition. *Let X' and X be smooth curves with the same function field K.*
pointsvalns **(i)** *A morphism $X' \xrightarrow{f} X$ that is compatible with the identity map on the function field K maps X' isomorphically to an open subvariety of X.*
(ii) *If X is projective, X' is isomorphic to an open subvariety of X.*
(iii) *If X' and X are both projective, they are isomorphic.*
(iv) *If X is projective, every valuation ring of K is the local ring at a point of X.*

proof. **(i)** Let p be the image in X of a point p' of X', let U be an affine open neighborhood of p in X, and let V be an affine open neighborhood of p' in X' that is contained in the inverse image of U. Say $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$. The morphism f gives us an injective homomorphism $A \to B$, and since p' maps to p, this homomorphism extends to an inclusion of local rings $A_p \subset B_{p'}$. These local rings are valuation rings with the same field of fractions, so they are equal. Since B is a finite-type algebra, there is an element s in A, with $s(p') \neq 0$, such that $A_s = B_s$. Then the open subsets $\operatorname{Spec} A_s$ of X and $\operatorname{Spec} B_s$ of X' are equal. Since p' is arbitrary, X' is a union of open subvarieties of X. So X' is an open subvariety of X.

(ii) The projective embedding $X \subset \mathbb{P}^n$ is defined by a point $(\alpha_0, ..., \alpha_n)$ with values in K. That point also defines a morphism $X' \to \mathbb{P}^n$. If $f(x_0, ..., x_n) = 0$ is a set of defining equations of X in $\mathbb{P}^n$, then $f(\alpha) = 0$

in K. Therefore f vanishes on X' too. So the image of X' is contained in the zero locus of f, which is X. Then **(i)** shows that X' is an open subvariety of X.

(iii) This follows from **(ii)**.

(iv) The local rings of X are normal and they have dimension one. They are valuation rings of K. Let R be any valuation ring of K, let v be the corresponding valuation, and let $\beta = (\beta_0, ..., \beta_n)$ be the point with values in K that defines the projective embedding of X. When we order the coordinates so that $\mathrm{v}(\beta_0)$ is minimal, the ratios $\gamma_j = \beta_j/\beta_0$ will be in R. The coordinate algebra A_0 of the affine variety $X^0 = X \cap \mathbb{U}^0$ is generated by the coordinate functions $\gamma_1, ..., \gamma_n$, so $A_0 \subset R$. Prposition 5.2.6 tells us that R is the local ring of X^0 at some point. □

5.2.8. Proposition. *Let p be a smooth point of an affine curve $X = \operatorname{Spec} A$, and let $\mathfrak{m}$ and v be the maximal ideal and valuation, respectively, at p. The valuation ring R of v is the local ring of A at p.* truncate-curve

(i) *The power $\mathfrak{m}^k$ consists of the elements of A whose values are at least k. If I is an ideal of A whose radical is $\mathfrak{m}$, then $I = \mathfrak{m}^k$ for some $k > 0$.*

(ii) *For every $n \geq 0$, the algebras $A/\mathfrak{m}^n$ and R/M^n are isomorphic to the truncated polynomial ring $\mathbb{C}[t]/(t^n)$.*

proof. **(i)** Proposition 5.1.8 tells us that the nonzero ideals of R are powers of its maximal ideal M, and that M^k is the set of elements of R with value $\geq k$. Let I be an ideal of A whose radical is $\mathfrak{m}$, and let k be the minimal value $\mathrm{v}(x)$ of the nonzero elements x of I. We will show that I is the set of all elements of A with value $\geq k$, i.e., that $I = M^k \cap A$. Since we can apply the same reasoning to $\mathfrak{m}^k$, it will follow that $I = \mathfrak{m}^k$.

We must show that if an element y of A has value $\mathrm{v}(y) \geq k$, then it is in I. We choose an element x of I with value k. Then x divides y in R, say $y/x = w$, with w in R. The element w will be a fraction a/s with s and a in A and s not in $\mathfrak{m}$: $sy = ax$. The element s will vanish at a finite set of points $q_1, ..., q_r$, but not at p. We choose an element z of A that vanishes at p but not at any of the points $q_1, ..., q_r$. Then z is in $\mathfrak{m}$, and since the radical of I is $\mathfrak{m}$, some power of z is in I. We replace z by such a power, so that z is in I. By our choice, z and s have no common zeros in X. They generate the unit ideal of A, say $1 = cs + dz$ with c and d in A. Then $y = csy + dzy = cax + dzy$. Since x and z are in I, so is y.

(ii) Since p is a smooth point, the local ring of A at p is the valuation ring R. Let s be an element of A that isn't in $\mathfrak{m}$. Then $A/\mathfrak{m}^k$ will be isomorphic to $A_s/\mathfrak{m}_s^k$. We may localize A by inverting s. Doing so suitably, we may suppose that $\mathfrak{m}$ is a principal ideal, say tA. Then $\mathfrak{m}^k = t^kA$. Let B be the subring $\mathbb{C}[t]$ of A, and let $\overline{B}_k = B/t^kB$, $\overline{A}_k = A/\mathfrak{m}^k = A/t^kA$, and $\overline{R}_k = R/M^k = R/t^kR$. The quotients $t^{k-1}B/t^kB$, $\mathfrak{m}^{k-1}/\mathfrak{m}^k$, and M^{k-1}/M^k are one-dimensional vector spaces. So the map labelled g_{k-1} in the diagram below is bijective.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & t^{k-1}B/t^kB & \longrightarrow & \overline{B}_k & \longrightarrow & \overline{B}_{k-1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle g_{k-1}} & & \Big\downarrow{\scriptstyle f_k} & & \Big\downarrow{\scriptstyle f_{k-1}} & & \\
0 & \longrightarrow & \mathfrak{m}^{k-1}/\mathfrak{m}^k & \longrightarrow & \overline{A}_k & \longrightarrow & \overline{A}_{k-1} & \longrightarrow & 0
\end{array}$$

By induction on k, we may assume that the map f_{k-1} is bijective, and then f_k is bijective too. So $\overline{B}_k$ and $\overline{A}_k$ are isomorphic. Analogous reasoning shows that $\overline{B}_k$ and $\overline{R}_k$ are isomorphic.

5.2.9. Proposition. *Let $X = \operatorname{Spec} A$ be a smooth affine curve. Every nonzero ideal I of A is a product of powers of maximal ideals: $I = \mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.* idealsin-curve

proof. Let I be a nonzero ideal of A. Because X has dimension one, the locus of zeros of I is a finite set $\{p_1, ..., p_k\}$. Therefore the radical of I is the intersection $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$ of the maximal ideals $\mathfrak{m}_j$ at p_j, which, by the Chinese Remainder Theorem, is the product ideal $\mathfrak{m}_1 \cdots \mathfrak{m}_k$, Moreover, I contains a power of that product, say $I \supset \mathfrak{m}_1^N \cdots \mathfrak{m}_k^N$. Let $J = \mathfrak{m}_1^N \cdots \mathfrak{m}_k^N$. The quotient algebra A/J is the product $B_1 \times \cdots \times B_k$, with $B_j = A/\mathfrak{m}_j^N$, and A/I is a quotient of A/J. Proposition 2.1.8 tells us that A/I is a product $\overline{A}_1 \times \cdots \times \overline{A}_k$, where $\overline{A}_j$ is a quotient of B_j. By Proposition 5.2.8**(ii)**, each B_j is a truncated polynomial ring, so the quotient $\overline{A}_j$ is also a truncated polynomial ring. The kernel of the map $A \to A_j$ is a power of $\mathfrak{m}_j$. The kernel I of the map $A \to \overline{A}_1 \times \cdots \times \overline{A}_k$ is a product of powers of $\mathfrak{m}_1, ..., \mathfrak{m}_k$. □

(5.2.10) isolated points, again jacobtwo

Recall that a point q of a topological space Y is an isolated point if the one-point set $\{q\}$ is open in Y.

noisolatedpt **5.2.11. Proposition.** *In the classical topology, a curve, smooth or not, contains no isolated point.*

This was proved for plane curves in Chapter 1 (Proposition 1.3.19).

isolptofy **5.2.12. Lemma.**

(i) *Let Y' be an open subvariety of a variety Y. A point q of Y' is an isolated point of Y if and only if it is an isolated point of Y'.*

(ii) *Let $Y' \xrightarrow{u'} Y$ be a nonconstant morphism of curves, let q' be a point of Y', and let q be its image in Y. If q is an isolated point of Y, then q' is an isolated point of Y'.*

proof. **(ii)** Because Y' has dimension one, the fibre over q will be a finite set. Say that the fibre is $\{q'\} \cup S$, where S is a finite set of points distinct form q. Let Y'' denote the open complement $Y'-S$ of S in Y', and let u'' be the restriction of u' to Y''. The fibre of Y'' over q is the point q'. If $\{q\}$ is open in Y, then because u'' is continuous, $\{q'\}$ will be open in Y''. By **(i)**, $\{q'\}$ is open in Y'. $\square$

proof of Proposition 5.2.11. Let q be a point of a curve Y. Part **(i)** of Lemma 5.2.12 allows us to replace Y by an affine neighborhood of q. Let $Y^\#$ be the normalization of Y. Part **(ii)** of that lemma allows us to replace Y by $Y^\#$. So we may assume that Y is a smooth affine curve, say $Y = \operatorname{Spec} B$. We can still replace Y by an open neighborhood of q, so we may assume that the maximal ideal $\mathfrak{m}_q$ of B is a principal ideal (5.2.6). Say that $B = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$, and that q is the origin in $\mathbb{A}^n$. Let f_0 be a polynomial whose residue in B generates $\mathfrak{m}_q$. Then $f_0, f_1, ..., f_k$ generate the maximal ideal of the polynomial ring $\mathbb{C}[x]$ at q.

Let f and x be the column vectors $(f_0, ..., f_k)^t$ and $(x_1, ..., x_n)^t$, respectively. Since f generates the maximal ideal at q, there is an $n\times(k+1)$ polynomial matrix P such that $Pf = x$. Let J be the $(k+1)\times n$ Jacobian matrix $\partial f_i/\partial x_j$, and let $\overline{J}$ and $\overline{P}$ denote the constant terms of J and P, respectively. Then $f = \overline{J}x + O(2)$, where $O(2)$ stands for a polynomial in x, all of whose terms have degree at least 2. Since $Pf = x$, $x = \overline{PJ}x + O(2)$, and therefore $x = \overline{PJ}x$. This shows that $\overline{PJ}$ is the identity matrix, and that the $(k+1)\times n$ matrix $\overline{J}$ has rank n. Adjusting coordinates, we may assume that the submatrix Q of J with the indices $1 \le i, j \le n-1$ is invertible at q. The Implicit Function Theorem tells us that the equations $f_1 = \cdots = f_{n-1} = 0$ can be solved for $x_1, ..., x_{n-1}$ as analytic functions of x_n. It follows that the locus of zeros Z of $f_1, ..., f_{n-1}$ is locally homeomorphic to the affine x_n-line (1.4.18), and it contains Y. Since Y has dimension 1, the component of Z that contains q is equal to Y. So Y is locally homeomorphic to $\mathbb{A}^1$, which has no isolated point. Therefore q isn't an isolated point of Y. $\square$

5.3 Constructible Sets

construct

In this section, X will denote a noetherian topological space. Every closed subset of X is a finite union of irreducible closed sets (2.2.16).

The intersection $L = Z \cap U$ of a closed set Z and an open set U is a *locally closed set*. Open sets and closed sets are locally closed. The following conditions on a subset L of X are equivalent.

- L is locally closed.
- L is a closed subset of an open subset U of X.
- L is an open subset of a closed subset Z of X.

A *constructible set* is a subset that is the union of finitely many locally closed sets.

constrincurve **5.3.1. Examples.**

(i) A subset S of a curve X is constructible if and only if it is either a finite set or the complement of a finite set. Thus S is constructible if and only if it is either closed or open.

(ii) In the affine plane $X = \operatorname{Spec} \mathbb{C}[x, y]$, let U be the complement of the line $\{y = 0\}$, and let p be the origin. The union $U \cup \{p\}$ is constructible, but not locally closed. $\square$

We use the following notation: Z will denote a closed set, U will denote an open set. and L will denote a locally closed set, such as $Z \cap U$.

5.3.2. Theorem. *The set $\mathbb{S}$ of constructible subsets of a noetherian topological space X is the smallest family of subsets that contains the open sets and is closed under the three operations of finite union, finite intersection, and complementation.* *defloc-closed*

By closure under complementation, we mean that if S is in $\mathbb{S}$, then its complement $S^c = X - S$ is in $\mathbb{S}$ too.

proof. Let $\mathbb{S}_1$ denote the family of subsets obtained from the open sets by the three operations mentioned in the statement. Open sets are constructible, and using those three operations, one can make any constructible set from the open sets. So $\mathbb{S} \subset \mathbb{S}_1$. To show that $\mathbb{S} = \mathbb{S}_1$, we show that the family of constructible sets is closed under the three operations.

It is obvious that a finite union of constructible sets is constructible. The intersection of two locally closed sets $L_1 = Z_1 \cap U_1$ and $L_2 = Z_2 \cap U_2$ is locally closed because $L_1 \cap L_2 = (Z_1 \cap Z_2) \cap (U_1 \cap U_2)$. If $S = L_1 \cup \cdots \cup L_k$ and $S' = L'_1 \cup \cdots \cup L'_r$ are constructible sets, the intersection $S \cap S'$ is the union of the locally closed intersections $L_i \cap L'_j$, so it is constructible.

Let S be the constructible set $L_1 \cup \cdots \cup L_k$. Its complement S^c is the intersection of the complements L_i^c of L_i: $S^c = L_1^c \cap \cdots \cap L_k^c$. We have shown that intersections of constructible sets are constructible. So to show that S^c is constructible, it suffices to show that the complement of any locally closed set is constructible. Let L be the locally closed set $Z \cap U$, and let Z^c and U^c be the complements of Z and U, respectively. Then Z^c is open and U^c is closed. The complement L^c of L is the union $Z^c \cup U^c$ of two constructible sets, so it is constructible. □

5.3.3. Proposition. *In a noetherian topological space X, every constructible set is a finite union of locally closed sets, $L_i = Z_i \cap U_i$, in which the irreducible sets Z_i are irreducible and distinct.* *contain-sopen*

proof. Let $L = Z \cap U$ be a locally closed set, and let $Z = Z_1 \cup \cdots \cup Z_r$ be the decomposition of Z into irreducible components. Then $L = (Z_1 \cap U) \cup \cdots \cup (Z_r \cap U)$, which is constructible. So every constructible set S is a union of locally closed sets $L_i = Z_i \cap U_i$ in which Z_i are irreducible. Next, suppose that two of the irreducible closed sets are equal, say $Z_1 = Z_2$. Then $L_1 \cup L_2 = (Z_1 \cap U_1) \cup (Z_1 \cap U_2) = Z_1 \cap (U_1 \cup U_2)$ is locally closed. So we can find an expression in which the irreducible closed sets are distinct. □

5.3.4. Lemma. *sinx*

(i) *Let X_1 be a closed subset of a variety X, and let X_2 be its open complement. A subset S of X is constructible if and only if $S \cap X_1$ and $S \cap X_2$ are constructible.*

(ii) *Let X' be an open or a closed subvariety of a variety X.*

a) *If S is a constructible subset of X, then $S \cap X'$ is a constructible subset of X'.*

b) *A subset S' of X' is a constructible subset of X' if and only if it is a constructible subset of X.*

proof. **(i)** This follows from Theorem 5.3.2.

(iia) It suffices to prove that, if L is a locally closed subset of X, the intersection $L' = L \cap X'$ is a locally closed subset of X'. If $L = Z \cap U$, then $Z' = Z \cap X'$ is closed in X', and $U' = U \cap X'$ is open in X'. So $L' = Z' \cap U'$ is locally closed.

(iib) It follows from **a)** that if a subset S' of X' is constructible in X, then it is constructible in X'. To show that a constructible subset of X' is contructible in X, it suffices to show that a locally closed subset $L' = Z' \cap U'$ of X' is locally closed in X. If X' is a closed subset of X, then Z' is a closed subset of X, and $U' = X \cap U$ for some open subset U of X. Since $Z' \subset X'$, $L' = Z' \cap U' = Z' \cap X' \cap U = Z' \cap U$, which is locally closed in X. If X' is open in X, then U' is open in X. Let Z be the closure of Z' in X. Then $L' = Z \cap U' = Z \cap X' \cap U' = Z' \cap U'$. Again, L' is locally closed in X. □

The next theorem illustrates a general fact, that many of the sets that arise in algebraic geometry are constructible.

5.3.5. Theorem. *Let $Y \xrightarrow{f} X$ be a morphism of varieties. The inverse image of a constructible subset of X is a constructible subset of Y. The image of a constructible subset of Y is a constructible subset of X.* *imagecon-str*

proof. The fact that a morphism is continuous implies that the inverse image of a constructible set is constructible. It is less obvious that the image of a constructible set is constructible. To prove that, we keep pecking away until there is nothing left to do. There may be a shorter proof.

Let S be a constructible subset of Y. Lemma 5.3.4 and Noetherian induction allow us to assume that the theorem is true when S is contained in a proper closed subset of Y, and also when its image $f(S)$ is contained in a proper closed subvariety of X.

Suppose that Y is the union of a proper closed subset Y_1 and its open complement Y_2. The sets $S_i = S \cap Y_i$ are constructible subsets of Y_i. It suffices to show that their images $f(S_i)$ are constructible, and Noetherian induction applies to Y_1. So we may replace Y by the open subvariety Y_2, which can be arbitrary.

Next, suppose that X is the union of a proper closed subset X_1 and its open complement X_2. Let Y_1 and Y_2 denote the inverse images of X_1 and X_2, respectively, and let $S_i = S \cap Y_i$. As before, it suffices to show that the images $f(S_i)$ are constructible. Here $f(S_i)$ is contained in X_i, and induction applies to X_1. So we may replace X by the arbitrary open subvariety X_2.

Summing up, we may replace X by any nonempty open subset X', and Y by any nonempty open subset of its inverse image Y'. We can do this finitely often.

Since a constructible set S is a finite union of locally closed sets, it suffices to show that the image of a locally closed subset S of Y is constructible. Moreover, we may suppose that S has the form $Z \cap U$, where U is open and Z is closed and irreducible. Then Y is the union of the closed set $Z = Y_1$ and its complement $Y_2 = (Y - Z)$, and $S \cap Y_2 = \emptyset$. We may replace Y by $Y_1 = Z$. Then $S = U$, and we may replace Y by U. We are thus reduced to the case that $S = Y$.

We may again replace X and Y by nonempty open subsets, so we may assume that they are affine, say $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, so that the morphism $Y \to X$ corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. If the kernel P of φ were nonzero, the image of Y would be contained in the proper closed subset $\operatorname{Spec} A/P$ of X, to which induction would apply. So we may assume that φ is injective.

Corollary 4.2.11 tells us that, for suitable nonzero element s in A, the localization B_s will be a finite module over a polynomial subring $A_s[y_1, ..., y_k]$. We may replace Y and X by the open subsets $Y_s = \operatorname{Spec} B_s$ and $X_s = \operatorname{Spec} A_s$. Then the maps $Y \to \operatorname{Spec} A[y]$ and $\operatorname{Spec} A[y] \to X$ are both surjective, so $Y = S$ maps surjectively to X. □

5.4 Closed Sets

usingcurves

Limits of sequences are often used to analyze subsets of a topological space. In the classical topology, a subset Y of $\mathbb{C}^n$ is closed if, whenever a sequence of points in Y has a limit in $\mathbb{C}^n$, the limit is in Y. In algebraic geometry, curves can be used as substitutes for sequences.

We use the following notation:

Cwithpoint

5.4.1. *C is a smooth affine curve, q is a point of C, and C' is the complement of q in C.*

The closure of C' will be C, and we think of q as a limit point. In fact, the closure will be C in the classical topology as well as in the Zariski topology, because C has no isolated point (5.2.11). Theorem 5.4.3, which is below, characterizes constructible subset of a variety in terms of such limit points.

The next theorem tells us that there are enough curves to do the job.

enoughcurves

5.4.2. Theorem. *(enough curves) Let Y be a constructible subset of a variety X, and let p be a point of its closure $\overline{Y}$. There exists a morphism $C \xrightarrow{f} X$ from a smooth affine curve to X and a point q of C, such that $f(q) = p$. and that the image of $C' = C - \{q\}$ is contained in Y.*

proof. If $X = p$, then $Y = p$. In this case, we may take for f the constant morphism from any curve C to p. So we may assume that X has dimension at least one. Next, we may replace X by an affine open subset X' that contains p, and Y by $Y' = Y \cap X'$. If $\overline{Y}$ denotes the losure of Y in X, the closure of Y' in X' will be $\overline{Y} \cap X'$, and it will contain p. So we may assume that X is affine, say $X = \operatorname{Spec} A$.

Since Y is constructible, it is a union $L_1 \cup \cdots \cup L_k$ of locally closed sets, say $L_i = Z_i \cap U_i$ where Z_i are irreducible closed sets and U_i are open. The closure of Y is the union $Z_1 \cup \cdots \cup Z_k$, and p will be in at least one of the closed sets, say $p \in Z_1$. We replace X by Z_1 and Y by U_1. This reduces us to the case that Y is a nonempty open subset of X.

We use Krull's Theorem to slice X down to dimension one. Suppose that the dimension n of X is at least two. Let $D = X - Y$ be the closed complement of the open set Y. The components of D have dimension at most $n-1$. We choose an element α of the coordinate algebra A of X that is zero at p and isn't identically zero on any component of D, except at p itself, if p happens to be a component. Krull's Theorem tells us that every component of the zero locus of α has dimension $n-1$, and at least one of those components, call it V, contains p. If V were contained in D, it would be a component of D because $\dim V = n-1$ and $\dim D \le n-1$. By our choice of α, this isn't the case. So $V \not\subset D$, and therefore $V \cap Y \neq \emptyset$. Let $W = V \cap Y$. Because V is irreducible and Y is open, W is a dense open subset of V, its closure is V, and p is a point of V. We replace X by V and Y by W. The dimension of X is thereby reduced to $n-1$.

Thus it suffices to treat the case that X has dimension one. Then X will be a curve that contains p, and Y will be a nonempty open subset of X. The normalization of X will be a smooth curve $X^{\#}$ that comes with an integral, and therefore surjective, morphism to X. Finitely many points of $X^{\#}$ will map to p. We choose for C an affine open subvariety of $X^{\#}$ that contains just one of those points, and we call that point q. □

5.4.3. Theorem *(curve criterion for a closed set) Let Y be a constructible subset of a variety X. The following conditions are equivalent:* *closed-crittwo*

(a) *Y is closed.*

(b) *For any morphism $C \xrightarrow{f} X$ from a smooth affine curve to X, the inverse image $f^{-1}Y$ is closed in C.*

(c) *Let q be a point of a smooth affine curve C, let $C' = C - \{q\}$, and let $C \xrightarrow{f} X$ be a morphism. If $f(C') \subset Y$, then $f(C) \subset Y$.*

The hypothesis that Y be constructible is necessary. For example, in the affine line X, the set W of points with integer coordinates isn't constructible, but it satisfies condition **(b)**. Any morphism $C' \to X$ whose image is in W will map C' to a single point, and therefore it will extend to C.

proof of Theorem 5.4.3. The implications **(a)** $\Rightarrow$ **(b)** $\Rightarrow$ **(c)** are obvious. We prove the contrapositive of the implication **(c)** $\Rightarrow$ **(a)**. Suppose that Y isn't closed. We choose a point p of the closure $\overline{Y}$ that isn't in Y, and we apply Theorem 5.4.2. There exists a morphism $C \xrightarrow{f} X$ from a smooth curve to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. Since $q \notin Y$, this morphism shows that **(c)** doesn't hold either. □

5.4.4. Theorem. *A constructible subset Y of a variety X is closed in the Zariski topology if and only if it is closed in the classical topology.* *class-closed*

proof. A Zariski closed set is closed in the classical topology because the classical topology is finer than the Zariski topology. Suppose that a constructible subset Y of X is closed in the classical topology. To show that Y is closed in the Zariski topology, we choose a point p of the Zariski closure $\overline{Y}$ of Y, and we show that p is a point of Y.

We use the notation (5.4.1). Theorem 5.4.2 tells us that there is a map $C \xrightarrow{f} X$ from a smooth curve C to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. Let C_1 denote the inverse image $f^{-1}(Y)$ of Y. Because C_1 contains C', either $C_1 = C'$ or $C_1 = C$. In the classical topology, a morphism is continuous. Since Y is closed, its inverse image C_1 is closed in C. If C_1 were C', then C' would closed as well as open. Its complement $\{q\}$ would be an isolated point of C. Because a curve contains no isolated point, the inverse image of Y is C, which means that $f(C) \subset Y$. In particular, p is in Y. Therefore Y is closed in the Zariski topology. □

5.5 Projective Varieties are Proper

proper

As has been noted before, an important property of projective space is that, in the classical topology, it is compact. A variety isn't compact in the Zariski topology unless it is a single point. However, in the Zariski topology, projective varieties have a property closely related to compactness: They are *proper*.

Before defining the concept of a proper variety, we explain an analogous property of compact spaces.

5.5.1. Proposition. *Let X be a compact space, let Z be a Hausdorff space, and let V be a closed subset of $Z \times X$. The image of V via the projection $Z \times X \to Z$ is closed in Z.* *proper-compact*

proof. Let W be the image of V in Z. We show that if a sequence of points z_i of the image W has a limit $\underline{z}$ in Z, then that limit is in W. For each i, we choose a point p_i of V that lies over z_i. So p_i is a pair (z_i, x_i), x_i being a point of X. Since X is compact, there is a subsequence of the sequence x_i that has a limit $\underline{x}$ in X. Passing to a subsequence of $\{p_i\}$, we may suppose that x_i has limit $\underline{x}$. Then p_i has limit $\underline{p} = (\underline{z}, \underline{x})$. Since V is closed, $\underline{p}$ is in V. Therefore $\underline{z}$ is in its image W. □

defproper **5.5.2. Definition.** A variety X is *proper* if it has the following property: Let $Z \times X$ be the product of X with another variety Z, let π_Z denote the projection $Z \times X \longrightarrow Z$, and let V be a closed subvariety of $Z \times X$. The image W of V is a closed subvariety of Z.

imclos (5.5.3)

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Z \times X \\ \downarrow & & \downarrow{\scriptstyle \pi_Z} \\ W & \xrightarrow{\subset} & Z \end{array}$$

If X is proper, then because every closed set is a finite union of closed subvarieties, the image of a closed subset of $Z \times X$ will be a closed subset of Z.

propim-closed **5.5.4. Corollary.** *Let X be a proper variety, let V be a closed subvariety of X, and let $X \xrightarrow{f} Y$ be a morphism. The image $f(V)$ of V is a closed subvariety of Y.*

proof. In $X \times Y$, the graph Γ_f of f is a closed set isomorphic to X, and V corresponds to a subset V' of Γ_f that is closed in Γ_f and in $X \times Y$. The points of V' are pairs (x, y) such that $x \in V$ and $y = f(x)$. The image of V' via the projection to $X \times Y \to Y$ is the same as the image of V. Since X is proper, the image of V' is closed. □

The next theorem is the most important application of the use of curves to characterize closed sets.

pnproper **5.5.5. Theorem.**

(i) *Projective varieties are proper.*

(ii) *If X is a projective variety and $X \to Y$ is a morphism, the image in Y of a closed subvariety of X is a closed subvariety of X.*

proof. Part **(ii)** follows from **(i)** and Corollary 5.5.4. Let X be a projective variety. Suppose we are given a closed subvariety V of the product $Z \times X$. We must show that its image W in Z is a closed subvariety of Z (see Diagram 5.5.3). Since V is irreducible, its image is irreducible, so it suffices to show that W is closed. Theorem 5.3.5 tells us that W is a constructible set, and since X is closed in projective space, it is compact in the classical topology. Proposition 5.5.1 tells us that W is closed in the classical topology, and 5.4.4 tells us that W is closed in the Zariski topology too. □

needcover **5.5.6. Note.** Since Theorem 5.5.5 is about algebra, an algebraic proof would be preferable. To make an algebraic proof, one could attempt to replace the limit argument used in the proof of Proposition 5.5.1 by the curve criterion, proceeding as follows: Given a closed subset V of $Z \times X$ with image W and a point p in the closure of W, one chooses a map $C \xrightarrow{f} Z$ from an affine curve C to Z such that $f(q) = p$ and $f(C') \subset W$, C' being the complement of q in C. Then one tries to lift this map by finding a morphism $C \xrightarrow{g} Z \times X$ such that $g(C') \subset V$ and $f = \pi \circ g$. Since V is closed, it would contain $g(q)$, and therefore $f(q) = \pi g(q)$ would be in $\pi(V) = W$. However, to find the lifting g, it may be necessary to replace C by a suitable covering. It isn't difficult to make this method work, but it takes longer. That is why we resorted to the classical topology. □

The next examples show how Theorem 5.5.5 can be used.

properex **5.5.7. Example.** (*singular curves*) We parametrize the plane projective curves of a given degree d. The number of monomials $x_0^i x_1^j x_2^k$ of degree $d = i+j+k$ is the binomial coefficient $\binom{d+2}{2}$. We label those monomials as $m_0, ..., m_r$, ordered arbitrarily, with $r = \binom{d+2}{2} - 1$. A homogeneous polynomial of degree d will be a combination $\sum z_i m_i$ with complex coefficients z_i, so the homogeneous polynomials f of degree d in x, taken up to scalar factors, are parametrized by the projective space of dimension r with coordinates z. Let's denote that projective space by Z. Points of Z correspond bijectively to divisors of degree d in the projective plane, as defined in (1.3.13).

The product variety $Z \times \mathbb{P}^2$ represents pairs (D, p), where D is a divisor of degree d and p is a point of $\mathbb{P}^2$. A variable homogeneous polynomial of degree d in x can be written as a bihomogeneous polynomial $f(z, x)$ of degree 1 in z and degree d in x. For example, in degree 2, f would be

$$z_0 x_0^2 + z_1 x_1^2 + z_2 x_2^2 + z_3 x_0 x_1 + z_4 x_0 x_2 + z_5 x_1 x_2$$

The locus Γ: $\{f(z,x)=0\}$ in $Z\times\mathbb{P}^2$ is closed. A point $z=c$, $x=a$ of Γ is a pair (D,p) such that D is the divisor $f(c,x)=0$ and p is the point (c,a) of D.

The set Σ of pairs (D,p) such that p is a singular point of D is also a closed set, because it is defined by the system of equations $f_0(z,x)=f_1(z,x)=f_2(z,x)=0$, where f_i are the partial derivatives $\frac{\partial f}{\partial x_i}$. (Euler's Formula shows that $f(x,z)=0$ follows from those equalities.) The partial derivatives f_i are bihomogeneous, of degree 1 in z and degree $d-1$ in x.

The next proposition isn't especially easy to verify directly, but the proof becomes easy when one uses the fact that projective space is proper.

5.5.8. Proposition *The singular divisors of degree d, those that contain at least one singular point, form a closed subset S of the projective space Z of all divisors of degree d.* *singclosed*

proof. The subset S is the projection of the closed subset Σ of $Z\times\mathbb{P}^2$. Since $\mathbb{P}^2$ is proper, the image of the closed set Σ is closed. $\square$

5.5.9. Example. (*surfaces that contain a line*) We go back to the discussion (**3.7.15**) of lines in a surface. Let $\mathbb{S}$ denote the projective space that parametrizes surfaces of degree d in $\mathbb{P}^3$. surface-line

5.5.10. Proposition *In $\mathbb{P}^3$, the surfaces of degree d that contain a line form a closed subset of the space $\mathbb{S}$.* *surfaceswithline*

The Grssmanian $\mathbb{G}=G(2,4)$ of lines in $\mathbb{P}^3$ is a projective variety (Corollary 3.7.13). Let Ξ be the subset of $\mathbb{G}\times\mathbb{S}$ of pairs of pairs $[\ell],[S]$ such that $\ell\subset S$. Lemma 3.7.17 tells us that Ξ is a closed subset of $\mathbb{G}\times\mathbb{S}$. Therefore its image in $\mathbb{S}$ is closed. $\square$

5.6 Fibre Dimension

semicont

A function $Y\xrightarrow{\delta}\mathbb{Z}$ from a variety to the integers is a *constructible* function if, for every integer n, the set of points of Y such that $\delta(p)=n$ is constructible, and δ is an *upper semicontinuous* functiion if for every n, the set of points such that $\delta(p)\geq n$ is closed. For brevity, we refer to an upper semicontinuous function as *semicontinuous*, though the term is ambiguous. A function might be lower semicontinuous.

A function δ on a curve Y is semicontinuous if and only if there exists an integer n and a nonempty open subset Y' of Y such that $\delta(p)=n$ for all points p of Y' and $\delta(p)\geq n$ for all points of Y not in Y'.

The next curve criterion for semicontinuous functions follows from the criterion for closed sets.

5.6.1. Proposition. *(curve criterion for semicontinuity) Let Y be a variety. A function $Y\xrightarrow{\delta}\mathbb{Z}$ is semicontinuous if and only if it is a constructible function, and for every morphism $C\xrightarrow{f}Y$ from a smooth curve C to Y, the composition $\delta\circ f$ is a semicontinuous function on C.* $\square$ *uppercrit*

Let $Y\xrightarrow{f}X$ be a morphism of varieties, let q be a point of Y, and let Y_p be the fibre of f over $p=f(q)$. The *fibre dimension* $\delta(q)$ of f at q is the maximum among the dimensions of the components of the fibre that contain q.

5.6.2. Theorem. *(semicontinuity of fibre dimension) Let $Y\xrightarrow{u}X$ be a morphism of varieties, and let $\delta(q)$ denote the fibre dimension at a point q of Y.* *uppersemi*

(i) *Suppose that X is a smooth curve, that Y has dimension n, and that u does not map Y to a single point. Then δ is constant — the nonempty fibres have constant dimension: $\delta(q)=n-1$ for all $q\in Y$.*

(ii) *Suppose that the image of Y is dense in X. Then it contains a nonempty open subset of X. Let the dimensions of X and Y be m and n, respectively. There is a nonempty open subset X' of X such that $\delta(q)=n-m$ for every point q in the inverse image of X'.*

(iii) *δ is a semicontinuous function on Y.*

The proof of this theorem is left as a long exercise. When you have done it, you will have understood the chapter.

5.7 Exercises

chapfiveex xevalzero **5.7.1.** Prove that, if v is a (discrete) valuation on a field K that contains the complex numbers, every nonzero complex number c has value zero.

xfinno **5.7.2.** Prove that a closed subset of dimension zero of a variety X is a finite set.

xblowups-ing **5.7.3.** Let $X = \operatorname{Spec} A$ be an affine curve, with $A = \mathbb{C}[x_0, ..., x_n]/P$, and let x_i also denote the residues of the variables in A. Let p be a point of X. We adjust coordinates so that p is the origin $(0, ..., 0)$, and are otherwise generic. Let $z_i = x_i/x_0, \quad i = 1, ..., n$, let $B = \mathbb{C}[x_0, z_1, z_2, ..., z_n]$, and let $Y = \operatorname{Spec} B$. The inclusion $A \subset B$ defines a morphism $Y \to X$ called the *blowup* of p in X. There will be finitely many points of Y in the fibre over p, and there will be at least one such point. We choose a point p_1 of the fibre, we replace X by Y and p by p_1 and repeat. Prove that this blowing up process yields a curve that is smooth above p in finitely many steps.

fseries **5.7.4.** Prove that the ring $k[[x, y]]$ of formal power series with coefficients in a field k is a local ring and a unique factorization domain.

xlocval-ring **5.7.5.** Let A be a normal finite-type domain. Prove that the localization A_P of A at a prime ideal P of codimension 1 is a valuation ring.

xnormaliz **5.7.6.** Let $X = \operatorname{Spec} A$, where $A = \mathbb{C}[x, y, z]/(y^2 - xz^2)$. Identify the normalization of X.

5.7.7. Let A be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, and let P be the principal ideal generated by an irreducible polynomial $f(x_1, ..., x_n)$. The local ring A_P consists of fractions g/h of polynomials in which g is arbitrary, and h can be any polynomial not divisible by f. Describe valuation v associated to this local ring.

xidemp-matr **5.7.8.** In the space $\mathbb{A}^{n\times n}$ of $n\times n$ matrices, let X be the locus of idempotent matrices: $P^2 = P$. The general linear group GL_n operates on X by conjugation.

(i) Decompose X into orbits for the operation of GL_n, and prove that the orbits are closed subsets of $\mathbb{A}^{n\times n}$.

(ii) Determine the dimensions of the orbits.

xcount-cover **5.7.9.** Prove that, when a variety X is covered by countably many constructible sets, a finite number of those sets will cover X.

xfconst **5.7.10.** Let $f(x, y)$ and $d(x, y)$ be polynomials. Show that if d divides the partial derivatives f_x and f_y, then f is constant on the locus $d = 0$.

xsimple-loc **5.7.11.** Let S be a multiplicative system in a finite-type domain R, and let A and B be finite-type domains that contain R as subring. Let R', A', B' be the rings of S-fractions of R, A, B, respectively. Prove:

(i) If a set of elements $\alpha_1, ..., \alpha_k$ generates A as R-algebra, it also generates A' as R'-algebra.

(ii) Let $A' \xrightarrow{\varphi'} B'$ be a homomorphism. For suitable s in S, there is a homomorphism $A_s \xrightarrow{\varphi_s} B_s$ whose localization is φ'. If φ' is injective, so is φ_s. If φ' is surjective or bijective, there will be an s such that φ_s is surjective or bijective, respectively.

(iii) If $A' \subset B'$ and if B' is a finite A'-module, then for suitable s in S, $A_s \subset B_s$, and B_s is a finite A_s-module.

xclosed-inGG **5.7.12.** Let G denote the Grassmanian $G(2, 4)$ of lines in $\mathbb{P}^3$, and let $[\ell]$ denote the point of G that corresponds to the line ℓ. In the product variety $G\times G$ of pairs of lines, let Z denote the set of pairs $[\ell_1], [\ell_2]$ whose intersection isn't empty. Prove that Z is a closed subset of $G\times G$.

exupper **5.7.13.** Is the constructibility hypothesis in 5.6.1 necessary?

singular-closed **5.7.14.** Prove Theorem 5.5.8 directly, without appealing to Theorem 5.5.5.

cantlift **5.7.15.** With reference to Note 5.5.6, let $X = \mathbb{P}^1$ and $Z = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. Find a closed subset V of $Z\times X$ whose image is Z, such that the identity map $Z \to Z$ can't be lifted to a map $Z \to V$.

xfibdim **5.7.16.** Let $f : Y \to X$ be a morphism of varieties. Suppose we know that the fibre dimension is a constructible function. Use the curve criterion to show that fibre dimension is semicontinuous. (This is a part of Theorem 5.6.2.)

5.7.17. Let $Y \xrightarrow{f} X$ be a morphism with finite fibres, and for p in X, let $N(p)$ be the number of points in the fibre $f^{-1}(p)$. Prove that N is a constructible function on X. xnumberofpoints

5.7.18. Prove that fibre dimension is a semicontinuous function. I recommend this outline: xsemicont

(i) We may assume that Y ane X are affine, $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.

(ii) The theorem is true when $A \subset B$ and B is an integral extension of a polynomial subring $A[y_1, ..., y_d]$.

(iii) The fibre dimension is a constructible function.

(iv) The theorem is true when X is a smooth curve.

(v) The theorem is true for all X.

5.7.19. ??? twisted cubic specializes to plane nodal cubic??? xtwistcubic

5.7.20. Prove that a (quasiprojective) variety X that is proper is projective. xpropproj

Chapter 6 MODULES

A brief review:

localization.

If s is a nonzero element of a domain A, the symbol A_s stands for the simple localization $A[s^{-1}]$, and if $X = \operatorname{Spec} A$, then $X_s = \operatorname{Spec} A_s$. This is what we will mean by the word 'localization'.

• Let s be a nonzero element of a domain A and let M be an A-module. The localized module M_s is the A_s-module whose elements are equivalence classes of fractions ms^{-k}, with m in M. The localized module M_s becomes an A-module by restriction of scalars. A homomorphism of A-modules $N \to M_s$ extends in a natural way to a homomorphism of A_s-modules $N_s \to M_s$.

• Let $X = \operatorname{Spec} A$ be an affine variety. The intersection of two localizations $X_s = \operatorname{Spec} A_s$ and $X_t = \operatorname{Spec} A_t$ is the localization $X_{st} = \operatorname{Spec} A_{st}$.

• Let $W \subset V \subset U$ be affine open subsets of a variety X. If V is a localization of U and W is a localization of V, then W is a localization of U.

• The affine open subsets form a basis for the topology on a variety X, and the localizations of an affine variety form a basis for its topology.

• If U and V are affine open subsets of X, the open sets W that are localizations of U as well as localizations of V, form a basis for the topology on $U \cap V$.

regular functions.

The *function field* of a variety X is the field of fractions of the coordinate algebra of any one of its affine open subsets, and a *rational function* on X is an element of its function field. A rational function f is *regular* on an affine open set $U = \operatorname{Spec} A$ if it is an element of A, and f is regular on any open set U that can be covered by affine open sets on which it is regular. Thus the function field contains the regular functions on every nonempty open subset, and the regular functions on an open subset are governed by the regular functions on its affine open subsets.

See Chapters 2 and 3 for these assertions. We will use them without further comment. We will also need the concepts of *category* and *functor*. If you aren't familiar with these concepts, please read about them. You won't need to know much. Learn the definitions and look at a few examples.

6.1 The Structure Sheaf.

strsh

We associate two categories to a variety X. The first is the category *(opens)*. Its objects are the open subsets of X, and its morphisms are inclusions. If U and V are open sets and if $V \subset U$, there is a unique morphism $V \to U$ in *(opens)*. If $V \not\subset U$ there is no morphism $V \to U$.

The other category, *(affines)*, is a subcategory of the category *(opens)*, and it is the more important category. Its objects are the affine open subsets of X, and its morphisms are localizations. A morphism $V \to U$ in *(opens)* is a morphism in *(affines)* if U and V are affine and V is a localization of U — a subset of the form U_s, where s is a nonzero element of the coordinate algebra of U.

The *structure sheaf* $\mathcal{O}_X$ on a variety X is the functor

$$\text{(6.1.1)} \qquad \textit{(affines)}^\circ \xrightarrow{\mathcal{O}_X} \textit{(algebras)}$$ strshdef

from affine open sets to algebras, that sends an affine open set $U = \operatorname{Spec} A$ to its coordinate algebra A. When speaking of the structure sheaf, the coordinate algebra of U will be denoted by $\mathcal{O}_X(U)$. If it is clear which variety is being studied, we may write $\mathcal{O}$ for $\mathcal{O}_X$.

Let $V \subset U$ be affine open subsets of a variety X, say $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$. Then $A \subset B$. If V is the localization U_s, then $B = A[s^{-1}]$. But if B isn't a localization of A, it won't be clear how to construct B from A., and The exact relationship between A and B will remain obscure.

A variety that isn't affine won't be determined by its regular functions. For instance, the only rational functions that are regular at all points of the projective line are the constants, which are useless. Nevertheless, the structure sheaf extends with little difficulty to all open sets (*see* Poposition 6.1.2 below). We will be interested in regular functions on non-affine open sets, especially in regular functions on the whole variety, but one should work with affine open sets and localizations, because the relation between the coordinate algebras of an affine variety and a localization is easy to understand.

6.1.2. Proposition. *Let X be a variety. Defining $\mathcal{O}_X(U)$ to be the algebra of regular functions on the open subset U extends the structure sheaf $\mathcal{O}_X$ to a functor* *extendOone*

$$\textit{(opens)}^\circ \xrightarrow{\mathcal{O}_X} \textit{(algebras)} \qquad \square$$

The regular functions on U, the elements of $\mathcal{O}_X(U)$, are the *sections* of the structure sheaf $\mathcal{O}_X$ on U. The elements of $\mathcal{O}_X(X)$, the rational functions that are regular everywhere, are the *global sections*.

Thus, if U is a nonempty open subset of a variety X, $\mathcal{O}_X(U)$ will be a subring of the function field of X, and when $V \to U$ is a morphism in *(opens)*, $\mathcal{O}_X(U)$ will be contained in $\mathcal{O}_X(V)$. This gives us the homomorphism, an inclusion,

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

that makes $\mathcal{O}_X$ into a functor. Note that arrows are reversed by $\mathcal{O}_X$. If $V \to U$, then $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A functor that reverses arrows is a *contravariant* functor. The superscript $^\circ$ in (6.1.1) and (6.1.2) is a customary notation to indicate that a functor is contravariant.

If $V \subset U \subset X$, then $\mathcal{O}_X(V) = \mathcal{O}_U(V) = \mathcal{O}_V(V)$.

6.1.3. Proposition *The extended structure sheaf has the following* ***sheaf property****:* *extendO*

- *Let Y be an open subset of X, and let $U^i = \operatorname{Spec} A_i$ be affine open subsets that cover Y. Then*

$$\mathcal{O}_X(Y) = \bigcap \mathcal{O}_X(U^i) \qquad \left(= \bigcap A_i \right)$$

The fact that regular functions are elements of the function field makes the statement of the sheaf property especially simple here.

By definition, f is a regular function on X if there is an affine covering U^i, a covering by affine open sets, such that f is in $\mathcal{O}_X(U^i)$ for every i. Therefore the next lemma proves the proposition.

6.1.4. Lemma. *Let Y be an open subset of a variety X. The intersection $\bigcap \mathcal{O}_X(U^i)$ is the same for every affine covering $\{U^i\}$ of Y.* *capsame*

We prove the lemma first in the case of a covering of an affine open set by localizations.

sheafforloc **6.1.5. Sublemma.** *Let $U = \operatorname{Spec} A$ be an affine variety, and let $\{U^i\}$ be a covering of U by localizations, say $U^i = \operatorname{Spec} A_i$, where A_i is a localization A_{s_i} of A. Then $A = \bigcap A_i$, i.e., $\mathcal{O}(U) = \bigcap \mathcal{O}(U^i)$.*

proof. It is clear that $A \subset \bigcap A_{s_i}$. We prove the opposite inclusion. A finite subset of the set $\{U^i\}$ will cover U, so we may assume that the index set is finite. Let α be an element of $\bigcap A_{s_i}$. So for every i, $\alpha = s_i^{-r} a_i$, or $s_i^r \alpha = a_i$ with a_i in A and r an integer. We can use the same r for every i. Because $\{U^i\}$ covers U, the elements s_i generate the unit ideal in A, and so do their powers s_i^r. There are elements b_i in A such that $\sum b_i s_i^r = 1$. Then $\alpha = \sum b_i s_i^r \alpha = \sum b_i a_i$ is an element of A. $\square$

proof of Lemma 6.1.4. Say that Y is covered by affine open sets $\{U^i\}$ and also by affine open sets $\{V^j\}$. We cover the intersections $U^i \cap V^j$ by open sets $W^{ij\nu}$ that are localizations of U^i and also localizations of V^j. Fixing i and letting j and ν vary, the set $\{W^{ij\nu}\}_{j,\nu}$ will be a covering of U^i by localizations, and the sublemma shows that $\mathcal{O}(U^i) = \bigcap_{j,\nu} \mathcal{O}(W^{ij\nu})$. Then $\bigcap_i \mathcal{O}(U^i) = \bigcap_{i,j,\nu} \mathcal{O}(W^{ij\nu})$. Similarly, $\bigcap_j \mathcal{O}(V^j) = \bigcap_{i,j,\nu} \mathcal{O}(W^{ij\nu})$. $\square$

6.2 $\mathcal{O}$–Modules

module On an affine variety $\operatorname{Spec} A$, one can work with A-modules. There is no need to do anything else. However, one can't do this when a variety X isn't affine. The best one can do is to work with modules on affine open subsets. An *$\mathcal{O}_X$-module* associates a module to every affine open subset.

To define an $\mathcal{O}$-module, we need notation for working when both the module and the ring are allowed to vary. Let R and R' be rings. A *homomorphism* from an R module M to an R'-module M' consists of a ring homomorphism $R \xrightarrow{f} R'$ and a homomorphism of abelian groups $M \xrightarrow{\varphi} M'$ that is *compatible* with f, in the sense that, if $m' = \varphi(m)$ and $r' = f(r)$, then $\varphi(rm) = r'm'$.

We use the symbol *(modules)* for the category whose objects are modules over rings, and whose morphisms are homomorphisms of modules, as defined above. Because the ring homomorphisms are usually clear from context, we suppress notation for them, denoting a module M over R and a homomorphism φ to a module M' over a ring R' by the symbols M and $M \xrightarrow{\varphi} M'$.

defOmodtwo **6.2.1. Definition.** An *$\mathcal{O}$-module* $\mathcal{M}$ on a variety X is a (contravariant) functor

$$\textit{(affines)}^\circ \xrightarrow{\mathcal{M}} \textit{(modules)}$$

such that, for every affine open set U, $\mathcal{M}(U)$ is an $\mathcal{O}(U)$-module, and when s is a nonzero element of $\mathcal{O}(U)$, the module $\mathcal{M}(U_s)$ is the localization $\mathcal{M}(U)_s$ of $\mathcal{M}(U)$. The map $\mathcal{M}(U) \to \mathcal{M}(U_s)$ that makes $\mathcal{M}$ into a functor is the canonical map from a module to a localization.

Thus if U is an affine open set, $\mathcal{M}(U)$ stands for a module over the ring $\mathcal{O}(U)$ of regular functions on U, and if $U' \to U$, the map $\mathcal{M}(U) \to \mathcal{M}(U')$ is compatible with the map $\mathcal{O}(U) \to \mathcal{O}(U')$. As was explained above, the compatibility means that, if $a \in \mathcal{O}(U)$ has image a' in $\mathcal{O}(U')$ and $m \in \mathcal{M}(U)$ has image m' in $\mathcal{M}(U')$, then the image of am in $\mathcal{O}(U')$ is $a'm'$.

isloc **6.2.2. Note.** To say that $\mathcal{M}(U_s)$ **is** the localization of $\mathcal{M}(U)$, isn't completely correct. One should say that $\mathcal{M}(U_s)$ and $\mathcal{M}(U)_s$ are *canonically isomorphic*. The map $\mathcal{M}(U) \to \mathcal{M}(U_s)$ induces a map from the localization $\mathcal{M}(U)_s$ to $\mathcal{M}(U_s)$, and that map should be an isomorphism. But let's not worry about this point. $\square$

Though the definition of an $\mathcal{O}$-module will seem complicated at first, perhaps too complicated for comfort, there is no need to worry. When a module has a natural definition, the data involved are taken care of automatically. This will become clear as we go along.

Some terminology:

• A *section* of an $\mathcal{O}$-module $\mathcal{M}$ on an affine open set U is an element of $\mathcal{M}(U)$, and an element of $\mathcal{M}(X)$ is a *global section* . The *module of sections* on U is $\mathcal{M}(U)$.

• When U_s is a localization of U, the image of a section m on U via the map $\mathcal{M}(U) \to \mathcal{M}(U_s)$ is the *restriction* of m to U_s.

• An $\mathcal{O}$-module $\mathcal{M}$ is a *finite $\mathcal{O}$-module* if $\mathcal{M}(U)$ is a finite $\mathcal{O}(U)$-module for every affine open set U.

• A *homomorphism* $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of $\mathcal{O}$-modules consists of homomorphisms of $\mathcal{O}(U)$-modules

$$\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)$$

for each affine open subset U, such that, when s is a nonzero element of $\mathcal{O}(U)$, the homomorphism $\varphi(U_s)$ is the localization of the homomorphism $\varphi(U)$.

• A sequence of homomorphisms

$$\mathcal{M} \to \mathcal{N} \to \mathcal{P} \tag{6.2.3}$$ exseqsheaves

of $\mathcal{O}$-modules on a variety X is *exact* if, for every affine open subset U of X, the sequence $\mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U)$ is exact. □

6.2.4. Lemma. *Let $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ be a homomorphism of $\mathcal{O}$-modules, and let U be an affine open set. The homomorphism $\mathcal{M}(U_s) \xrightarrow{\varphi(U_s)} \mathcal{N}(U_s)$ can be obtained by localizing the homomorphism $\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)$.* *homomisloc*

proof. This follows from the fact that φ is a functor, which tells us that the diagram

$$\begin{array}{ccc} \mathcal{M}(U_s) & \longrightarrow & \mathcal{N}(U_s) \\ \uparrow & & \uparrow \\ \mathcal{M}(U) & \longrightarrow & \mathcal{M}(U) \end{array}$$

commutes. □

6.3 Some $\mathcal{O}$-Modules

6.3.1. The *free module* $\mathcal{O}^k$ is an $\mathcal{O}$-module. Its sections on an affine open set U are the elements of the free $\mathcal{O}(U)$-module $\mathcal{O}(U)^k$. In particular, $\mathcal{O}$ is an $\mathcal{O}$-module. freemod moduleexamples

6.3.2. The *kernel, image*, and *cokernel* of a homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ are among the operations that can be made on $\mathcal{O}$-modules. The kernel $\mathcal{K}$ of φ is the $\mathcal{O}$-module defined by $\mathcal{K}(U) = \ker\big(\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)\big)$ for every *affine* open set U, and the image and cokernel are defined analogously. The reason that we work with localizations is that many operations, including these, are compatible with localization. kerim

6.3.3. modules on a point sheafonpoint

Let's denote the affine variety $\operatorname{Spec}\mathbb{C}$, a point, by p. The point has just one nonempty open set, the whole space p. It is an affine open set, and $\mathcal{O}_p(p) = \mathbb{C}$. To define an $\mathcal{O}_p$-module $\mathcal{M}$, the vector space $\mathcal{M}(p)$ can be assigned arbitrarily. One may say that a module on the point **is** a complex vector space.

6.3.4. the residue field module κ_p. refldmod

Let p be a point of a variety X. The residue field module κ_p is defined as follows: If an affine open subset U of X contains p, then $\mathcal{O}(U)$ has a residue field $k(p)$ at p, and $\kappa_p(U) = k(p)$. If U doesn't contain p, then $\kappa_p(U) = 0$.

For example, let p be the point at infinity of $X = \mathbb{P}^1$ and let $\mathbb{U}^0$ and $\mathbb{U}^1$ be the standard affine open sets. Then $\kappa_p(\mathbb{U}^0) = 0$ and $\kappa_p(\mathbb{U}^1) = \mathbb{C}$.

6.3.5. torsion modules.

An $\mathcal{O}$-module $\mathcal{M}$ is a *torsion module* if $\mathcal{M}(U)$ is a torsion $\mathcal{O}(U)$-module for every affine open set U (see (**2.1.24**)).

6.3.6. ideals.

A *submodule* $\mathcal{N}$ of an $\mathcal{O}$-module $\mathcal{M}$ is an $\mathcal{O}$-module such that $\mathcal{N}(U)$ is a submodule of $\mathcal{M}(U)$ for every affine open set U. An ideal $\mathcal{I}$ of the structure sheaf is an $\mathcal{O}$-submodule of $\mathcal{O}$. If Y is a closed subvariety of a variety X, the *ideal of Y* is the submodule of $\mathcal{O}$ whose sections on an affine open subset U of X are the rational functions on X that are regular on U and that vanish on $Y \cap U$.

Let p be a point of a variety X. The *maximal ideal* at p, which we denote by $\mathfrak{m}_p$, is an ideal. If an affine open subset U contains p, its coordinate algebra $\mathcal{O}(U)$ will have a maximal ideal whose elements are the regular functions that vanish at p. That maximal ideal is the module of sections $\mathfrak{m}_p(U)$ on U. If U doesn't contain p, then $\mathfrak{m}_p(U) = \mathcal{O}(U)$.

We use the notation $V(\mathcal{I})$ for the zero set in a variety X of an ideal $\mathcal{I}$ in the structure sheaf $\mathcal{O}_X$. A point p of X is in $V(\mathcal{I})$ if, whenever U is an affine open subset of X that contains p, all elements of $\mathcal{I}(U)$ vanish at p. When $\mathcal{I}$ is the ideal of functions that vanish on a closed subvariety Y, $V(\mathcal{I}) = Y$.

defker **6.3.7. some homomorphisms**

• Let κ_p be the residue field module at a point p of X. There is a homomorphism of $\mathcal{O}$-modules $\mathcal{O} \to \kappa_p$ whose kernel is the maximal ideal $\mathfrak{m}_p$.

• Homomorphisms $\mathcal{O}^n \to \mathcal{O}^m$ of free $\mathcal{O}$-modules correspond to $m \times n$-matrices of global sections of $\mathcal{O}$.

• Multiplication by a global section f of $\mathcal{O}$ defines a homomorphism $\mathcal{M} \xrightarrow{f} \mathcal{M}$.

• Let $\mathcal{M}$ be an $\mathcal{O}$-module. Homomorphisms of $\mathcal{O}$-modules $\mathcal{O} \xrightarrow{\varphi} \mathcal{M}$ correspond bijectively to global sections of $\mathcal{M}$.

This last example is analogous to the fact that, when M is a module over a ring A, homomorphisms $A \to M$ correspond to elements of M. If m is a global section of $\mathcal{M}$, the homomorphism $\mathcal{O}(U) \xrightarrow{\varphi(U)} \mathcal{M}(U)$ is multiplication by the restriction of m to U. If we denote that restriction by the same letter m, then $\varphi(f) = fm$. Similarly, **(c)** is analogoues to the fact that multipliexation by an element a of R defines a homomorphism $M \to M$.

6.4 The Sheaf Property

nonaffine-sections In this section, we extend an $\mathcal{O}$-module $\mathcal{M}$ on a variety X to a functor *(opens)*$^\circ \xrightarrow{\widetilde{\mathcal{M}}}$ *(modules)* on all open subsets of X with these properties:

• $\widetilde{\mathcal{M}}(Y)$ is an $\mathcal{O}(Y)$-module for every open subset Y.

• When U is an affine open set, $\widetilde{\mathcal{M}}(U) = \mathcal{M}(U)$.

• $\widetilde{\mathcal{M}}$ has the *sheaf property* that is described below.

The tilde $\sim$ is used for clarity. When we have finished with the discussion, we will use the same notation for the functor on *(affines)* and for its extension to *(opens)*.

defsect **6.4.1.** Let *(opens)*$^\circ \xrightarrow{\widetilde{\mathcal{M}}}$ *(modules)* be a functor. If U is an open subset of X, an element of $\widetilde{\mathcal{M}}(U)$ is a *section* of $\widetilde{\mathcal{M}}$ on U. If $V \xrightarrow{j} U$ is an inclusion of open subsets, the associated homomorphism $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is the *restriction* from U to V.

When $V \xrightarrow{j} U$ is an inclusion of open sets, the restriction to V of a section m on U may be denoted by $j^\circ m$. However, the restriction operation occurs very often. Because of this, we usually abbreviate, using the same symbol m for a section and for its restriction. If an open set V is contained in two open sets U and U', and if m and m' are sections of $\widetilde{\mathcal{M}}$ on U and U', respectively, we may say that m and m' are *equal on* V if their restrictions to V are equal. For example, if the restriction of a setcion m to V is zero, we may say $m = 0$ on V. $\square$

extendO-mod **6.4.2. Theorem.** *An $\mathcal{O}$-module $\mathcal{M}$ extends uniquely to a functor*

$$\textit{(opens)}^\circ \xrightarrow{\widetilde{\mathcal{M}}} \textit{(modules)}$$

that has the sheaf property (6.4.4) below. Moreover, for every open set U, $\widetilde{\mathcal{M}}(U)$ is an $\mathcal{O}(U)$-module, and for every inclusion $V \to U$ of nonempty open sets, the map $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is compatible with the map $\mathcal{O}(U) \to \mathcal{O}(V)$.

The proof of this theorem isn't especially difficult, but it is lengthy because there are several things to check. In order not to break up the discussion, we have put the proof into Section 6.9 at the end of the chapter.

Though the theorem describes the sections of an $\mathcal{O}$-module on every affine open set, one always works with the affine open sets. Sometimes, we will want to look at sections of an $\mathcal{O}$-module on a non-affine open set, but most of the time, the non-affine open sets are just along for the ride.

(6.4.3) the sheaf property sheafprop

The sheaf property is the key requirement that determines the extension of an $\mathcal{O}$-module $\mathcal{M}$ to a functor $\widetilde{\mathcal{M}}$ on *(opens)*.

Let Y be an open subset of X, and let $\{U^i\}$ be an affine covering of Y. The intersections $U^{ij} = U^i \cap U^j$ are also affine open sets, so $\mathcal{M}(U^i)$ and $\mathcal{M}(U^{ij})$ are defined. The sheaf property asserts that an element m of $\widetilde{\mathcal{M}}(Y)$ corresponds to a set of elements m_i in $\mathcal{M}(U^i)$ such that the restrictions of m_j and m_i to U^{ij} are equal.

If the affine open subsets U^i are indexed by $i = 1, ..., n$, the sheaf property asserts that an element of $\widetilde{\mathcal{M}}(Y)$ is determined by a vector $(m_1, ..., m_n)$ with m_i in $\mathcal{M}(U^i)$, such that the restrictions of m_i and m_j to U^{ij} are equal. This means that $\widetilde{\mathcal{M}}(Y)$ is the kernel of the difference map β:

(6.4.4)
$$\prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^{ij})$$
sheafker

that sends the vector $(m_1, ..., m_n)$ to the $n \times n$ matrix (z_{ij}), where z_{ij} is the difference $m_j - m_i$ of restrictions of the sections m_j and m_i to U^{ij}. The analogous description is true when the index set is infinite.

In short, the sheaf property tells us that sections of $\widetilde{\mathcal{M}}$ are determined locally: A section on an open set Y is determined by its restrictions to the open subsets U^i of any affine covering of Y.

Note. With notation as above, there is a morphism $U^{ij} \to U^i$ in *(opens)* because U^{ij} is contained in U^i. However, this morphism needn't be a localization, and if it isn't a localization, it won't be a morphism in *(affines)*. Then the restriction maps $\mathcal{M}(U^i) \to \mathcal{M}(U^{ij})$ won't be a part of the structure of an $\mathcal{O}$-module. We need a definition of the restriction map for an arbitrary inclusion $V \to U$ of affine open subsets. This point will be taken care of in the proof of Theorem 6.4.2. (See Step 2 in Section 6.9.) So we don't need to worry about it here. $\square$

We drop the tilde now, and denote the extension of an $\mathcal{O}$-module $\mathcal{M}$ to all open sets by the same symbol $\mathcal{M}$. The sheaf property for $\mathcal{M}$ is the statement that, when $\{U^i\}$ is an affine covering of an open set U, the sequence

(6.4.5)
$$0 \to \mathcal{M}(U) \xrightarrow{\alpha} \prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^{ij})$$
sheaf-proptwo

is exact, where α is the product of the restriction maps, and β is the difference map described in (6.4.4). So $\mathcal{M}(U)$ is mapped isomorphically to the kernel of β. Elements of $\mathcal{M}(U)$ correspond bijectively to vectors $(m_1, ..., m_n)$, with m_i in $\mathcal{M}(U^i)$, such that the restrictions of m_i and m_j to $\mathcal{M}(U^{ij})$ are equal.

The next corollary follows from Theorem 6.4.2.

6.4.6. Corollary. *Let $\{U^i\}$ be an affine covering of a variety X.* *injoncover*
(i) *An $\mathcal{O}$-module $\mathcal{M}$ is the zero module if and only if $\mathcal{M}(U^i) = 0$ for every i.*
(ii) *A homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of $\mathcal{O}$-modules is injective, surjective, or bijective if and only if the maps $\mathcal{M}(U^i) \xrightarrow{\varphi(U^i)} \mathcal{N}(U^i)$ are injective, surjective, or bijective, respectively, for every i.*

proof. **(i)** Let V be an open subset of X. We cover each intersection $V \cap U^i$ by affine open sets $V^{i\nu}$ that are localizations of U^i. These sets, taken together, cover V. If $\mathcal{M}(U^i) = 0$, then the localizations $\mathcal{M}(V^{i\nu})$ are zero too. The sheaf property shows that the map $\mathcal{M}(V) \to \prod \mathcal{M}(V^{i\nu})$ is injective, and therefore that $\mathcal{M}(V) = 0$.

(ii) This follows from **(i)** because a homomorphism φ is injective or surjective if and only if its kernel or its cokernel is zero. $\square$

(6.4.7) families of open sets omod

It is convenient to have a more compact notation for the sheaf property (6.4.5), and for this, one can introduce symbols to represent families of open sets. Say that $\mathbf{U}$ and $\mathbf{V}$ represent families of open sets $\{U^i\}$ and $\{V^\nu\}$, respectively. A *morphism* of families $\mathbf{V} \to \mathbf{U}$ consists of a morphism from each V^ν to one of the subsets U^i. Such a morphism will be given by a map of index sets sending $\nu \rightsquigarrow i_\nu$, such that $V^\nu \subset U^{i_\nu}$.

There may be more than one morphism $\mathbf{V} \to \mathbf{U}$, because a subset V^ν may be contained in more than one of the subsets U^i. To define a morphism, one must make a choice among those subsets. For example, let $\mathbf{U} = \{U^i\}$ be a family of open sets, and let V be another open set. For each i such that $V \subset U^i$, there is a morphism $V \to \mathbf{U}$ that sends V to U^i. In the other direction, there is a unique morphism $\mathbf{U} \to V$ provided that $U^i \subset V$ for all i.

We extend a functor *(opens)*$^\circ \xrightarrow{\mathcal{M}}$ *(modules)* to families $\mathbf{U} = \{U^i\}$, defining

section-sonfamily (6.4.8)
$$\mathcal{M}(\mathbf{U}) = \prod \mathcal{M}(U^i).$$

Then a morphism of families $\mathbf{U} \xrightarrow{f} \mathbf{V}$ defines a map $\mathcal{M}(\mathbf{V}) \xleftarrow{f^\circ} \mathcal{M}(\mathbf{U})$ in a way that is fairly obvious, though our notation for it is clumsy. Say that $\mathbf{V} = \{V^\nu\}$, that $\mathbf{U} = \{U^i\}$, and that f is given by a map $\nu \rightsquigarrow i_\nu$ of index sets, such that $V^\nu \to U^{i_\nu}$. A section of $\mathcal{M}$ on $\mathbf{U}$, an element of $\mathcal{M}(\mathbf{U})$, can be thought of as a vector $u = (u_i)$ with $u_i \in \mathcal{M}(U^i)$, and a section of $\mathcal{M}(\mathbf{V})$ as a vector $v = (v_\nu)$ with $v_\nu \in \mathcal{M}(V^\nu)$. If v_ν denotes the restriction of u_{i_ν} to V^ν, the restriction $f^\circ(u)$ of $u = \{u_i\}$ to $\mathbf{V}$ is $v = \{v_\nu\}$.

We write the sheaf property in terms of families of open sets: Let $\mathbf{U}_0 = \{U^i\}$ be an affine covering of an open set Y, and let $\mathbf{U}_1$ denote the family $\{U^{ij}\}$ of intersections: $U^{ij} = U^i \cap U^j$. The intersections are also affine, and there are two sets of inclusions

$$U^{ij} \subset U^i \quad \text{and} \quad U^{ij} \subset U^j$$

They give us two morphisms of families $\mathbf{U}_1 \xrightarrow{d_0, d_1} \mathbf{U}_0$ of affine open sets: $U^{ij} \xrightarrow{d_0} U^j$ and $U^{ij} \xrightarrow{d_1} U^i$. We also have a morphism $\mathbf{U}_0 \to Y$, and the composed morphisms $\mathbf{U}_1 \xrightarrow{d_i} \mathbf{U}_0 \to Y$ are equal. These maps form what we all a *covering diagram*

covdiagr (6.4.9)
$$Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$$

When we apply a functor *(opens)* $\xrightarrow{\mathcal{M}}$ *(modules)* to this diagram, we obtain a sequence

defbeta (6.4.10)
$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha_{\mathbf{U}}} \mathcal{M}(\mathbf{U}_0) \xrightarrow{\beta_{\mathbf{U}}} \mathcal{M}(\mathbf{U}_1)$$

where $\alpha_{\mathbf{U}}$ is the restriction map and $\beta_{\mathbf{U}}$ is the difference $d_0 - d_1$ of the maps induced by the two morphisms $\mathbf{U}_1 \rightrightarrows \mathbf{U}_0$. The sheaf property for the covering $\mathbf{U}_0$ of Y (6.4.5) is the assertion that this sequence is exact, which means that $\alpha_{\mathbf{U}}$ is injective, and that its image is the kernel of $\beta_{\mathbf{U}}$.

twoopensets **6.4.11. Note.** Let $\{U^i\}$ be an affine covering of Y. Then, with $U^{ij} = U^i \cap U^j$, we will have $U^{ii} = U^i$ and $U^{ij} = U^{ji}$. These coincidences lead to redundancy in the statement (6.4.10) of the sheaf property. If the indices are $i = 1, ..., k$, we only need to look at intersections U^{ij} with $i < j$. The product $\mathcal{M}(\mathbf{U}_1) = \prod_{i,j} \mathcal{M}(U^{ij})$ that appears in the sheaf property can be replaced by the product with increasing pairs of indices $\prod_{i<j} \mathcal{M}(U^{ij})$. For instance, if an open set Y is covered by two affine open sets U and V, the sheaf property for this covering is an exact sequence

$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha} \big[\mathcal{M}(U) \times \mathcal{M}(V)\big] \xrightarrow{\beta} \big[\mathcal{M}(U \cap U) \times \mathcal{M}(U \cap V) \times \mathcal{M}(V \cap U) \times \mathcal{M}(V \cap V)\big]$$

The exact sequence

twoopens (6.4.12)
$$0 \to \mathcal{M}(Y) \to \big[\mathcal{M}(U) \times \mathcal{M}(V)\big] \xrightarrow{-,+} \mathcal{M}(U \cap V)$$

is equivalent. $\square$

6.4.13. Example. strsheafPn

Let A denote the polynomial ring $\mathbb{C}[x,y]$, and let V be the complement of a point p in the affine space $X = \operatorname{Spec} A$. This is an open set, but it isn't affine. We cover V by two localizations of X: $X_x = \operatorname{Spec} A[x^{-1}]$ and $X_y = \operatorname{Spec} A[y^{-1}]$. The sheaf property (6.4.12) for $\mathcal{O}_X$ and for this covering is equivalent to an exact sequence

$$0 \to \mathcal{O}_X(V) \to A[x^{-1}] \times A[y^{-1}] \to A[(xy)^{-1}]$$

It shows that a regular function on V is in the intersection $A[x^{-1}] \cap A[y^{-1}]$, which is equal to A. Therefore the sections of the structure sheaf $\mathcal{O}_X$ on V are the elements of A. They are the same as the sections on X. $\square$

We have been working (tacitly) with nonempty open sets. This isn't much of a problem, but when a module $\mathcal{M}$ on *(affines)* is extended to a module on all open sets, the empty set should be included. The next lemma takes care of the empty set.

6.4.14. Lemma. *The only section of an $\mathcal{O}$-module $\mathcal{M}$ on the empty set is the zero section: $\mathcal{M}(\emptyset) = \{0\}$. In particular, $\mathcal{O}(\emptyset)$ is the zero ring.* *emptyset*

proof. This follows from the sheaf property. The empty set $\emptyset$ is covered by the empty covering, the covering indexed by the empty set. Therefore $\mathcal{M}(\emptyset)$ is contained in an empty product. We want the empty product to be a module, and we have no choice but to define it to be zero. Then $\mathcal{M}(\emptyset)$ is zero too.

If you find this reasoning pedantic, you can take $\mathcal{M}(\emptyset) = \{0\}$ as an axiom. $\square$

6.5 Some More Modules

moremodules

6.5.1. kernel kershm

As we have noted, many operations that one makes on modules over a ring are compatible with localization, and therefore can be made on $\mathcal{O}$-modules. However, for sections over a non-affine open set one must use the sheaf property. The sections over a non-affine open set are almost *never* determined by an operation. The kernel of a homomorphism is among the few exceptions.

6.5.2. Proposition. *Let X be a variety, and let $\mathcal{K}$ be the kernel of a homomorphism of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{N}$, so that the there is an exact sequence $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{N}$. For every open subset Y of X, the sequence of sections* *leftexact-section*

(6.5.3) $$0 \to \mathcal{K}(Y) \to \mathcal{M}(Y) \to \mathcal{N}(Y)$$ *lexsect*

is exact.

proof. We choose a covering diagram $Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$, and inspect the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}(\mathbf{U}_0) & \longrightarrow & \mathcal{M}(\mathbf{U}_0) & \longrightarrow & \mathcal{N}(\mathbf{U}_0) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}(\mathbf{U}_1) & \longrightarrow & \mathcal{M}(\mathbf{U}_1) & \longrightarrow & \mathcal{N}(\mathbf{U}_1) \end{array}$$

where the vertical maps are the difference maps $\beta_{\mathbf{U}}$ described in (6.4.10). The rows are exact because $\mathbf{U}_0$ and $\mathbf{U}_1$ are families of affines, and the sheaf property asserts that the kernels of the vertical maps form the sequence (6.5.3). The sequence of kernels is exact because taking kernels is a left exact operation (2.1.20). $\square$

The section functor isn't right exact. When $\mathcal{M} \to \mathcal{N}$ is a surjective homomorphism of $\mathcal{O}$-modules and Y is a non-affine open set, the map $\mathcal{M}(Y) \to \mathcal{N}(Y)$ often fails to be surjective. There is an example below. Cohomology, which will be discussed in the next chapter, is a substitute for right exactness.

6.5.4. modules on the projective line coverpone

The projective line $\mathbb{P}^1$ is covered by the standard open sets $\mathbb{U}^0$ and $\mathbb{U}^1$, and the intersection $\mathbb{U}^{01} = \mathbb{U}^0 \cap \mathbb{U}^1$ is a localization of $\mathbb{U}^0$ and of $\mathbb{U}^1$. The coordinate algebras of these affine open sets are $\mathcal{O}(\mathbb{U}^0) = A_0 = \mathbb{C}[u]$, $\mathcal{O}(\mathbb{U}^1) = A_1 = \mathbb{C}[u^{-1}]$, and $\mathcal{O}(\mathbb{U}^{01}) = A_{01} = \mathbb{C}[u, u^{-1}]$. The form (6.4.12) of the sheaf property asserts that a global section of $\mathcal{O}$ is determined by polynomials $f(x)$ and $g(x)$ such that $f(u) = g(u^{-1})$ in A_{01}. The only such polynomials are the constants. So the constants are the only rational functions that are regular everywhere on $\mathbb{P}^1$. I think we knew this.

If $\mathcal{M}$ is an $\mathcal{O}$-module, then $M_0 = \mathcal{M}(\mathbb{U}^0)$ and $M_1 = \mathcal{M}(\mathbb{U}^1)$ will be modules over the algebras A_0 and A_1, respectively. The A_{01}-module $\mathcal{M}(\mathbb{U}^{01}) = M_{01}$ will be obtained by localizing M_0 and also by localizing M_1. Let $v = u^{-1}$. Then

$$M_0[u^{-1}] \approx M_{01} \approx M_1[v^{-1}]$$

As (6.4.12) tells us, a global section of $\mathcal{M}$ is determined by a pair of elements m_1 and m_2 in M_1 and M_2, respectively, that become equal in the common localization M_{01}. The next lemma shows that this data determines the module $\mathcal{M}$.

modon-pone **6.5.5. Lemma.** *With notation as above, let M_0, M_1, and M_{01} be modules over the algebras A_0, A_1, and A_{01}, respectively, and let $M_0[u^{-1}] \xrightarrow{\varphi_0} M_{01}$ and $M_1[v^{-1}] \xrightarrow{\varphi_1} M_{01}$ be A_{01}-isomorphisms. There is an $\mathcal{O}_X$-module $\mathcal{M}$, unique up to isomorphism, such that $\mathcal{M}(\mathbb{U}^0)$ and $\mathcal{M}(\mathbb{U}^1)$ are isomorphic to M_0 and M_1, respectively, and such that the diagram below commutes.*

$$\begin{tikzcd}
\mathcal{M}(\mathbb{U}^0) \arrow[r] \arrow[d] & \mathcal{M}(\mathbb{U}^{01}) \arrow[d] & \mathcal{M}(\mathbb{U}^1) \arrow[l] \arrow[d] \\
M_0 \arrow[r, "\varphi_0"] & M_{01} & M_1 \arrow[l, "\varphi_1"']
\end{tikzcd}$$

The proof is at the end of this section.

Suppose that M_0 and M_1 are free modules over A_0 and A_1. The common localization M_{01} will be a free A_{01}-module. A basis $\mathbf{B}_0$ of the free A_0-module M_0 will also be a basis of the A_{01}-module M_{01}, and a basis $\mathbf{B}_1$ of M_1 will be a basis of M_{01}. When regarded as bases of M_{01}, $\mathbf{B_0}$ and $\mathbf{B_1}$ will be related by an invertible A_{01}-matrix P, and as Lemma 6.5.5 tells us, that matrix determines $\mathcal{M}$ up to isomorphism. When M_i have rank one, P will be an invertible 1×1 matrix in the Laurent polynomial ring $A_{01} = \mathbb{C}[u, u^{-1}]$, a unit of that ring. The units in A_{01} are scalar multiples of powers of u. Since the scalar can be absorbed into one of the bases, an $\mathcal{O}$-module of rank 1 is determined, up to isomorphism, by a power of u. It is one of the twisting modules that will be described below, in Section 6.8.

The Birkhoff-Grothendieck Theorem, which will be proved in Chapter 8, describes the $\mathcal{O}$-modules on the projective line whose sections on $\mathbb{U}^0$ and on $\mathbb{U}^1$ are free, as direct sums of free $\mathcal{O}$-modules of rank one. This means that by changing the bases $\mathbf{B}_0$ and $\mathbf{B}_1$, one can diagonalize the matrix P. The changes of basis will be given by an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1, respectively. In down-to-Earth terms, the Birkhoff-Grothendieck Theorem asserts that, for any invertible A_{01}-matrix P, there exist an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1 , such that $Q_0^{-1}PQ_1$ is diagonal, and its diagonal entries are powers of u. $\square$

tensprod-module **6.5.6. tensor products**

Tensor products are compatible with localization. If M and N are modules over a domain A and s is a nonzero element of A, the canonical map $(M \otimes_A N)_s \to M_s \otimes_{A_s} N_s$ is an isomorphism (Corollary 2.1.32). Therefore the tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ of $\mathcal{O}$-modules $\mathcal{M}$ and $\mathcal{N}$ is defined. On an affine open set U, $[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](U)$ is the tensor product $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules, let $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ be the tensor product module, and let V be an arbitrary open subset of X. There is a canonical map

ten-sprodmap (6.5.7) $$\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \to [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)$$

By definition of the tensor product module, this map is an equality when V is affine. To describe the map for arbitrary V, we cover V by a family $\mathbf{U}_0$ of affine open sets and form a diagram

$$\begin{tikzcd}
 & \mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \arrow[r, "f"] \arrow[d, "a"] & \mathcal{M}(\mathbf{U}_0) \otimes_{\mathcal{O}(\mathbf{U}_0)} \mathcal{N}(\mathbf{U}_0) \arrow[r, "g"] \arrow[d, "b"] & \mathcal{M}(\mathbf{U}_1) \otimes_{\mathcal{O}(\mathbf{U}_1)} \mathcal{N}(\mathbf{U}_1) \arrow[d, "c"] \\
0 \arrow[r] & {[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)} \arrow[r] & {[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](\mathbf{U}_0)} \arrow[r, "g"] & {[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](\mathbf{U}_1)}
\end{tikzcd}$$

The family $\mathbf{U}_1$ of intersections consists of affine open sets, as does $\mathbf{U}_0$, so the vertical maps b and c are equalities. The bottom row is exact, and the composition gf is zero. So f maps $\mathcal{M}(V)\otimes_{\mathcal{O}(V)}\mathcal{N}(V)$ to the kernel of g, which is equal to $[\mathcal{M}\otimes_{\mathcal{O}}\mathcal{N}](V)$. The map a is (6.5.7). When V isn't affine, this map needn't be either injective or surjective.

6.5.8. Examples. These examples illustrate the failure of bijectivity of (6.5.7). residue-fieldmodule

(i) Let p and q be distinct points of the projective line X, and let κ_p and κ_q be the residue field modules on X. Then $\kappa_p(X)=\kappa_q(X)=\mathbb{C}$, so $\kappa_p(X)\otimes_{\mathcal{O}(X)}\kappa_q(X)=\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}=\mathbb{C}$. But $\kappa_p\otimes_{\mathcal{O}}\kappa_q=0$. The map (6.5.7) with $V=X$,

$$\kappa_p(X)\otimes_{\mathcal{O}(X)}\kappa_q(X)\ \to\ [\kappa_p\otimes_{\mathcal{O}}\kappa_q](X)$$

is the zero map. It isn't injective.

(ii) Let p a point of a variety X, and let $\mathfrak{m}_p$ and κ_p be the maximal ideal and residue field modules at p. There is an exact sequence of $\mathcal{O}$-modules

(6.5.9) $$0\to\mathfrak{m}_p\to\mathcal{O}\xrightarrow{\pi_p}\kappa_p\to 0$$ ideal-sheafatp

In this case, the sequence of global sections is exact.

(iii) Let p and q be the points $(1,0)$ and $(0,1)$ of the projective line $\mathbb{P}^1$. We form a homomorphism

$$\mathfrak{m}_p\times\mathfrak{m}_q\xrightarrow{\varphi}\mathcal{O}$$

φ being the map $(a,b)\to b-a$. On the open set $\mathbb{U}^0$, $\mathfrak{m}_q\to\mathcal{O}$ is bijective and therefore surjective. Similarly, $\mathfrak{m}_p\to\mathcal{O}$ is surjective on $\mathbb{U}^1$. Since $\mathbb{U}^0$ and $\mathbb{U}^1$ cover $\mathbb{P}^1$, φ is surjective (6.4.6)**(ii)**. The only global section of $\mathfrak{m}_p\times\mathfrak{m}_q$ is zero, while $\mathcal{O}$ has the nonzero global section 1. The map on global sections determined by φ isn't surjective. □

6.5.10. the function field module ffldmod

Let F be the function field of a variety X. The *function field module* $\mathcal{F}$ is defined as follows: The module of sections $\mathcal{F}(U)$ on any nonempty open set U is the field F. It is called a *constant* $\mathcal{O}$-module because $\mathcal{F}(U)$ is the same for every nonempty U. It won't be a finite module unless X is a point.

Tensoring with the function field module: Let $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X, and let $\mathcal{F}$ be the function field module. We describe the tensor product module $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{F}$.

If $U=\operatorname{Spec}A$ is an affine open set and $M=\mathcal{M}(U)$, the module of sections of on U is the F-vector space $M\otimes_A F$. If S is the multipicative system of nonzero elements of A, then $M\otimes_A F$ is the localization MS^{-1}. On a simple localization U_s, the module of sections will be $M_s\otimes_{A_s}F$, which is the same as $M\otimes_A F$, because s is invertible in F. The vector space $M\otimes_A F$ is independent of the affine open set U. So $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{F}$ is a constant $\mathcal{O}$-module. If $\mathcal{M}$ is a torsion module, the tensor product $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{F}$ will be zero.

(6.5.11) $\mathcal{O}$-modules on affine varieties modaff

The next proposition shows that, on an affine variety $\operatorname{Spec}A$, $\mathcal{O}$-modules correspond to A-modules .

6.5.12. Proposition. *Let $X=\operatorname{Spec}A$ be an affine variety. Sending an $\mathcal{O}$-module $\mathcal{M}$ to the A-module $\mathcal{M}(X)$ of its global sections defines a bijective correspondence between $\mathcal{O}$-modules and A-modules.* *moduleon-affines*

proof. We must invert the functor $\mathcal{O}$*–(modules)* $\to$ A*–(modules)* that sends $\mathcal{M}$ to $\mathcal{M}(X)$. Given an A-module M, the corresponding $\mathcal{O}$-module $\mathcal{M}$ is defined as follows: Let $U=\operatorname{Spec}B$ be an affine open subset of X. The inclusion $U\subset X$ corresponds to an algebra homomorphism $A\to B$. We define $\mathcal{M}(U)$ to be the B-module $B\otimes_A M$. If s is a nonzero element of B, then $B_s\otimes_A M$ is canonically isomorphic to the localization $(B\otimes_A M)_s$ of $B\otimes_A M$. Therefore $\mathcal{M}$ is an $\mathcal{O}$-module, and $\mathcal{M}(X)=M$.

Conversely, let $\mathcal{M}$ be an $\mathcal{O}$-module such that $\mathcal{M}(X)=M$. Then, with notation as above, the map $M=\mathcal{M}(X)\to\mathcal{M}(U)$ induces a homomorphism of B-modules $M\otimes_A B\to\mathcal{M}(U)$. When U is a localization X_s of X, so that $B=A_s$, both $M\otimes_A A_s$ and $\mathcal{M}(X_s)$ are localizations of M, so they are isomorphic. Therefore the module $\mathcal{M}$ is determined up to isomorphism. □

delete-point **6.5.13. Example.**

This example shows that, when an open set isn't affine, defining $\mathcal{M}(V) = B \otimes_A M$, as in the proof of Proposition 6.5.12, may be wrong. Let X be the affine plane $\operatorname{Spec} A$, $A = \mathbb{C}[x,y]$, let V be the complement of the origin in X, and let M be the A-module A/yA. This module can be identified with $\mathbb{C}[x]$, which becomes an A-module when scalar multiplication by y is defined to be zero. Here $\mathcal{O}(V) = \mathcal{O}(X) = A$ (6.4.13). If we followed the method used for affine open sets, we would set $\mathcal{M}(V) = A \otimes_A M = \mathbb{C}[x]$. To identify $\mathcal{M}(V)$ correctly, we cover V by the two affine open sets $X_x = \operatorname{Spec} A[x^{-1}]$ and $X_y = \operatorname{Spec} A[y^{-1}]$. Then $\mathcal{M}(X_x) = M[x^{-1}] = \mathbb{C}[x, x^{-1}]$, while $\mathcal{M}(X_y) = 0$. The sheaf property of $\mathcal{M}$ shows that $\mathcal{M}(V) \approx \mathcal{M}(X_x) = \mathbb{C}[x, x^{-1}]$. □

omodlim **(6.5.14) limits of $\mathcal{O}$-modules**

A directed set $M_\bullet$ of modules over a ring R is a sequence of homomorphisms $M_0 \to M_1 \to M_2 \to \cdots$. Its *limit* $\varinjlim M_\bullet$ is the R-module whose elements are equivalence classes on the elements of the union $\bigcup M_k$, the equivalence relation being that elements m in M_i and m' in M_j are equivalent if they have the same image in M_n when n is sufficiently large. An element of $\varinjlim M_\bullet$ will be represented by an element of M_i for some i.

limexample **6.5.15. Example.** Let $R = \mathbb{C}[x]$ and let $\mathfrak{m}$ be the maximal ideal xR. Repeated multiplication by x defines a directed set

$$R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow \cdots$$

Its limit is isomorphic to the Laurent polynomial ring $R[x^{-1}] = \mathbb{C}[x, x^{-1}]$. □

limits **6.5.16.** A *directed set of $\mathcal{O}$-modules* on a variety X is an infinite sequence $\mathcal{M}_\bullet = \{\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \cdots\}$ of homomorphisms of $\mathcal{O}$-modules. For every affine open set U, the $\mathcal{O}(U)$-modules $\mathcal{M}_n(U)$ will form a directed set, as defined above. The *direct limit* $\varinjlim \mathcal{M}_\bullet$ is defined simply, by taking the limit for each affine open set: $[\varinjlim \mathcal{M}_\bullet](U) = \varinjlim [\mathcal{M}_\bullet(U)]$. This limit operation is compatible with localization, so $\varinjlim \mathcal{M}_\bullet$ is an $\mathcal{O}$-module. In fact, the equality $[\varinjlim \mathcal{M}_\bullet](U) = \varinjlim [\mathcal{M}_\bullet(U)]$ will be true for every open set, not only for affine open sets.

A map of directed sets of $\mathcal{O}$-modules $\mathcal{M}_\bullet \to \mathcal{N}_\bullet$ is a diagram

$$\begin{array}{ccccc} \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \mathcal{N}_1 & \longrightarrow & \mathcal{N}_2 & \longrightarrow & \cdots \end{array}$$

A sequence $\mathcal{M}_\bullet \to \mathcal{N}_\bullet \to \mathcal{P}_\bullet$ of maps of directed sets is exact if the sequences $\mathcal{M}_i \to \mathcal{N}_i \to \mathcal{P}_i$ are exact for every i.

jstarlimit **6.5.17. Lemma. (i)** *The limit operation is exact. If $\mathcal{M}_\bullet \to \mathcal{N}_\bullet \to \mathcal{P}_\bullet$ is an exact sequence of directed sets of $\mathcal{O}$-modules, the limits form an exact sequence.*

(ii) *Tensor products are compatible with limits: If $\mathcal{N}_\bullet$ is a directed set of $\mathcal{O}$-modules and $\mathcal{M}$ is another $\mathcal{O}$-module, then $\varinjlim [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}_\bullet] \approx \mathcal{M} \otimes_{\mathcal{O}} [\varinjlim \mathcal{N}_\bullet]$.* □

module-Hom **(6.5.18) the module of homomorphisms**

Let M and N be modules over a ring A. The set of homomorphisms $M \to N$, which is often denoted by $\operatorname{Hom}_A(M, N)$, becomes an A-module with some fairly obvious laws of composition: If φ and ψ are homomorphisms and a is an element of A, then $\varphi{+}\psi$ and $a\varphi$ are defined by

homis-module (6.5.19) $$[\varphi{+}\psi](m) = \varphi(m) + \psi(m) \quad \text{and} \quad [a\varphi](m) = a\varphi(m)$$

Because φ is a module homomorphism, it is also true that $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$, and that $a\varphi(m) = \varphi(am)$.

6.5.20. Lemma. (i) *An A-module N is canonically isomorphic to the module $\mathrm{Hom}_A(A,N)$. The homomorphism $A \xrightarrow{\varphi} N$ that corresponds to an element v of N is multiplication by v: $\varphi(a)=av$. The element of N that corresponds to a homomorphism $A \xrightarrow{\varphi} N$ is $v=\varphi(1)$.* Nequal-sHom
(ii) *$\mathrm{Hom}_A(A^k,N)$ is isomorphic to N^k, and $\mathrm{Hom}_A(A^k,A^\ell)$ is isomorphic to the module $A^{\ell\times k}$ of $k\times\ell$ A-matrices.* □

6.5.21. Lemma. *As a functor in two variables, Hom_A is left exact and* ***contravariant*** *in the first variable: For any A-module N, an exact sequence $M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_3 \to 0$ of A-modules induces an exact sequence* homfinite

$$0 \to \mathrm{Hom}_A(M_3,N) \xrightarrow{\circ b} \mathrm{Hom}_A(M_2,N) \xrightarrow{\circ a} \mathrm{Hom}_A(M_1,N)$$

Hom_A is left exact and **covariant** *in the second variable.* □

6.5.22. Corollary. *If M and N are finite A-modules over a notherian ring A, then $\mathrm{Hom}_A(M,N)$ is a finite A-module.* homfinitetwo

proof. Because M is finitely generated, there is a surjective map $A^k \to M$, which gives us an injective map $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(A^k,N) = N^k$. So $\mathrm{Hom}_A(M,N)$ is isomorphic to a submodule of the finite module N^k. Therefore it is a finite module. □

6.5.23. Lemma. *Let M and N be modules over a noetherian domain A, and suppose that M is a finite module. Let s be a nonzero element of A. The localization $\big(\mathrm{Hom}_A(M,N)\big)_s$ is canonically isomorphic to $\mathrm{Hom}_{A_s}(M_s,N_s)$. The analogous statement is true for localization by a multiplicative system S.* localize-Hom

proof. Since $\mathrm{Hom}_A(A,M) \approx M$, it is true that $(\mathrm{Hom}_A(A,M))_s \approx M_s \approx \mathrm{Hom}_{A_s}(A_s,M_s)$ and that $(\mathrm{Hom}_A(A^k,M))_s \approx M_s^k \approx \mathrm{Hom}_{A_s}(A_s^k,M_s)$.

We choose a presentation $A^\ell \to A^k \to M \to 0$ of the A-module M (**2.1.21**). Its localization $A_s^\ell \to A_s^k \to M_s \to 0$, is a presentation of the A_s-module M_s. The sequence

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(A^k,N) \to \mathrm{Hom}_A(A^\ell,N)$$

is exact, and so is its localization. So the lemma follows from the case that $M=A^k$. □

This lemma shows that, when $\mathcal{M}$ and $\mathcal{N}$ are finite $\mathcal{O}$-modules on a variety X, an $\mathcal{O}$-module of homomorphisms $\mathcal{M}\to\mathcal{N}$ is defined. This $\mathcal{O}$-module may be denoted by $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. When $U=\operatorname{Spec} A$ is an affine open set, $M=\mathcal{M}(U)$, and $N=\mathcal{N}(U)$, the module of sections of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ on U is the A-module $\mathrm{Hom}_A(M,N)$.

The analogues of Lemma 6.5.20 and lemma 6.5.21 are true for $\underline{\mathrm{Hom}}$:

6.5.24. Corollary. (i) *An $\mathcal{O}$-module $\mathcal{M}$ on a smooth curve Y is isomorphic to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O},\mathcal{M})$.* leftexco
(ii) *The functor $\underline{\mathrm{Hom}}$ is left exact and* ***contravariant*** *in the first variable, and it is left exact and* ***covariant*** *in the second variable.* □

6.5.25. Note. The notations Hom and $\underline{\mathrm{Hom}}$ are cumbersome as well as confusing. It seems permissible to drop the symbol Hom, and to write $_A(M,N)$ for $\mathrm{Hom}_A(M,N)$. Similarly, if $\mathcal{M}$ and $\mathcal{M}$ are $\mathcal{O}$-modules on a variety X, we may write $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ or $_X(\mathcal{M},\mathcal{N})$ for $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. dropHom

(6.5.26) the dual module dualmod

Let $\mathcal{M}$ be a locally free $\mathcal{O}$-modules on a variety X. The *dual module* $\mathcal{M}^*$ is the $\mathcal{O}$-module of homomorphisms $\mathcal{M}\to\mathcal{O}$: $\mathcal{M}^* = {}_{\mathcal{O}}(\mathcal{M},\mathcal{O})$. A section of $\mathcal{M}^*$ on an affine open set U is an $\mathcal{O}(U)$-module homomorphism $\mathcal{M}(U)\to\mathcal{O}(U)$. The dualizing operation is contravariant. A homomorphism $\mathcal{M}\to\mathcal{N}$ of locally free $\mathcal{O}$-modules induces a homomorphism $\mathcal{M}^*\leftarrow\mathcal{N}^*$.

The dual $\mathcal{O}^*$ of the structure sheaf $\mathcal{O}$ is $\mathcal{O}$. If $\mathcal{M}$ is a free module with basis $v_1,\dots,v_k$, then $\mathcal{M}^*$ is also free, with the *dual basis* $v_1^*,\dots,v_k^*$, defined by

$$v_i^*(v_i)=1 \quad\text{and}\quad v_i^*(v_j)=0 \;\text{ if }\; i\neq j$$

When $\mathcal{M}$ is locally free, $\mathcal{M}^*$ is also locally free.

bidualm **6.5.27. Corollary.** **(i)** *Let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules. The module $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ of homomorphisms is canonically isomorphic to the tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}$.*

(ii) *A locally free $\mathcal{O}$-module $\mathcal{M}$ is canonically isomorphic to its bidual: $(\mathcal{M}^*)^* \approx \mathcal{M}$.*

(iii) *If $\mathcal{M}$ and $\mathcal{N}$ are locally free $\mathcal{O}$-modules, the tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}'^*$ is isomorphic to $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})^*$.*

*proof.***(i)** We identify $\mathcal{N}$ with the module $_{\mathcal{O}}(\mathcal{O},\mathcal{N})$ (6.5.24). Given sections φ of $\mathcal{M}^* = {}_{\mathcal{O}}(\mathcal{M},\mathcal{O})$ and γ of $\mathcal{N} = {}_{\mathcal{O}}(\mathcal{O},\mathcal{N})$, the composition $\gamma\varphi$ is a homomorphism $\mathcal{M} \to \mathcal{N}$, a section of $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. This composition is bilinear, so it defines a map $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N} \to {}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. To show that this map is an isomorphism is a local problem, so we may assume that $Y = \operatorname{Spec} A$ is affine and that $\mathcal{M}$ and $\mathcal{N}$ are free modules of ranks k an ℓ, respectively. Then both $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}$ and $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ are the modules of $k \times \ell$ matrices with entries in A. □

dualseq **6.5.28. Proposition.** *Let X be a variety.*

(i) *Let $\mathcal{P} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$ be homomorphisms of $\mathcal{O}$-modules such that the composition gf is the identity map $\mathcal{P} \to \mathcal{P}$. So f is injective and g is surjective. Then $\mathcal{N}$ is the direct sum of the image of f, which is isomorphic to $\mathcal{P}$, and the kernel $\mathcal{K}$ of g: $\mathcal{N} \approx \mathcal{P} \oplus \mathcal{K}$.*

(ii) *Let $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ be an exact sequence of $\mathcal{O}$-modules. If $\mathcal{P}$ is locally free, the dual modules form an exact sequence $0 \to \mathcal{P}^* \to \mathcal{N}^* \to \mathcal{M}^* \to 0$.*

proof. **(i)** This follows from the analogous statement about modules over a ring.

(ii) The sequence $0 \to \mathcal{P}^* \to \mathcal{N}^* \to \mathcal{M}^*$ is exact whether or not the modules are locally free (6.5.21) **(ii)**. The zero on the right comes from the fact that, when $\mathcal{P}$ is locally free, there is an affine covering on which it is free. When $\mathcal{P}$ is free, the given sequence splits: $\mathcal{N}$ is isomorphic to $\mathcal{M} \oplus \mathcal{P}$ (**2.1.22**). Therefore the map $\mathcal{N}^* \to \mathcal{M}^*$ is locally surjective. □

6.5.29. proof of Proposition 6.5.5.

With notation as in the statement of the proposition, we suppose given modules M_0, M_1 and an isomorphism $M_0[u^{-1}] \to M_1[v^{-1}]$. We are to show that this data comes from an $\mathcal{O}$-module $\mathcal{M}$. Proposition 6.5.12 shows that M_i defines $\mathcal{O}$-modules $\mathcal{M}_i$ on $\mathbb{U}^i$ for $i = 0, 1$, and the restrictions of $\mathcal{M}_0$ and $\mathcal{M}_1$ to $\mathbb{U}^{01}$ are isomorphic. Let's denote all of these modules by $\mathcal{M}$. Then $\mathcal{M}$ is defined on any open set that is contained in $\mathbb{U}^0$ or in $\mathbb{U}^1$.

Let V be an arbitrary open set V, and let $V^i = V \cap \mathbb{U}^i$, for $i = 0, 1, 01$. We define $\mathcal{M}(V)$ to be the kernel of the map $[\mathcal{M}(V^0) \times \mathcal{M}(V^1)] \to \mathcal{M}(V^{01})$. With this definition, $\mathcal{M}$ becomes a functor. We must verify the sheaf property, and the notation gets confusing. We suppose given an affine covering $\{W^\nu\}$ of V. We denote this covering by $\mathbf{W}_0$, and we denote $\{W^\nu \cap W^\mu\}$ by $\mathbf{W}_1$, so that the corresponding covering diagram is $V \leftarrow \mathbf{W}_0 \leftleftarrows \mathbf{W}_1$.

For $i = 0, 1, 01$, let $\mathbf{W}_0^i = \mathbf{W}_0 \cap \mathbb{U}^i$ and $\mathbf{W}_1^i = \mathbf{W}_1 \cap \mathbb{U}^i$. We form a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \longrightarrow & \mathcal{M}(\mathbf{W}_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V^0)\times\mathcal{M}(V^1) & \longrightarrow & \mathcal{M}(\mathbf{W}_0^0)\times\mathcal{M}(\mathbf{W}_0^1) & \longrightarrow & \mathcal{M}(\mathbf{W}_1^0)\times\mathcal{M}(\mathbf{W}_1^1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V^{01}) & \longrightarrow & \mathcal{M}(\mathbf{W}_0^{01}) & \longrightarrow & \mathcal{M}(\mathbf{W}_1^{01})
\end{array}$$

The columns are exact by our definition of $\mathcal{M}$, and the second and third rows are exact because the open sets involved are contained in $\mathbb{U}^0$ or $\mathbb{U}^1$. Since kernel is a left exact operation, the top row is exact too. This is the sheaf property. □

6.6 Direct Image

directimage affinemorph **(6.6.1) affine morphisms**

6.6.2. Definition. An *affine morphism* is a morphism $Y \xrightarrow{f} X$ of varieties with the property that the inverse image $f^{-1}(U)$ of every affine open subset U of X is an affine open subset of Y. $\square$ de-faffmorph

The following are examples of affine morphisms:

- the inclusion of an affine open subset Y into X,
- the inclusion of a closed subvariety Y into X,
- a finite morphism, an integral morphism.

However, if Y is the complement of a point of the projective plane X, the inclusion of Y into X isn't an affine morphism.
As one sees from these examples, affine morphisms form a rather miscellaneous collection. However, the concept is convenient.

6.6.3. Definition. Let $Y \xrightarrow{u} X$ be an affine morphism and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The *direct image* $f_*\mathcal{N}$ of $\mathcal{N}$ is the $\mathcal{O}_X$-module such that if U is an affine open subset of X and V is its inverse image in Y, then $[f_*\mathcal{N}](U) = \mathcal{N}(V)$, or, $[f_*\mathcal{N}](U) = \mathcal{N}(f^{-1}U)$, i.e., $[f_*\mathcal{N}](U) = \mathcal{N}(f^{-1}U)$. defdirim

The direct image generalizes restriction of scalars in modules over rings. Recall that, if $A \xrightarrow{\varphi} B$ is an algebra homomorphism and ${}_BN$ is a B-module, one can *restrict scalars* to make N into an A-module. Scalar multiplication by an element a of A on the restricted module ${}_AN$ is defined to be scalar multiplication by the image $\varphi(a)$ of a.

6.6.4. Lemma. *Let $Y \xrightarrow{f} X$, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, be the morphism defined by an algebra homomorphism $A \xrightarrow{\varphi} B$. If $\mathcal{N}$ is the $\mathcal{O}_Y$-module determined by the B-module ${}_BN$, its direct image $f_*\mathcal{N}$ is the $\mathcal{O}_X$-module determined by the A-module ${}_AN$.* $\square$ *dirim-restsc*

6.6.5. Lemma. *Let $Y \xrightarrow{f} X$ be an affine morphism of varieties, and let$\mathcal{N}$ be an $\mathcal{O}_Y$-module. The direct image $f_*\mathcal{N}$ is an $\mathcal{O}_X$-module.* *fstarcoh*

proof. Let $U' \to U$ be an inclusion of affine open subsets of X, and let $V = f^{-1}U$ and $V' = f^{-1}U'$. These inverse images are affine open subsets of Y. The inclusion $V' \to V$ gives us a homomorphism $\mathcal{N}(V) \to \mathcal{N}(V')$, and therefore a homomorphism $f_*\mathcal{N}(U) \to f_*\mathcal{N}(U')$. Composition with f defines a homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$, and $\mathcal{N}(V)$ is an $\mathcal{O}_Y(V)$-module. Restriction of scalars to $\mathcal{O}_X(U)$ makes $[f_*\mathcal{N}](U) = \mathcal{N}(V)$ into an $\mathcal{O}_X(U)$-module.

To verify that $f_*\mathcal{N}$ is an $\mathcal{O}_X$-module, one must show that if U is an affine open subset of X and s is a nonzero element of $\mathcal{O}_X(U)$, then $[f_*\mathcal{N}](U_s)$ is obtained by localizing $[f_*\mathcal{N}](U)$. Let V be the inverse image of U and let s' be the image of s in $\mathcal{O}_Y(V)$. Then $[f_*\mathcal{N}](U_s) = \mathcal{N}(V_{s'}) = \mathcal{N}(V)_{s'}$, provided that $s' \neq 0$. If $s' = 0$, then both $f_*\mathcal{N}](U_s)$ and $\mathcal{N}(V_{s'})$ will be zero. $\square$

It isn't difficult to extend the definition of direct image to an arbitrary morphism of varieties, but since we will use direct images only for affine morphisms, we leave the extension as an exercise.

6.6.6. Lemma. *Let $Y \xrightarrow{f} X$ be an affine morphism and let $\mathcal{N} \to \mathcal{N}' \to \mathcal{N}''$ be an exact sequence of $\mathcal{O}_Y$-modules. The direct images form an exact sequence of $\mathcal{O}_X$-modules $f_*\mathcal{N} \to f_*\mathcal{N}' \to f_*\mathcal{N}''$.* $\square$ *fs-taraffmorph*

6.6.7. Lemma. *Direct images are compatible with limits: If $\mathcal{M}_\bullet$ is a directed set of $\mathcal{O}$-modules, then $\varinjlim (f_*\mathcal{M}_\bullet) \approx f_*(\varinjlim \mathcal{M}_\bullet)$.* $\square$ *dirimlim*

Two important cases of direct image will be that f is the inclusion of a closed subvariety or an affine open subvariety. We discuss those special cases now.

(6.6.8) extension by zero – the inclusion of a closed subset extbyzero

When $Y \xrightarrow{i} X$ is the inclusion of a **closed** subvariety into a variety X and $\mathcal{N}$ is an $\mathcal{O}_Y$-module, the direct image $i_*\mathcal{N}$ is also called the *extension* of $\mathcal{N}$ *by zero*. If U is an affine open subset of X then, because i is an inclusion map, $i^{-1}U = U \cap Y$. Therefore

$$[i_*\mathcal{N}](U) = \mathcal{N}(U \cap Y)$$

The term "extension by zero" refers to the fact that, when an affine open set U of X doesn't meet Y, the intersection $U \cap Y$ is empty, and the module of sections of $[i_*\mathcal{N}](U)$ is zero. So $i_*\mathcal{N}$ is zero on the complement of Y.

extfrompoint **6.6.9. Examples.**

(i) Let $p \xrightarrow{i} X$ be the inclusion of a point into a variety. When we view the residue field $k(p)$ as a module on the point p, its extension by zero is the residue field module κ_p.

(ii) Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety, and let $\mathcal{I}$ be the ideal of Y in $\mathcal{O}_Y$. The extension by zero of the structure sheaf on Y fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

So $i_*\mathcal{O}_Y$ is isomorphic to the quotient module $\mathcal{O}_X/\mathcal{I}$. $\square$

izero **6.6.10. Proposition.** *Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X, and let $\mathcal{I}$ be the ideal of Y. Let $\mathscr{M}$ denote the subcategory of the category of $\mathcal{O}_X$-modules s that are annihilated by $\mathcal{I}$. Extension by zero defines an equivalence of categories*

$$\mathcal{O}_Y - modules \xrightarrow{i_*} \mathscr{M}$$

proof. Let U be an affine open subset of X. The intersection $U \cap Y = V$ is a closed subvariety of U. Let α be a section of $i_*\mathcal{N}(U) \ \big(= \mathcal{N}(V)\big)$. Scalar multiplication on $i_*\mathcal{N}$ is defined by restriction of scalars from $\mathcal{O}_X$ to it quotient $\mathcal{O}_Y$. If f is a section of $\mathcal{O}_X$ on U and $\overline{f}$ is its restriction to V, then $f\alpha = \overline{f}\alpha$. If f is in $\mathcal{I}(U)$, then $\overline{f} = 0$ and therefore $f\alpha = \overline{f}\alpha = 0$. So the extension by zero of an $\mathcal{O}_Y$-module is annihilated by $\mathcal{I}$. The direct image $i_*\mathcal{N}$ is an object of $\mathscr{M}$.

To complete the proof, we construct an inverse to the direct image. Starting with an $\mathcal{O}_X$-module $\mathcal{M}$ that is annihilated by $\mathcal{I}$, we construct an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $i_*\mathcal{N}$ is isomorphic to $\mathcal{M}$.

Let V be an open subset of Y. The topology on Y is induced from the topology on X, so $V = X_1 \cap Y$ for some open subset X_1 of X. We try to set $\mathcal{N}(V) = \mathcal{M}(X_1)$. To show that this is well-defined, we show that if X_2 is another open subset of X and if $V = X_2 \cap Y$, then $\mathcal{M}(X_2)$ is isomorphic to $\mathcal{M}(X_1)$. Let $X_3 = X_1 \cap X_2$. Then it is also true that $V = X_3 \cap Y$. Since $X_3 \subset X_1$, we have a map $\mathcal{M}(X_1) \to \mathcal{M}(X_3)$. It suffices to show that this map is an isomorphism, because the same reasoning will give us an isomorphism $\mathcal{M}(X_2) \to \mathcal{M}(X_3)$.

The complement $U = X_1 - V$ of V in X_1 is an open subset of X_1 and of X, and $U \cap Y = \emptyset$. We cover U by a set $\{U^i\}$ of affine open sets. Then X_1 is covered by the open sets $\{U^i\}$ together with X_3. The restriction of $\mathcal{I}$ to each of the sets U^i is the unit ideal, and since $\mathcal{I}$ annihilates $\mathcal{M}$, $\mathcal{M}(U^i) = 0$. The sheaf property shows that $\mathcal{M}(X_1)$ is isomorphic to $\mathcal{M}(X_3)$.

The rest of the proof, checking localization and verifying that $\mathcal{N}$ is determined up to isomorphism, is boring. $\square$

includeopen **(6.6.11) inclusion of an affine open subset**

Let $Y \xrightarrow{j} X$ be the inclusion of an affine **open** subvariety Y into a variety X.

Before going to the direct image, we mention a rather trivial operation, the *restriction* of an $\mathcal{O}_X$- module from X to Y. By definition, the sections of the restricted module on a subset U of Y are simply the elements of $\mathcal{M}(U)$. This makes sense because open subsets of Y are open subsets of X too. We can use subscript notation for restriction, writing $\mathcal{M}_Y$ for the restriction of an $\mathcal{O}_X$-module $\mathcal{M}$ to Y, and denoting the given module $\mathcal{M}$ by $\mathcal{M}_X$ when that seems advisable for clarity. If U is an open subset of Y,

restrtoopen (6.6.12) $$\mathcal{M}_Y(U) = \mathcal{M}_X(U)$$

This subscript notation is permissible becuse the restriction of the structure sheaf $\mathcal{O}_X$ to the open set Y is the structure sheaf $\mathcal{O}_Y$ on Y.

Now the direct image: Let $Y \xrightarrow{j} X$ be the inclusion of an affine open subvariety Y, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The inverse image of an open subset U of X is the intersection $Y \cap U$, which is affine. So by definition, the direct image $j_*\mathcal{N}$ s defined by

$$[j_*\mathcal{N}](U) = \mathcal{N}(Y \cap U)$$

For example, $[j_*\mathcal{O}_Y](U)$ is the algebra of rational functions on X that are regular on $Y \cap U$.

6.6.13. Example. Let $X_s \xrightarrow{j} X$ be the inclusion of a localization X_s into an affine variety $X = \operatorname{Spec} A$. Modules on X correspond to their global sections, which are A-modules. Similarly, modules on X_s correspond to A_s-modules. When we restrict the $\mathcal{O}_X$-module $\mathcal{M}_X$ that corresponds to an A-module M to the open set X_s, we obtain the $\mathcal{O}_{X_s}$-module $\mathcal{M}_{X_s}$ that corresponds to the A_s-module M_s. The module M_s is also the module of global sections of $j_*\mathcal{M}_{X_s}$ on X: exam-pledirec-timage

$$[j_*\mathcal{M}_{X_s}](X) \overset{def}{=} \mathcal{M}_{X_s}(X_s) = M_s$$

The localization M_s is made into an A-module by restriction of scalars. □

6.6.14. Proposition. *Let $Y \xrightarrow{j} X$ be the inclusion of an affine open subvariety Y into a variety X.* *fstarexact*

(i) *The restriction $\mathcal{O}_X$-modules $\to \mathcal{O}_Y$-modules is an exact operation.*

(ii) *The direct image functor j_* is exact.*

(iii) *Let $\mathcal{M} = \mathcal{M}_X$ be an $\mathcal{O}_X$-module. There is a canonical homomorphism from $\mathcal{M}_X$ to the direct image of its restriction: $\mathcal{M}_X \to j_*[\mathcal{M}_Y]$.*

(iv) *Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The restriction of the direct image $j_*\mathcal{N}$ to Y is equal to $\mathcal{N}$: $[j_*\mathcal{N}]_Y = \mathcal{N}$.*

proof. **(ii)** Let U be an affine open subset of X, and let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be an exact sequence of $\mathcal{O}_Y$-modules. The sequence $j_*\mathcal{M}(U) \to j_*\mathcal{N}(U) \to j_*\mathcal{P}(U)$ is the same as the sequence $\mathcal{M}(U \cap Y) \to \mathcal{N}(U \cap Y) \to \mathcal{P}(U \cap Y)$, though the scalars have changed. Since U and Y are affine, $U \cap Y$ is affine. By definition of exactness, this sequence is exact.

(iii) Let U be open in X. Then $j_*\mathcal{M}_Y(U) = \mathcal{M}(U \cap Y)$. Since $U \cap Y \subset U$, $\mathcal{M}(U)$ maps to $\mathcal{M}(U \cap Y)$.

(iv) An open subset V of Y is also open in X, and $[j_*\mathcal{N}]_Y(V) = [j_*\mathcal{N}](V) = \mathcal{N}(V \cap Y) = \mathcal{N}(V)$. □

6.6.15. Example. Let $X = \mathbb{P}^n$, and let j denote the inclusion of the standard affine open subset $\mathbb{U}^0$ into X. The direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ is the algebra of rational functions that are allowed to have poles on the hyperplane at infinity. The inverse image of an open subset W of X is its intersection with $\mathbb{U}^0$: $j^{-1}W = W \cap \mathbb{U}^0$, and the sections of the direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ on an open subset W of X are the regular functions on $W \cap \mathbb{U}^0$: dirim-standaffine

$$[j_*\mathcal{O}_{\mathbb{U}^0}](W) \;=\; \mathcal{O}_{\mathbb{U}^0}(W \cap \mathbb{U}^0) \;=\; \mathcal{O}_X(W \cap \mathbb{U}^0)$$

Say that we write a rational function α on X as a fraction g/h of relatively prime polynomials. Then α is a section of $\mathcal{O}_X$ on W if h doesn't vanish at any point of W, and α is a section of $[j_*\mathcal{O}_{\mathbb{U}^0}]$ on W if h doesn't vanish on $W \cap \mathbb{U}^0$. Arbitrary powers of x_0 can appear in the denominator of a section of $j_*\mathcal{O}_{\mathbb{U}^0}$. □

6.7 Support

annand-supp

Annihilators. Let M be a module over a ring A. The *annihilator* I of an element m of M is the set of elements α of A such that $\alpha m = 0$. It is an ideal of A that is often denoted by $\operatorname{ann}(m)$. The *annihilator* of an A-module M is the set of elements of A such that $aM = 0$. It is an ideal too.

Support. Let A be a finite-type domain and let $X = \operatorname{Spec} A$. The *support* of a finite A-module M is the locus $C = V(I)$ of zeros in X of its annihilator I, the set of points p of X such that $p \in C$, or $I \subset \mathfrak{m}_p$ (2.4.2). The support of a finite module is a closed subset of X.

The next lemma allows us to extend the concepts of annihilator and support to finite $\mathcal{O}$-modules on a variety X.

localize-support

6.7.1. Lemma. *Let $X = \operatorname{Spec} A$ be an affine variety, let I be the annihilator of an element m of an A-module M, and let s be a nonzero element of A. The annihilator of m in the localized module M_s is the localized ideal I_s. If M is a finite module with support C, the support of M_s is the intersection $C \cap X_s$ of C with X_s.* $\square$

If $\mathcal{I}$ is the annihilator of a finite $\mathcal{O}$-module $\mathcal{M}$ on a variety X, the *support* of $\mathcal{M}$ is defined to be the closed subset $V(\mathcal{I})$ of zeros of the ideal. For example, the support of the residue field module κ_p is the point p. The support of the maximal ideal $\mathfrak{m}_p$ at p is the whole variety X.

support-dimzero

(6.7.2) $\mathcal{O}$-modules with support of dimension zero

suppfinite

6.7.3. Proposition. *Let X be a variety.*

(i) *Suppose that the support of a finite $\mathcal{O}$-module $\mathcal{M}$ is a single point p, let $M = \mathcal{M}(X)$, and let U be an affine open subset of X. If U contains p, then $\mathcal{M}(U) = M$, and if U doesn't contain p, then $\mathcal{M}(U) = 0$.*

(ii) *(Chinese Remainder Theorem) If the support of a finite $\mathcal{O}$-module $\mathcal{M}$ is a finite set $\{p_1, ..., p_k\}$, then $\mathcal{M}$ is the direct sum $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ of $\mathcal{O}$-modules supported at the points p_i.*

proof. **(i)** Let $\mathcal{I}$ be the annihilator of an $\mathcal{O}$-module $\mathcal{M}$ with support p. The locus $V(\mathcal{I})$ is the support p. If p isn't contained in U, then when we restrict $\mathcal{M}$ to U, we obtain an $\mathcal{O}_U$-module whose support is empty. So $\mathcal{I}(U)$ is the unit ideal, and the restriction of $\mathcal{M}$ to U is the zero module.

Next, suppose that p is contained in U, and let V denote the complement of p in X. We cover X by a finite set $\{U^i\}$ of affine open sets such that $U = U^1$, and such that $U^i \subset V$ if $i > 1$. By what has been shown, $\mathcal{M}(U^i) = 0$ if $i > 0$. The sheaf axiom for this covering shows that $\mathcal{M}(X) \approx \mathcal{M}(U)$. $\square$

Note. If i denotes the inclusion of a point p into a variety X, it is natural to suppose that an $\mathcal{O}$-module $\mathcal{M}$ supported at p will be the extension by zero of a module on the point p (a vector space). However, this won't be true unless $\mathcal{M}$ is annihilated by the maximal ideal $\mathfrak{m}_p$. $\square$

6.8 Twisting

twisting-modules

The twisting modules that we define here are among the most important modules on projective space.

As before, a homogeneous fraction of degree n in $x_0, ..., x_n$ is a fraction g/h of homogeneous polynomials, such that $\deg g - \deg h = n$. When g and h are relatively prime, the fraction g/h is regular on an open subset V of $\mathbb{P}^n_x$ if and only if h isn't zero at any point of V.

The definition of the twisting module $\mathcal{O}(n)$ is this: The sections of $\mathcal{O}(n)$ on an open subset V of $\mathbb{P}^n$ are the homogeneous fractions of degree n that are regular on V. In particular, $\mathcal{O}(0) = \mathcal{O}$.

Odismodule

6.8.1. Proposition.

(i) *Let V be an affine open subset of $\mathbb{P}^n$ that is contained in the standard affine open set $\mathbb{U}^0$. The sections of the twisting module $\mathcal{O}(n)$ on V form a free module of rank one with basis x_0^n, over the coordinate algebra $\mathcal{O}(V)$.*

(ii) *The twisting module $\mathcal{O}(n)$ is an $\mathcal{O}$-module.*

proof. **(i)** Let V be an open set contained in $\mathbb{U}^0$, and let $\alpha = g/h$ be a section of $\mathcal{O}(n)$ on V, with g, h relatively prime. Then $f = \alpha x_0^{-n}$ has degree zero. It is a rational function. Since $V \subset \mathbb{U}^0$, x_0 doesn't vanish at any point of V. Since α is regular on V, f is a regular function on V, and $\alpha = f x_0^n$.

(ii) It is clear that $\mathcal{O}(n)$ is a contravariant functor. We verify compatibility with localization. Let $V = \operatorname{Spec} A$ be an affine open subset of X and let s be a nonzero element of A. We must show that $[\mathcal{O}(n)](V_s)$ is the localization of $[\mathcal{O}(n)](V)$. We must show that, if β is a section of $\mathcal{O}(n)$ on V_s, then $s^k\beta$ is a section on V when k is sufficiently large.

We cover V by the affine open sets $V^i = V \cap \mathbb{U}^i$. It suffices to show that $s^k\beta$ is a section on V^i for every i. For the case $i = 0$, we apply **(i)**. Since V^0_s is contained in $\mathbb{U}^0$, β can be written uniquely in the form $f x_0^n$, where f is a regular function on V^0_s and n is an integer. Then $s^k f$ is a regular function on V^0 when k is large, and then $s^k\alpha = s^k f x_0^n$ is a section of $\mathcal{O}(n)$ on V^0. The analogous statement is true for every index i. $\square$

As part **(i)** of the proposition shows, $\mathcal{O}(n)$ is quite similar to the structure sheaf. However, $\mathcal{O}(n)$ is only *locally* free.

6.8.2. Proposition. *When $d \geq 0$, the global sections of the twisting module $\mathcal{O}(n)$ on $\mathbb{P}^n$ are the homogeneous polynomials of degree n. When $n < 0$, the only global section of $\mathcal{O}(n)$ is the zero section.* *sectionsofOn*

proof. A nonzero global section u of $\mathcal{O}(n)$ will restrict to a section on the standard affine open set $\mathbb{U}^0$. Since elements of $\mathcal{O}(\mathbb{U}^0)$ are homogeneous fractions of degree zero whose denominators are powers of x_0, and since $[\mathcal{O}(n)](\mathbb{U}^0)$ is a free module over $\mathcal{O}(\mathbb{U}^0)$ with basis x_0^d, we will have $u = g/x_0^k$ for some some homogeneous polynomial g_0 not divisible by x_0 and some k. Similarly, restriction to $\mathbb{U}^1$ shows that u has the form g_1/x_1^ℓ. It follows that $k = \ell = 0$ and that $u = g_0$. Since u has degree n, so does g_0. $\square$

6.8.3. Examples. prodhomfr

(i) The product uv of homogeneous fractions of degrees r and s is a homogeneous fraction of degree $r+s$, and if u and v are regular on an open set V, so is their product uv. Multiplication defines a homomorphism of $\mathcal{O}$-modules

(6.8.4) $$\mathcal{O}(r) \oplus \mathcal{O}(s) \to \mathcal{O}(r+s)$$ oros

(ii) Multiplication by a homogeneous polynomial f of degree n defines an injective homomorphism

(6.8.5) $$\mathcal{O}(k) \xrightarrow{f} \mathcal{O}(k+n)$$ $\square$ multbyfone

The twisting modules $\mathcal{O}(n)$ on projective space $\mathbb{P}^d$ have a second interpretation. They are isomorphic to the modules that we denote by $\mathcal{O}(nH)$, of rational functions with poles of order at most n on the hyperplane $H : \{x_0 = 0\}$ at infinity.

By definition, the sections of $\mathcal{O}(nH)$ on an open set V are the rational functions f such that $x_0^n f$ is a section of $\mathcal{O}(n)$ on V. Thus multiplication by x_0^n defines an isomorphism

(6.8.6) $$\mathcal{O}(nH) \xrightarrow{x_0^n} \mathcal{O}(n)$$ OnOnH

If f is a section of $\mathcal{O}(nH)$ on an open set V, and if we write f as a homogeneous fraction g/h of degree zero, with g, h relatively prime, the denominator h may have x_0^k, with $k \leq n$, as factor. The other factors of h cannot vanish anywhere on V. If $f = g/h$ is a *global* section of $\mathcal{O}(nH)$, then the denominator h has the form cx_0^k, with $c \in \mathbb{C}$ and $k \leq n$. A global section of $\mathcal{O}(nH)$ can be represented as a homogeneous fraction g/x_0^k of degree zero.

Since x_0 doesn't vanish at any point of the standard open set $\mathbb{U}^0$, the sections of $\mathcal{O}(nH)$ on an open subset V of $\mathbb{U}^0$ are simply the regular functions on V. Using the subscript notation (**6.6.11**) for restriction to an open set,

(6.8.7) $$\mathcal{O}(nH)_{\mathbb{U}^0} = \mathcal{O}_{\mathbb{U}^0}$$ jstarOH

Let V be an open subset of one of the other standard affine open sets, say of $\mathbb{U}^1$. The ideal of $H \cap \mathbb{U}^1$ in $\mathbb{U}^1$ is the principal ideal generated by $v_0 = x_0/x_1$, and the element v_0 generates the ideal of $H \cap V$ in V too. If f is a rational function, then because x_1 doesn't vanish on $\mathbb{U}^1$, the function fv_0^n will be regular on V if and only if the homogeneous fraction fx_0^n is regular there, i.e., if an only if f is a section of $\mathcal{O}(nH)$ on V. We say that such a function f has a pole of order at most n on H because v_0 generates the ideal of H in V.

The isomorphic $\mathcal{O}$-modules $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ are interchangeable. The twisting module $\mathcal{O}(n)$ is often better because its definition is independent of coordinates. On the other hand, $\mathcal{O}(nH)$ can be convenient because its restriction to $\mathbb{U}^0$ is the structure sheaf.

6.8.8. Proposition. *Let Y be a hypersurface of degree n in $\mathbb{P}^n$, the zero locus of an irreducible homogeneous polynomial f of degree n. Let $\mathcal{I}$ be the ideal of Y, and let $\mathcal{O}(-n)$ be the twisting module. Multiplication by f defines an isomorphism $\mathcal{O}(-n) \xrightarrow{f} \mathcal{I}$.* *curveideal* *proof.* We choose coordinates so that f isn't isn't divisible by any of the coordinate variables x_i.

If α is a section of $\mathcal{O}(-n)$ on an open set V, then $f\alpha$ will be a regular function on V that vanishes on $Y \cap V$. Therefore the image of the multiplication map $\mathcal{O}(-n) \xrightarrow{f} \mathcal{O}$ is contained in the ideal $\mathcal{I}$. The multiplication map is injective because $\mathbb{C}[x_0, ..., x_n]$ is a domain. To show that it is an isomorphism, it suffices to show that its restrictions to the standard affine open sets $\mathbb{U}^i$ are surjective (6.4.6). We work with $\mathbb{U}^0$, as usual.

Because x_0 desn't divide f, $Y \cap \mathbb{U}^0$ will be a nonempty, and therefore dense, open subset of Y. The sections of $\mathcal{O}$ on $\mathbb{U}^0$ are the homogeneous fractions g/x_0^k of degree zero. Such a fraction is a section of $\mathcal{I}$ on $\mathbb{U}^0$ if and only if g vanishes on $Y \cap \mathbb{U}^0$. If so, then since $Y \cap \mathbb{U}^0$ is dense in Y and since the zero set of g is closed, g will vanish on Y, and therefore it will be divisible by f: $g = fq$. The sections of $\mathcal{I}$ on $\mathbb{U}^0$ have the form fq/x_0^k. They are in the image of the map $\mathcal{O}(-n) \xrightarrow{f} \mathcal{I}$. □

The proposition has an interesting corollary:

idealsisom **6.8.9. Corollary.** *When regarded as $\mathcal{O}$-modules, the ideals of all hypersurfaces of degree n are isomorphic.*

twistmodule **(6.8.10) twisting a module**

deftwistm **6.8.11. Definition** Let $\mathcal{M}$ be an $\mathcal{O}$-module on projective space $\mathbb{P}^d$, and let $\mathcal{O}(n)$ be the twisting module. The (nth) *twist* of $\mathcal{M}$ is defined to be the tensor product $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(n)$. Similarly, $\mathcal{M}(nH) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(nH)$. If X is a closed subvariety of $\mathbb{P}^d$ and $\mathcal{M}$ is an $\mathcal{O}_X$-module, $\mathcal{M}(n)$ and $\mathcal{M}(nH)$ are obtained by twisting the extension of $\mathcal{M}$ by zero. (See the equivalence of categories (6.6.10).)

Since x_0^n is a basis of $\mathcal{O}(n)$ on $\mathbb{U}^0$, a section of $\mathcal{M}(n)$ on an open subset V of $\mathbb{U}^0$ can be written in the form $\alpha = m \otimes g x_0^n$, where g is a regular function on V and m is a section of $\mathcal{M}$ on V (6.8.1). The function g can be moved over to m, so α can also be written in the form $\alpha = m \otimes x_0^n$. This expression for α is unique because the operation of tensoring with x_0^n is injective.

The modules $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ form directed sets that are related by a diagram

Oninclusions (6.8.12)
$$\begin{array}{ccccccc}
\mathcal{O} & \xrightarrow{\subset} & \mathcal{O}(H) & \xrightarrow{\subset} & \mathcal{O}(2H) & \xrightarrow{\subset} & \cdots \\
\| & & \downarrow{\scriptstyle x_0} & & \downarrow{\scriptstyle x_0^2} & & \\
\mathcal{O} & \xrightarrow{x_0} & \mathcal{O}(1) & \xrightarrow{x_0} & \mathcal{O}(2) & \longrightarrow & \cdots
\end{array}$$

In this diagram, the vertical arrows are bijections and the horizontal arrows are injections. The limit of the upper directed set is the module whose sections on an open set V are rational functions that can have arbitrary poles on $H \cap V$, and are otherwise regular. This is also the module $j_*\mathcal{O}_{\mathbb{U}^0}$, where j denotes the inclusion of the standard affine open set $\mathbb{U}^0$ into X (see (6.6.14) **(iii)**):

limequalsjstar (6.8.13)
$$\varinjlim_n \mathcal{O}(nH) = j_*\mathcal{O}_{\mathbb{U}^0}$$

Tensoring (6.8.12) with $\mathcal{M}$ give us the diagram

Mnmaps (6.8.14)
$$\begin{array}{ccccccc}
\mathcal{M} & \longrightarrow & \mathcal{M}(H) & \longrightarrow & \mathcal{M}(2H) & \longrightarrow & \cdots \\
\| & & \downarrow{\scriptstyle x_0} & & \downarrow{\scriptstyle x_0^2} & & \\
\mathcal{M} & \xrightarrow{1\otimes x_0} & \mathcal{M}(1) & \xrightarrow{1\otimes x_0} & \mathcal{M}(2) & \longrightarrow & \cdots
\end{array}$$

The vertical maps here are bijective, but because $\mathcal{M}$ may have torsion, the horizontal maps needn't be injective.

Let $\mathbb{U} = \mathbb{U}^0$. Since tensor products are compatible with limits,

limjstar (6.8.15)
$$\varinjlim_n \mathcal{M}(nH) \overset{(1)}{\approx} \mathcal{M} \otimes_{\mathcal{O}} \big(\varinjlim_n \mathcal{O}(nH)\big) \approx \mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}} \overset{(2)}{\approx} j_*\mathcal{M}_{\mathbb{U}}$$

The isomorphism (1) comes from the fact that tensor produts are compatible with limits, and (2) is part **(ii)** of the next lemma.

6.8.16. Lemma. *Let $\mathcal{M}$ be an $\mathcal{O}$-module on $\mathbb{P}^n$, and let j be the inclusion of $\mathbb{U} = \mathbb{U}^0$.* *Mjstarequaljs-tarM*

(i) *For every k, the restriction of $\mathcal{M}(kH)$ to $\mathbb{U}$ is the same as the restriction of $\mathcal{M}$, which is $\mathcal{M}_{\mathbb{U}}$, and the restriction of $j_*\mathcal{M}_{\mathbb{U}}$ to $\mathbb{U}$ is also $\mathcal{M}_{\mathbb{U}}$. The restriction of the map $\mathcal{M}(kH) \to j_*(\mathcal{M}_{\mathbb{U}})$ to $\mathbb{U}$ is the identity map.*

(ii) *The direct image $j_*\mathcal{M}_{\mathbb{U}}$ is isomorphic to $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}}$.*

proof. **(i)** Because $H \cap \mathbb{U}$ is empty, the restrictions of $\mathcal{M}(kH)$ and $\mathcal{M}$ to $\mathbb{U}$ are equal. The fact that the restriction of $j_*\mathcal{M}_{\mathbb{U}}$ is also equal to $\mathcal{M}_{\mathbb{U}}$ is Proposition 6.6.14 **(iv)**.

(ii) Suppose given a section of $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}}$ of the form $\alpha \otimes f$ on an open set V, where α is a section of $\mathcal{M}$ on V and and f is a section of $j_*\mathcal{O}_{\mathbb{U}}$ on V, a regular function on $V \cap \mathbb{U}$. We denote the restriction of α to $V \cap \mathbb{U}$ by the same symbol α. Then αf will be a section of $\mathcal{M}$ on $V \cap \mathbb{U}$ and therefore a section of $j_*\mathcal{M}_{\mathbb{U}}$ on V. The map $(\alpha, f) \to \alpha f$ is $\mathcal{O}$-bilinear, so it corresponds to a homomorphism $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}} \to j_*\mathcal{M}_{\mathbb{U}}$. To show that this homomorphism is an isomorphism, it suffices to verify that it restricts to an isomorphism on each of the standard affine open sets $\mathbb{U}^i$. The restrictions of $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}}$ and $j_*\mathcal{M}_{\mathbb{U}}$ to $\mathbb{U}^0$ are both equal to $\mathcal{M}_{\mathbb{U}}$. So that case is trivial. We look at $\mathbb{U}^1$. On that open set, $[j_*\mathcal{M}_{\mathbb{U}}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^{01})$, and with $v_0 = x_0/x_1$, $[j_*\mathcal{O}_{\mathbb{U}}](\mathbb{U}^1) = \mathcal{O}(\mathbb{U}^{01}) = \mathcal{O}(\mathbb{U}^1)[v_0^{-1}]$. By definition of the tensor product, $[\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^1) \otimes_{\mathcal{O}(\mathbb{U}^1)} \mathcal{O}(\mathbb{U}^{01})$ and

$$\mathcal{M}(\mathbb{U}^1) \otimes_{\mathcal{O}(\mathbb{U}^1)} \mathcal{O}(\mathbb{U}^{01}) = \mathcal{M}(\mathbb{U}^1)[v_0^{-1}] = \mathcal{M}(\mathbb{U}^{01}) = [j_*\mathcal{M}_{\mathbb{U}}](\mathbb{U}^1) \qquad \square$$

(6.8.17) generating an $\mathcal{O}$-module generators

A set of global sections $m = (m_1, ..., m_k)$ of an $\mathcal{O}$-module on a variety X defines a map

$$\mathcal{O}^k \xrightarrow{m} \mathcal{M} \qquad (6.8.18)$$ gen

that sends a section $(\alpha_1, ..., \alpha_k)$ of $\mathcal{O}^k$ on an open set to the combination $\sum \alpha_i m_i$. The global sections $m_1, ..., m_k$ are said to *generate* $\mathcal{M}$ if this map is surjective. If the sections generate $\mathcal{M}$, then they (or to be precise, their restrictions), generate the $\mathcal{O}(U)$-module $\mathcal{M}(U)$ for every affine open set U. When U isn't affine, they may fail to generate $\mathcal{M}(U)$.

6.8.19. Example. Let $X = \mathbb{P}^1$. For $n \geq 0$, the global sections of the twisting module $\mathcal{O}(n)$ are the polynomials of degree n in the coordinate variables x_0, x_1 (6.8.2). Consider the map $\mathcal{O} \oplus \mathcal{O} \xrightarrow{(x_0^n, x_1^n)} \mathcal{O}(n)$. On $\mathbb{U}^0$, $\mathcal{O}(n)$ has basis x_0^n. Therefore the map is surjective on $\mathbb{U}^0$. Similarly, it is surjective on $\mathbb{U}^1$. So it is a surjective map on all of X (6.4.6). The global sections x_0^n, x_1^n generate $\mathcal{O}(n)$. However, the global sections of $\mathcal{O}(n)$ are the homogeneous polynomials of degree n. When $n > 1$, the two sections x_0^n, x_1^n don't span the space of global sections. $\square$ generate-example

The next theorem explains the importance of the twisting operation.

6.8.20. Theorem. *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. For sufficiently large k, the twist $\mathcal{M}(k)$ is generated by global sections.* *gentwist*

proof. We may assume that X is the projective space $\mathbb{P}^n$. We are to show that if $\mathcal{M}$ is a finite $\mathcal{O}$-module, and k is sufficiently large, $\mathcal{M}(k)$ is generated by its global sections. It suffices to show that for each $i = 0, ..., n$, the restrictions of the global sections generate the $\mathcal{O}(\mathbb{U}^i)$-module $[\mathcal{M}(k)](\mathbb{U}^i)$ (6.4.6). We work with the index $i = 0$.

We replace $\mathcal{M}(k)$ by the isomorphic module $\mathcal{M}(kH)$. Recall that the maps $\varinjlim_k \mathcal{M}(kH) = j_*\mathcal{M}_{\mathbb{U}^0}$ (6.8.15) and that the $\mathcal{M}(kH) \to j_*\mathcal{M}_{\mathbb{U}^0}$ restrict to bijections on $\mathbb{U}^0$ for every k (6.8.16 **(i)**).

We have maps $\mathcal{M} \xrightarrow{1 \otimes x_0^k} \mathcal{M}(kH) \to j_*\mathcal{M}_{\mathbb{U}^0}$, that are isomorphisms on $\mathbb{U}^0$. Let $A_0 = \mathcal{O}(\mathbb{U}^0)$ and $M_0 = \mathcal{M}(\mathbb{U}^0)$. Then M_0 is a finite A_0-module because $\mathcal{M}$ is a finite $\mathcal{O}$-module. We choose a finite set of generators $m_1, ..., m_r$ for the A_0-module M_0. The elements of M_0, and in particular, the chosen generators, are global sections of $j_*\mathcal{M}_{\mathbb{U}^0}$. Since $\varinjlim_k \mathcal{M}(kH) = j_*\mathcal{M}_{\mathbb{U}^0}$, they are represented by global sections $m_1', ..., m_r'$ of $\mathcal{M}(kH)$ when k is large. The restrictions of $\mathcal{M}(kH)$ and $\mathcal{M}$ to $\mathbb{U}_0$ are equal (6.8.16), and the restrictions of m_i' to $\mathbb{U}^0$ is equal to the restriction of m_i. So the restrictions of $m_1', ..., m_r'$ generate M_0 too. Therefore M_0 is generated by global sections of $\mathcal{M}(kH)$, as was to be shown. $\square$

6.9 Extending a Module: proof

proveex-tension

We prove Theorem 6.4.2 here.

The statement to be proved is that an $\mathcal{O}$-module $\mathcal{M}$ on a variety X has a unique extension to a functor

$$\textit{(opens)} \xrightarrow{\widetilde{\mathcal{M}}} \textit{(modules)}$$

with the sheaf property (6.4.5), and that a homomorphism of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{N}$ has a unique extension to a homomorphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$.

The proof has the following steps:

1. Verification of the sheaf property for a covering of an affine open set by localizations.
2. Extension of the functor $\mathcal{M}$ to all morphisms between affine open sets.
3. Definition of $\widetilde{\mathcal{M}}$.

Step 1. *the sheaf property for a covering of an affine open set by localizations*

Suppose that an affine open subset $Y = \operatorname{Spec} A$ of X is covered by a family of localizations $\mathbf{U}_0 = \{U_{s_i}\}$, and let $\mathcal{M}$ be an $\mathcal{O}$-module. Let M, M_i, and M_{ij} denote the modules of sections $\mathcal{M}(Y), \mathcal{M}(U_{s_i})$, and $\mathcal{M}(U_{s_is_j})$, respectively. The exact sequence that expresses the sheaf property for the covering diagram $Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$ becomes

localize-module

(6.9.1) $$0 \to M \xrightarrow{\alpha} \prod M_i \xrightarrow{\beta} \prod M_{ij}$$

where α sends an element m of M to the vector $(m, ..., m)$ of its images in $\prod_i M_i$, and the difference map β sends a vector $(m_1, ..., m_k)$ in $\prod_i M_i$ to the matrix (z_{ij}), with $z_{ij} = m_j - m_i$ in M_{ij} (6.4.5). We must show that the sequence (6.9.1) is exact.

exactness at M: Since the open sets U^i cover Y, the elements $s_1, ..., s_k$ generate the unit ideal. Let m be an element of M that maps to zero in every M_i. Then there exists an n such that $s_i^n m = 0$, and we can use the same exponent n for all i. The elements s_i^n generate the unit ideal. Writing $\sum a_i s_i^n = 1$, we have $m = \sum a_i s_i^n m = \sum a_i 0 = 0$.

exactness at $\prod M_i$: Let m_i be elements of M_i such that $m_j = m_i$ in M_{ij} for all i, j. We must find an element w in M that maps to m_j in M_j for every j.

We write m_i as a fraction: $m_i = s_i^{-n} x_i$, or $x_i = s_i^n m_i$, with x_i in M, using the same integer n for all i. The equation $m_j = m_i$ in M_{ij} tells us that $s_i^n x_j = s_j^n x_i$ in M_{ij}. Since M_{ij} is the localization $M[(s_is_j)^{-1}]$ $(s_is_j)^r s_i^n x_j = (s_is_j)^r s_j^n x_i$ will be true in M, if r is large.

The exponents here confuse the argument, so we adjust the notation. Let $\widetilde{x}_i = s_i^r x_i$, and $\widetilde{s}_i = s_i^{r+n}$. Then in M, $\widetilde{x}_i = \widetilde{s}_i m_i$ and $\widetilde{s}_j \widetilde{x}_i = \widetilde{s}_i \widetilde{x}_j$. The elements $\widetilde{s}_i$ generate the unit ideal. So there is an equation in A, of the form $\sum a_i \widetilde{s}_i = 1$.

Let $w = \sum a_i \widetilde{x}_i$. This is an element of M, and

$$\widetilde{x}_j = \Big(\sum_i a_i \widetilde{s}_i\Big)\widetilde{x}_j = \sum_i a_i \widetilde{s}_j \widetilde{x}_i = \widetilde{s}_j w$$

So $\widetilde{x}_j = \widetilde{s}_j w$ and also $\widetilde{x}_j = \widetilde{s}_j m_j$. Since $\widetilde{s}_j$ is invertible on $\mathbb{U}^j$, $w = \widetilde{s}_j^{-1} \widetilde{x}_j = m_j$, in M_j. Since j is arbitrary, w is the required element of M. $\square$

Step 2. *extending an $\mathcal{O}$-module to all morphisms between affine open sets*

The $\mathcal{O}$-module $\mathcal{M}$ comes with localization maps $\mathcal{M}(U) \to \mathcal{M}(U_s)$. It doesn't come with homomorphisms $\mathcal{M}(U) \to \mathcal{M}(V)$ when $V \to U$ is an arbitrary inclusion of affine open sets. We define those maps here.

Let $\mathcal{M}$ be an $\mathcal{O}$-module and let $V \to U$ be an inclusion of affine open sets. To describe the homomorphism $\mathcal{M}(U) \to \mathcal{M}(V)$, we cover V by a family $\mathbf{V}_0 = \{V^1, ..., V^r\}$ of open sets that are localizations of U and therefore also localizations of V. Then V^{ij} are localizations of V^i and of V^j. So we have a covering diagram $\mathbf{V}_1 \rightrightarrows \mathbf{V}_0 \to V$. Composing with the map $V \to U$ gives us a map $\mathbf{V}_0 \to U$ such that the two maps

$\mathbf{V}_1 \to U$ obtained by composition are equal. Therefore image of the map $\mathcal{M}(U) \to \mathbf{V}_0(U)$ is contained in the kernel of the difference map $\mathcal{M}(\mathbf{V}_0) \xrightarrow{\beta} \mathcal{M}(\mathbf{V}_1)$, and by Step 1, that kernel is $\mathcal{M}(V)$. This defines the map $\mathcal{M}(U) \to \mathcal{M}(V)$. We must show that it is independent of the choice of the covering $\mathbf{V}_0$.

One can go from one affine covering to another in a finite number of steps, each of which adds or deletes a single affine open set. So to prove independence of the map $\mathcal{M}(U) \to \mathcal{M}(V)$ defined above, it suffices to relate $\mathbf{V}_0$ to the family $\mathbf{W}_0 = \{V^1, ..., V^r, W\}$ obtained by adding one localization W of U to the covering $\mathbf{V}$. Let $\mathbf{W}_1$ be the family of intersections of pairs of elements of $\mathbf{W}_0$. The inclusion $\mathbf{V}_0 \subset \mathbf{W}_0$ defines a map $\mathcal{M}(\mathbf{V}_0) \to \mathcal{M}(\mathbf{W}_0)$, and similarly, we have a map $\mathcal{M}(\mathbf{V}_1) \to \mathcal{M}(\mathbf{W}_1)$. This gives us a diagram

(6.9.2)
$$\begin{array}{ccccc}
\mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{V}_0) & \xrightarrow{\beta_{\mathbf{V}}} & \mathcal{M}(\mathbf{V}_1) \\
\| & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow \\
\mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \xrightarrow{\beta_{\mathbf{W}}} & \mathcal{M}(\mathbf{W}_1)
\end{array}$$
compare-covtwo

in which $\mathcal{M}(U)$ is mapped to the kernels of $\beta_{\mathbf{V}}$ and $\beta_{\mathbf{W}}$, both of which are equal to $\mathcal{M}(V)$. Looking at the diagram, one sees that the maps $\mathcal{M}(U) \to \mathcal{M}(V)$ defined using the two coverings $\mathbf{V}_0$ and $\mathbf{W}_0$ are the same.

To show that this extended functor has the sheaf property for an arbitrary affine covering $\mathbf{V}_0 = \{V^i\}$ of an affine variety U, we let $\mathbf{W}_0$ be the affine covering of U that is obtained by covering each V^i by localizations of U. We substitute the covrings $\mathbf{V}_i$ and $\mathbf{W}_i$ into the diagram (**??**), and add zeros on the left to each row of the diagram. The sheaf property to be verified is that the top row of this diagram is exact. Since $\mathbf{W}_0$ is an affine covering of the affine variety U, the bottom row is exact. Because $\mathbf{W}_0$ covers $\mathbf{V}_0$, $\mathbf{W}_1$ covers $\mathbf{V}_1$ as well. So the maps a and b are injective. It follows that the top row is exact.

Step 3. *definition of* $\widetilde{\mathcal{M}}$

We introduce some temporary notation: Suppose that a covering diagram $U \leftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$ and an $\mathcal{O}$-module $\mathcal{M}$ are given, and let $\mathcal{M}(\mathbf{U}_0) \xrightarrow{\beta_{\mathbf{U}}} \mathcal{M}(\mathbf{U}_1)$ be the difference map (6.4.10). We denote the kernel of $\beta_{\mathbf{U}}$ by $\mathbf{K}_{\mathbf{U}}$.

Let Y be an open subset of X. We define $\widetilde{\mathcal{M}}(Y)$ in a fairly obvious way: We choose an affine covering $\mathbf{V}_0 = \{V^1, ...V^r\}$ of Y, and we define $\widetilde{\mathcal{M}}(Y) = \mathbf{K}_{\mathbf{V}}$. When we show that $\mathbf{K}_{\mathbf{V}}$ doesn't depend on the covering $\mathbf{V}_0$, it will follow that $\widetilde{\mathcal{M}}$ is well-defined, and that it has the sheaf property.

As explained in Step 2, it suffices to relate the covering $\mathbf{V}$ to a covering $\mathbf{W}_0 = \{V^1, ..., V^r, W\}$ that is obtained by adding one affine open subset W of Y to $\mathbf{V}_0$. The inclusion $\mathbf{V}_0 \subset \mathbf{W}_0$ defines a map $\mathcal{M}(\mathbf{V}_0) \to \mathcal{M}(\mathbf{W}_0)$, and a map $\mathcal{M}(\mathbf{V}_1) \to \mathcal{M}(\mathbf{W}_1)$. It gives us a map $\mathbf{K}_{\mathbf{W}} \to \mathbf{K}_{\mathbf{V}}$. We will show that, for any element $(v_1, ..., v_r)$ in the kernel $\mathbf{K}_{\mathbf{V}}$, there is a unique element w in $\mathcal{M}(W)$ such that $(v_1, ..., v_r, w)$ is in the kernel $\mathbf{K}_{\mathbf{W}}$. This will show that $\mathbf{K}_{\mathbf{W}}$ and $\mathbf{K}_{\mathbf{V}}$ are isomorphic.

Let $W^i = V^i \cap W,\ i = 1, ..., r$. Since $\mathbf{V}_0$ is an affine covering of Y, $\mathbf{W}_0 = \{W^i\}$ is an affine covering of W. Let w_i denote the restriction of the section v_i to W^i. Since $(v_1, ..., v_r)$ is in the kernel of $\beta_{\mathbf{V}}$, i.e., $v_i = v_j$ on V^{ij}. Then it is also true that $w_i = w_j$ on the smaller open set W^{ij}. So $(w_1, ..., w_r)$ is in the kernel $\mathbf{K}_{\mathbf{W}}$, and since $\mathbf{W}_0$ is an affine covering of the affine variety W, Step 2 tells us that $\mathbf{K}_{\mathbf{W}} = \mathcal{M}(W)$. So there is a unique element w in $\mathcal{M}(W)$ that restricts to w_i on W^i for each i. We show that, with this element w, $(v_1, ..., v_r, w)$ is in the kernel of $\beta_{\mathbf{W}}$.

When the subsets in the family $\mathbf{W}_1$ are listed in the order

$$\mathbf{W}_1 = \{V^i \cap V^j\}_{ij}, \{W \cap V^j\}_j, \{V^i \cap W\}_i, \{W \cap W\}$$

the difference map $\beta_{\mathbf{W}}$ sends $(v_1, ..., v_r, w)$ to $\big[(v_j - v_i), (v_j - w), (w - v_i),\ 0\big]$, the sections being restricted appropriately. Here $v_i = v_j$ on $V^i \cap V^j$ because $(v_1, ..., v_r)$ is in the kernel $\mathbf{K}_{\mathbf{V}}$. By definition, $v_j = w_j = w$ on $V^j \cap W = W^j$.

It remains to prove that $\widetilde{\mathcal{M}}$ is a functor. This proof has no interesting features, and we won't use the functorality, so we omit it. $\square$

6.10 Exercises

chapsixex xsectsnotfin **6.10.1.** Let U be the complement of the origin in the affine plane $X = \operatorname{Spec} A$, $A = \mathbb{C}[x,y]$.
(i) Let $\mathcal{M}$ be the $\mathcal{O}_X$-module that correponds to the A-module $M = A/yA$. Show that $\mathcal{M}$ is a finite $\mathcal{O}$-module, but that $\mathcal{M}(U)$ isn't a finite module over the ring $\mathcal{O}(U)$.

(ii) Show that, for any $k \geq 1$, the homomorphism $\mathcal{O}\times \xrightarrow{(x,y)^t} \mathcal{O}$ is surjective on U, but that the map of sections on U isn't surjective.

xsimplemod **6.10.2.** An R-modue is *simple* if it is nonzero and if it has no proper submodules. Prove that a simple module over a finite type $\mathbb{C}$-algebra has dimension 1.

xcohpropsheaf **6.10.3.** Prove that if an $\mathcal{O}$-module has the coherence property for affine open sets, then it has the sheaf property for affine coverings of affine open sets.

xcomplvin **6.10.4.** Let V be the complement of a finite set in $\mathbb{P}^d$. Determine $\mathcal{O}_{\mathbb{P}}(V)$.

xMistensor **6.10.5.** Let $U' \subset U$ be affine open subsets of a variety X, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Say that $\mathcal{O}(U) = A$, $\mathcal{O}(U') = A'$, $\mathcal{M}(U) = M$, and $\mathcal{M}(U') = M'$. Prove that $M' = M \otimes_A A'$.

xinterscoh **6.10.6.** Show that if $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{O}$, so is $\mathcal{I} \cap \mathcal{J}$.

xlimM **6.10.7.** Let s be an element of a domain A, and let M be an A-module. Identify the limit of the directed set $M \xrightarrow{s} M \xrightarrow{s} \cdots$ is isomorphic.

xgenrel **6.10.8.** Let $R = \mathbb{C}[x_0, x_1, x_2]$, and let $f = x_0^2 - x_1x_2$.
(i) Determine generators and defining relations for the ring $R_{\{f\}}$ of homogeneous fractions of degree zero whose denominators are powers of f.
(ii) Prove that the twisting module $\mathcal{O}(1)$ isn't a free module on the open subset $\mathbb{U}_{\{f\}}$ of $\mathbb{P}^2$ at which $f \neq 0$.

xascchcd **6.10.9.** Let X be a variety. Prove that every strictly ascending chain of submodules of a finite $\mathcal{O}$-module $\mathcal{M}$ is finite.

notfree **6.10.10.** Let $R = \mathbb{C}[x,y,z]$, let $X = \mathbb{P}^2$, and let $s = z^2 - xy$. Determine the degree one part of R_s, and prove that $\mathcal{O}(1)$ is not free on X_s.

xsectOn **6.10.11.** What are the sections of $\mathcal{O}(nH)$ on an open set V that isn't contained in any $\mathbb{U}^i$?

xtwistsisom **6.10.12.** In the description (6.5.4) of modules over the projective line, we considered the standard affine open sets U^0 and U^1. Interchanging these open sets changes the variable t to t^{-1}, and it changes the matrix P accordingly. Does it follow, when the rank is 1, that the $\mathcal{O}$-modules defined by t^k and by t^{-k} are isomorphic?

xmultf **6.10.13.** Describe the kernel and cokernel of the multiplication mmap $\mathcal{M}(k) \xrightarrow{f} \mathcal{M}(k+d)$ when f is a homogeneous polynomial of degree d.

sectwist **6.10.14.** Let $X = \mathbb{P}^2$. What are the sections of the twisting module $\mathcal{O}_X(n)$ on the open complement of the line $\{x_1 + x_2 = 0\}$?

xmultinject **6.10.15.** Let M be a finite module over a finite-type domain A, and let α be a nonzero element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s = \alpha - c$ defines an injective map $M \xrightarrow{s} M$.

coherprop **6.10.16.** Prove the following *coherence property* of an $\mathcal{O}$-module: Let Y be an open subset of a variety X, let s be a nonzero regular function on Y, and let Y_s be a localization. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $\mathcal{M}(Y_s)$ is the localization $\mathcal{M}(Y)_s$ of $\mathcal{M}(Y)$. In particular, $\mathcal{O}_X(U_s)$ is the localization $\mathcal{O}_X(U)_s$. (This is a requirement for an $\mathcal{O}$-module, when Y is affine.)

gendirim **6.10.17.** Using Exercise 6.10.16, extend the definition of direct image to an arbitrary morphism of varieties.

Chapter 7 COHOMOLOGY

cohomology

7.1 Cohomology

cohqcoh

This chapter is adapted from Serre's classic 1956 paper "Faisceaux Algébriques Cohérents", in which Serre showed how the Zariski topology could be used to define cohomology of $\mathcal{O}$-modules.

To save time, we define cohomology only for $\mathcal{O}$-modules. Anyway, the Zariski topology has limited use for cohomology with other coefficients. For instance, the constant coefficient cohomology $H^q(X,\mathbb{Z})$ is zero for all $q>0$, when X is given the Zariski topology.

Let $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X. The *zero-dimensional cohomology* of $\mathcal{M}$ is the space $\mathcal{M}(X)$ of its global sections. When speaking of cohomology, one denotes that space by $H^0(X,\mathcal{M})$.

The functor

$$\textit{(}\mathcal{O}\textit{-modules)} \xrightarrow{H^0} \textit{(vector spaces)}$$

that carries an $\mathcal{O}$-module $\mathcal{M}$ to $H^0(X,\mathcal{M})$ is left exact: If

SS (7.1.1) $$0\to\mathcal{M}\to\mathcal{N}\to\mathcal{P}\to 0$$

is a short exact sequence of $\mathcal{O}$-modules, the associated sequence of global sections

globalsections (7.1.2) $$0\to H^0(X,\mathcal{M})\to H^0(X,\mathcal{N})\to H^0(X,\mathcal{P})$$

is exact, but unless X is affine, the map $H^0(X,\mathcal{N})\to H^0(X,\mathcal{P})$ needn't be surjective (6.5.8). The *cohomology* on X is a sequence of functors *(*$\mathcal{O}$*-modules)* $\xrightarrow{H^q}$ *(vector spaces)*,

$$H^0,\ H^1,\ H^2,\ \dots$$

beginning with H^0, one for each *dimension*, that compensates for the lack of exactness in the way that is explained in **(a)** and **(b)** below:

(a) To every short exact sequence (7.1.1) of $\mathcal{O}$-modules, there is an associated long exact *cohomology sequence*

HSS (7.1.3) $$0\to H^0(X,\mathcal{M})\to H^0(X,\mathcal{N})\to H^0(X,\mathcal{P})\xrightarrow{\delta^0}$$

$$\xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to H^1(X,\mathcal{N}) \to H^1(X,\mathcal{P}) \xrightarrow{\delta^1} \dots$$
$$\dots \xrightarrow{\delta^{q-1}} H^q(X,\mathcal{M}) \to H^q(X,\mathcal{N}) \to H^q(X,\mathcal{P}) \xrightarrow{\delta^q} \cdots$$

The maps δ^q in this sequence are called the *coboundary maps*.

(b) A map of exact sequences of $\mathcal{O}$-modules, a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{M}' \arrow[r] & \mathcal{N}' \arrow[r] & \mathcal{P}' \arrow[r] & 0 \end{tikzcd}$$

whose rows are short exact sequences of $\mathcal{O}$-modules, induces a map of cohomology sequences

(7.1.4) deltadiagram

$$\begin{tikzcd} \cdots \arrow[r] & H^q(X,\mathcal{N}) \arrow[r] \arrow[d] & H^q(X,\mathcal{P}) \arrow[r, "\delta^q"] \arrow[d] & H^{q+1}(X,\mathcal{M}) \arrow[r] \arrow[d] & H^{q+1}(X,\mathcal{N}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^q(X,\mathcal{N}') \arrow[r] & H^q(X,\mathcal{P}') \arrow[r, "\delta^q"] & H^{q+1}(X,\mathcal{M}') \arrow[r] & H^{q+1}(X,\mathcal{N}) \arrow[r] & \cdots \end{tikzcd}$$

A sequence of functors H^q , $q = 0, 1, 2, ...$ from $\mathcal{O}$-modules to vector spaces that comes with long cohomology sequences **(a)** for every short exact sequence of $\mathcal{O}$-modules and that satisfies **(b)** is called a *cohomological functor*. Cohomology is a cohomological functor.

Most of Diagram 7.1.4 comes from the fact that the H^q are functors. The only additional property is that the squares

(7.1.5) deltadiagramtwo

$$\begin{tikzcd} H^q(X,\mathcal{P}) \arrow[r, "\delta^q"] \arrow[d] & H^{q+1}(X,\mathcal{M}) \arrow[d] \\ H^q(X,\mathcal{P}') \arrow[r, "\delta^q"] & H^{q+1}(X,\mathcal{M}') \end{tikzcd}$$

that involve the coboundary maps δ commute.

Unfortunately, there is no canonical construction of cohomology. We present a construction in Section 7.4, but it isn't canonical. One needs to look at an explicit construction sometimes, but most of the time, it is best to work with the characteristic properties of cohomology that are described below, in Section 7.3.

The one-dimensional cohomology H^1 has an interesting interpretation that you can read about if you like. We won't use it. The cohomology in dimension greater than one has no useful direct interpretation.

7.2 Complexes

complexes

Complexes are used in the construction of cohomology, so we discuss them here.

A *complex* $V^\bullet$ of vector spaces is a sequence of homomorphisms of vector spaces

(7.2.1) complexone
$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

indexed by the integers, such that the composition $d^n d^{n-1}$ of adjacent maps is zero, which means that, for every n, the image of d^{n-1} is contained in the kernel of d^n. The q-dimensional *cohomology* of the complex $V^\bullet$ is the quotient

(7.2.2) cohcoplxone
$$\mathbf{C}^q(V^\bullet) = (\ker\, d^q)/(\text{im}\, d^{q-1}).$$

A complex whose cohomology is zero is an exact sequence.

A finite sequence of homomorphisms $V^k \xrightarrow{d^k} V^{k+1} \to \cdots \xrightarrow{d^{r-1}} V^r$ such that the compositions $d^i d^{i-1}$ are zero for $i = k, ..., r-1$, can be made into a complex, defining $V^n = 0$ for all other integers n. For example, a homomorphism of vector spaces $V^0 \xrightarrow{d^0} V^1$ can be made into the complex

$$\cdots \to 0 \to V^0 \xrightarrow{d^0} V^1 \to 0 \to \cdots$$

For this complex, the cohomology $\mathbf{C}^0$ is the kernel of d^0, $\mathbf{C}^1$ is its cokernel, and $\mathbf{C}^q$ is zero for all other q.

In the complexes that arise here, V^q will be zero when $q < 0$.

A *map* $V^\bullet \xrightarrow{\varphi} V'^\bullet$ of complexes is a collection of homomorphisms $V^n \xrightarrow{\varphi^n} V'^n$ making a diagram

$$\begin{tikzcd}
{} \arrow[r] & V^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & V^n \arrow[r, "d^n"] \arrow[d, "\varphi^n"'] & V^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"'] & \cdots \\
{} \arrow[r] & V'^{n-1} \arrow[r, "d'^{n-1}"] & V'^n \arrow[r, "d'^n"] & V'^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

A map of complexes induces maps on the cohomology

$$\mathbf{C}^q(V^\bullet) \to \mathbf{C}^q(V'^\bullet)$$

because $\ker d^q$ maps to $\ker d'^q$ and $\operatorname{im} d^q$ maps to $\operatorname{im} d'^q$.

An exact sequence of complexes

exseqcplx (7.2.3) $$\cdots \to V^\bullet \xrightarrow{\varphi} V'^\bullet \xrightarrow{\psi} V''^\bullet \to \cdots$$

is a sequence of maps in which the sequences

(7.2.4) $$\cdots \to V^q \xrightarrow{\varphi^q} V'^q \xrightarrow{\psi^q} V''^q \to \cdots$$

are exact for every q.

cohcplx **7.2.5. Proposition.**
Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ *be a short exact sequence of complexes. For every* q*, there are maps* $\mathbf{C}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{C}^{q+1}(V^\bullet)$ *such that the sequence*

$$0 \to \mathbf{C}^0(V^\bullet) \to \mathbf{C}^0(V'^\bullet) \to \mathbf{C}^0(V''^\bullet) \xrightarrow{\delta^0} \mathbf{C}^1(V^\bullet) \to \mathbf{C}^1(V'^\bullet) \to \mathbf{C}^1(V''^\bullet) \xrightarrow{\delta^1} \mathbf{C}^2(V^\bullet) \to \cdots$$

is exact.

The proof of the proposition is below.

This long exact sequence is the *cohomology sequence* associated to the short exact sequence of complexes. The set of functors $\{\mathbf{C}^q\}$ is a cohomological functor on the category of complexes.

snakeco-homology **7.2.6. Example.** We make the Snake Lemma 2.1.20 into a cohomology sequence. Suppose given a diagram

$$\begin{tikzcd}
 & V \arrow[r, "u"] \arrow[d, "f"'] & V' \arrow[r] \arrow[d, "f'"'] & V'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & W \arrow[r] & W' \arrow[r, "v"'] & W'' &
\end{tikzcd}$$

with exact rows. We form the complex $V^\bullet :\ 0 \to V \xrightarrow{f} W \to 0$ with V in degree zero, so that $\mathbf{C}^0(V^\bullet) = \ker f$ and $\mathbf{C}^1(V^\bullet) = \operatorname{coker} f$, and we do the analogous thing for the maps f' and f''. Having done that, the Snake Lemma becomes an exact sequence

$$\mathbf{C}^0(V^\bullet) \to \mathbf{C}^0(V'^\bullet) \to \mathbf{C}^0(V''^\bullet) \to \mathbf{C}^1(V^\bullet) \to \mathbf{C}^1(V'^\bullet) \to \mathbf{C}^1(V''^\bullet)$$ $\square$

proof of Proposition 7.2.5. Given a complex $V^\bullet$:

$$\cdots \to V^{q-1} \xrightarrow{d^{q-1}} V^q \xrightarrow{d^q} V^{q+1} \xrightarrow{d^{q+1}} \cdots$$

let B^q be the image of d^{q-1}, let D^q be the cokernel of d^{q-1}, the quotient V^q/B^q, and let Z^q be the kernel of d^q. The cohomology $\mathbf{C}^q(V^\bullet)$ is is Z^q/B^q.

7.2.7. Lemma. (i) *With the above notation, there is a map $D^q \xrightarrow{f^q} Z^{q+1}$ such that $V^q \xrightarrow{d^q} V^{q+1}$ becomes a composition of the three maps* *factordq*

$$V^q \xrightarrow{\pi^q} D^q \xrightarrow{f^q} Z^{q+1} \xrightarrow{i^{q+1}} V^{q+1}$$

where π^q is the projection from V^q to its quotient D^q and i^{q+1} is the inclusion of Z^{q+1} into V^{q+1}.
(ii) *With f^q as in* **(i)***,*

$$\mathbf{C}^q(V^\bullet) = \ker f^q \quad \textit{and} \quad \mathbf{C}^{q+1}(V^\bullet) = \operatorname{coker} f^q$$

proof. **(i)** The image B^{q+1} of d^q is contained in Z^{q+1}, and the kernel Z^q of d^q contains B^q. So d^q factors as indicated.

(ii) Since the kernel Z^q of d^q contains B^q, and since $D^q = V^q/B^q$, the kernel of f^q is $Z^q/B^q = \mathbf{C}^q$. The image B^{q+1} of d^q is also the image of f^q in Z^{q+1}. So the cokernel of f^q is $Z^{q+1}/B^{q+!} = \mathbf{C}^{q+1}$. □

Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ be a short exact sequence of complexes, as in Proposition 7.2.5. We apply Lemma 7.2.7. In the diagram below, the top row is exact because D^q, D'^q, D''^q are cokernels, and cokernel is a right exact operation. The bottom row is exact because Z^q, Z'^q, Z''^q are kernels, and kernel is left exact:

$$\begin{tikzcd}
 & D^q \arrow[r] \arrow[d, "f^q"'] & D'^q \arrow[r] \arrow[d, "f'^q"'] & D''^q \arrow[r] \arrow[d, "f''^q"'] & 0 \\
0 \arrow[r] & Z^{q+1} \arrow[r] & Z'^{q+1} \arrow[r] & Z''^{q+1} &
\end{tikzcd}$$

The Snake Lemma gives us an exact sequence

$$\mathbf{C}^q(V^\bullet) \to \mathbf{C}^q(V'^\bullet) \to \mathbf{C}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{C}^{q+1}(V^\bullet) \to \mathbf{C}^{q+1}(V'^\bullet) \to \mathbf{C}^{q+1}(V''^\bullet)$$

The cohomology sequence associated to the short exact sequence of complexes is obtained by splicing these sequences together. □

The coboundary maps δ^q in cohomology sequences are related in a natural way. If

$$\begin{tikzcd}
0 \arrow[r] & U^\bullet \arrow[r] \arrow[d] & U'^\bullet \arrow[r] \arrow[d] & U''^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V^\bullet \arrow[r] & V'^\bullet \arrow[r] & V''^\bullet \arrow[r] & 0
\end{tikzcd}$$

is a diagram of maps of complexes whose rows are short exact sequences of complexes, the diagrams

$$\begin{tikzcd}
\mathbf{C}^q(U''^\bullet) \arrow[r, "\delta^q"] \arrow[d] & \mathbf{C}^{q+1}(U^\bullet) \arrow[d] \\
\mathbf{C}^q(V''^\bullet) \arrow[r, "\delta^q"] & \mathbf{C}^{q+1}(V^\bullet)
\end{tikzcd}$$

commute. It isn't difficult to check this. Thus a map of short exact sequences induces a map of cohomology sequences.

7.3 Characteristic Properties

charprop

The cohomology of $\mathcal{O}$-modules, which is a sequence of functors $H^0,\ H^1,\ H^2,\ \cdots$

$$\textit{(}\mathcal{O}\textit{-modules)} \xrightarrow{H^q} \textit{(vector spaces)}$$

is characterized by the three properties below. The first two have already been mentioned.

charpro-pone

(7.3.1) Characteristic Properties of Cohomology

1. $H^0(X, \mathcal{M})$ is the space $\mathcal{M}(X)$ of global sections of $\mathcal{M}$.
2. The sequence $H^0, H^1, H^2, \cdots$ is a cohomological functor on $\mathcal{O}$-modules: A short exact sequence of $\mathcal{O}$-modules produces a long exact cohomology sequence.
3. Let $Y \xrightarrow{f} X$ be the inclusion of an affine open subset Y into X, let $\mathcal{N}$ be an $\mathcal{O}_Y$-module, and let $f_*\mathcal{N}$ be its direct image on X. The cohomology $H^q(X, f_*\mathcal{N})$ is zero for all $q > 0$.

Note. When X is an affine variety, the global section functor is exact: If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is a short exact sequence of $\mathcal{O}$-modules on X, the sequence

$$0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{N}) \to H^0(X, \mathcal{P}) \to 0$$

is exact (6.2.3). There is no need for the higher cohomology $H^1,\ H^2, \cdots$ when X is affine. One may as well define $H^q(X, \cdot) = 0$ when X is affine and $q > 0$. This is the third characteristic property for the identity map $X \to X$, and the third property is based on this observation. Intuitively, the third property tells us that allowing poles on the complement of an affine open set kills cohomology in positive dimension.

existco-hom

7.3.2. Theorem. *There exists a cohomology theory with the properties (**7.3.1**), and it is unique up to unique isomorphism.*

The proof is in the next section.

cohze-roaffine

7.3.3. Corollary. *If X is an affine variety, $H^q(X, \mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.*

This follows when one applies the third characteristic property to the identity map $X \to X$. □

We begin with an example, which shows how the third characteristic property can be used.

thirdex

7.3.4. Example. let j be inclusion of the standard affine $\mathbb{U}^0$ into $X = \mathbb{P}$. Then $\varinjlim_n \mathcal{O}(nH) \approx j_*\mathcal{O}_{\mathbb{U}^0}$, where $\mathcal{M}_{\mathbb{U}^0}$ is the restriction of $\mathcal{M}$ to $\mathbb{U}^0$ (6.8.13). The third property tells us that the cohomology H^q of the direct image $j_*\mathcal{O}_{\mathbb{U}}^0$ is zero when $q > 0$. We will see below (7.4.24) that cohomology commutes with direct limits. Therefore $\varinjlim_n H^q(X, \mathcal{O}_X(nH))$ and $\varinjlim_n H^q(X, \mathcal{O}_X(n))$ are zero when $q > 0$. □

cohdirsum

7.3.5. Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules on a variety X. The cohomology of the direct sum $\mathcal{M} \oplus \mathcal{N}$ is canonically isomorphic to the direct sum $H^q(X, \mathcal{M}) \oplus H^q(X, \mathcal{N})$.*

In this statement, one could substitute just about any functor for H^q. And, since the direct sum and the direct product are equal, one could substitute $\times$ for $\oplus$

proof. We have homomorphisms of $\mathcal{O}$-modules $\mathcal{M} \xrightarrow{i_1} \mathcal{M} \oplus \mathcal{N} \xrightarrow{\pi_1} \mathcal{M}$ and analogous homomorphisms $\mathcal{N} \xrightarrow{i_2} \mathcal{M} \oplus \mathcal{N} \xrightarrow{\pi_2} \mathcal{N}$. The direct sum can be characterized by these maps, together with the relations $\pi_1 i_1 = id_{\mathcal{M}}$, $\pi_2 i_2 = id_{\mathcal{N}}$, $\pi_2 i_1 = 0$, $\pi_1 i_2 = 0$, and $i_1\pi_1 + i_2\pi_2 = id_{\mathcal{M}\oplus\mathcal{N}}$. The proof of this is an exercise. Applying the functor H^q gives analogous homomorphisms relating $H^q(\mathcal{M})$, $H^q(\mathcal{N})$, and $H^q(\mathcal{M} \oplus \mathcal{N})$. Therefore $H^q(\mathcal{M} \oplus \mathcal{N}) \approx H^q(\mathcal{M}) \oplus H^q(\mathcal{N})$. □

7.4 Existence of Cohomology

constrcoh

The proof of existence and uniqueness of cohomology are based on the following facts:

• The intersection of two affine open subsets of a variety is an affine open set.

• A sequence $\cdots \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to \cdots$ of $\mathcal{O}$-modules on a variety X is exact if and only if, for every affine open subset U, the sequence of sections $\cdots \to \mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U) \to \cdots$ is exact. (This is the definition of exactness.)

We begin by choosing an arbitrary affine covering $\mathbf{U} = \{U^\nu\}$ of our variety X by finitely many affine open sets U^ν, and we use this covering to describe the cohomology. When we have shown that the cohomology is unique, we will know that it doesn't depend on our choice of covering.

Let $\mathbf{U} \xrightarrow{j} X$ denote the family of inclusions $U^\nu \xrightarrow{j^\nu} X$ of our chosen affine open sets into X. If $\mathcal{M}$ is an $\mathcal{O}$-module, $\mathcal{R}_\mathcal{M}$ will denote the $\mathcal{O}$-module $\prod j^\nu_* \mathcal{M}_{U^\nu}$, where $\mathcal{M}_{U^\nu}$ denotes the restriction of $\mathcal{M}$ to U^ν (6.6.12). We can also write $\prod j^\nu_* \mathcal{M}_{U^\nu}$ as $j_*\mathcal{M}_\mathbf{U}$. As has been noted, there is a canonical map $\mathcal{M} \to j^\nu_*\mathcal{M}_{U^\nu}$, and therefore a canonical map $\mathcal{M} \to \mathcal{R}_\mathcal{M}$ (6.6.14).

7.4.1. Lemma. **(i)** *Let X' be an open subset of X. The module $\mathcal{R}_\mathcal{M}(X')$ of sections of $\mathcal{R}_\mathcal{M}$ on X' is the product $\prod_\nu \mathcal{M}(X' \cap U^\nu)$. The space of global sections $\mathcal{R}_\mathcal{M}(X)$, which is $H^0(X, \mathcal{R}_\mathcal{M})$, is the product $\prod_\nu \mathcal{M}(U^\nu)$.* defcalr
(ii) *The canonical map $\mathcal{M} \to \mathcal{R}_\mathcal{M}$ is injective. Thus, if $\mathcal{S}_\mathcal{M}$ denotes the cokernel of that map, there is a short exact sequence of $\mathcal{O}$-modules*

(7.4.2) $$0 \to \mathcal{M} \to \mathcal{R}_\mathcal{M} \to \mathcal{S}_\mathcal{M} \to 0$$ MRzeroMone

(iii) *For any cohomology theory with the characteristic properties and for any $q > 0$, $H^q(X, \mathcal{R}_\mathcal{M}) = 0$.*

proof. **(i)** This is seen by going through the definitions:

$$\mathcal{R}(X') \;=\; \textstyle\prod_\nu [j^\nu_*\mathcal{M}_{U^\nu}](X') = \prod_\nu \mathcal{M}_{U^\nu}(X' \cap U^\nu) = \prod_\nu \mathcal{M}(X' \cap U^\nu).$$

(ii) Let X' be an open subset of X. The map $\mathcal{M}(X') \to \mathcal{R}_\mathcal{M}(X')$ is the product of the restriction maps $\mathcal{M}(X') \to \mathcal{M}(X' \cap U^\nu)$. Because the open sets U^ν cover X, the intersections $X' \cap U^\nu$ cover X'. The sheaf property of $\mathcal{M}$ tells us that the map $\mathcal{M}(X') \to \prod_\nu \mathcal{M}(X' \cap U^\nu)$ is injective.

(iii) This follows from the third characteristic property. □

7.4.3. Lemma. **(i)** *A short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ of $\mathcal{O}$-modules embeds into a diagram* Rcminjective

(7.4.4) $$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_\mathcal{M} & \longrightarrow & \mathcal{R}_\mathcal{N} & \longrightarrow & \mathcal{R}_\mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_\mathcal{M} & \longrightarrow & \mathcal{S}_\mathcal{N} & \longrightarrow & \mathcal{S}_\mathcal{P} \end{array}$$ ttdiagr

whose rows and columns are short exact sequences. (We have suppressed the surrounding zeros.)
(ii) *The sequence of global sections $0 \to \mathcal{R}_\mathcal{M}(X) \to \mathcal{R}_\mathcal{N}(X) \to \mathcal{R}_\mathcal{P}(X) \to 0$ is exact.*

proof. **(i)** We are given that the top row of the diagram is a short exact sequence, and we have seen that the columns are short exact sequences. To show that the middle row

(7.4.5) $$0 \to \mathcal{R}_\mathcal{M} \to \mathcal{R}_\mathcal{N} \to \mathcal{R}_\mathcal{P} \to 0$$ Rsequence

is exact, we must show that if X' is an affine open subset of X, the sections on X' form a short exact sequence. The sections are explained in Lemma 7.4.1 **(i)**. Since products of exact sequences are exact, we must show that the sequence

$$0 \to \mathcal{M}(X' \cap U^\nu) \to \mathcal{N}(X' \cap U^\nu) \to \mathcal{P}(X' \cap U^\nu) \to 0$$

is exact. This is true because $X' \cap U^\nu$ is an intersection of affine opens, and is therefore affine.

Now that we know that the first two rows of the diagram are short exact sequences, the Snake Lemma tells us that the bottom row is a short exact sequence.

(ii) The sequence of of global sections referred to in the statement is the product of the sequences

$$0 \to \mathcal{M}(U^\nu) \to \mathcal{N}(U^\nu) \to \mathcal{P}(U^\nu) \to 0$$

These sequences are exact because the open sets U^ν are affine. □

uniquecoh **(7.4.6) uniqueness of cohomology**

Suppose that a cohomology with the characteristic properties (**7.3.1**) is given, and let $\mathcal{M}$ be an $\mathcal{O}$-module. The cohomology sequence associated to the sequence $0 \to \mathcal{M} \to \mathcal{R}_\mathcal{M} \to \mathcal{S}_\mathcal{M} \to 0$ is

$$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{R}_\mathcal{M}) \to H^0(X,\mathcal{S}_\mathcal{M}) \xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to H^1(X,\mathcal{R}_\mathcal{M}) \to \cdots$$

Lemma 7.4.1 **(iii)** tells us that $H^q(X,\mathcal{R}_\mathcal{M}) = 0$ when $q > 0$. So this cohomology sequence breaks up into an exact sequence

hone (7.4.7) $$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{R}_\mathcal{M}) \to H^0(X,\mathcal{S}_\mathcal{M}) \xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to 0$$

and isomorphisms

hq (7.4.8) $$0 \to H^q(X,\mathcal{S}_\mathcal{M}) \xrightarrow{\delta^q} H^{q+1}(X,\mathcal{M}) \to 0$$

for every $q > 0$. The first three terms of the sequence (7.4.7), and the arrows connecting them, depend on our choice of covering of X, but the important point is that they don't depend on the cohomology. So that sequence determines $H^1(X,\mathcal{M})$ up to unique isomorphism as the cokernel of a map that is independent of the cohomology. This this is true for every $\mathcal{O}$-module $\mathcal{M}$, including for the module $\mathcal{S}_\mathcal{M}$. Therefore it is also true that $H^1(X,\mathcal{S}_\mathcal{M})$ is determined uniquely. This being so, $H^2(X,\mathcal{M})$ is determined uniquely for every $\mathcal{M}$, by the isomorphism (7.4.8), with $q = 1$. The isomorphisms (7.4.8) determine the rest of the cohomology up to unique isomorphism by induction on q. □

existcoh **(7.4.9) construction of cohomology**

One can use the sequence (7.4.2) and induction to construct cohomology, but it seems clearer to proceed by iterating the construction of $\mathcal{R}_\mathcal{M}$.

Let $\mathcal{M}$ be an $\mathcal{O}$-module. We rewrite the exact sequence (7.4.2), labeling $\mathcal{R}_\mathcal{M}$ as $\mathcal{R}^0_\mathcal{M}$, and $\mathcal{S}_\mathcal{M}$ as $\mathcal{M}^1$:

MtoR-toMone (7.4.10) $$0 \to \mathcal{M} \to \mathcal{R}^0_\mathcal{M} \to \mathcal{M}^1 \to 0$$

and we repeat the construction with $\mathcal{M}^1$. Let $\mathcal{R}^1_\mathcal{M} = \mathcal{R}^0_{\mathcal{M}^1}$ $(= j_*\mathcal{M}^1_{\mathbb{U}})$, so that there is an exact sequence

cmonesequence (7.4.11) $$0 \to \mathcal{M}^1 \to \mathcal{R}^1_\mathcal{M} \to \mathcal{M}^2 \to 0$$

analogous to the sequence (7.4.10), with $\mathcal{M}^2 = \mathcal{R}^1_\mathcal{M}/\mathcal{M}^1$. We combine the sequences (7.4.10) and (7.4.11) into an exact sequence

RMone (7.4.12) $$0 \to \mathcal{M} \to \mathcal{R}^0_\mathcal{M} \to \mathcal{R}^1_\mathcal{M} \to \mathcal{M}^2 \to 0$$

and we let $\mathcal{R}^2_\mathcal{M} = \mathcal{R}^0_{\mathcal{M}^2}$. Continuing in this way, we construct modules $\mathcal{R}^k_\mathcal{M}$ that form an exact sequence

RM (7.4.13) $$0 \to \mathcal{M} \to \mathcal{R}^0_\mathcal{M} \to \mathcal{R}^1_\mathcal{M} \to \mathcal{R}^2_\mathcal{M} \to \cdots$$

The next lemma follows by induction from Lemma 7.4.1 **(iii)** and Lemma 7.4.3 **(i,ii)**.

7.4.14. Lemma. *Rcmexact*
(i) *Let* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *be a short exact sequence of* $\mathcal{O}$*-modules. For every* n*, the sequences*

$$0 \to \mathcal{R}^n_{\mathcal{M}} \to \mathcal{R}^n_{\mathcal{N}} \to \mathcal{R}^n_{\mathcal{P}} \to 0$$

are exact, and so are the sequences of global sections

$$0 \to \mathcal{R}^n_{\mathcal{M}}(X) \to \mathcal{R}^n_{\mathcal{N}}(X) \to \mathcal{R}^n_{\mathcal{P}}(X) \to 0$$

(ii) *If* $H^0,\ H^1,\ \ldots$ *is a cohomology theory, then* $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ *for all* n *and all* $q > 0$. $\square$

An exact sequence such as (7.4.13) is called a *resolution* of $\mathcal{M}$, and because $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ when $q > 0$, it is an *acyclic resolution*.

Continuing with the proof of existence, we consider the complex of $\mathcal{O}$-modules that is obtained by omitting the term $\mathcal{M}$ from (7.4.13). Let $\mathcal{R}^\bullet_{\mathcal{M}}$ denote that complex:

(7.4.15) $$\mathcal{R}^\bullet_{\mathcal{M}} \ = \ 0 \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{R}^2_{\mathcal{M}} \to \cdots$$ Rsequencetwo

The complex $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ of its global sections

(7.4.16) $$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X) \to \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$ RMX

can also be written as

$$0 \to H^0(X, \mathcal{R}^0_{\mathcal{M}}) \to H^0(X, \mathcal{R}^1_{\mathcal{M}}) \to H^0(X, \mathcal{R}^2_{\mathcal{M}}) \to \cdots$$

The complex $\mathcal{R}^\bullet_{\mathcal{M}}$ becomes the resolution (7.4.13) when the module $\mathcal{M}$ is inserted. So it is an exact sequence except at $\mathcal{R}^0_{\mathcal{M}}$. However, the global section functor is only left exact, and the sequence (7.4.16) of global sections $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ needn't be exact anywhere. It is a complex though, because $\mathcal{R}^\bullet_{\mathcal{M}}$ is a complex. The composition of adjacent maps is zero.

Recall that the cohomology of a complex $0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots$ of vector spaces is $\mathbf{C}^q(V^\bullet) = (\ker\ d^q)/(\mathrm{im}\ d^{q-1})$, and that $\{\mathbf{C}^q\}$ is a cohomological functor on complexes (7.2.5).

7.4.17. Definition. The cohomology of an $\mathcal{O}$-module $\mathcal{M}$ is the cohomology of the complex $\mathcal{R}^\bullet_{\mathcal{M}}(X)$: definecoh

$$H^q(X, \mathcal{M}) = \mathbf{C}^q(\mathcal{R}^\bullet_{\mathcal{M}}(X))$$

Thus if we denote the maps in the complex (7.4.16) by d^q:

$$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \xrightarrow{d^0} \mathcal{R}^1_{\mathcal{M}}(X) \xrightarrow{d^1} \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$

then $H^q(X, \mathcal{M}) = (\ker\ d^q)/(\mathrm{im}\ d^{q-1})$.

7.4.18. Lemma. *Let* X *be an affine variety. With cohomology defined as above,* $H^q(X, \mathcal{M}) = 0$ *for all* $\mathcal{O}$*-modules* $\mathcal{M}$ *and all* $q > 0$. *affinecohzero*

proof. When X is affine, the sequence of global sections of the exact sequence (7.4.13) is exact. $\square$

To show that our definition gives the unique cohomology, we verify the three characteristic properties. Since the sequence (7.4.13) is exact and since the global section functor is left exact, $\mathcal{M}(X)$ is the kernel of the map $\mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X)$. This kernel is also equal to $\mathbf{C}^0(\mathcal{R}^\bullet_{\mathcal{M}}(X))$, so our cohomology has the first property: $H^0(X, \mathcal{M}) = \mathcal{M}(X)$.

To show that we obtain a cohomological functor, we apply Lemma 7.4.14 to conclude that, for a short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$, the spaces of global sections

(7.4.19) $$0 \to \mathcal{R}^\bullet_{\mathcal{M}}(X) \to \mathcal{R}^\bullet_{\mathcal{N}}(X) \to \mathcal{R}^\bullet_{\mathcal{P}}(X) \to 0,$$ crtwo

form an exact sequence of complexes. The cohomology $H^q(X, \cdot\,)$ is a cohomological functor because cohomology of complexes is a cohomological functor. This is the second characteristic property.

We make a digression before verifying the third characteristic property.

Let $Y \xrightarrow{f} X$ be a morphism of varieties. Let $U \xrightarrow{j} X$ be the inclusion of an open subvariety into X and let V be the inverse image $f^{-1}U$, which is an open subvariety of Y. These varieties and maps form a diagram

jstarfstar (7.4.20)
$$\begin{array}{ccc} V & \xrightarrow{i} & Y \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ U & \xrightarrow{j} & X \end{array}$$

When we restrict the direct image $f_*\mathcal{N}$ of $\mathcal{N}$ to U, we obtain an $\mathcal{O}_U$-module $[f_*\mathcal{N}]_U$. We can obtain an $\mathcal{O}_U$-module in a second way: First restrict the module $\mathcal{N}$ to the open subset V of Y, and then take its direct image. This gives us the $\mathcal{O}_U$-module $g_*[\mathcal{N}_V]$.

jstarfstartwo **7.4.21. Lemma.** *The $\mathcal{O}_U$-modules $g_*[\mathcal{N}_V]$ and $[f_*\mathcal{N}]_U$ are equal.*

proof. Let U' be an open subset of U, and let $V' = g^{-1}U'$. Then

$$[f_*\mathcal{N}]_U(U') = [f_*\mathcal{N}](U') = \mathcal{N}(V') = \mathcal{N}_V(V') = [g_*[\mathcal{N}_V]](U')$$ □

affinedirectimage **7.4.22. Proposition.** *Let $Y \xrightarrow{f} X$ be an affine morphism, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Let $H^q(X, \cdot\,)$ be the cohomology defined as in (7.4.17), and let $H^q(Y, \cdot\,)$ be the cohomology that is defined in the analogous way, using the covering $\mathbf{V}$ of Y. Then $H^q(X, f_*\mathcal{N})$ is isomorphic to $H^q(Y,\mathcal{N})$.*

proof. This proof requires untangling the notation. Except for that, it is easy.

To compute the cohomology of $f_*\mathcal{N}$ on X, we substitute $\mathcal{M} = f_*\mathcal{N}$ into (7.4.17):

$$H^q(X, f_*\mathcal{N}) = \mathbf{C}^q(\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)).$$

To compute the cohomology of $\mathcal{N}$ on Y, we let

$$\mathcal{R}'^0_{\mathcal{N}} = i_*[\mathcal{N}_{\mathbf{V}}]$$

where i is as in Diagram 7.4.20, and we continue, to construct a resolution $0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{R}'^1_{\mathcal{N}} \to \cdots$. The prime is there to remind us that $\mathcal{R}'$ is defined using the covering $\mathbf{V}$ of Y. Let $\mathcal{R}'^\bullet_{\mathcal{N}}$ be the complex that is obtained by replacing the term $\mathcal{N}$ by zero. Then

$$H^q(Y,\mathcal{N}) = \mathbf{C}^q(\mathcal{R}'^\bullet_{\mathcal{N}}(Y)).$$

It suffices to show that the complexes of global sections $\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)$ and $\mathcal{R}'^\bullet_{\mathcal{N}}(Y)$ are isomorphic. If so, we will have

$$H^q(X, f_*\mathcal{N}) = \mathbf{C}^q(\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)) \approx \mathbf{C}^q(\mathcal{R}'^\bullet_{\mathcal{N}}(Y)) = H^q(Y,\mathcal{N})$$

as required.

By definition of the direct image, $[f_*\mathcal{R}'^q_{\mathcal{N}}](X) = \mathcal{R}'^q_{\mathcal{N}}(Y)$. So we must show that $[\mathcal{R}^q_{f_*\mathcal{N}}](X)$ is isomorphic to $[f_*\mathcal{R}'^q_{\mathcal{N}}](X)$, and it suffices to show that $\mathcal{R}^q_{f_*\mathcal{N}} \approx f_*\mathcal{R}'^q_{\mathcal{N}}$. We look back at the definition of the modules $\mathcal{R}^0$ in the form (7.4.10). On Y, the analogous sequence for $\mathcal{N}$ is

$$0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{N}^1 \to 0$$

where $\mathcal{R}'^0_{\mathcal{N}} = i_*[\mathcal{N}_{\mathbf{V}}]$, i being the map $\mathbf{V} \to Y$. When f is an affine morphism, the direct image of this sequence

$$0 \to f_*\mathcal{N} \to f_*\mathcal{R}'^0_{\mathcal{N}} \to f_*\mathcal{N}^1 \to 0$$

is exact. We substitute $U = \mathbf{U}$ nd $V = \mathbf{V}$ into Diagram 7.4.20. Then

$$f_*\mathcal{R}'^0_{\mathcal{N}} = f_*i_*[\mathcal{N}_{\mathbf{V}}] = j_*g_*[\mathcal{N}_{\mathbf{V}}] \xrightarrow{(1)} j_*[f_*\mathcal{N}]_{\mathbb{U}} = \mathcal{R}^0_{f_*\mathcal{N}}$$

the equality (1) being Lemma 7.4.21. So $f_*\mathcal{R}'^0_{\mathcal{N}} = \mathcal{R}^0_{f_{\mathcal{N}}}$. Now induction on q completes the proof. $\square$

We go back to verify the third characteristic property of cohomology, which is that when $Y \xrightarrow{f} X$ is the inclusion of an affine open subset, $H^q(X, f_*\mathcal{N}) = 0$ for all $\mathcal{O}_Y$-modules $\mathcal{N}$ and all $q > 0$. The inclusion of an affine open set is an affine morphism, so $H^q(Y,\mathcal{N}) = H^q(X, f_*\mathcal{N})$ (7.4.22), and since Y is affine, $H^q(Y,\mathcal{N}) = 0$ for all $q > 0$ (7.4.18). $\square$

7.4.23. Corollary. *Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. With cohomology defined as above, $H^q(Y,\mathcal{N})$ and $H^q(X, i_*\mathcal{N})$ are isomorphic for every q.* $\square$ *cohextbyO*

Proposition 7.4.22 is one of the places where a specific construction of cohomology is used. The characteristic properties don't apply directly. The next proposition is another such place.

7.4.24. Lemma. *Cohomology is compatible with limits of directed sets of $\mathcal{O}$-modules: $H^q(X, \varinjlim \mathcal{M}_\bullet) \approx \varinjlim H^q(X, \mathcal{M}_\bullet)$ for all q.* *cohlimit*

proof. The direct and inverse image functors and the global section functor are all compatible with direct limits, and $\varinjlim$ is exact (6.5.17). So the module $\mathcal{R}^q_{\varinjlim \mathcal{M}_\bullet}$ that is used to compute the cohomology of $\varinjlim \mathcal{M}_\bullet$ is isomorphic to $\varinjlim [\mathcal{R}^q_{\mathcal{M}_\bullet}]$, and $\mathcal{R}^q_{\varinjlim \mathcal{M}_\bullet}(X)$ is isomorphic to $\varinjlim [\mathcal{R}^q_{\mathcal{M}_\bullet}](X)$. $\square$

7.5 Cohomology of the Twisting Modules

cohprojsp

As we will see, the cohomology $H^q(\mathbb{P}^n, \mathcal{O}(d))$ of the twisting modules $\mathcal{O}(d)$ on $\mathbb{P}^n$ is zero for most values of q. This fact will help to determine the cohomology of other modules.

Lemma 7.4.18 about vanishing of cohomology on an affine variety, and Lemma 7.4.22 about the direct image via an affine morphism, were stated using a particular affine covering. Since we know that cohomology is unique, that particular covering is irrelevant. Though it isn't strictly necessary, we restate those lemmas here as a corollary:

7.5.1. Corollary. (i) *On an affine variety X, $H^q(X,\mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.* *affineco-hzerotwo*

(ii) *Let $Y \xrightarrow{f} X$ be an affine morphism. If $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then $H^q(X, f_*\mathcal{N})$ and $H^q(Y,\mathcal{N})$ are isomorphic. If Y is an affine variety, $H^q(X, f_*\mathcal{N}) = 0$ for all $q > 0$.* $\square$

One case to which **(ii)** applies is that f is the inclusion of a closed subvariety Y into a variety X:

7.5.2. Corollary. *Let $X \xrightarrow{i} \mathbb{P}^n$ be the embedding of a projective variety into projective space, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. For all q, the cohomology $H^q(X,\mathcal{M})$ of $\mathcal{M}$ on X is isomorphic to the cohomology $H^q(\mathbb{P}^n, i_*\mathcal{M})$ of its extension by zero to $\mathbb{P}^n$.* $\square$ *cohXcohP*

Recall also that, on projective space, $\mathcal{M}(d) \approx \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(d)$. If $\mathcal{M}$ is an $\mathcal{O}_X$-module on a projective variety X, its twist $\mathcal{M}(d)$ is defined to be the $\mathcal{O}_X$-module that corresponds to the twist of its extension by zero $i_*\mathcal{M}$, which is $[i_*\mathcal{M}] \otimes_{\mathcal{O}} \mathcal{O}(d)$.

Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on projective space $\mathbb{P}^n$. The twisting modules $\mathcal{O}(d)$ and the twists $\mathcal{M}(d) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(d)$ are isomorphic to $\mathcal{O}(dH)$ and $\mathcal{M}(dH)$, respectively (6.8.11) and there are maps of directed sets

$$\begin{array}{ccccccc} \mathcal{O} & \xrightarrow{\subset} & \mathcal{O}(H) & \xrightarrow{\subset} & \mathcal{O}(2H) & \xrightarrow{\subset} & \cdots \\ {\scriptstyle 1}\downarrow & & {\scriptstyle x_0}\downarrow & & {\scriptstyle x_0^2}\downarrow & & \\ \mathcal{O} & \xrightarrow{x_0} & \mathcal{O}(1) & \xrightarrow{x_0} & \mathcal{O}(2) & \xrightarrow{x_0} & \cdots \end{array} \quad , \quad \begin{array}{ccccccc} \mathcal{M} & \longrightarrow & \mathcal{M}(H) & \longrightarrow & \mathcal{M}(2H) & \longrightarrow & \cdots \\ {\scriptstyle 1}\downarrow & & {\scriptstyle x_0}\downarrow & & {\scriptstyle x_0^2}\downarrow & & \\ \mathcal{M} & \xrightarrow{x_0} & \mathcal{M}(1) & \xrightarrow{x_0} & \mathcal{M}(2) & \xrightarrow{x_0} & \cdots \end{array}$$

The second diagram is obtained from the first one by tensoring with $\mathcal{M}$.

Let $\mathbb{U}$ denote the standard affine open subset $\mathbb{U}^0$ of $\mathbb{P}^n$, and let j be the inclusion of $\mathbb{U}$ into $\mathbb{P}^n$. Then $\varinjlim_d \mathcal{O}(dH) \approx j_*\mathcal{O}_{\mathbb{U}}$ and $\varinjlim_d \mathcal{M}(dH) \approx j_*\mathcal{M}_{\mathbb{U}}$ (6.8.15).

Onse-quencetwo **7.5.3. Corollary.** **(i)** *Let j denote the inclusion $\mathbb{U} \xrightarrow{j} \mathbb{P}^n$. For all $q > 0$, $H^q(\mathbb{P}^n, j_*\mathcal{O}_{\mathbb{U}}) = 0$ and $H^q(\mathbb{P}^n, j_*\mathcal{M}_{\mathbb{U}}) = 0$.*

(ii) *For all projective varieties X, for all $\mathcal{O}$-modules $\mathcal{M}$ and for all $q > 0$, $\varinjlim_d H^q(X, \mathcal{O}(d)) = 0$ and $\varinjlim_d H^q(X, \mathcal{M}(d)) = 0$.*

proof. **(i)** This follows from the facts that the inclusion j is an affine morphism and that $\mathbb{U}$ is affine.

(ii) This follows from **(i)** because $\mathcal{M}(d)$ is isomorphic to $\mathcal{M}(dH)$, and cohomology is compatible with direct limits (7.4.24). $\square$

dimnotation **7.5.4. Notation.** If $\mathcal{M}$ is an $\mathcal{O}$-module, we denote the dimension of $H^q(X, \mathcal{M})$ by $\mathbf{h}^q\mathcal{M}$, or by $\mathbf{h}^q(X, \mathcal{M})$ if there is ambiguity about the variety. We can write $\mathbf{h}^q\mathcal{M} = \infty$ if the dimension is infinite. However, in Section 7.7, we will see that, when $\mathcal{M}$ is a finite $\mathcal{O}$-module on a projective variety X, the dimension $\mathbf{h}^q(X, \mathcal{M})$ will be finite for every q. $\square$

cohOd **7.5.5. Theorem.**

(i) *For $d \geq 0$,* $\mathbf{h}^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{d+n}{n}$ *and* $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}(d)) = 0$ *if $q \neq 0$.*

(ii) *For $r > 0$,* $\mathbf{h}^n(\mathbb{P}^n, \mathcal{O}(-r)) = \binom{r-1}{n}$ *and* $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}(-r)) = 0$ *if $q \neq n$.*

Note that, in **(i)**, the case $d = 0$ asserts that $\mathbf{h}^0(\mathbb{P}^n, \mathcal{O}) = 1$ and $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}) = 0$ for all $q > 0$, while **(ii)** asserts that $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}(-1)) = 0$ for all q, if $1 \leq q$.

proof. We have described the global sections of $\mathcal{O}(d)$ before: If $d \geq 0$, $H^0(X, \mathcal{O}(d))$ is the space of homogeneous polynomials of degree d in the coordinate variables. Its dimension is $\binom{d+n}{n}$, and $H^0(\mathbb{P}^n, \mathcal{O}(d)) = 0$ if $d < 0$. (See (6.8.2).)

Let $X = \mathbb{P}^n$, and let Y be the hyperplane at infinity $\{x = 0\}$, and let $Y \xrightarrow{i} X$ be the inclusion of Y into X.

(i) *the case* $d \geq 0$.

By induction on n, we may assume that the theorem has been proved for Y, which is a projective space of dimension $n-1$. We consider the exact sequence

basecase (7.5.6)
$$0 \to \mathcal{O}_X(-1) \xrightarrow{x_0} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

and its twists

Od (7.5.7)
$$0 \to \mathcal{O}_X(d-1) \xrightarrow{x_0} \mathcal{O}_X(d) \to i_*\mathcal{O}_Y(d) \to 0$$

The twisted sequences are exact because they are obtained by tensoring (7.5.6) with the invertible $\mathcal{O}$-modules $\mathcal{O}(d)$. Because the inclusion i is an affine morphism, $H^q(X, i_*\mathcal{O}_Y(d)) \approx H^q(Y, \mathcal{O}_Y(d))$.

The monomials of degree d in $n+1$ variables form a basis of the space of global sections of $\mathcal{O}_X(d)$. Setting $x_0 = 0$ and deleting terms that become zero gives us a basis of $\mathcal{O}_Y(d)$. Every global section of $\mathcal{O}_Y(d)$ is the restriction of a global section of $\mathcal{O}_X(d)$. So the sequence of global sections

$$0 \to H^0(X, \mathcal{O}_X(d-1)) \xrightarrow{x_0} H^0(X, \mathcal{O}_X(d)) \to H^0(Y, \mathcal{O}_Y(d)) \to 0$$

is exact. The cohomology sequence associated to (7.5.7) tells us that the map $H^1(X, \mathcal{O}_X(d-1)) \longrightarrow H^1(X, \mathcal{O}_X(d))$ is injective.

By induction on the dimension of X, $H^q(Y, \mathcal{O}_Y(d)) = 0$ for $d \geq 0$ and $q > 0$. When combined with the injectivity noted above, the cohomology sequence of (7.5.7) shows that the maps $H^q(X, \mathcal{O}_X(d-1)) \to H^q(X, \mathcal{O}_X(d))$ are bijective for every $q > 0$. Since the limits are zero (7.5.3), $H^q(X, \mathcal{O}_X(d)) = 0$ for all $d \geq 0$ and all $q > 0$.

(ii) *the case $d < 0$, or $r > 0$.*

We use induction on the integers r and n. We suppose the theorem proved for a given r, and we substitute $d = -r$ into the sequence (7.5.7):

Or (7.5.8)
$$0 \to \mathcal{O}_X(-(r+1)) \xrightarrow{x_0} \mathcal{O}_X(-r) \to i_*\mathcal{O}_Y(-r) \to 0$$

For $r=0$, the exact sequence is $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$. In the cohomology sequence associated to that sequence, the terms $H^q(X,\mathcal{O}_X)$ and $H^q(Y,\mathcal{O}_Y)$ are zero when $q>0$, and $H^0(X,\mathcal{O}_X) = H^0(Y,\mathcal{O}_Y) = \mathbb{C}$. Therefore $H^q(X,\mathcal{O}_X(-1)) = 0$ for every q. This proves **(ii)** for $r=1$.

Our induction hypothesis is that, $\mathbf{h}^n(X,\mathcal{O}(-r)) = \binom{r-1}{n}$ and $\mathbf{h}^q = 0$ if $q \neq n$. By induction on n, we may suppose that $\mathbf{h}^{n-1}(Y,\mathcal{O}(-r)) = \binom{r-1}{n-1}$ and that $\mathbf{h}^q = 0$ if $q \neq n-1$.

Instead of displaying the cohomology sequence associated to (7.5.8), we assemble the dimensions of cohomology into a table, in which the asterisks stand for entries that are to be determined:

(7.5.9) co-hdimstwo

	$\mathcal{O}_X(-(r+1))$	$\mathcal{O}_X(-r)$	$i_*\mathcal{O}_Y(-r)$
$\mathbf{h}^0$:	$*$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{h}^{n-2}$:	$*$	0	0
$\mathbf{h}^{n-1}$:	$*$	0	$\binom{r-1}{n-1}$
$\mathbf{h}^n$:	$*$	$\binom{r-1}{n}$	0

The second column is our induction hypothesis, and the third column is determined by induction on n. The exact cohomology sequence shows that that

$$\mathbf{h}^n(X,\mathcal{O}(-(r+1))) = \binom{r-1}{n-1} + \binom{r-1}{n}$$

and that the other entries labeled with an asterisk are zero. The right side of this equation is equal to $\binom{r}{n}$. $\square$

7.6 Cohomology of Hypersurfaces

cohhyper

We begin with the cohomology of a plane projective curve. Let X be the projective plane $\mathbb{P}^2$ and let $C \xrightarrow{i} X$ denote the inclusion of a plane curve C of degree k. The ideal of functions that vanish on C is isomorphic to the twisting module $\mathcal{O}_X(-k)$ (6.8.8) so one has an exact sequence

(7.6.1) $$0 \to \mathcal{O}_X(-k) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0$$ coh-planecurvetwo

The table below displays the dimensions of the cohomology. Theorem 7.5.5 determines the first two columns, and the cohomology sequence determines the last column.

(7.6.2) cohdims

	$\mathcal{O}_X(-k)$	$\mathcal{O}_X$	$i_*\mathcal{O}_C$
$\mathbf{h}^0$:	0	1	1
$\mathbf{h}^1$:	0	0	$\binom{k-1}{2}$
$\mathbf{h}^2$:	$\binom{k-1}{2}$	0	0

Since the inclusion of C into X is an affine morphism, $\mathbf{h}^q(C,\mathcal{O}_C) = \mathbf{h}^q(X,i_*\mathcal{O}_C)$. Therefore

(7.6.3) $$\mathbf{h}^0(C,\mathcal{O}_C) = 1, \quad \mathbf{h}^1(C,\mathcal{O}_C) = \binom{k-1}{2}, \quad \text{and} \quad \mathbf{h}^q = 0 \quad \text{when} \quad q>1$$ coh-planecurvethre

The dimension of $H^1(C,\mathcal{O}_C)$, which is $\binom{k-1}{2}$, is called the *arithmetic genus* of C. It is usually denoted by p_a or $p_a(C)$. We will see later (8.8.2) that the arithmetic genus of smooth curve is equal to its topological genus: $p_a = g$. But the arithmetic genus of a plane curve of degree k is equal to $\binom{k-1}{2}$ when the curve C is singular too.

We restate the results as a corollary.

7.6.4. Corollary. *For a plane curve C of degree k, $\mathbf{h}^0\mathcal{O}_C = 1$, $\mathbf{h}^1\mathcal{O}_C = \binom{k-1}{2} = p_a$, and $\mathbf{h}^q = 0$ if $q \neq 0,1$.* $\square$ *coh-planecurve*

The fact that $\mathbf{h}^0\mathcal{O}_C = 1$ tells us that the only rational functions that are regular everywhere on C are the constants. It follows that a plane curve is connected in the Zariski topology, and it hints at a fact to be proved later, that a plane curve is connected in the classical topology, but it isn't a proof of that fact.

In the next section we will see that the cohomology of any $\mathcal{O}$-module on a projective curve is zero except in dimensions 0 and 1. To determine cohomology of a curve that is embedded in a higher dimensional projective space, we will need to know that its cohomology is finite-dimensional, which is Theorem 7.7.3 below, and that it is zero in dimension greater than one, which is Theorem 7.7.1, also below. The cohomology of projective curves will be studied again, in Chapter 8.

One can make a similar computation for the hypersurface Y in $X = \mathbb{P}^n$ defined by an irreducible homogeneous polynomial f of degree k. The ideal of such a hypersurface Y is isomorphic to $\mathcal{O}_X(-k)$ (6.8.8), so there is an exact sequence

$$0 \to \mathcal{O}_X(-k) \xrightarrow{f} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

Since we know the cohomology of $\mathcal{O}_X(-k)$ and of $\mathcal{O}_X$, and since $H^q(X, i_*\mathcal{O}_Y) \approx H^q(Y, \mathcal{O}_Y)$, we can use this sequence to compute the dimensions of the cohomology of $\mathcal{O}_Y$.

cohhyper-surface **7.6.5. Corollary.** *Let Y be a hypersurface of dimension d and degree k in a projective space of dimension $d+1$. Then $\mathbf{h}^0(Y, \mathcal{O}_Y) = 1$, $\mathbf{h}^d(Y, \mathcal{O}_Y) = \binom{k-1}{d+1}$, and $\mathbf{h}^q(Y, \mathcal{O}_Y) = 0$ for all other q.* $\square$

In particular, when S is the *surface* in $\mathbb{P}^3$ defined by an irreducible polynomial of degree k, $\mathbf{h}^0(S, \mathcal{O}_S) = 1$, $\mathbf{h}^1(S, \mathcal{O}_S) = 0$, $\mathbf{h}^2(S, \mathcal{O}_S) = \binom{k-1}{3}$, and $\mathbf{h}^q = 0$ if $q > 2$. When a projective surface S is embedded into a higher dimensional projective space, it is still true that $\mathbf{h}^q = 0$ if $q > 2$, but $\mathbf{h}^1(S, \mathcal{O}_S)$ may be nonzero. The dimensions $\mathbf{h}^1(S, \mathcal{O}_S)$ and $\mathbf{h}^2(S, \mathcal{O}_S)$ are invariants of a surface S that are somewhat analogous to the genus of a curve. In classical terminology, $\mathbf{h}^2(S, \mathcal{O}_S)$ is the *geometric genus* p_g and $\mathbf{h}^1(S, \mathcal{O}_S)$ is the *irregularity* q . The *arithmetic genus* p_a of S is defined to be

patwo (7.6.6)
$$p_a \;=\; \mathbf{h}^2(S, \mathcal{O}_S) - \mathbf{h}^1(S, \mathcal{O}_S) = p_g - q$$

Therefore the irregularity of S is $q = p_g - p_a$. When S is a surface in $\mathbb{P}^3$, the irregularity is zero, and $p_g = p_a$.

In modern terminology, it might seem more natural to replace the arithmetic genus by the Euler characteristic of the structure sheaf $\chi(\mathcal{O}_S)$, which is defined to be $\sum_q (-1)^q \mathbf{h}^q \mathcal{O}_S$ (see (**7.7.7**) below). The Euler characteristic of the structure sheaf on a curve is

$$\chi(\mathcal{O}_C) = \mathbf{h}^0(C, \mathcal{O}_C) - \mathbf{h}^1(C, \mathcal{O}_C) = 1 - p_a$$

and on a surface S it is

$$\chi(\mathcal{O}_S) = \mathbf{h}^0(S, \mathcal{O}_S) - \mathbf{h}^1(S, \mathcal{O}_S) + \mathbf{h}^2(S, \mathcal{O}_S) = 1 + p_a$$

But because of tradition, the arithmetic genus is still used quite often.

7.7 Three Theorems about Cohomology

threethms *consup port* **7.7.1. Theorem.** *Let X be a projective variety, and let $\mathcal{M}$ be a finite $\mathcal{O}_X$-module. If the support of $\mathcal{M}$ has dimension k, then $H^q(X, \mathcal{M}) = 0$ for all $q > k$. In particular, if X has dimension n, then $H^q(X, \mathcal{M}) = 0$ for all $q > n$.*

See Section 6.7 for the definition of support.

largetwist **7.7.2. Theorem.** *Let $\mathcal{M}(d)$ be the twist of a finite $\mathcal{O}_X$-module $\mathcal{M}$ on a projective variety X. For sufficiently large d and for all $q > 0$, $H^q(X, \mathcal{M}(d)) = 0$.*

findim **7.7.3. Theorem.** *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. The cohomology $H^q(X, \mathcal{M})$ is a finite-dimensional vector space for every q.*

descind **7.7.4. Notes.** **(a)** As the first theorem asserts, the highest dimension in which cohomology of an $\mathcal{O}_X$-module on a projective variety X can be nonzero is the dimension of X. It is also true that, on a projective variety X of dimension n, there will be $\mathcal{O}_X$-modules $\mathcal{M}$ such that $H^n(X, \mathcal{M}) \neq 0$. In contrast, in the classical topology on a projective variety X of dimnsion n, the constant coefficient cohomology $H^{2n}(X_{class}, \mathbb{Z})$ isn't zero. As we have mentioned, in the Zariski topology, the cohomology $H^q(X_{zar}, \mathbb{Z})$ with constant coefficients is zero for every $q > 0$. When X is an affine variety, the cohomology of any $\mathcal{O}_X$-module is zero for all $q > 0$.

(b) The third theorem tells us that the space of global sections $H^0(X, \mathcal{M})$ of a finite $\mathcal{O}$-module on a projective variety is finite-dimensional. This is one of the most important consequences of the theorem, and it isn't easy to prove directly. Cohomology needn't be finite-dimensional on a variety that isn't projective. For example, on an affine variety $X = \operatorname{Spec} A$, $H^0(X, \mathcal{O}) = A$ isn't finite-dimensional unless X is a point. When X is the complement of a point in $\mathbb{P}^2$, $H^1(X, \mathcal{O})$ isn't finite-dimensional.

(c) The proofs have an interesting structure. The first theorem allows us to use *descending* induction to prove the second and third theorems, beginning with the fact that $H^k(X, \mathcal{M}) = 0$ when k is greater than the dimension of X. □

In these theorems, we are given that X is a closed subvariety of a projective space $\mathbb{P}^n$. We can replace an $\mathcal{O}_X$-module by its extension by zero to $\mathbb{P}^n$, since this doesn't change the cohomology or the dimension of support. The twist $\mathcal{M}(d)$ of an $\mathcal{O}_X$-module that is referred to in the second theorem is defined in terms of the extension by zero. So we may assume that X is a projective space.

The proofs are based on the cohomology of the twisting modules (7.5.5) and on the vanishing of the limit $\varinjlim_d H^q(X, \mathcal{M}(d))$ for $q > 0$ (7.5.3).

proof of Theorem 7.7.1 (*vanishing in large dimension*)
Here $\mathcal{M}$ is a finite $\mathcal{O}$-module whose support S has dimension at most k. We are to show that $H^q(X, \mathcal{M}) = 0$ when $q > k$. We choose coordinates so that the hyperplane $H : x_0 = 0$ doesn't contain any component of the support S. Then $H \cap S$ has dimension at most $k-1$. We inspect the multiplication map $\mathcal{M}(-1) \xrightarrow{x_0} \mathcal{M}$. The kernel $\mathcal{K}$ and cokernel $\mathcal{Q}$ are annihilated by x_0, so the supports of $\mathcal{K}$ and $\mathcal{Q}$ are contained in H. Since they are also in S, the supports have dimension at most $k-1$. We can apply induction on k to them. In the base case $k = 0$, the supports of $\mathcal{K}$ and $\mathcal{Q}$ will be empty, and their cohomology will be zero.

We break the exact sequence $0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{M} \to \mathcal{Q} \to 0$ into two short exact sequences by introducing the kernel $\mathcal{N}$ of the map $\mathcal{M} \to \mathcal{Q}$:

(7.7.5) $$0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{N} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0$$ kercoker

The induction hypothesis applies to $\mathcal{K}$ and to $\mathcal{Q}$. It tells us that $H^q(X, \mathcal{K}) = 0$ and $H^q(X, \mathcal{Q}) = 0$, when $q \geq k$. For $q > k$, the cohomology sequences associated to the two exact sequences give us bijections

$$H^q(X, \mathcal{M}(-1)) \to H^q(X, \mathcal{N}) \quad \text{and} \quad H^q(X, \mathcal{N}) \to H^q(X, \mathcal{M})$$

Therefore the composed map $H^q(X, \mathcal{M}(-1)) \to H^q(X, \mathcal{M})$ is bijective, and this is true for every $\mathcal{O}$-module whose support has dimension $\leq k$, including for the $\mathcal{O}$-module $\mathcal{M}(d)$. For every $\mathcal{O}$-module whose support has dimension at most k, every d, and every $q > k$, the canonical map $H^q(X, \mathcal{M}(d-1)) \to H^q(X, \mathcal{M}(d))$ is bijective. According to (7.5.3), the limit $\varinjlim_d H^q(X, \mathcal{M}(d))$ is zero. It follows that $H^q(X, \mathcal{M}(d)) = 0$ for all d when $q > 0$, and in particular, $H^q(X, \mathcal{M}) = 0$.

proof of Theorem 7.7.2 (*vanishing for a large twist*)
Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. We recall that $\mathcal{M}(r)$ is generated by global sections when r is sufficiently large (6.8.20). Choosing generators gives us a surjective map $\mathcal{O}^n \to \mathcal{M}(r)$. Let $\mathcal{N}$ be the kernel of this map. When we twist the exact sequence $0 \to \mathcal{N} \to \mathcal{O}^n \to \mathcal{M}(r) \to 0$, we obtain short exact sequences

(7.7.6) $$0 \to \mathcal{N}(d) \to \mathcal{O}(d)^n \to \mathcal{M}(d+r) \to 0$$ threetwos

for every $d \geq 0$. These sequences are useful because $H^q(X, \mathcal{O}(d)) = 0$ when $d \geq 0$ and $q > 0$ (7.5.5).

To prove Theorem 7.7.2, we must show this:

(*) *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module. For sufficiently large d and for all $q > 0$, $H^q(X, \mathcal{M}(d)) = 0$.*

Let n be the dimension of X. By Theorem 7.7.1, $H^q(X, \mathcal{M}) = 0$ for any $\mathcal{O}$-module $\mathcal{M}$, when $q > n$, In particular, $H^q(X, \mathcal{M}(d)) = 0$ when $q > n$. This leaves a finite set of integers $q = 1, ..., n$ to consider, and it suffices to consider them one at a time. If (*) is true for each individual q, there will be a sufficiently large d such that it is true for each of the integers $q = 1, ..., n$ at the same time, and therefore for all positive integers q, as the theorem asserts.

We use descending induction on q, the base case being $q = n + 1$, for which (*) is true with $d = 0$. We suppose that (*) is true for every finite $\mathcal{O}$-module $\mathcal{M}$ when $q = p + 1$, and that $p > 0$, and we show that (*) is true for every finite $\mathcal{O}$-module $\mathcal{M}$ when $q = p$.

We substitute $q = p$ into the cohomology sequence associated to the sequence (7.7.6). The relevant part of that sequence is

$$\to H^p(X, \mathcal{O}(d)^n) \to H^p(X, \mathcal{M}(d+r)) \xrightarrow{\delta^p} H^{p+1}(X, \mathcal{N}(d)) \to$$

Since p is positive, $H^p(X, \mathcal{O}(d)) = 0$ for all $d \geq 0$. The map δ^p is injective. We note that $\mathcal{N}$ is a finite $\mathcal{O}$-module. So our induction hypothesis applies to it. The induction hypothesis tells us that, when d is large, $H^{p+1}(X, \mathcal{N}(d)) = 0$ and therefore $H^p(X, \mathcal{M}(d+r)) = 0$. The particular form of the integer $d+r$ isn't useful. Our conclusion is that, for every finite $\mathcal{O}$-module $\mathcal{M}$, $H^p(X, \mathcal{M}(d_1)) = 0$ when d_1 is large enough. □

proof of Theorem 7.7.3 (*finiteness of cohomology*)

This proof uses ascending induction on the dimension of support and descending induction on the degree d of a twist. As has been mentioned, it isn't easy to prove directly that the space $H^0(X, \mathcal{M})$ of global sections is finite-dimensional.

Let $\mathcal{M}$ be an $\mathcal{O}$-module whose support has dimension at most k. We go back to the sequences (7.7.5) and their cohomology sequences, in which the supports of $\mathcal{K}$ and $\mathcal{Q}$ have dimension $\leq k-1$. Ascending induction on the dimension of support allows us to assume that $H^r(X, \mathcal{K})$ and $H^r(X, \mathcal{Q})$ are finite-dimensional for all r. Denoting finite-dimensional spaces ambiguously by $finite$, the two cohomology sequences become

$$\cdots \to finite \to H^q(X, \mathcal{M}(-1)) \to H^q(X, \mathcal{N}) \to finite \to \cdots$$

and

$$\cdots \to finite \to H^q(X, \mathcal{N}) \to H^q(X, \mathcal{M}) \to finite \to \cdots$$

The first of these sequences shows that if $H^q(X, \mathcal{M}(-1))$ has infinite dimension, then $H^q(X, \mathcal{N})$ has infinite dimension, and the second sequence shows that if $H^q(X, \mathcal{N})$ has infinite dimension, then $H^q(X, \mathcal{M})$ has infinite dimenson. Therefore either $H^q(X, \mathcal{M}(-1))$ and $H^q(X, \mathcal{M})$ are both finite-dimensional, or else they are both infinite-dimensional. This applies to the twisted modules $\mathcal{M}(d)$ as well as to $\mathcal{M}$: $H^q(X, \mathcal{M}(d-1))$ and $H^q(X, \mathcal{M}(d))$ are both finite-dimensional or both infinite-dimensional.

Suppose that $q > 0$. Then $H^q(X, \mathcal{M}(d)) = 0$ when d is large enough (Theorem 7.7.2). Since the zero space is finite-dimensional, we can use the sequence together with descending induction on d, to conclude that $H^q(X, \mathcal{M}(d))$ is finite-dimensional for every finite module $\mathcal{M}$ and every d. In particular, $H^q(X, \mathcal{M})$ is finite-dimensional.

This leaves the case that $q = 0$. To prove that $H^0(X, \mathcal{M})$ is finite-dimensional, we put $d = -r$ with $r > 0$ into the sequence (7.7.6):

$$0 \to \mathcal{N}(-r) \to \mathcal{O}(-r)^m \to \mathcal{M} \to 0$$

The corresponding cohomology sequence is

$$0 \to H^0(X, \mathcal{N}(-r)) \to H^0(X, \mathcal{O}(-r))^m \to H^0(X, \mathcal{M}) \xrightarrow{\delta^0} H^1(X, \mathcal{N}(-r)) \to \cdots.$$

Here $H^0(X, \mathcal{O}(-r))^m = 0$, and we've shown that $H^1(X, \mathcal{N}(-r))$ is finite-dimensional. It follows that $H^0(X, \mathcal{M})$ is finite-dimensional, and this completes the proof.

Notice that the finiteness of H^0 comes out only at the end. The higher cohomology is essential for the proof. □

eulerchar **(7.7.7) Euler characteristic**

Theorems 7.7.1 and 7.7.3 allow us to define the Euler characteristic of a finite module on projective variety.

7.7.8. Definition. Let X be a projective variety. The *Euler characteristic* of a finite $\mathcal{O}$-module $\mathcal{M}$ is the alternating sum of the dimensions of its cohomology: defeuler

$$\chi(\mathcal{M}) = \sum (-1)^q \mathbf{h}^q(X, \mathcal{M}). \tag{7.7.9}$$ chi

This makes sense because $\mathbf{h}^q(X, \mathcal{M})$ is finite for every q, and is zero when q is large.

Try not to confuse the Euler characterstic of an $\mathcal{O}$-module with the topological Euler characteristic of the variety X.

7.7.10. Proposition. (i) *If* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *is a short exact sequence of finite* $\mathcal{O}$*-modules on a projective variety* X*, then* $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P}) = 0$.
(ii) *If* $0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \cdots \to \mathcal{M}_n \to 0$ *is an exact sequence of finite* $\mathcal{O}$*-modules on* X*, the alternating sum* $\sum (-1)^i \chi(\mathcal{M}_i)$ *is zero.*

7.7.11. Lemma. *Let* $0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ *be an exact sequence of finite dimensional vector spaces. The alternating sum* $\sum (-1)^q \dim V^q$ *is zero.* □ *altsum*

proof of Proposition 7.7.10. **(i)** Let n be the dimension of X. The cohomology sequence associated to the given sequence is

$$0 \to H^0(\mathcal{M}) \to H^0(\mathcal{N}) \to H^0(\mathcal{P}) \to H^1(\mathcal{M}) \to \quad \ldots \quad \to H^n(\mathcal{N}) \to H^n(\mathcal{P}) \to 0$$

and the lemma tells us that the alternating sum of its dimensions is zero. That alternating sum is also equal to $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P})$.
(ii) Let's denote the given sequence by $\mathbb{S}_0$ and the alternating sum $\sum_i (-1)^i \chi(\mathcal{M}_i)$ by $\chi(\mathbb{S}_0)$. Let $\mathcal{N} = \mathcal{M}_1/\mathcal{M}_0$. The sequence $\mathbb{S}_0$ decomposes into the two exact sequences

$$\mathbb{S}_1 : \ 0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{N} \to 0 \quad \text{and} \quad \mathbb{S}_2 : \ 0 \to \mathcal{N} \to \mathcal{M}_2 \to \cdots \to \mathcal{M}_k \to 0$$

One sees directly that $\chi(\mathbb{S}_0) = \chi(\mathbb{S}_1) - \chi(\mathbb{S}_2)$, so the assertion follows from **(i)** by induction on n. □

7.8 Bézout's Theorem

bezout

As an application of cohomology, we use it to prove Bézout's Theorem. We restate it here:

7.8.1. Bézout's Theorem. *Let* Y *and* Z *be distinct curves, of degrees* m *and* n*, respectively, in the projective plane* X*. The number of intersection points* $Y \cap Z$*, when counted with an appropriate multiplicity, is equal to* mn*. Moreover, the multiplicity is* 1 *at a point at which* Y *and* Z *intersect transversally.* *bezoutrestated*

The definition of the multiplicity will emerge during the proof.

Note. Let f and g be relatively prime homogeneous polynomials. When one replaces Y and Z by their divisors of zeros (1.3.13), the theorem remains true whether or not they are irreducible. The proof isn't signifiantly different from the one we give here, except that it requires setting up some notation. For example, suppose that f and g are products of linear polynomials, so that Y is the union of m lines and Z is the union of n lines, and suppose that those lines are distinct. Since distinct lines intersect transversally in a single point, there are mn intersection points of multiplicity 1. □ intersectlines

proof of Bézout's Theorem. We suppress notation for the extension by zero from Y or Z to the plane X, denoting the direct images of $\mathcal{O}_Y$ and $\mathcal{O}_Z$ by the same symbols. Let f and g be the irreducible homogeneous polynomials whose zero loci are Y and Z. Multiplication by f defines a short exact sequence

$$0 \to \mathcal{O}_X(-m) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

This exact sequence describes $\mathcal{O}_X(-m)$ as the ideal $\mathcal{I}$ of regular functions that vanish on Y, and there is a similar sequence describing the module $\mathcal{O}_X(-n)$ as the ideal $\mathcal{J}$ of Z. The zero locus of the ideal $\mathcal{I}+\mathcal{J}$ is the intersection $Y \cap Z$, which is a finite set of points $\{p_1, ..., p_k\}$.

Let $\overline{\mathcal{O}}$ denote the quotient $\mathcal{O}_X/(\mathcal{I}{+}\mathcal{J})$. Its support is the finite set $Y\cap Z$, and therefore $\overline{\mathcal{O}}$ is isomorphic to a direct sum $\bigoplus \overline{\mathcal{O}}_i$, where each $\overline{\mathcal{O}}_i$ is a finite-dimensional algebra whose support is p_i (**6.7.2**). The *intersection multiplicity* of Y and Z at p_i is defined to be the dimension of $\overline{\mathcal{O}}_i$, which is also the dimension of the space of its global sections. Let's denote the intersection multiplicity by μ_i. The dimension of $H^0(X,\overline{\mathcal{O}})$ is the sum $\mu_1{+}\cdots{+}\mu_k$, and $H^q(X,\overline{\mathcal{O}})=0$ for all $q>0$ (Theorem 7.7.1). The Euler characteristic $\chi(\overline{\mathcal{O}})$ is equal to $\mathbf{h}^0(X,\overline{\mathcal{O}})$. We'll show that $\chi(\overline{\mathcal{O}}) \ = \ mn$, and therefore that $\mu_1{+}\cdots{+}\mu_k \ = mn$. This will prove Bézout's Theorem.

We form a sequence, in which $\mathcal{O}$ stands for $\mathcal{O}_X$:

bezoutres-olution (7.8.2)
$$0\to\mathcal{O}(-m-n)\xrightarrow{(g,f)^t}\mathcal{O}(-m)\times\mathcal{O}(-n)\xrightarrow{(-f,g)}\mathcal{O}\xrightarrow{\pi}\overline{\mathcal{O}}\to 0$$

In order to interpret the maps in this sequence as matrix multiplication, with homomorphisms acting on the left, a section of $\mathcal{O}(-m)\times\mathcal{O}(-n)$ should be represented as a column vector $(u,v)^t$, u and v being sections of $\mathcal{O}(-m)$ and $\mathcal{O}(-n)$, respectively.

bresolex-act **7.8.3. Lemma.** *The sequence (7.8.2) is exact.*

proof. We suppose that coordinates have been chosen so that none of the points making up $Y\cap Z$ lie on the coordinate axes.

To prove exactness, it suffices to show that the sequence of sections on each of the standard open sets is exact. We look at the index 0 as usual, denoting $\mathbb{U}^0$ by $\mathbb{U}$. Let A be the algebra of regular functions on $\mathbb{U}$, which is the polynomial algebra $\mathbb{C}[u_1,u_2]$, with $u_i=x_i/x_0$. We identify $\mathcal{O}(k)$ with $\mathcal{O}(kH)$, H being the hyperplane at infinity. The restriction of the module $\mathcal{O}(kH)$ to $\mathbb{U}$ is isomorphic to the restriction $\mathcal{O}_{\mathbb{U}}$ of $\mathcal{O}$. Its sections on $\mathbb{U}$ are the elements of A. Let $\overline{A}$ be the algebra of sections of $\overline{\mathcal{O}}$ on $\mathbb{U}$. Since f and g are relatively prime, so are their dehomogenizations $F=f(1,u_1,u_2)$ and $G=g(1,u_1,u_2)$. The sequence of sections of (7.8.2) on $\mathbb{U}$ is

$$0\to A\xrightarrow{(G,F)^t}A\times A\xrightarrow{(-F,G)}A\to\overline{A}\to 0$$

and the only place at which exactness of this sequence isn't obvious is at $A\times A$. Suppose that $(u,v)^t$ is in the kernel of the map $(-F,G)$, i.e., that $Fu=Gv$. Since F and G are relatively prime, F divides v, G divides u, and $v/F=u/G$. Let $w=v/F=u/G$. Then $(u,v)^t=(G,F)^tw$. So $(u,v)^t$ is the image of w. □

We go back to the proof of Bézout's Theorem. Proposition 7.7.10 **(ii)**, applied to the exact sequence (7.8.2), tells us that the alternating sum

chiforbe-zout (7.8.4)
$$\chi(\mathcal{O}(-m-n))\ -\ \chi(\mathcal{O}(-m))\ -\ \chi(\mathcal{O}(-n))\ +\ \mathcal{O}\ -\ \chi(\overline{\mathcal{O}})$$

is zero. Since cohomology is compatible with products(7.3.5), $\chi(\mathcal{M}\times\mathcal{N})=\chi(\mathcal{M})+\chi(\mathcal{N})$. Solving for $\chi(\overline{\mathcal{O}})$ and applying Theorem 7.5.5,

$$\chi(\overline{\mathcal{O}})\ =\ \binom{n+m-1}{2}-\binom{m-1}{2}-\binom{n-1}{2}+1$$

The right side of this equation evaluates to mn. This completes the proof. □

We still need to explain the assertion that the multiplicity at a transversal intersection p is equal to 1. The intersection at p will be transversal if and only if $\mathcal{I}{+}\mathcal{J}$ generates the maximal ideal $\mathfrak{m}$ of $A=\mathbb{C}[y,z]$ at p locally. If so, then the component of $\overline{\mathcal{O}}$ supported at p will have dimension 1, and the intersection multiplicity at p will be 1.

When Y and Z are lines, we may choose affine coordinates so that p is the origin in the plane $X=\operatorname{Spec}A$ and the curves are the coordinate axes $\{z=0\}$ and $\{y=0\}$. The variables y,z generate the maximal ideal at the origin.

Suppose that Y and Z intersect transverally at p, but that they aren't lines. We choose affine coordinates so that p is the origin and that the tangent directions of Y and Z at p are the coordinate axes. The affine equations of Y and Z will have the form $y_1=0$ and $z_1=0$, where $y_1=y+g(y,z)$ and $z_1=z+h(y,z)$, g and h being polynomials all of whose terms have degree at least 2. Because Y and Z may intersect at points other than p, the elements y_1 and z_1 may fail to generate the maximal ideal $\mathfrak{m}$ at p. However, they do generate the maximal ideal locally. To show this, it suffices to show that they generate the maximal ideal M in the local ring R at p. By Corollary 5.1.2, it suffices to show that y_1 and z_1 generate M/M^2, and this is true because y_1 and z_1 are congruent to y and z modulo M^2. □

7.9 Uniqueness of the Coboundary Maps

unique-cobound

In Section 7.4, we constructed a cohomology $\{H^q\}$ that has the characteristic properties, and we showed that the functors H^q are unique. We didn't show that the coboundary maps δ^q that appear in the cohomology sequences are unique. We go back to do this now.

To make it clear that there is something to show, we note that the cohomology sequence (7.1.3) remains exact when a coboundary map δ^q is multiplied by -1. Why can't we define a new collection of coboundary maps by changing some signs? The reason we can't do this is that we used the coboundary maps δ^q in (7.4.7) and (7.4.8), to identify $H^q(X,\mathcal{M})$. Having done that, we aren't allowed to change δ^q for the particular short exact sequences (7.4.2). We show that the coboundary maps for those sequences determine the coboundary maps for every short exact sequence of $\mathcal{O}$-modules

$$(A) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \to 0$$

The sequences (7.4.2) were rewritten as (7.4.10):

$$(B) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{R}^0_{\mathcal{M}} \longrightarrow \mathcal{M}^1 \to 0$$

To show that the coboundaries for the sequence (A) are determined uniquely, we relate it to the sequence (B), for which the coboundary maps are fixed. We map the sequences (A) and (B) to a third exact sequence

$$(C) \qquad 0 \to \mathcal{M} \xrightarrow{\psi} \mathcal{R}^0_{\mathcal{N}} \longrightarrow \mathcal{Q} \to 0$$

where ψ is the composition of the injective maps $\mathcal{M} \to \mathcal{N} \to \mathcal{R}^0_{\mathcal{N}}$ and $\mathcal{Q}$ is the cokernel of ψ.

First, we inspect the diagram

$$\begin{tikzcd}[column sep=small]
(A) & \mathcal{M} \arrow[r] \arrow[d, equal] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\
(C) & \mathcal{M} \arrow[r, "\psi"] & \mathcal{R}^0_{\mathcal{N}} \arrow[r] & \mathcal{Q}
\end{tikzcd}$$

and its diagram of coboundary maps

$$\begin{tikzcd}
(A) & H^q(X,\mathcal{P}) \arrow[r, "\delta^q_A"] \arrow[d] & H^{q+1}(X,\mathcal{M}) \arrow[d, equal] \\
(C) & H^q(X,\mathcal{Q}) \arrow[r, "\delta^q_C"] & H^{q+1}(X,\mathcal{M})
\end{tikzcd}$$

This diagram shows that the coboundary map δ^q_A for the sequence (A) is determined by the coboundary map δ^q_C for (C).

Next, we inspect the diagram

$$\begin{tikzcd}
(B) & \mathcal{M} \arrow[r] \arrow[d, equal] & \mathcal{R}^0_{\mathcal{M}} \arrow[r] \arrow[d, "u"'] & \mathcal{M}^1 \arrow[d, "v"'] \\
(C) & \mathcal{M} \arrow[r, "\psi"] & \mathcal{R}^0_{\mathcal{N}} \arrow[r] & \mathcal{Q}
\end{tikzcd} \tag{7.9.1}$$

BtoC

and its diagram of coboundary maps

$$\begin{tikzcd}
(B) & H^q(X,\mathcal{M}^1) \arrow[r, "\delta^q_B"] \arrow[d] & H^{q+1}(X,\mathcal{M}) \arrow[d, equal] \\
(C) & H^q(X,\mathcal{Q}) \arrow[r, "\delta^q_C"] & H^{q+1}(X,\mathcal{M})
\end{tikzcd}$$

When $q > 0$, δ^q_C and δ^q_B are bijective because the cohomology of $\mathcal{R}^0_{\mathcal{M}}$ and $\mathcal{R}^0_{\mathcal{N}}$ is zero in positive dimension. Then δ^q_C is determined by δ^q_B, and so is δ^q_A.

We have to look more closely to settle the case $q = 0$. The map labeled u in (7.9.1) is injective. The Snake Lemma shows that v is injective, and that the cokernels of u and v are isomorphic. We denote both of those cokernels by $\mathcal{R}^0_{\mathcal{P}}$. When we add the cokernels to the diagram, and pass to cohomology, we obtain a diagram whose relevant part is

$$\begin{array}{cccccc}
(B) & H^0(X, \mathcal{R}^0_{\mathcal{M}}) & \longrightarrow & H^0(X, \mathcal{M}^1) & \xrightarrow{\delta^0_B} & H^1(X, \mathcal{M}) \\
& \downarrow u & & \downarrow v & & \| \\
(C) & H^0(X, \mathcal{R}^0_{\mathcal{N}}) & \xrightarrow{\beta} & H^0(X, \mathcal{Q}) & \xrightarrow{\delta^0_C} & H^1(X, \mathcal{M}) \\
& \downarrow \gamma & & \downarrow & & \\
& H^0(X, \mathcal{R}^0_{\mathcal{P}}) & = & H^0(X, \mathcal{R}^0_{\mathcal{P}}) & &
\end{array}$$

Its rows and columns are exact. We want to show that the map δ^0_C is determined uniquely by δ^0_B. It is determined by δ^0_B on the image of v and it is zero on the image of β. To show that δ^0_C is determined by δ^0_B, it suffices to show that the images of v and β together span $H^0(X, \mathcal{Q})$. This follows from the fact that γ is surjective (7.4.3). Thus δ^0_C is determined by δ^0_B, and so is δ^0_A. $\square$

7.10 Exercises

7.10.1. Let X be the complement of the point $(0,0,1)$ in $\mathbb{P}^2$. Use the covering of X by the two standard affine open sets $\mathbb{U}^0, \mathbb{U}^1$ to compute the cohomology $H^q(X, \mathcal{O}_X)$. chapsev xcohcomplpt

7.10.2. Let $0 \to V_0 \to \cdots \to V_n \to 0$ be a complex of finite-dimensional vector spaces. Prove that $\sum_i (-1)^i \dim V_i = \sum (-1)^q \mathbf{C}^q(V^\bullet\}$. xeuler-complex

7.10.3. Let $0 \to \mathcal{M}_0 \to \cdots \to \mathcal{M}_k \to 0$ be an exact sequence of $\mathcal{O}$-modules on a variety X. Prove that if $H^q(\mathcal{M}_i) = 0$ for all $q > 0$ and all i, the sequence of global sections is exact. xglobsec-sexact

7.10.4. *the Cousin Problem.* Let X be a projective variety. xcousin

(i) Detemine the cohomology of the function field module $\mathcal{F}$ (6.5.10).

(ii) Let X be a projective space, and let $\{V^i\}$, $i = 1, ..., k$ be an open covering of X. Suppose that rational functions f_i are given, such that $f_i - f_j$ is a regular function on $V^i \cap V^j$ for all i and j. The Cousin Problem asks for a rational function $\widetilde{f}$ such that $\widetilde{f} - f_i$ is a regular function on V^i for every i. Analyze this problem making use of the exact sequence $0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{Q} \to 0$, where $\mathcal{Q}$ is the quotient $\mathcal{F}/\mathcal{O}$.

(iii) What can one say for other varieties X?

7.10.5. Prove that if a variety X is covered by two affine open sets, then $H^q(X, \mathcal{M}) = 0$ for every $\mathcal{O}$-module $\mathcal{M}$ and every $q > 1$. xcohdi-mone

7.10.6. Let C be a plane curve of degree d with δ nodes and κ cusps, and let C' be its normalization. Determine the genus of C'. nodes-cusps

7.10.7. Let $f(x_0, x_1, x_2)$ be an irreduible homogeneous polynomial of degree $2d$, and let Y be the projective double plane $y^2 = f(x_0, x_1, x_2)$. Compute the cohomology $H^q(Y, \mathcal{O}_Y)$. xcohd-blplane

7.10.8. Let A, B be $2{\times}2$ variable matrices, let P be the polynomial ring $\mathbb{C}[a_{ij}, b_{ij}]$. and let R be the quotient of P by the ideal that expresses the condition $AB = BA$. Show that R has a resolution as P-module of the form $0 \to P^2 \to P^3 \to P \to R \to 0$. (Hint: Write the equations in terms of $a_{11} - a_{22}$ and $b_{11} - b_{22}$.) xABBA

7.10.9. Prove that a regular function on a projective variety is constant. xregfn-const

7.10.10. *an algebraic version of Bézout's Theorem.* Let $R = \mathbb{C}[x, y, z]$, and let f and g be homogeneous polynomials in R, of degrees m and n, respectively. The quotient falgebra $A = R/(f, g)$ inherits a grading by degree: $A = A_0 \oplus A_1 \oplus \cdots$, where A_n is the image of the space of homogeneous polynomials of degree n, together with 0. xalgbez

(i) Show that the sequence

$$0 \to R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)^t} R \to A \to 0$$

is exact.

(ii) Prove that $\dim A_k = mn$ for all sufficiently large k.

(iii) Explain in what way this is an algebraic version of Bézout's Theorem.

7.10.11. Let p_1, p_2, p_3 and q_1, q_2, q_3 be distinct points on a conic C, and let L_{ij} be the line through p_i and q_j. xpascal

(i) Let g and h be the homogeneous cubic polynomials whose zero loci are $L_{12} \cup L_{13} \cup L_{23}$ and $L_{21} \cup L_{31} \cup L_{32}$, respectively, and let x be another point on C. Show that for some scalar c, the cubic $f = g + ch$ vanishes at x as well as at the six given points p_i and q_i. What does Bézout's Theorem tell us about this cubic f?

(ii) Prove *Pascal's Theorem,* which asserts that the three intersection points $r_1 = L_{23} \cap L_{32}$, $r_2 = L_{31} \cap L_{13}$, and $r_3 = L_{12} \cap L_{21}$ lie on a line.

(iii) Let six lines $Z_1, ..., Z_6$ be given, and suppose that a conic C is tangent to each of those lines. Let $p_{12} = Z_1 \cap Z_2$, $p_{23} = Z_2 \cap Z_3$, $p_{34} = Z_3 \cap Z_4$, $p_{45} = Z_4 \cap Z_5$, $p_{56} = Z_5 \cap Z_6$, and $p_{61} = Z_6 \cap Z_1$. We think of the six lines as sides of a 'hexagon', whose vertices are $p_{ij} = L_i \cap L_j$ for $ij = 12, 23, 34, 45, 56$, and 61. The 'main diagonals' are the lines D_1 through p_{12} and p_{45}, D_2 through p_{23} and p_{56}, and D_3 through p_{61} and p_{34}. *Brianchon's Theorem* asserts that the main diagonals have a common point. Prove this by studying the dual configuration in $\mathbb{P}^*$.

7.10.12. Let $X = \mathbb{P}^d$ and let $Y \xrightarrow{\pi} X$ be a finite morphism. Prove that Y is a projective variety. Do this by showing that the global sections of $\mathcal{O}_Y(nH) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(nH)$ define a map to projective space whose image is isomorphic to Y. xfiniteproj xmod-fandg **7.10.13.** **(i)** Let R be the polynomial ring $\mathbb{C}[x,y,z]$, let $f(x,y,z)$ and $g(x,y,z)$ be homogeneous polynomials of degrees m and n, and with no common factor, and let $A = R/(f,g)$. Show that the sequence

$$0 \to R \overset{(-g,f)}{\longrightarrow} R^2 \overset{(f,g)^t}{\longrightarrow} R \to A \to 0$$

is exact.

(ii) Let Y be an affine variety with integrally closed coordinate ring B. Let I be an ideal of B generated by two elements u, v, and let X be the locus $V(I)$ in Y. Suppose that $\dim X \le \dim Y - 2$. Use the fact that $B = \bigcup B_Q$ where Q ranges over prime ideals of codimension 1 to prove that this sequence is exact:

$$0 \to B \overset{(v,-u)^t}{\longrightarrow} B^2 \overset{(u,v)}{\longrightarrow} B \to B/I \to 0.$$

xmodfg **7.10.14.** Let Ibe the ideal of $\mathbb{C}[x_0,x_1,x_2,x_3]$ generated by two homogeneous polynomials f, g, of dimensions d, e respectively, and assume that the locus $X = V(\mathcal{I})$ in $\mathbb{P}^3$ has dimension 1. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$. Multiplication by f and g defines a map $\mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O}$. Let $\mathcal{A}$ be the cokernel of this map.

(i) Construct an exact sequence

$$0 \to \mathcal{O}(-d-e) \to \mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O} \to \mathcal{A} \to 0.$$

(ii) Show that X is a connected subset of $\mathbb{P}^3$ in the Zariski topology, i.e., that it is not the union of two proper disjoint Zariski-closed subsets.

(iii) Determine the genus of X in the case that X is a smooth curve.

xcplt-inttwo **7.10.15.** A curve in $\mathbb{P}^3$ that is the zero locus of a homogeneous prime ideal generated by two elements is a *complete intersection*. Determine the genus of a smooth complete intersection when the generators have degrees r and s.

xafplusbg **7.10.16.** *a theorem of Max Noether.* (Max Noether was Emmy Noether's father.) Let f and g be homogeneous polynomials in $x_0, ..., x_k$, of degrees r and s, respectively, with $k \ge 2$. Suppose that the locus $X : \{f = g = 0\}$ in $\mathbb{P}^k$ consists of rs distinct points if $k = 2$, or is a closed subvariety of codimension 2 if $k > 2$. A theorem that is called the AF+BG Theorem, asserts that, if a homogeneous polynomial p of degree n vanishes on X, there are homogeneous polynomials a and b such that $p = af + bg$. Prove this theorem.

xmi-norstwo **7.10.17.** Let

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix}$$

be a 3×2 matrix whose entries are homogeneous quadratic polynomials in four variables $x_0, ..., x_3$. Let $M = (m_1, m_2, m_3)$ be the 1×3 matrix of minors

$$m_1 = u_{21}u_{32} - u_{22}u_{31}, \quad m_2 = -u_{11}u_{32} + u_{12}u_{31}, \quad m_3 = u_{11}u_{22} - u_{12}u_{21}$$

The matrices U and M give us a sequence

$$0 \to \mathcal{O}(-6)^2 \overset{U}{\longrightarrow} \mathcal{O}(-4)^3 \overset{M}{\longrightarrow} \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0$$

where $\mathcal{I}$ is the ideal generated by the minors.

(i) Suppose that the above sequence is exact, and that the locus of zeros of I in $\mathbb{P}^3$ is a curve. Determine the genus of that curve.

(ii) Prove that, if the locus is a curve, the sequence is exact.

Chapter 8 THE RIEMANN-ROCH THEOREM FOR CURVES

rrcurves

In this chapter, we investigate a classical problem of algebraic geometry, to determine the rational functions with given poles on a smooth projective curve. This is often difficult. The rational functions whose poles have orders at most r_i at p_i, for $i = 1, ..., k$, form a vector space, and one is happy when one can determine the dimension of that space. The most important tool for determining the dimension is the Riemann-Roch Theorem.

8.1 Divisors

divtwo Smooth affine curves were discussed in Chapter 5. An affine curve is smooth if its local rings are valuation rings, or if its coordinate ring is a normal domain. An arbitrary curve is smooth if it has an open covering by smooth affine curves.

We take a brief look at modules on a smooth curve. Recall that a module M over a domain A is said to be torsion-free if its only torsion element is zero: If $a \in A$ and $m \in M$ are nonzero, then $am \neq 0$. This definition is extended to $\mathcal{O}$-modules by applying it to the affine open subsets.

lfree **8.1.1. Lemma.** *Let Y be a smooth curve.*
(i) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is locally free if and only if it is torsion-free.*
(ii) *An $\mathcal{O}_Y$-module $\mathcal{M}$ that isn't torsion-free has a nonzero global section.*

proof. **(i)** We may assume that Y is affine, $Y = \operatorname{Spec} B$, and that $\mathcal{M}$ is the $\mathcal{O}$-module associated to a B-module M. Let $\widetilde{B}$ be the local ring of B at a point q, and let $\widetilde{M}$ be the localization of M at q. It is isomorphic to the tensor product $M \otimes_B \widetilde{B}$. If M is a torsion-free B-module, then $\widetilde{M}$ is a torsion-free module over the ring $\widetilde{B}$, which is a valuation ring. It suffices to show that, for every point q of Y, $\widetilde{M}$ is a free $\widetilde{B}$-module (**2.7.13**). The next sublemma does this.

vringpid **8.1.2. Sublemma.** *A finite, torsion-free module $\widetilde{M}$ over a valuation ring $\widetilde{B}$ is a free module.*

proof. It is easy to prove this directly. Or, one can use the fact that every finite, torsion-free module over a principal ideal domain is free. A valuation ring is a principal ideal domain because its nonzero ideals are powers of its maximal ideal, and the maximal ideal is a principal ideal. □

proof of Lemma 8.1.1 **(ii)** If $\mathcal{M}$ isn't torsion-free, then for some affine open subset U, there will be nonzero elements m in $\mathcal{M}(U)$ and a in $\mathcal{O}(U)$, such that $am = 0$. Let Z be the finite set of zeros of a in U, and let $V = Y - Z$ be the open complement of Z in Y. Then a is invertible on the intersection $W = U \cap V$, and since $am = 0$, the restriction of m to W is zero.

The open sets U and V cover Y, and the sheaf property for this covering can be written as an exact sequence

$$0 \to \mathcal{M}(Y) \to \mathcal{M}(U)\times\mathcal{M}(V) \xrightarrow{-,+} \mathcal{M}(W)$$

(see Lemma 6.4.11). In this sequence, the section $(m,0)$ of $\mathcal{M}(U)\times\mathcal{M}(V)$ maps to zero in $\mathcal{M}(W)$. Therefore it is the image of a nonzero global section of $\mathcal{M}$. □

8.1.3. Lemma. *Let Y be a smooth curve. Every nonzero ideal $\mathcal{I}$ of $\mathcal{O}_Y$ is a product of powers of maximal ideals: $\mathfrak{m}_1^{e_1}\cdots\mathfrak{m}_k^{e_k}$.* *idealprod-max*

proof. This follows for any smooth curve from the case that the curve is affine, which is Proposition 5.2.9. □

(8.1.4) divisors divsect

A *divisor* on a smooth curve Y is a finite combination

$$D = r_1q_1 + \cdots + r_kq_k$$

where r_i are integers and q_i are points. The terms whose integer coefficients r_i are zero can be omitted or not, as desired.

The *degree* of the divisor D is the sum $r_1+\cdots+r_k$ of the coefficients. Its *support* the set of points q_i of Y such that $r_i \neq 0$.

The *restriction* of a divisor $D = r_1q_1 + \cdots + r_kq_k$ to an open subset of Y' of Y is the divisor obtained from D by deleting points of the support that aren't in Y'. For example, let $D = q$. The restriction to Y' is q if q is in Y', and it is zero if q is not in Y'.

A divisor $D = \sum r_iq_i$ is *effective* if all of its coefficients r_i are non-negative, and D is *effective on an open set* Y' if its restriction to Y' is effective — if $r_i \geq 0$ for every i such that q_i is a point of Y'.

Let $D = \sum r_ip_i$ and $E = \sum s_ip_i$ be divisors. We my write $E \geq D$ if $s_i \geq r_i$ for all i, or if $E - D$ is effctive. With this notation, $D \geq 0$ means that D is effective.

(8.1.5) the divisor of a function divfn

The *divisor of* a nonzero rational function f on a smooth curve Y is

$$\text{div}(f) = \sum_{q\in Y} \text{v}_q(f)\, q$$

where, as usual, v_q denotes the valuation of K that corresponds to the point q. The divisor of the zero function is defined to be the zero divisor. The divisor of f is written here as a sum over all points q, but it becomes a finite sum when we disregard terms with coefficient zero, because f has finitely many zeros and poles. The coefficients will be zero at all other points.

The map

$$K^\times \xrightarrow{\text{div}} (divisors)^+ \tag{8.1.6}$$ Ktodiv

that sends a nonzero rational function to its divisor is a homomorphism from the multiplicative group $K^\times$ of nonzero elements of K to the additive group of divisors:

$$\text{div}(fg) = \text{div}(f) + \text{div}(g)$$

The divisor of a rational function is called a *principal divisor*. The image of the map (8.1.6) is the set of principal divisors.

As before, if r is a positive integer, a nonzero rational function f has a *zero of order* r at q if $\text{v}_q(f) = r$, and it has a *pole of order* r at q if $\text{v}_q(f) = -r$. Thus the divisor of f is the difference of two effective divisors:

$$\text{div}(f) = \; zeros(f) - \; poles(f)$$

A rational function f is regular on Y if and only if $\operatorname{div}(f)$ is effective — if and only if $poles(f) = 0$.

Every divisor is locally principal: There is an affine covering $\{Y^i\}$ of Y such that the restriction of D is a principal divisor on each Y^i. This is true because the maximal ideals of Y are locally principal. If f is a generator of the maximal ideal at a point q, then $\operatorname{div}(f) = q$.

Two divisors D and E are *linearly equivalent* if their difference $D - E$ is a principal divisor. For instance, the divisors $zeros(f)$ and $poles(f)$ of a rational function f are linearly equivalent.

levelsets **8.1.7. Lemma.** *Let f be a rational function on a smooth curve Y. For all complex numbers c, the divisors $zeros(f-c)$, the level sets of f, are linearly equivalent.*

proof. The functions $f-c$ have the same poles as f. □

divterm **(8.1.8) review of terminology**

divisor: a (finite) integer combination of points: $D = r_1q_1 + \cdots + r_kq_k$.

divisor of a function: The divisor of the rational function f is the sum $\sum_q \mathrm{v}_q(f)q$.

effective divisor: The divisor D is effective if $r_i \geq 0$ for all i.

linearly equivalent divisors: Two divisors D and E are linearly equivalent if $D - E$ is a principal divisor.

principal divisor: The divisor of a rational function.

restriction of the divisor D to an open set: The restriction to the open set U is the sum of the terms r_iq_i such that q_i is a point of U.

support of the divisor D: the points q_i such that $r_i \neq 0$.

zeros and poles of a divisor: The zeros of D are the points q_i such that $r_i > 0$. The poles are the points q_i such that $r_i < 0$.

mod-uleOD **(8.1.9) the module $\mathcal{O}(D)$**

To analyze functions with given poles on a smooth curve Y, we associate an $\mathcal{O}$-module $\mathcal{O}(D)$ to a divisor D. The module $\mathcal{O}(D)$ is a submodule of the function field module $\mathcal{F}$ 6.5.10. Its nonzero sections on an open subset V of Y are the nonzero rational functions f such that the the divisor $\operatorname{div}(f)+D$ is effective on V — such that its restriction to V is effective.

ODV (8.1.10)
$$\begin{aligned}[\mathcal{O}(D)](V) &= \{f \mid \operatorname{div}(f)+D \text{ is effective on } V\} \cup \{0\} \\ &= \{f \mid poles(f) \leq D \text{ on } V\} \cup \{0\}\end{aligned}$$

When D is effective, the global sections of $\mathcal{O}(D)$ are the solutions of the classical problem, to determine the rational functions whose poles are bounded by D.

Points that aren't in an open set V impose no conditions on the sections of $\mathcal{O}(D)$ on V. A section on V can have arbitrary zeros or poles at points not in V.

Let D be the divisor $\sum r_iq_i$. If q_i is a point of an open set V and if $r_i > 0$, a section of $\mathcal{O}(D)$ on V may have a pole of order at most r_i at q_i, and if $r_i < 0$ a section must have a zero of order at least $-r_i$ at q_i. For example, the sections of the module $\mathcal{O}(-q)$ on an open set V that contains q are the regular functions on V that are zero at q. So $\mathcal{O}(-q)$ is the maximal ideal $\mathfrak{m}_q$. Similarly, the sections of $\mathcal{O}(q)$ on an open set V that contains q are the rational functions that have a pole of order at most 1 at q and are regular at every other point of V. The sections of $\mathcal{O}(-q)$ and of $\mathcal{O}(q)$ on an open set V that doesn't contain p are the regular functions on V.

The fact that a section of $\mathcal{O}(D)$ is allowed to have a pole at q_i when $r_i > 0$ contrasts with the divisor of a function. If $\operatorname{div}(f) = \sum r_iq_i$, then $r_i > 0$ means that f has a zero at q_i. If $\operatorname{div}(f) = D$, then f will be a global section of $\mathcal{O}(-D)$.

8.1.11. Lemma. **(i)** *If D is the principal divisor $\operatorname{div}(g)$, then $\mathcal{O}(D)$ is the free $\mathcal{O}$-module $g^{-1}\mathcal{O}$ of rank 1.* ODinvert
(ii) *For any divisor D on a smooth curve, $\mathcal{O}(D)$ is a locally free module of rank one.*

proof. **(i)** Let D be the divisor of a rational function g. The sections of $\mathcal{O}(D)$ on an open set U are the rational functions f such that $\operatorname{div}(f) + D = \operatorname{div}(f) + \operatorname{div}(g) \geq 0$ on U. These are the functions f such that fg is a section of $\mathcal{O}$ on U, or such that f is a section of $g^{-1}\mathcal{O}$.
(ii) This follows from **(i)** because every divisor is locally principal, □

8.1.12. Proposition. *Let D and E be divisors on a smooth curve Y.* LisOD
(i) *The map $\mathcal{O}(D)\otimes_{\mathcal{O}}\mathcal{O}(E) \to \mathcal{O}(D+E)$ that sends $f\otimes g$ to the product fg is an isomorphism.*
(ii) *$\mathcal{O}(D) \subset \mathcal{O}(E)$ if and only if $E \geq D$.*

proof. **(i)** It is enough to verify this locally, so we may assume that Y is affine and that the supports of D and E contain just one point, say $D = rq$ and $E = sq$. We may also assume that the maximal ideal at q is a principal ideal, generated by an element x. Then $\mathcal{O}(D)$, $\mathcal{O}(E)$, and $\mathcal{O}(D+E)$ are free modules with bases x^r, x^s and x^{r+s}, respectively. □

8.1.13. Proposition. *Let Y be a smooth curve.* idealOD
(i) *The nonzero ideals of $\mathcal{O}_Y$ are the modules $\mathcal{O}(-E)$, where E is an effective divisor.*
(ii) *The modules $\mathcal{O}(D)$ are the finite $\mathcal{O}$-submodules of the function field module $\mathcal{F}$ of Y.*
(iii) *The function field module $\mathcal{F}$ is the union of the modules $\mathcal{O}(D)$.*

proof. **(i)** Say that $E = r_1q_1 + \cdots + r_kq_k$, with $r_i \geq 0$ for all i. A rational function f is a section of $\mathcal{O}(-E)$ if $\operatorname{div}(f) - E$ is effective, which happens when $poles(f) = 0$, and $zeros(f) \geq E$. The same condition describes the elements of the ideal $\mathcal{I} = \mathfrak{m}_1^{r_1}\cdots\mathfrak{m}_k^{r_k}$.

(ii) First, if D_1 and D_2 are divisors, and if $D_1 \neq D_2$, then $\mathcal{O}(D_1) \neq \mathcal{O}(D_2)$. Let $\mathcal{L}$ be a finite $\mathcal{O}$-submodule of $\mathcal{F}$. The local ring R of Y at a point q is a valuation ring. Since $\mathcal{L}$ is a finite $\mathcal{O}$-module, it will be generated by one element, a rational function f, in some open neighborhood U of q. (**2.7.13**). If D is the divisor of f^{-1} on U, then $\mathcal{L} = \mathcal{O}(D)$ on U, and this determines the divisor D uniquely. So when $\mathcal{L} = \mathcal{O}(D)$ on U and $\mathcal{L} = \mathcal{O}(D')$ on U', then D and D' agree on $U \cap U'$. Therefore there is a divisor D on the whole curve Y such that $\mathcal{L} = \mathcal{O}(D)$ in a suitable neighbohood U of any point q. This implies that $\mathcal{L} = \mathcal{O}(D)$. □

8.1.14. Proposition. *Let D and E be divisors on a smooth curve Y. Multiplication by a rational function f such that $\operatorname{div}(f)+E-D \geq 0$ defines a homomorphism of $\mathcal{O}$-modules $\mathcal{O}(D) \to \mathcal{O}(E)$, and every homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ is multiplication by such a function.* mapOD-toOE

proof. For any $\mathcal{O}$-module $\mathcal{M}$, a homomorphism $\mathcal{O} \to \mathcal{M}$ is multiplication by a global section of $\mathcal{M}$ (6.3.7) **(b)**. So a homomorphism $\mathcal{O} \to \mathcal{O}(E–D)$ will be multiplication by a rational function f such that $\operatorname{div}(f)+E–D \geq 0$. If f is such a function, one obtains a homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ by tensoring with $\mathcal{O}(D)$. □

8.1.15. Corollary. *Let D and E be divisors on a smooth curve Y.* ODOE
(i) *The modules $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are isomorphic if and only if D and E are linearly equivalent divisors.*
(ii) *Let f be a rational function on Y, and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$.*

proof. If a rational function f defines an isomorphism, the inverse morphism is defined by f^{-1}. Then $\operatorname{div}(f)+E-D \geq 0$ and also $\operatorname{div}(f^{-1}) + D - E = -\operatorname{div}(f) + D - E \geq 0$, so $\operatorname{div}(f) = D - E$. This proves **(i)**, and **(ii)** is the special case that $E = 0$. □

(**8.1.16**) invertible modules invertmod

An *invertible* $\mathcal{O}$-module is a locally free module of rank one, a module that is isomorphic to the module $\mathcal{O}$ in a neighborhood of any point.

The tensor product $\mathcal{L}\otimes_{\mathcal{O}}\mathcal{M}$ of invertible modules is invertible. The dual $\mathcal{L}^*$ of an invertible module $\mathcal{L}$ is invertible. If D is a divisor on a smooth curve Y, then $\mathcal{O}(D)$ is an invertible module. Its dual is the module $\mathcal{O}(-D)$.

inverse-mod **8.1.17. Lemma.** *Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module.*

(i) *Let $\mathcal{L}^*$ be the dual module. The canonical map $\mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{L} \to \mathcal{O}$ defined by $\gamma \otimes \alpha \mapsto \gamma(\alpha)$ is an isomorphism. Thus $\mathcal{L}^*$ may be thought of as an inverse to $\mathcal{L}$. (This is the reason for the term 'invertible'.)*

(ii) *The map $\mathcal{O} \to {}_{\mathcal{O}}(\mathcal{L}, \mathcal{L}) \; \big(= \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{L}, \mathcal{L})\big)$ that sends a regular function α to the operation of multiplication by α is an isomorphism.*

(iii) *Every nonzero homomorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ to a locally free module $\mathcal{M}$ is injective.*

proof. **(i,ii)** It is enough to verify these assertions in the case that $\mathcal{L}$ is free, isomorphic to $\mathcal{O}$, in which case they are clear.

(iii) The problem is local, so we may assume that the variety is affine, say $Y = \operatorname{Spec} A$, and that $\mathcal{L}$ and $\mathcal{M}$ are free. Then φ becomes a nonzero homomorphism $A \to A^k$, which is injective because A is a domain. $\square$

Lemma 8.1.11 shows that the only difference between an invertible module $\mathcal{L}$ and a module $\mathcal{O}(D)$ is that $\mathcal{O}(D)$ is a submodule of the function field module $\mathcal{F}$, while $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{F}$ can be any one-dimensional K-vector space.

Lequal-sOD **8.1.18. Corollary.** *Every invertible $\mathcal{O}$-module $\mathcal{L}$ on a smooth curve Y is isomorphic to a module of the form $\mathcal{O}(D)$.* $\square$

We will use the next lemma in the proof of Theorem 8.6.15 below.

thetaspole **8.1.19. Lemma.** *Let $\mathcal{L} \subset \mathcal{M}$ be an inclusion of invertible modules on a smooth curve Y, let q be a point in the support of $\mathcal{M}/\mathcal{L}$, and let V be an affine open subset of Y that contains q. Suppose that a rational function f has a simple pole at q and is regular at all other points of V. If α is a section of $\mathcal{L}$ on V, then $f^{-1}\alpha$ is a section of $\mathcal{M}$ on V.*

proof. Working locally, we may assume that $\mathcal{L} = \mathcal{O}$. Then $\mathcal{M} = \mathcal{O}(D)$ for some effective divisor D. Since q is in the support of $\mathcal{M}/\mathcal{L}$, the coefficient of q in D is positive. Therefore $\mathcal{L} = \mathcal{O} \subset \mathcal{O}(q) \subset \mathcal{O}(D) = \mathcal{M}$. With this notation, α will be a section of $\mathcal{O}$, while f^{-1} is a section of $\mathcal{O}(q)$. Then $f^{-1}\alpha$ will be a section of $\mathcal{O}(q)$, and therefore a section of $\mathcal{O}(D) = \mathcal{M}$. $\square$

8.2 The Riemann-Roch Theorem I

rrone

Let Y be a smooth projective curve, and let $\mathcal{M}$ be a finite $\mathcal{O}_Y$-module. In Chapter 7, we learned that the cohomology $H^q(Y, \mathcal{M})$ is a finite-dimensional vector space for $q = 0, 1$, and is zero when $q > 1$. As before, we denote the dimension of $H^q(Y, \mathcal{M})$ by $\mathbf{h}^q\mathcal{M}$ or by $\mathbf{h}^q(Y, \mathcal{M})$.

The *Euler characteristic* (7.6.6) of a finite $\mathcal{O}$-module $\mathcal{M}$ is

chicurve (8.2.1)
$$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^1\mathcal{M}$$

In particular,
$$\chi(\mathcal{O}_Y) \;=\; \mathbf{h}^0\mathcal{O}_Y - \mathbf{h}^1\mathcal{O}_Y$$

The dimension $\mathbf{h}^1\mathcal{O}_Y$ is called the *arithmetic genus* of Y. It is denoted by p_a. This is a notation that was used before, for plane curves. We will see below, in (8.2.9)**(iv)**, that $\mathbf{h}^0\mathcal{O}_Y = 1$. So

chi-curvetwo (8.2.2)
$$\chi(\mathcal{O}) \;=\; 1 - p_a$$

RRcurve **8.2.3. Riemann-Roch Theorem (version 1).** *Let $D = \sum r_i p_i$ be a divisor on a smooth projective curve Y. Then*
$$\chi(\mathcal{O}(D)) \;=\; \chi(\mathcal{O}) + \deg\, D \qquad \big(= \deg\, D + 1 - p_a\big)$$

proof. To analyze the effect on cohomology when a divisor is changed by adding or subtracting a point, we inspect the inclusion $\mathcal{O}(D-p) \subset \mathcal{O}(D)$. The cokernel ϵ of the inclusion map is a one-dimensional vector space supported at p. So there is a short exact sequence

(8.2.4) $$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \epsilon \to 0$$ addpoint

Because $\mathfrak{m}_p = \mathcal{O}(-p)$, this sequence can be obtained by tensoring the sequence

(8.2.5) $$0 \to \mathfrak{m}_p \to \mathcal{O} \to \kappa_p \to 0$$ kappaseq

with the invertible module $\mathcal{O}(D)$.

Since ϵ is a one-dimensional module supported at p, $\mathbf{h}^0\epsilon = 1$ and $\mathbf{h}^1\epsilon = 0$. Let's denote the one-dimensional vector space $H^0(Y,\epsilon)$ by [1]. Then the cohomology sequence associated to (8.2.4) is

(8.2.6) $$0 \to H^0(Y,\mathcal{O}(D-p)) \to H^0(Y,\mathcal{O}(D)) \xrightarrow{\gamma} [1] \xrightarrow{\delta} H^1(Y,\mathcal{O}(D-p)) \to H^1(Y,\mathcal{O}(D)) \to 0$$ add-pointtwo

In this exact sequence, one of the two maps, γ or δ, must be zero. Either

(1) γ is zero and δ is injective. In this case

$$\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D) + 1, \quad \text{or}$$

(2) δ is zero and γ is surjective. In this case

$$\mathbf{h}^0\mathcal{O}(D)-p) = \mathbf{h}^0\mathcal{O}(D) - 1 \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D)$$

In either case,

(8.2.7) $$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D-p)) + 1$$ chichange

The Riemann-Roch theorem follows, because $\deg D = \deg\,(D-p) + 1$, and because we can get from $\mathcal{O}$ to $\mathcal{O}(D)$ by a finite number of operations, each of which changes the divisor by adding or subtracting a point. □

Because $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$, this version of the Riemann-Roch Theorem gives reasonably good control of H^0. It is less useful for controlling H^1. For that, one wants the full Riemann-Roch Theorem (version 2), which identifies H^1. The full theorem requires some preparation, so we have put it into Section 8.7. However, version 1 has important consequences:

8.2.8. Corollary. *Let p be a point of a smooth projective curve Y. The dimension $\mathbf{h}^0(Y,\mathcal{O}(np))$ tends to infinity with n. Therefore there exist rational functions on Y that have a pole of sufficiently large order at a single point p and no other poles.* *onepole*

proof. When we go from $\mathcal{O}(np)$ to $\mathcal{O}((n{+}1)p)$, either $\mathbf{h}^0$ increases or else $\mathbf{h}^1$ decreases. Since $\mathbf{h}^1(Y,\mathcal{O}(np))$ is finite, the second possibility can occur only finitely many times, as x tends to ∞. □

8.2.9. Corollary. *Let Y be a smooth projective curve.* *RRcor*

(i) *The divisor of a rational function on Y has degree zero: The number of zeros is equal to the number of poles.*

(ii) *Linearly equivalent divisors on Y have equal degrees.*

(iii) *A nonconstant rational function on Y takes every value, including infinity, the same number of times (counted with multiplicity).*

(iv) *A rational function on Y that is regular at every point of Y is a constant: $H^0(Y,\mathcal{O}) = \mathbb{C}$.*

proof. **(i)** Let f be a nonzero rational function and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$ (8.1.15), so $\chi(\mathcal{O}(D)) = \chi(\mathcal{O})$. On the other hand, by Riemann-Roch, $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D$. Therefore $\deg D = 0$.

(ii) If D and E are linearly equivalent divisors, say $D-E = \operatorname{div}(f)$, then, according to **(i)**, $D-E$ has degree zero, and $\deg D = \deg E$.

(iii) The divisor of zeros of the function $f - c$ is linearly equivalent to the divisor of poles of f.

(iv) According to **(iii)**, a nonconstant function must have a pole. □

RRcorwo **8.2.10. Corollary.** *Let D be a divisor on Y. If $\deg D \geq p_a$, then $\mathbf{h}^0\mathcal{O}(D) > 0$. If $\mathbf{h}^0\mathcal{O}(D) > 0$, then $\deg D \geq 0$.*

proof. If $\deg D \geq p_a$, then $\chi(\mathcal{O}(D)) = \deg D + 1 - p_a \geq 1$, and $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$. If $\mathcal{O}(D)$ has a nonzero global section f, a rational function such that $\operatorname{div}(f) + D$ is effective, then $\deg(\operatorname{div}(f) + D) \geq 0$, and because the degree of $\operatorname{div}(f)$ is zero, $\deg D \geq 0$. □

curveconn **8.2.11. Theorem.** *With its classical topology, a smooth projective curve Y is a connected, compact, orientable two-dimensional manifold.*

proof. We prove connectedness here. The other points have been discussed before (see Theorem 1.8.20).

A nonempty topological space is connected if it isn't the union of two disjoint, nonempty, closed subsets. Suppose that, in the classical topology, Y is the union of disjoint, nonempty closed subsets Y_1 and Y_2. Both Y_1 and Y_2 will be compact, two-dimensional manifolds. Let p be a point of of Y_1. Corollary 8.2.8 shows that there is a nonconstant rational function f whose only pole is at p. Then f will be a regular function on the complement of p, and therefore a regular function on the entire compact manifold Y_2.

For review: Any point q of the smooth curve Y has a neighborhood V that is analytically equivalent to an open subset U of the affine line X. If a function g on V is analytic, the function on U that corresponds to g is an analytic function of one variable on U. The maximum principle for analytic functions asserts that a nonconstant analytic function on an open region of the complex plane has no maximal absolute value in the region. This applies to the open set U and therefore also to the neighborhood V of q. Since q can be any point of Y_2, a nonconstant function g that is analytic on Y_2 cannot have a maximum anywhere on Y_2. On the other hand, since Y_2 is compact, a continuous function does have a maximum. So an analytic function g on Y_2 must be a constant.

Going back to the rational function f with a single pole p, the restriction of f to Y_2 will be analytic, and therefore constant. When we subtract that constant from f, we obtain a nonconstant rational function on Y that is zero on Y_2. But the zero locus of a rational function on a curve is a finite set. This is a contradiction. □

8.3 The Birkhoff-Grothendieck Theorem

birkgroth This theorem describes finite, torsion-free modules on the projective line $X = \mathbb{P}^1$.

BGtheorem **8.3.1. Birkhoff-Grothendieck Theorem.** *A finite, torsion-free $\mathcal{O}$-module on the projective line X is isomorphic to a direct sum of twisting modules: $\mathcal{M} \approx \bigoplus \mathcal{O}(n_i)$.*

(Because X is a smooth curve, an $\mathcal{O}$-module $\mathcal{M}$ is locally free if and only if it is torsion-free.)

This theorem was proved by Grothendieck in 1957 using cohomology. It had been proved by Birkhoff in 1909, in the following equivalent form:

Birkhoff Factorization Theorem. Let $A_0 = \mathbb{C}[u]$, $A_1 = \mathbb{C}[u^{-1}]$, and $A_{01} = \mathbb{C}[u, u^{-1}]$. Let P be an invertible A_{01}-matrix. There exist an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1 such that $Q_0^{-1}PQ_1$ is diagonal, and its diagonal entries are integer powers of u.

proof of the Birkhoff-Grothendieck Theorem. This is Grothendieck's proof.

According to Theorem 7.5.5, the cohomology of the twisting modules on X is $\mathbf{h}^0\mathcal{O} = 1$, $\mathbf{h}^1\mathcal{O} = 0$, and if r is a positive integer,

$$\mathbf{h}^0\mathcal{O}(r) = r+1, \quad \mathbf{h}^1\mathcal{O}(r) = 0, \quad \mathbf{h}^0\mathcal{O}(-r) = 0, \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(-r) = r-1$$

mapstoMbounded **8.3.2. Lemma.** *Let $\mathcal{M}$ be a finite, torsion-free $\mathcal{O}$-module on the projective line X. For sufficiently large r,*
(i) *the only homomorphism $\mathcal{O}(r) \to \mathcal{M}$ is the zero map, and*
(ii) $\mathrm{h}^0(X, \mathcal{M}(-r)) = 0$.

proof. **(i)** A nonzero homomorphism $\mathcal{O}(r) \xrightarrow{\varphi} \mathcal{M}$ from the twisting module $\mathcal{O}(r)$ to the locally free module $\mathcal{M}$ will be injective (8.1.17), and the associated map $H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{M})$ will be injective too, so $\mathbf{h}^0(X, \mathcal{O}(r)) \leq \mathbf{h}^0(X, \mathcal{M})$. Since $\mathbf{h}^0(X, \mathcal{O}(r)) = r+1$, r is bounded by the integer $\mathbf{h}^0(X, \mathcal{M}) - 1$.

(ii) A global section of $\mathcal{M}(-r)$ defines a map $\mathcal{O} \to \mathcal{M}(-r)$. Its twist by r will be a map $\mathcal{O}(r) \to \mathcal{M}$. By **(i)**, r is bounded. □

We go to the proof now.

As Lemma 8.1.1 tells us, $\mathcal{M}$ is locally free. We use induction on its rank. We suppose that $\mathcal{M}$ has rank r, that $r > 0$, and that the theorem has been proved for locally free $\mathcal{O}$-modules of rank less than r. The plan is to show that $\mathcal{M}$ has a twisting module as a direct summand, so that $\mathcal{M} = \mathcal{W} \oplus \mathcal{O}(n)$ for some $\mathcal{W}$. Then induction on the rank, applied to $\mathcal{W}$, will prove the theorem.

Twisting is compatible with direct sums, so we may replace $\mathcal{M}$ by a twist $\mathcal{M}(n)$. Instead of showing that $\mathcal{M}$ has a twisting module $\mathcal{O}(n)$ as a direct summand, we show that, after we replace $\mathcal{M}$ by a suitable twist, the structure sheaf $\mathcal{O}$ will be a direct summand.

The twist $\mathcal{M}(n)$ will have a nonzero global section when n is sufficiently large (6.8.20), and it will have no nonzero global section when n is sufficiently negative (8.3.2) **(ii)**). Therefore, when we replace $\mathcal{M}$ by a suitable twist, we will have $H^0(X, \mathcal{M}) \neq 0$ but $H^0(X, \mathcal{M}(-1)) = 0$. We assume that this is true for $\mathcal{M}$.

We choose a nonzero global section m of $\mathcal{M}$ and consider the injective multiplication map $\mathcal{O} \xrightarrow{m} \mathcal{M}$. Let $\mathcal{W}$ be its cokernel, so that we have a short exact sequence

(8.3.3) $$0 \to \mathcal{O} \xrightarrow{m} \mathcal{M} \xrightarrow{\pi} \mathcal{W} \to 0$$ cvcwsequence

8.3.4. Lemma. *Let $\mathcal{W}$ be the $\mathcal{O}$-module that appears in the sequence (8.3.3).* *sectionbasic*
(i) $H^0(X, \mathcal{W}(-1)) = 0$.
(ii) *$\mathcal{W}$ is torsion-free, and therefore locally free.*
(iii) *$\mathcal{W}$ is a direct sum $\bigoplus_{i=1}^{r-1} \mathcal{O}(n_i)$ of twisting modules on $\mathbb{P}^1$, with $n_i \leq 0$.*

proof. **(i)** This follows from the cohomology sequence associated to the twisted sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{M}(-1) \to \mathcal{W}(-1) \to 0$$

because $H^0(X, \mathcal{M}(-1)) = 0$ and $H^1(X, \mathcal{O}(-1)) = 0$.

(ii) If the torsion submodule of $\mathcal{W}$ were nonzero, the torsion submodule of $\mathcal{W}(-1)$ would also be nonzero, and then $\mathcal{W}(-1)$ would have a nonzero global section (8.1.1).

(iii) The fact that $\mathcal{W}$ is a direct sum of twisting modules follows by induction on the rank: $\mathcal{W} \approx \bigoplus \mathcal{O}(n_i)$. Since $H^0(X, \mathcal{W}(-1)) = 0$, we must have $H^0(X, \mathcal{O}(n_i - 1)) = 0$. Therefore $n_i - 1 < 0$, and $n_i \leq 0$. $\square$

We go back to the proof of Theorem 8.3.1. Because $\mathcal{O}^* = \mathcal{O}$, the dual of the sequence (8.3.3) is an exact sequence

$$0 \longleftarrow \mathcal{O} \xleftarrow{m^*} \mathcal{M}^* \xleftarrow{\pi^*} \mathcal{W}^* \longleftarrow 0$$

and $\mathcal{W}^* \approx \bigoplus \mathcal{O}(-n_i)$ with $-n_i \geq 0$. Therefore $\mathbf{h}^1 \mathcal{W}^* = 0$. The map $H^0(\mathcal{M}^*) \to H^0(\mathcal{O})$ is surjective. Let α be a global section of $\mathcal{M}^*$ whose image in $\mathcal{O}$ is 1. Multiplication by α defines a map $\mathcal{O} \xrightarrow{\alpha} \mathcal{M}^*$ that splits the sequence: $m^* \alpha$ is the identity map on $\mathcal{O}$, and $\mathcal{M}^*$ is the direct sum $im(\alpha) \oplus ker(m^*) \approx \mathcal{O} \oplus \mathcal{W}^*$. Therefore $\mathcal{M} \approx \mathcal{W} \oplus \mathcal{O}$. $\square$

8.4 Differentials

diff

We introduce some terminology, differentials and branched coverings, that will be used in version II of the Riemann-Roch theorem. Why differentials enter into the Riemann-Roch Theorem is something of a mystery, but one important fact is the *Residue Theorem*, which controls the poles of a rational differential. Proofs of Reimann-Roch are often based on that theorem. We recommend reading about the Residue Theorem, though we won't use it. [1]

Try not to get bogged down in the preliminary disussions. Give the next pages a quick read to learn the terminology. You can look back as needed. Begin to read more carefully when you get to Section 8.6.

Let A be an algebra and let M be an A-module. A *derivation* $A \xrightarrow{\delta} M$ is a $\mathbb{C}$-linear map that satisfies the product rule for differentiation, a map that has these properties:

[1] See one of the books by Fulton, Miranda, or Mumford in the bibliography, or for a general treatment, Tate, J., Residues of differentials on curves, Ann Sci ENS 1968.

deriv (8.4.1) $$\delta(ab) = a\,\delta b + b\,\delta a\ , \quad \delta(a{+}b) = \delta a + \delta b\ , \quad \text{and} \quad \delta c = 0$$

for all a, b in A and all c in $\mathbb{C}$. The fact that δ is $\mathbb{C}$-linear, i.e., that it is a homomorphism of vector spaces, follows: Since $dc = 0$, $\delta(cb) = c\,\delta b + b\,\delta c = c\,\delta b$.

For example, differentiation $\frac{d}{dt}$ is a derivation $\mathbb{C}[t] \to \mathbb{C}[t]$.

compde **8.4.2. Lemma.** **(i)** *Let B be an algebra, let $M \xrightarrow{g} N$ be a homomorphism of B-modules, and let $B \xrightarrow{\delta} M$ be a derivation. The composition $B \xrightarrow{g\delta} N$ is a derivation.*

(ii) *Let $A \xrightarrow{\varphi} B$ be an algebra homomorphism, and let $B \xrightarrow{\delta} M$ be a derivation. The composition $A \xrightarrow{\delta\varphi} M$ is a derivation.*

(iii) *Let $A \xrightarrow{\varphi} B$ be a surjective algebra homomorphism, let $B \xrightarrow{h} M$ be a map to a B-module M, and let $d = h\varphi$. If $A \xrightarrow{d} M$ is a derivation, then h is a derivation.* $\square$

The *module of differentials* Ω_A of an algebra A is an A-module generated by elements denoted by da, one for each element a of A. The elements of Ω_A are (finite) combinations $\sum b_i\, da_i$, with a_i and b_i in A. The defining relations among the generators da are the ones that make the map $A \xrightarrow{d} \Omega_A$ that sends a to da a derivation: For all a, b in A and all c in $\mathbb{C}$,

defdiff (8.4.3) $$d(ab) = a\,db + b\,da\ , \quad d(a{+}b) = da + db\ , \quad \text{and} \quad dc = 0$$

The elements of Ω_A are called *differentials*.

homomderiv **8.4.4. Lemma.**

(i) *When we compose a homomorphism $\Omega_A \xrightarrow{f} M$ of $\mathcal{O}$-modules with the derivation $A \xrightarrow{d} \Omega_A$, we obtain a derivation $A \xrightarrow{fd} M$. Composition with d defines a bijection between module homomorphisms $\Omega_A \to M$ and derivations $A \xrightarrow{\delta} M$.*

(ii) *Ω is a functor: An algebra homomorphism $A \xrightarrow{u} B$ induces a homomorphism $\Omega_A \xrightarrow{v} \Omega_B$ that is compatible with the ring homomorphism u, and that makes a diagram*

$$\begin{tikzcd}
B \arrow[r, "d"] & \Omega_B \\
A \arrow[u, "u"] \arrow[r, "d"] & \Omega_A \arrow[u, "v"']
\end{tikzcd}$$

Recall that, when ω is an element of Ω_A and α is an element of A, compatibility of v with u means that $v(\alpha\omega) = u(\alpha)v(\omega)$.

proof. **(ii)** When Ω_B is made into an A-module by restriction of scalars, the composed map $A \xrightarrow{u} B \xrightarrow{d} \Omega_B$ will be a derivation to which **(i)** applies. $\square$

omegafree **8.4.5. Lemma.** *Let R be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$. The module of differentials Ω_R is a free R-module with basis $dx_1, ..., dx_n$.*

proof. The formula $df = \sum \frac{df}{dx_i} dx_i$ follows from the defining relations. It shows that the elements $dx_1, ..., dx_n$ generate the R-module Ω_R. Let V be a free R-module with basis $v_1, ..., v_n$. The product rule for differentiation shows that the map $\delta : R \to V$ defined by $\delta(f) = \frac{\partial f}{\partial x_i} v_i$ is a derivation. It induces a module homomorphism $\Omega_A \xrightarrow{\varphi} V$ ssu that sends dx_i to v_ib (8.4.4). Since $dx_1, ..., dx_n$ generate Ω_R and since $v_1, ..., v_n$ is a basis of V, φ is an isomorphism. $\square$

8.4.6. Proposition. *Let I be an ideal of an algebra R, let A be the quotient algebra R/I, and let dI denote*
omegafreetwo *the set of differentials df with f in I. The subset $N = dI{+}I\Omega_R$ of Ω_R is a submodule, and Ω_A is isomorphic to the quotient Ω_R/N.*

The proposition can be interpreted this way: Suppose that the ideal I is generated by elements $f_1, ..., f_r$ of R. Then Ω_A is the quotient of Ω_R that is obtained by introducing these two rules:

- $df_i = 0,$ and
- multiplication by f_i is zero.

8.4.7. Example. Let R be the polynomial ring $\mathbb{C}[y]$ in one variable. So Ω_R is a free module with basis dy. Let I be the principal ideal (y^2), and let A be the quotient R/I. In this case, dI is an R-module generated by te element $2y\,dy$, and y^2dy generates $I\Omega_A$. The R-module N is generated by ydy. If $\overline{y}$ denotes the residue of y in A, $\Omega_A = \Omega_R/N$ is generated by an element $d\overline{y}$, with the relation $\overline{y}\,d\overline{y} = 0$. It isn't the zero module. $\square$ diffmod-square

proof of Proposition 8.4.6. First, $I\Omega_R$ is a submodule of Ω_R, and dI is an additive subgroup of Ω_R. To show that N is a submodule, we must show that scalar multiplication by an element of R maps dI to N, i.e., that if g is in R and f is in I, then $g\,df$ is in N. By the product rule, $g\,df = d(fg) - f\,dg$. Since I is an ideal, fg is in I. Then $d(fg)$ is in dI, and $f\,dg$ is in $I\Omega_R$. So $g\,df$ is in N.

The rules displayed above hold in Ω_A because the generators f_i of I are zero in A. Therefore N is contained in the kernel of the surjective map $\Omega_R \xrightarrow{v} \Omega_A$ defined by the homomorphism $R \to A$. Let $\overline{\Omega}$ denote the quotient module Ω_R/N. This is an A-module, and because $N \subset \ker v$, v defines a surjective map of A-modules $\overline{\Omega} \xrightarrow{\overline{v}} \Omega_A$. We show that $\overline{v}$ is bijective. Let r be an element of R, let a be its image in A, and let $\overline{dr}$ be its image in $\overline{\Omega}$. The composed map $R \xrightarrow{d} \Omega_R \xrightarrow{\pi} \overline{\Omega}$ is a derivation that sends r to $\overline{dr}$, and I is in its kernel. It defines a derivation $R/I = A \xrightarrow{\delta} \overline{\Omega}$ that sends a to $\overline{dr}$. This derivation corresponds to a homomorphism of A-modules $\Omega_A \to \overline{\Omega}$ that sends da to $\overline{dr}$, and that inverts $\overline{v}$. $\square$

8.4.8. Corollary. *If A is a finite-type algebra, then Ω_A is a finite A-module.* *Omegafinite*

This follows from Proposition 8.4.6, because the module of differentials of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is a finite module. $\square$

8.4.9. Lemma. *Let S be a multiplicative system in a domain A. The module $\Omega_{S^{-1}A}$ of differentials of $S^{-1}A$ is canonically isomorphic to the module of fractions $S^{-1}\Omega_A$. In particular, if K is the field of fractions of A, then $K \otimes_A \Omega_A \approx \Omega_K$.* *localizeomega*

We have moved the symbol S^{-1} to the left for clarity.

proof of Lemma 8.4.9. The composed map $A \to S^{-1}A \xrightarrow{d} \Omega_{S^{-1}A}$ is a derivation. It defines an A-module homomorphism $\Omega_A \to \Omega_{S^{-1}A}$ which extends to an $S^{-1}A$-homomorphism $S^{-1}\Omega_A \xrightarrow{\varphi} \Omega_{S^{-1}A}$ because scalar multiplication by the elements of S is invertible in $\Omega_{S^{-1}A}$. The relation $ds^{-k} = -ks^{-k-1}ds$ follows from the definition of a differential, and it shows that φ is surjective. The quotient rule

$$\delta(s^{-k}a) = -ks^{-k-1}a\,ds + s^{-k}da$$

defines a derivation $S^{-1}A \xrightarrow{\delta} S^{-1}\Omega_A$, that corresponds to a homomorphism $\Omega_{S^{-1}A} \to S^{-1}\Omega_A$ and that inverts φ. Here, one must show that δ is well-defined, that $\delta(s_1^{-k}a_1) = \delta(s_2^{-\ell}a_2)$ if $s_1^{-\ell}a_1 = s_2^{-k}a_2$, and that δ is a derivation. You will be able to do this. $\square$

Lemma 8.4.9 shows that a finite $\mathcal{O}$-module Ω_Y of differentials on a variety Y is defined such that, when $U = \operatorname{Spec} A$ is an affine open subset of Y, $\Omega_Y(U) = \Omega_A$.

8.4.10. Proposition. *Let y be a local generator for the maximal ideal at a point q of a smooth curve Y. In a suitable neighborhood of q, the module Ω_Y of differentials is a free $\mathcal{O}$-module with basis dy. Therefore Ω_Y is an invertible module.* *omegafunct*

proof. We may assume that Y is affine, say $Y = \operatorname{Spec} B$. Let q be a point of Y, and let y be an element of B with $\mathrm{v}_q(y) = 1$. To show that dy generates Ω_B locally, we may localize, so we may suppose that y generates the maximal ideal $\mathfrak{m}$ at q. We must show that after we localize B once more, every differential df with f in B will be a multiple of dy. Let $c = f(q)$, so that $f = c + yg$ for some g in B, and because $dc = 0$, $df = g\,dy + y\,dg$. Here $g\,dy$ is in $B\,dy$ and $y\,dy$ is in $\mathfrak{m}\Omega_B$. This shows that

$$\Omega_B = B\,dy + \mathfrak{m}\Omega_B$$

An element β of Ω_B can be written as $\beta = b\,dy + \gamma$, with b in B and γ in $\mathfrak{m}\Omega_B$. If W denotes the quotient module $\Omega_B/(B\,dy)$, then $W = \mathfrak{m}W$. The Nakayama Lemma tells us that there is an element z in $\mathfrak{m}$ such that $s = 1 - z$ annihilates W. When we replace B by its localization B_s, we will have $W = 0$ and $\Omega_B = B\,dy$, as required.

We must still verify that the generator dy of Ω_B isn't a torsion element. Suppose that $b\,dy = 0$, with $b \neq 0$, then Ω_B will be zero except at the finite set of zeros of b in Y. We replace the point q by a point at which Ω_B is zero, keeping the rest of the notation unchanged. Let and $A = \mathbb{C}[y]/(y^2)$. As was noted in Example 8.4.7, Ω_A isn't the zero module. Proposition 5.2.8 tells us that, at our point q, the algebra $B/\mathfrak{m}_q^2$ is isomorphic to A, and Proposition 8.4.6 tells us that Ω_A is a quotient of Ω_B. Since Ω_A isn't zero, neither is Ω_B. Therefore dy isn't a torsion element. □

8.5 Branched Coverings

cover-curve By a *branched covering*, we mean an integral morphism $Y \xrightarrow{\pi} X$ of smooth curves. Chevalley's Finiteness Theorem 4.6.8 shows that, when a smooth curve Y is projective, every nonconstant morphism $Y \to X$ will be a branched covering, unless it maps Y to a point.

Let $Y \to X$ be a branched covering. The function field K of Y will be a finite extension of the function field F of X. The *degree* $[Y:X]$ of the covering is defined to be the degree $[K:F]$ of that field extension. If $X' = \operatorname{Spec} A$ is an affine open subset of X, its inverse image Y' will be an affine open subset $Y' = \operatorname{Spec} B$ of Y, and B will be a locally free A-module whose rank is $[Y:X]$.

To describe the fibre of a branched covering $Y \xrightarrow{\pi} X$ over a point p of X, we may localize. So we may assume that X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that the maximal ideal $\mathfrak{m}_p$ of A at a point p is a principal ideal, generated by an element x of A.

If a point q of Y lies over p, the *ramification index* at q is defined to be $\mathrm{v}_q(x)$, where v_q is the valuation of the function field K that corresponds to q. We usually denote the ramification index by e. Then, if y is a local generator for the maximal ideal $\mathfrak{m}_q$ of B at q, we will have

$$x = uy^e$$

where u is a *local unit* — a rational function on Y that is regular and invertible on some open neighborhood of q.

Points of Y whose ramification indices are greater than one are called *branch points*. We will also call a point p of X a *branch point* of the covering Y if there is a branch point of Y that lies over p.

mostpts **8.5.1. Lemma. (i)** *A branched covering $Y \to X$ has finitely many branch points.*

(ii) *Let n denote the degree $[Y:X]$. If a point p of X isn't a branch point, the fibre over p consists of n points with ramification indices equal to* 1.

proof. This is very simple. We may delete finite sets of points, so we may suppose that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then B is a finite A-module of rank n. Let F and K be the fraction fields of A and B, respectively, and let β be an element of B that generates the field extension K/F. Then $A[\beta] \subset B$, and since these two rings have the same fraction field, there will be a nonzero element s in A such that $A_s[\beta] = B_s$. We may replace A and B by A_s and B_s, so that $B = A[\beta]$. Let g be the monic irreducible polynomial for β over A. The discriminant of g isn't the zero ideal (1.7.22). So for all but finitely many points p of X, the discriminant will be nonzero, and there will be n points of Y over p with ramification indices equal to 1. □

degoneisom **8.5.2. Corollary.** *A branched covering $Y \xrightarrow{\pi} X$ of degree one is an isomorphism.*

proof. When $[Y:X] = 1$, the function fields of Y and X will be equal. Then, because $Y \to X$ is an integral morphism and X is normal, $Y = X$. □

The next lemma follows from Lemma 8.1.3 and the Chinese Remainder Theorem.

extendedidealisprod **8.5.3. Lemma.** *Let $Y \xrightarrow{\pi} X$ be a branched covering, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Suppose that the maximal ideal $\mathfrak{m}_p$ at p is a principal ideal, generated by an element x. Let $q_1, ..., q_k$ be the points of Y that lie over a point p of X and let $\mathfrak{m}_i$ and e_i be the maximal ideal and ramification index at q_i, respectively.*

(i) *The extended ideal $\mathfrak{m}_p B = xB$ is the product ideal $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.*
(ii) *Let $\overline{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\overline{B} = B/xB$ is isomorphic to the product $\overline{B}_1 \times \cdots \times \overline{B}_k$.*
(iii) *The degree $[Y:X]$ of the covering is the sum $e_1 + \cdots + e_k$ of the ramification indices at the points q_i.* □

(8.5.4) local analytic structure locanstr

The local analytic structure of a branched covering $Y \xrightarrow{\pi} X$ in the classical topology is very simple. We explain it here because it is useful and helpful for intuition.

8.5.5. Proposition. *Locally in the classical topology, Y is analytically isomorphic to an e-th root covering $y^e = x$.* *yeth-roottwo*

proof. Let q be a point of Y, let p be its image in X, let x and y be local generators for the maximal ideals $\mathfrak{m}_p$ of $\mathcal{O}_X$ and $\mathfrak{m}_q$ of $\mathcal{O}_Y$, respectively. Let e be the ramification index at q. So $x = uy^e$, where u is a local unit at q. In a neighborhood of q in the classical topology, u will have an analytic e-th root w. The element $y_1 = wy$ also generates $\mathfrak{m}_q$ locally, and $x = y_1^e$. We replace y by y_1. Then the implicit function theorem tells us that that x and y are local analytic coordinate functions on X and Y (see (1.4.18)). □

8.5.6. Proposition. *Let $Y \xrightarrow{\pi} X$ be a branched covering, let $\{q_1, ..., q_k\}$ be the fibre over a point p of X, and let e_i be the ramification index at q_i. As a point p' of X approaches p, e_i points that lie over p' approach q_i.* □ *pprimeap-proachesp*

(8.5.7) suppressing notation for the direct image workonX

When considering a branched covering $Y \xrightarrow{\pi} X$ of smooth curves, we will often pass between an $\mathcal{O}_Y$-module $\mathcal{M}$ and its direct image $\pi_*\mathcal{M}$, and it will be convenient to work primarily on X. Recall that if X' is an open subset X' of X and Y' is its invere image, then

$$[\pi_*\mathcal{M}](X') = \mathcal{M}(Y')$$

One can think of the direct image $\pi_*\mathcal{M}$ as working with $\mathcal{M}$, but looking only at the open subsets Y' of Y that are inverse images of open subsets of X. If we look only at such subsets, the only significant difference between $\mathcal{M}$ and its direct image will be that, when X' is open in X and $Y' = \pi^{-1}X'$, the $\mathcal{O}_Y(Y')$-module $\mathcal{M}(Y')$ is made into an $\mathcal{O}_X(X')$-module by restriction of scalars.

To simplify notation, we will often drop the symbol π_*, and write $\mathcal{M}$ instead of $\pi_*\mathcal{M}$. If X' is an open subset of X, $\mathcal{M}(X')$ will stand for $\mathcal{M}(\pi^{-1}X')$. When denoting the direct image of an $\mathcal{O}_Y$-module $\mathcal{M}$ by the same symbol $\mathcal{M}$, we may refer to it as an $\mathcal{O}_X$-module. In accordance with this convention, we may also write $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$, but we must include the subscript Y.

This abbreviated terminology is analogous to the one used for restriction of scalars in a module. When $A \to B$ is an algebra homomorphism and M is a B-module, the B-module $_BM$ and the A-module $_AM$ obtained from it by restriction of scalars are usually denoted by the same letter M.

8.5.8. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, of degree $n = [Y:X]$. With notation as above,* *BrankArank*
(i) *The direct image of $\mathcal{O}_Y$, which we may also denote by $\mathcal{O}_Y$, is a locally free $\mathcal{O}_X$-module and of rank n.*
(ii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a torsion $\mathcal{O}_Y$-module if and only if its direct image (also denoted by $\mathcal{M}$) is a torsion $\mathcal{O}_X$-module.*
(iii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module if and only if its direct image is a locally free $\mathcal{O}_X$-module. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module of rank r, then its directc image is a locally free $\mathcal{O}_X$-module of rank nr.* □

8.6 Trace of a Differential

tracediff

tracefn **(8.6.1) trace of a function**

Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, and let F and K be the function fields of X and Y, respectively. The trace map $K \xrightarrow{\text{trace}} F$ for a field extension of finite degree has been defined before (4.3.11). If α is an element of K, multiplication by α on the F-vector space K is a linear operator, and $\text{trace}(k)$ is the trace of that operator. The trace is F-linear: If f_i are in F and α_i are in K, then $\text{trace}(\sum f_i\alpha_i) = \sum f_i \,\text{trace}(\alpha_i)$. Moreover, the trace carries regular functions to regular functions: If $X' = \operatorname{Spec} A'$ is an affine open subset of X, with inverse image $Y' = \operatorname{Spec} B'$, then because A' is a normal algebra, the trace of an element of B' will be in A' (4.3.7). Using our abbreviated notation $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$ (**8.5.7**), the trace defines a homomorphism of $\mathcal{O}_X$-modules

trace-global (8.6.2)
$$\mathcal{O}_Y \xrightarrow{\text{trace}} \mathcal{O}_X$$

Analytically, this trace can be described as a sum over the sheets of the covering. Let $n = [Y : X]$. When a point p of X isn't a branch point, there will be n points $q_1, ..., q_n$ of Y lying over p. If U is a small neighborhood of p in X in the classical topology, its inverse image V will consist of disjoint neighborhoods V_i of q_i, each of which maps bijectively to U. The ring of analytic functions on V_i will be isomorphic to the ring $\mathcal{A}$ of analytic functions on U. So the ring of analytic functions on V is isomorphic to a direct sum $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ of n copies of $\mathcal{A}$. If a rational function g on Y is regular on V, its restriction to V can be written as $g = g_1 \oplus \cdots \oplus g_n$, with g_i in $\mathcal{A}_i$. The matrix of left multiplication by g on $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ is the diagonal matrix with entries $g_1, ..., g_n$, and

trsum (8.6.3)
$$\text{trace}(g) = g_1 + \cdots + g_n$$

tracesum **8.6.4. Lemma.** *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, let p be a point of X, let $q_1, ..., q_k$ be the fibre over p, and let e_i be the ramification index at q_i. If a rational function g on Y is regular at the points $q_1, ..., q_k$, its trace is regular at p. Its value at p is $[\text{trace}(g)](p) = e_1g(q_1) + \cdots + e_kg(q_k)$.*

proof. The regularity was discussed above. If p isn't a branch point, we will have $k = n$ and $e_i = 1$ for all i. In this case, the lemma follows by evaluating (8.6.3). It follows by continuity for any point p. As a point p' approaches p, e_i points q' of Y approach q_i (8.5.6). For each point q' that approaches q_i, the limit of $g(q')$ will be $g(q_i)$. □

traced **(8.6.5) trace of a differential**

The structure sheaf is naturally contravariant. A branched covering $Y \xrightarrow{\pi} X$ corresponds to a homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{O}_Y$. The trace map for functions is a homomorphism in the opposite direction: $\mathcal{O}_Y \xrightarrow{\text{trace}} \mathcal{O}_X$.

Differentials are also naturally contravariant. A morphism $Y \xrightarrow{\pi} X$ induces an $\mathcal{O}_X$-module homomorphism $\Omega_X \to \Omega_Y$ that sends a differential dx on X to a differential on Y that we may also denote by dx (8.4.4) **(ii)**. As is true for functions, there is a trace map for differentials in the opposite direction. It is defined below, in (8.6.7), and it will be denoted by τ:

$$\Omega_Y \xrightarrow{\tau} \Omega_X$$

But first, a lemma about the natural contravariant map $\Omega_X \to \Omega_Y$:

dxydy **8.6.6. Lemma.** *Let $Y \to X$ be a branched covering.*

(i) *Let p be the image in X of a point q of Y, let x and y be local generators for the maximal ideals of X and Y at p and q, respectively, and let e be the ramification index at q. As a differential on Y, $dx = vy^{e-1}dy$, where v is a local unit at q.*

(ii) *The canonical homomorphism $\Omega_X \to \Omega_Y$ is injective.*

proof. **(i)** As we have noted before, $x = uy^e$, for some local unit u. Since dy generates Ω_Y locally, there is a rational function z that is regular at q, such that $du = zdy$. Let $v = yz + eu$. Then

$$dx \;=\; d(uy^e) \;=\; y^e z\, dy + ey^{e-1}u\, dy \;=\; vy^{e-1}dy$$

Since yz is zero at q and eu is a local unit, v is a local unit.
See (8.1.17) **(iv)** for the proof of part **(ii)**. □

To define the trace for differentials, we begin with differentials of the functions fields F and K of X and Y, respectively. The $\mathcal{O}_Y$-module Ω_Y is invertible (8.4.10), and the module Ω_K of K-differentials is a localization of Ω_Y. So Ω_K is a free K-module of rank one. Any nonzero differential will form a K-basis. We choose as basis a nonzero F-differential α. Its image in Ω_K, which we denote by α too, will be a K-basis for Ω_K. We could take $\alpha = dx$, where x is a local coordinate function on X, for instance.

Since α is a basis, any element β of Ω_K can be written uniquely, as

$$\beta = g\alpha$$

where g is an element of K. The *trace* $\Omega_K \xrightarrow{\tau} \Omega_F$ is defined by

(8.6.7) $$\tau(\beta) = \mathrm{trace}(g)\alpha$$ deftrdif

where $\mathrm{trace}(g)$ is the trace of the function g. Since the trace for functions is F-linear, τ is also F-linear.

We need to check that τ is independent of the choice of α. Let α' be another nonzero F-differential. Then $f\alpha' = \alpha$ for some nonzero element f of F, and $gf\alpha' = g\alpha$. Since trace is F-linear, $\mathrm{trace}(gf) = f\,\mathrm{trace}(g)$. Then

$$\tau(gf\alpha') = \mathrm{trace}(gf)\alpha' = f\,\mathrm{trace}(g)\alpha' = \mathrm{trace}(g)f\alpha' = \mathrm{trace}(g)\alpha = \tau(g\alpha)$$

Using α' in place of α gives the same value for the trace.

A differenial of the function field K is called a *rational differential*. A rational differential β is *regular* at a point q of Y if there is an affine open neighborhood $Y' = \operatorname{Spec} B$ of q such that β is an element of Ω_B. If y is a local generator for the maximal ideal $\mathfrak{m}_q$ and if $\beta = g\,dy$, the differential β is regular at q if and only if the rational function g is regular at q.

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties, and let p be a point of X. Suppose that the maximal ideal at p is a principal ideal, generated by an element x of A, and that the differential dx generates Ω_A. Let $q_1, ..., q_k$ be the points of Y that lie over p, and let e_i be the ramification index at q_i.

8.6.8. Corollary. *With notation as above,* *poleorder*

(i) *When viewed as a differential on Y, dx has a zero of order e_i-1 at q_i.*

(ii) *If a differential β on Y that is regular at q_i is written as $\beta = g\,dx$, where g is a rational function on Y, then g has a pole of order at most e_i-1 at q_i.*

This follows from Lemma 8.6.6 **(i)**.

8.6.9. Main Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering, let p be a point of X, let $q_1, ..., q_k$ be the points of Y that lie over p, and let β be a rational differential on Y.* *tracereg*

(i) *If β is regular at the points $q_1, ..., q_k$, its trace $\tau(\beta)$ is regular at p.*

(ii) *If β has a simple pole at q_i and is regular at q_j for all $j \neq i$, then $\tau(\beta)$ is not regular at p.*

proof. Let x be a local generator for the maximal ideal at p. We write β as $g\,dx$, where g is a rational function on Y.

(i) Suppose that β is regular at the points q_i. Corollary 8.6.8 tells us that g has poles of orders at most e_i-1 at the points q_i. Since x has a zero of order e_i at q_i, the function xg is regular at q_i, and its value there is zero. Then $\mathrm{trace}(xg)$ is regular at p, and its value at p is zero (8.6.4). So $x^{-1}\,\mathrm{trace}(xg)$ is a regular function at p. Since trace is F-linear and x is in F, $x^{-1}\,\mathrm{trace}(xg) = \mathrm{trace}(g)$. Therefore $\mathrm{trace}(g)$ and $\tau(\beta) = \mathrm{trace}(g)dx$ are regular at p.

(ii) In this case, g has poles of orders at most e_j-1 at the points q_j when $j \neq i$, and it has a pole of order e_i at q_i. So xg will be regular at the points q_i. It will be zero at q_j when $j \neq i$, and nonzero at q_i. The function xg will be regular at all of the points q_j. Its value at q_j will be zero when $j \neq i$, and not zero when $j = i$. Then $\mathrm{trace}(xg)$ will be regular at p, but not zero there (8.6.4). Therefore $\tau(\beta) = x^{-1}\,\mathrm{trace}(xg)dx$ won't be regular at p. $\square$

8.6.10. Corollary. *The trace map (8.6.7) defines a homomorphism of $\mathcal{O}_X$-modules* $\Omega_Y \xrightarrow{\tau} \Omega_X$. $\square$

omegaYandX

sumroot-cover **8.6.11. Example.** **(i)** Let x be a local generator for the maximal ideal $\mathfrak{m}_p$ at a point p of X. If the degree $[Y:X]$ of Y over X is n, then when we regard dx as a differential on Y,

tracedx (8.6.12)
$$\tau(dx) = n\,dx$$

(ii) Let Y be the locus $y^e = x$ in $\mathbb{A}^2_{x,y}$. Multiplying y by $\zeta = e^{2\pi i/e}$ permutes the sheets of Y over X. The trace of an integer power y^k is

sumzeta (8.6.13)
$$\mathrm{trace}(y^k) \;=\; \sum_{j=0}^{e-1} \zeta^{jk} y^k$$

The sum $\sum_j \zeta^{kj}$ is zero unless $k \equiv 0$ modulo e. Then $\mathrm{trace}(dy)$ is not regular at $x = 0$, but dy/y is regular there. $\square$

Let $Y \xrightarrow{\pi} X$ be a branched covering. As is true for any $\mathcal{O}_Y$-module, Ω_Y is isomorphic to the module of homomorphisms ${}_{\mathcal{O}_Y}(\mathcal{O}_Y, \Omega_Y)$. The homomorphism $\mathcal{O}_Y \to \Omega_Y$ that corresponds to a section β of Ω_Y on an open set U sends a regular function f on U to $f\beta$. We denote that homomorphism by β too: $\mathcal{O}_Y \xrightarrow{\beta} \Omega_Y$.

compwtau **8.6.14. Lemma.** *Composition with the trace $\Omega_Y \xrightarrow{\tau} \Omega_X$ defines a homomorphism of $\mathcal{O}_X$-modules*

$$\Omega_Y \approx {}_{\mathcal{O}_Y}(\mathcal{O}_Y, \Omega_Y) \xrightarrow{\tau} {}_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega_X)$$

proof. An $\mathcal{O}_Y$-linear map becomes an $\mathcal{O}_X$-linear map by restriction of scalars. When we compose an $\mathcal{O}_Y$-linear map β with τ, then because τ is $\mathcal{O}_X$-linear, the result will be $\mathcal{O}_X$-linear. It will be a homomorphism of $\mathcal{O}_X$-modules. $\square$

8.6.15. Theorem. **(i)** *The map (8.6.14) is bijective.*

traceomega **(ii)** *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. Composition with the trace defines a bijection*

compw-trace (8.6.16)
$${}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \xrightarrow{\tau\circ} {}_{\mathcal{O}_X}(\mathcal{M}, \Omega_{\mathcal{O}_X})$$

This theorem follows from the Main Lemma, when one looks carefully.

Note. The domain and range (8.6.16) are to be interpreted as modules on X. When we put the symbols $\underline{\mathrm{Hom}}$ and π_* that we have suppressed into the notation, the map (8.6.16) becomes a bijection

$$\pi_*\big(\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_Y)\big) \xrightarrow{\tau\circ} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\pi_*\mathcal{M}, \Omega_X)$$

It suffices to verify the theorem locally, because it concerns modules on X. So we may suppose that X and Y are affine, say $X = \mathrm{Spec}\,A$ and $Y = \mathrm{Spec}\,B$. When the theorem is stated in terms of algebras and modules, it becomes this:

8.6.17. Theorem. *Let $Y \to X$ be a branched covering, with $Y = \mathrm{Spec}\,B$ and $X = \mathrm{Spec}\,A$.*

traceomegaalgebra **(i)** *The trace map $\Omega_B = {}_B(B, \Omega_B) \xrightarrow{\tau\circ} {}_A(B, \Omega_A)$ is bijective.*

(ii) *For any locally free B-module M, composition with the trace defines a bijection ${}_B(M, \Omega_B) \xrightarrow{\tau\circ} {}_A(M, \Omega_A)$.*

When we write ${}_A(M, \Omega_A)$ here, we are interpreting the B-module M as an A-module by restriction of scalars.

8.6.18. Lemma. *Let $A \subset B$ be rings, let M be a B-module, and let N be an A-module. Then the module ${}_A(M, N)$ of homomorphisms has the structure of a B-module.* homisB-mod

proof. We must define scalar multiplication of a homomorphism $M \xrightarrow{\varphi} N$ of A-modules by an element b of B. The definition is: $[b\varphi](m) = \varphi(bm)$. One must show that the map $b\varphi$ is a homomorphism of A-modules $M \to N$, and that the axioms for a B-module are true. You will be able to check those things. $\square$

proof of Theorem 8.6.15 **(i)**. Since the theorem is local, we are still allowed to localize. We use the algebra version 8.6.17 of the theorem. Both B and Ω_B are torsion-free, and therefore locally free A-modules. Localizing as needed, we may assume that they are free A-modules, and that Ω_A is a free module of rank one with basis of the form dx. Then ${}_A(B, \Omega_A)$ will be a free A-module too.

Let's denote ${}_A(B, \Omega_A)$ by Θ. Lemma 8.6.18 tells us that Θ is a B-module. Because B and Ω_A are free A-modules, Θ is a free A-module and a locally free B-module. Since Ω_A has A-rank 1, the A-rank of Θ is the same as the A-rank of B. Then the B-rank of Θ is 1, the same as the B-rank of B too (see (8.5.8)). Therefore Θ is an invertible B-module.

If x is a local coordinate on X, then $\tau\, dx \neq 0$ (8.6.12). The trace map $\Omega_B \xrightarrow{\tau} \Theta$ isn't the zero map. Since domain and range are invertible B-modules, it is an injective homomorphism. Its image, which is isomorphic to Ω_B, is an invertible submodule of the invertible B-module Θ.

To show that $\Omega_B = \Theta$, we will apply Lemma 8.1.19 to show that the quotient $\overline{\Theta} = \Theta/\Omega_B$ is the zero module. Suppose not, and let q be a point in the support of $\overline{\Theta}$. Let p be the image of q in X and let $q_1, ..., q_k$ be the fibre over p, with $q = q_1$.

We choose a differential α that is regular at all of the points q_i. If y is a local generator for the maximal ideal at q_1, then $\alpha = g\, dy$, where g is a regular function at q_1. We assume also that α has been chosen so that that $g(q_1) \neq 0$.

Let f be a rational function that is regular on an affine open set V of Y that contains the points $q_1, ..., q_k$, and such that $f(q_1) = 0$ and $f(q_i) \neq 0$ when $i > 1$. Lemma 8.1.19 tells us that $\beta = f^{-1}\alpha$ is a section of Θ on V, but the Main Lemma 8.6.9 tells us that $\tau(\beta)$ isn't regular at p. This contradicion proves the theorem. $\square$

proof of Theorem 8.6.15 **(ii)**. We are to show that if $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, composition with the trace defines a bijective map ${}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \to {}_{\mathcal{O}_X}(\mathcal{M}, \Omega_{\mathcal{O}_X})$. Part **(i)** of the theorem tells us that this is true in when $\mathcal{M} = \mathcal{O}_Y$. Therefore it is also true when $\mathcal{M}$ is a free module $\mathcal{O}_Y^k$. And, since **(ii)** is a statement about $\mathcal{O}_X$-modules, it suffices to prove it locally on X.

8.6.19. Lemma. *Let $q_1, ..., q_k$ be points of a smooth curve Y, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. There is an open set V that contains the points $q_1, ..., q_k$, such that $\mathcal{M}$ is free on V.* local-lyfreeonX

We assume the lemma and complete the proof of the theorem. Let $\{q_1, ..., q_k\}$ be the fibre over a point p of X and let be V as in the lemma. The complement $D = Y - V$ is a finite set whose image Z in X is also finite, and Z doesn't contain p. If U is the complement of Z in X, its inverse image W will be a subset of V that contains the points of the fibre, on which $\mathcal{M}$ is free. $\square$

proof of the lemma. We may assume that Y is affine, $Y = \operatorname{Spec} B$, and that the $\mathcal{O}$-module $\mathcal{M}$ corresponds to a locally free B-module M.

With terminology as in Lemma 8.5.3, let $\mathfrak{m}_i$ be the maximal ideal of B at q_i, and let $\overline{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\overline{B} = B/xB$ is isomorphic to the product $\overline{B}_1 \times \cdots \times \overline{B}_k$. Since $\mathcal{M}$ is locally free, $M/\mathfrak{m}_i M = \overline{M}_i$ is a free $\overline{B}_i$-module. Its dimension is the rank r of the B-module M.

If M has rank r, there will be a set of elements $m = (m_1, ..., m_r)$ in M whose residues form a basis of $\overline{M}_i$ for every i. This follows from the Chinese Remainder Theorem. Therefore m generates M locally at each of the points. Let M' be the B-submodule of M generated by m. The cokernel of the map $M' \to M$ is zero at the points $q_1, ..., q_k$, and therefore it's support, which is a finite set, is disjoint from those points. When we localize to delete this finite set from X, the set m becomes a basis for M. $\square$

Note. Theorem 8.6.15 is subtle. Unfortunately the proof, though understandable, doesn't give an intuitive explanation of the fact that Ω_B is isomorphic to ${}_A(B, \Omega_A)$. To get more insight into that, we would need a better understanding of differentials. My father Emil Artin said: "One doesn't really understand differentials, but one can learn to work with them."

8.7 The Riemann-Roch Theorem II

rroch
serredual **(8.7.1) the Serre dual**

Let Y be a smooth projective curve, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. The *Serre dual* of $\mathcal{M}$, is the module

defSerred-ual (8.7.2)
$$\mathcal{M}^S \;=\; {}_Y(\mathcal{M},\Omega_Y) \qquad \big(= \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(M,\Omega_Y)\big)$$

Its sections on an open subset U are the homomorphisms of $\mathcal{O}_Y(U)$-modules $\mathcal{M}(U) \to \Omega_Y(U)$, and it can also be written as $\mathcal{M}^S = \mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y$, where $\mathcal{M}^*$ is the ordinary dual ${}_Y(\mathcal{M},\mathcal{O}_Y)$. Since the module Ω_Y is invertible, it is locally isomorphic to $\mathcal{O}_Y$. So the Serre dual $\mathcal{M}^S$ is locally isomorphic to the ordinary dual $\mathcal{M}^*$. It is a locally free module of the same rank as $\mathcal{M}$, and the Serre bidual $(\mathcal{M}^S)^S$ is isomorphic to $\mathcal{M}$:

$$(\mathcal{M}^S)^S \approx (\mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y)^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \otimes_{\mathcal{O}} \Omega_Y^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \approx \mathcal{M}$$

(See (8.1.17) **(i)**.) For example, $\mathcal{O}_Y^S = \Omega_Y$ and $\Omega_Y^S = \mathcal{O}_Y$.

dualco-hom **8.7.3. Riemann-Roch Theorem, version 2.** *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module on a smooth projective curve Y, and let $\mathcal{M}^S$ be its Serre dual. Then $\mathbf{h}^0\mathcal{M} = \mathbf{h}^1\mathcal{M}^S$ and $\mathbf{h}^1\mathcal{M} = \mathbf{h}^0\mathcal{M}^S$.*

The two assertions are equivalent, because $\mathcal{M}$ and $(\mathcal{M}^S)^S$ are isomorphic. The second one follows from the first when one replaces $\mathcal{M}$ by $\mathcal{M}^S$. For example, $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$ and $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y$ is the arithmetic genus p_a.

If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then

chitwo (8.7.4)
$$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^0\mathcal{M}^S$$

A more precise statement of the Riemann-Roch Theorem is that $H^1(Y,\mathcal{M})$ and $H^0(Y,\mathcal{M}^S)$ are dual vector spaces. This becomes important when one wants to apply the theorem to a cohomology sequence, but we omit the proof. The fact that the dimensions are equal is enough for many applications.

Our plan is to prove Theorem 8.7.3 directly for the projective line $\mathbb{P}^1$. This will be easy, because the structure of locally free modules on $\mathbb{P}^1$ is very simple. We derive it for an arbitrary smooth projective curve Y by projection to $\mathbb{P}^1$. Projection to projective space is the method that was used by Grothendieck in his proof of the general Riemann-Roch Theorem.

Let $X = \mathbb{P}^1$, let Y be a smooth projective curve, and let $Y \xrightarrow{\pi} X$ be a branched covering. Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module, and let the Serre dual of $\mathcal{M}$, as defined in (8.7.2), be

$$\mathcal{M}_1^S \;=\; {}_Y(\mathcal{M},\Omega_Y)$$

The direct image of $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module that we are denoting by $\mathcal{M}$ too, and we can form the Serre dual on X. Let

$$\mathcal{M}_2^S \;=\; {}_X(\mathcal{M},\Omega_X)$$

abouthom **8.7.5. Corollary.** *The direct image $\pi_*\mathcal{M}_1^S$, which we also denote by $\mathcal{M}_1^S$, is isomorphic to $\mathcal{M}_2^S$.*

proof. This is Theorem 8.6.15. □

The corollary allows us to drop the subscripts from $\mathcal{M}^S$. Because a branched covering $Y \xrightarrow{\pi} X$ is an affine morphism, the cohomology of $\mathcal{M}$ and of its Serre dual $\mathcal{M}^S$ can be computed, either on Y or on X. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then $H^q(Y,\mathcal{M}) \approx H^q(X,\mathcal{M})$ and $H^q(Y,\mathcal{M}^S) \approx H^q(X,\mathcal{M}^S)$ (7.4.22).

Thus it is enough to prove Riemann-Roch for the projective line.

dualityfor
pone **(8.7.6) Riemann-Roch for the projective line**

The Riemann-Roch Theorem for the projective line $X = \mathbb{P}^1$ is a consequence of the Birkhoff-Grothendieck Theorem, which tells us that a locally free $\mathcal{O}_X$-module $\mathcal{M}$ on X is a direct sum of twisting modules. To prove Riemann-Roch for the projective line X, it suffices to prove it for the twisting modules $\mathcal{O}_X(k)$.

8.7.7. Lemma. *The module of differentials Ω_X on the projective line X is isomorphic to the twisting module $\mathcal{O}_X(-2)$.* *omegapone*

proof. Let $\mathbb{U}^0 = \operatorname{Spec}\mathbb{C}[x]$, and $\mathbb{U}^1 = \operatorname{Spec}\mathbb{C}[z]$ be the standard open subsets of $\mathbb{P}^1$, with $z = x^{-1}$. On $\mathbb{U}^0$, the module of differentials is free, with basis dx, and $dx = d(z^{-1}) = -z^{-2}dz$ describes the differential dx on $\mathbb{U}^1$. Since the point p_∞ at infinity is $\{z = 0\}$, dx has a pole of order 2 there. It is a global section of $\Omega_X(2p_\infty)$, and as a section of that module, it isn't zero anywhere. Multiplication by dx defines an isomorphism $\mathcal{O} \to \Omega_X(2p_\infty)$ that sends 1 to dx. Tensoring with $\mathcal{O}(-2p_\infty)$ shows that $\mathcal{O}(-2p_\infty)$ is isomorphic to Ω_X. $\square$

8.7.8. Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules on the projective line X. Then $\;_X(\mathcal{M}(r),\mathcal{N})$ is canonically isomorphic to $\;_X(\mathcal{M},\mathcal{N}(-r))$.* *twisthom*

proof. When we tensor a homomorphism $\mathcal{M}(r) \xrightarrow{\varphi} \mathcal{N}$ with $\mathcal{O}(-r)$, we obtain a homomorphism $\mathcal{M} \to \mathcal{N}(-r)$, and tensoring with $\mathcal{O}(r)$ is the inverse operation. $\square$

The Serre dual $\mathcal{O}(n)^S$ of $\mathcal{O}(n)$ is therefore

$$\mathcal{O}(n)^S = \;_X(\mathcal{O}(n),\mathcal{O}(-2)) \approx \mathcal{O}(-2-n)$$

To prove Riemann-Roch for $X = \mathbb{P}^1$, we must show that

$$\mathbf{h}^0\mathcal{O}(n) = \mathbf{h}^1\mathcal{O}(-2-n) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(n) = \mathbf{h}^0\mathcal{O}(-2-n)$$

This follows from Theorem 7.5.5, which computes the cohomology of the twisting modules. As we've noted before, the two assertions are equivalent, so it suffices to verify the first one. If $n < 0$, then $-2-n \geq -1$. In this case $\mathbf{h}^0\mathcal{O}(n)$ and $\mathbf{h}^1\mathcal{O}(-2-n) = 0$ are both zero. If $n \geq 0$, then $-2-n < -2$, and then $\mathbf{h}^0\mathcal{O}(n) = \mathbf{h}^1\mathcal{O}(-2-n) = n+1$.

8.8 Using Riemann-Roch

appl

(8.8.1) **genus** genussm-curve

Three closely related numbers associated to a smooth projective curve Y are: the *topological genus* g, the *arithmetic genus* $p_a = \mathbf{h}^1\mathcal{O}_Y$, and the *degree* δ of the module of differentials Ω_Y.

8.8.2. Theorem. *The topological genus g and the arithmetic genus p_a of a smooth projective curve Y are equal, and the degree δ of the module Ω_Y is $2g-2$, which is equal to $2p_a - 2$.* *genus-genus*

Thus the Riemann-Roch Theorem 8.2.3 can we written as

(8.8.3) $$\chi(\mathcal{O}(D)) = \deg\, D + 1 - g$$ dplusone-minusg

We'll write it this way when the theorem is proved.

###By the way, the fact that the genus is always ≥ 0 has applications. ###

proof. Let $Y \xrightarrow{\pi} X$ be a branched covering with $X = \mathbb{P}^1$. The topological Euler characteristic $e(Y)$, which is $2-2g$, can be computed in terms of the branching data for the covering, as in (1.8.23). Let q_i be the ramification points in Y, and let e_i be the ramification index at q_i. Then e_i sheets of the covering come together at q_i. (One might say that e_i-1 points are lacking.) If the degree of Y over X is n, then since $e(X) = 2$,

(8.8.4) $$2-2g = e(Y) \;=\; ne(X) - \sum(e_i-1) \;=\; 2n - \sum(e_i-1)$$ degdelta

We compute the degree δ of Ω_Y in two ways. First, the Riemann-Roch Theorem tells us that $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = p_a$ and $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$. So $\chi(\Omega_Y) = -\chi(\mathcal{O}_Y) = p_a - 1$. The Riemann-Roch Theorem also tells us that $\chi(\Omega_Y) = \delta + 1 - p_a$. Therefore

(8.8.5) $$\delta = 2p_a - 2$$ delta

Next, we compute δ by computing the divisor of the differential dx on Y, x being a coordinate on the projective line X. Let q_i be one of the ramification points in Y, and let e_i be the ramification index at q_i. Then dx has a zero of order e_i-1 at q_i. At the point of X at infinity, dx has a pole of order 2. Let's choose coordinates so that the point at infinity isn't a branch point. Then there will be n points of Y at which dx has a pole of order 2, n being the degree of Y over X. The degree of Ω_Y is therefore

deltatwo (8.8.6) $$\delta = \mathit{zeros} - \mathit{poles} = \sum(e_i-1) - 2n$$

Combining (8.8.6) with (8.8.4), one sees that $\delta = 2g-2$. Since we also have $\delta = 2p_a - 2$, we conclude that $g = p_a$. □

canondiv **(8.8.7)** **canonical divisors**

Because the module Ω_Y of differentials on a smooth curve Y is invertible, it is isomorphic to $\mathcal{O}(K)$ for some divisor K that is called a *canonical divisor* (Proposition 8.1.13). The degree of a canonical divisor is $2g-2$, the same as the degree of Ω_Y. It is often convenient to represent Ω_Y as a module $\mathcal{O}(K)$, though the canonical divisor K isn't unique. It is determined only up to linear equivalence (8.1.15).

When written in terms of a canonical divisor K, the Serre dual of an invertible module $\mathcal{O}(D)$ will be

degK (8.8.8) $$\mathcal{O}(D)^S \approx \mathcal{O}(K-D)$$

(**8.1.16**), (8.1.12). With this notation, the Riemann-Roch Theorem for $\mathcal{O}(D)$ becomes

RRforOD (8.8.9) $$\mathbf{h}^0\mathcal{O}(D) = \mathbf{h}^1\mathcal{O}(K-D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(K-D)$$ □

genuszero **(8.8.10)** **curves of genus zero**

Let Y be a smooth projective curve Y of genus $g=0$, and let p be a point of Y. The exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(p) \to \epsilon \to 0$$

where ϵ is a one-dimensional module supported at p, gives us an exact cohomology sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(p)) \to H^0(Y, \epsilon) \to 0$$

The zero on the right is due to the fact that $\mathbf{h}^1\mathcal{O}_Y = g = 0$. We also have $\mathbf{h}^0\mathcal{O}_Y = 1$ and $\mathbf{h}^0\epsilon = 1$, so when Y has genus zero, $\mathbf{h}^0\mathcal{O}_Y(p) = 2$. We choose a basis $(1, x)$ for $H^0(Y, \mathcal{O}_Y(p))$, 1 being the constant function and x being a nonconstant function with a single pole of order 1 at p. This basis defines a point of $\mathbb{P}^1$ with values in the function field K of Y, and therefore a morphism $Y \xrightarrow{\varphi} \mathbb{P}^1$ (5.2.4). Because x has just one pole of order 1, it takes every value exactly once (8.2.9)**(iii)**. Therefore φ is bijective. It is a map of degree 1, and therefore an isomorphism (8.5.2).

gzero **8.8.11. Corollary.** *A smooth projective curve of genus zero is isomorphic to the projective line* $\mathbb{P}^1$. □

A *rational curve* is a curve, smooth or not, whose function field is isomorphic to the field $\mathbb{C}(t)$ of rational functions in one variable. A smooth projective curve of genus zero is a rational curve.

genusone **(8.8.12)** **curves of genus one**

A smooth projective curve of genus $g=1$ is called an *elliptic curve*. The Riemann-Roch Theorem tells us that on an elliptic curve Y,

$$\chi(\mathcal{O}(D)) = \deg\, D$$

Since $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = 1$, Ω_Y has a nonzero global section ω. Since Ω_Y has degree zero (8.8.2), ω doesn't vanish anywhere. Multiplication by ω defines an isomorphism $\mathcal{O} \to \Omega_Y$. So Ω_Y is a free module of rank one.

8.8.13. Lemma. *Let D be a divisor of degree $r > 0$ on an elliptic curve Y. Then $\mathbf{h}^0\mathcal{O}(D) = r$, and $\mathbf{h}^1\mathcal{O}(D) = 0$.* *hrpforelliptic*

This follows from Riemann-Roch. Since $\Omega)Y$ is free, isomorphic to $co\ K = 0$ is a canonical divisor, so the Serre dual of $\mathcal{O}(D)$ is $\mathcal{O}(-D)$, and $\mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(-D)$, which is zero when the degree of D is positive. □

Now, since $H^0(Y, \mathcal{O}_Y) \subset H^0(Y, \mathcal{O}_Y(p))$, and since both spaces have dimension one, they are equal. So (1) is a basis for $H^0(Y, \mathcal{O}_Y(p))$. We choose a basis $(1, x)$ for the two-dimensional space $H^1(Y, \mathcal{O}_Y(2p))$. Then x isn't a section of $\mathcal{O}(p)$. It has a pole of order precisely 2 at p and no other pole. Next, we choose a basis $(1, x, y)$ for $H^1(Y, \mathcal{O}_Y(3p))$. So y has a pole of order 3 at p, and no other pole. The point $(1, x, y)$ of $\mathbb{P}^2$ with values in K determines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^2$.

Let u, v, w be coordinates in $\mathbb{P}^2$. The map φ sends a point q distinct from p to $(u, v, w) = (1, x(q), y(q))$. Since Y has dimension one, φ is a finite morphism. Its image Y' is closed (5.5.4). It is a plane curve.

To determine the image of the point p, we multiply $(1, x, y)$ by $\lambda = y^{-1}$, obtaining the equivalent vector $(y^{-1}, xy^{-1}, 1)$. The rational function y^{-1} has a zero of order 3 at p, and xy^{-1} has a simple zero there. Evaluating at p, we see that the image of p is the point $(0, 0, 1)$.

Let ℓ be a generic line $\{au+bv+cw = 0\}$ in $\mathbb{P}^2$. The rational function $a+bx+cy$ on Y has a pole of order 3 at p and no other pole. It takes every value, including zero, three times, and the three points of Y at which $a + bx + cy$ is zero form the inverse image of ℓ. The only possibilities for the degree of Y' are 1 and 3. Since $1, x, y$ are independent, they don't satisfy a homogeneous linear equation. So Y' isn't a line. It is a cubic curve (see Corollary 1.3.10).

To determine the image, we look for a cubic relation among the functions $1, x, y$ on Y. The seven monomials $1, x, y, x^2, xy, x^3, y^2$ have poles at p, of orders $0, 2, 3, 4, 5, 6, 6$, respectively, and no other poles. They are sections of $\mathcal{O}_Y(6p)$. Riemann-Roch tells us that $\mathbf{h}^0\mathcal{O}_Y(6p) = 6$. So those seven functions are linearly dependent. The dependency relation gives us a cubic equation among x and y, which we may write in the form

$$cy^2 + (a_1x+a_3)y + (a_0x^3+a_2x^2+a_4x+a_6) = 0$$

There can be no linear relation among functions whose orders of pole at p are distinct. So when we delete either x^3 or y^2 from the list of monomials, we obtain an independent set of six functions — a basis for the six-dimensional space $H^0(Y, \mathcal{O}(6p))$. In the cubic relation, the coefficients c and a_0 aren't zero. We normalize c and a_0 to 1. Next, we eliminate the linear term in y from the relation by substituting $y - \frac{1}{2}(a_1x+a_3)$ for y, and we eliminate the quadratic term in x in the resulting polynomial by substituting $x - \frac{1}{3}a_2$ for x. Bringing the terms in x to the other side of the equation, we are left with a cubic relation of the form

$$y^2 = x^3 + a_4x + a_6$$

The coefficients a_4 and a_6 have been changed, of course.

The cubic curve Y' defined by the homogenized equation $y^2z = x^3 + a_4xz^2 + a_6z^3$ is the image of Y. This curve meets a generic line $ax + by + cz = 0$ in three points and, as we saw above, its inverse image in Y consists of three points too. Therefore the morphism $Y \xrightarrow{\varphi} Y'$ is generically injective, and Y is the normalization of Y'. Let's denote the direct image $\varphi_*(\mathcal{O}_Y)$ by $\mathcal{O}_Y$, and let $\mathcal{F}$ be the $\mathcal{O}_{Y'}$-module $\mathcal{O}_Y/\mathcal{O}_{Y'}$. Since Y is the normalization of Y', $\mathcal{F}$ is a torsion module, and $H^1(\mathcal{F}) = 0$. We assemble the dimensions of cohomology into a table:

	$\mathcal{O}_{Y'}$	$\mathcal{O}_Y$	$\mathcal{F}$
$\mathbf{h}^0:$	1	1	$*$
$\mathbf{h}^1:$	1	1	0

The table shows that $\mathbf{h}^0\mathcal{F} = 0$. So $\mathcal{F}$ is torsion module with no global sections. So $\mathcal{F} = 0$, and $Y \approx Y'$.

8.8.14. Corollary. *Every elliptic curve is isomorphic to a cubic curve in $\mathbb{P}^2$.* □ *genusonecubic*

(8.8.15) the group law on an elliptic curve

genusonegroup

The points of an elliptic curve Y form an abelian group, once one fixes a point as the identity element. We choose a point and label it o. Let p and q be points of Y. We write the law of composition as $p \oplus q$, to make a clear distinction between the product in the group, which is a point of Y, and the divisor $p+q$. To define $p \oplus q$, we compute the cohomology of $\mathcal{O}_Y(p+q-o)$. Lemma 8.8.13 shows that $\mathbf{h}^0\mathcal{O}_Y(p+q-o) = 1$. So there is a nonzero function f, unique up to scalar factor, with simple poles at p and q and a simple zero at o. This function has exactly one additional zero. That zero is defined to be the sum $p \oplus q$ in the group. In terms of linearly equivalent divisors, $p \oplus q$ is the unique point s such that the divisor $p+q$ is linearly equivalent to $o+s$.

grplaw **8.8.16. Proposition.** *With the law of composition $\oplus$ defined above, an elliptic curve mibecomes an abelian group.*

The proof is left as an exercise. $\square$

maptoP (8.8.17) maps to projective space

Let Y be a smooth projective curve. We have seen that any set $(f_0, ..., f_n)$ of rational functions on Y, not all zero, defines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ (5.2.4). As a reminder, let q be a point of Y and let $g_j = f_j/f_i$, where i is an index such that the value $\mathrm{v}_q(f_i)$ is a minimum, for $k = 0, ..., n$. The rational functions g_j are regular at q for all j, and the morphism φ sends the point q to $(g_0(q), ..., g_n(q))$. For example, the inverse image $\varphi^{-1}(\mathbb{U}^0)$ of the standard open set $\mathbb{U}^0$ is the set of points of Y at which the functions $g_j = f_j/f_0$ are regular. If q is such a point, then $\varphi(q) = (1, g_1(q), ..., g_n(q))$.

The *degree* d of a nonconstant morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ from a projective curve Y (smooth or not) to projective space is defined to be the number of points of the inverse image $\varphi^{-1}H$ of a generic hyperplane H in $\mathbb{P}^n$.

mapcurve **8.8.18. Lemma.** *Let Y be a smooth projective curve, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism to projective space defined by a set $(f_0, ..., f_n)$ of rational functions on Y.*
(i) *If the space spanned by $\{f_0, ..., f_n\}$ has dimension at least two, then φ isn't a constant morphism to a point.*
(ii) *If $f_0, ..., f_n$ are linearly independent, the image of Y isn't contained in a hyperplane.* $\square$

basept (8.8.19) base points

Let D be a divisor on the smooth projective curve Y, and suppose that $\mathbf{h}^0\mathcal{O}(D) = k > 1$. A basis $(f_0, ..., f_k)$ of global sections of $\mathcal{O}(D)$ defines a morphism $Y \to \mathbb{P}^{k-1}$. This is a common way to construct a morphism to projective space, though one could use any set of rational functions that aren't all zero.

If a global section of $\mathcal{O}(D)$ vanishes at a point p of Y, it is also a global section of $\mathcal{O}(D-p)$. A *base point* of $\mathcal{O}(D)$ is a point of Y at which every global section of $\mathcal{O}(D)$ vanishes. A base point can be described in terms of the usual exact sequence (8.2.4)

$$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \epsilon \to 0$$

where ϵ is a one-dimensional module whose support is p. The point p is a base point if $\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D)$, or if $\mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D) - 1$.

deg-nobasept **8.8.20. Lemma.** *Let D be a divisor on a smooth projective curve Y with $\mathbf{h}^0\mathcal{O}(D) = n > 1$, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism defined by a basis of global sections.*
(i) *The image of φ isn't contained in any hyperplane.*
(ii) *If $\mathcal{O}(D)$ has no base point, the degree of φ is equal to degree of D. If there are base points, the degree is lower.* $\square$

exbase-points **8.8.21. Proposition.** *Let K be a canonical divisor on a smooth projective curve Y of genus $g > 0$.*
(i) *$\mathcal{O}(K)$ has no base point.*
(ii) *Every point p of Y is a base point of $\mathcal{O}(K+p)$.*

proof. **(i)** Let p be a point of Y. We apply Riemann-Roch to the exact sequence

$$0 \to \mathcal{O}(K-p) \to \mathcal{O}(K) \to \epsilon \to 0$$

The Serre duals of $\mathcal{O}(K)$ and $\mathcal{O}(K-p)$ are $\mathcal{O}(K)^S = \mathcal{O}$ and $\mathcal{O}(K-p)^S = \mathcal{O}(p)$, respectively. They form an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p) \to \epsilon' \to 0$$

Because Y has positive genus, there is no rational function on Y with just one simple pole. So $\mathbf{h}^0\mathcal{O} = \mathbf{h}^0\mathcal{O}(p) = 1$. Riemann-Roch tells us that $\mathbf{h}^1\mathcal{O}(K-p) = \mathbf{h}^1\mathcal{O}(K) = 1$. The cohomology sequence

$$0 \to H^0(\mathcal{O}(K-p)) \to H^0(\mathcal{O}(K)) \to [1] \to H^1(\mathcal{O}(K-p)) \to H^1(\mathcal{O}(K)) \to 0$$

shows that $\mathbf{h}^0\mathcal{O}(K-p) = \mathbf{h}^0\mathcal{O}(K) - 1$. So p is not a base point.

(ii) Here, the relevant sequence is

$$0 \to \mathcal{O}(K) \to \mathcal{O}(K+p) \to \epsilon'' \to 0$$

The Serre dual of $\mathcal{O}(K+p)$ is $\mathcal{O}(-p)$, which has no global section. Therefore $\mathbf{h}^1\mathcal{O}(K+p) = 0$, while $\mathbf{h}^1\mathcal{O}(K) = \mathbf{h}^0\mathcal{O} = 1$. The cohomology sequence

$$0 \to \mathbf{h}^0\mathcal{O}(K) \to \mathbf{h}^0\mathcal{O}(K+p) \to [1] \to \mathbf{h}^1\mathcal{O}(K) \to \mathbf{h}^1\mathcal{O}(K+p) \to 0$$

shows that $H^0(\mathcal{O}(K+p)) = H^0(\mathcal{O}(K))$. So p is a base point of $\mathcal{O}(K+p)$. $\square$

(8.8.22) hyperelliptic curves

hyper

A *hyperelliptic curve* Y is a smooth projective curve of genus $g \geq 2$ that can be represented as a branched double covering of the projective line — such that there exists a morphism $Y \xrightarrow{\pi} X = \mathbb{P}^1$ of degree two. The term 'hyperelliptic' comes from the fact that every elliptic curve can be represented (not uniquely) as a double cover of $\mathbb{P}^1$. The global sections of $\mathcal{O}(2p)$, where p can be any point of an elliptic curve, define a map to $\mathbb{P}^1$ of degree 2.

The topological Euler characteristic of a hyperelliptic curve Y can be computed in terms of the double covering $Y \to X$, which will be branched at a finite set, say of n points, of Y. Since π has degree two, the ramification index at a branch point will be 2. The Euler characteristic is therefore $e(Y) = 2e(X) - n = 4 - n$. Since we know that $e(Y) = 2 - 2g$, the number of branch points is $n = 2g + 2$. When $g = 3$, $n = 8$.

It would take some experimentation to guess that the next remarkable theorem might be true, and some time to find a proof.

8.8.23. Theorem. *Let Y be a hyperelliptic curve, let $Y \xrightarrow{\pi} X = \mathbb{P}^1$ be a branched covering of degree 2.* *hyper-canon*

The morphism $Y \xrightarrow{\psi} \mathbb{P}^{g-1}$ defined by the global sections of $\Omega_Y = \mathcal{O}(K)$ factors through π. There is a unique morphism $X \xrightarrow{u} \mathbb{P}^{g-1}$ such that ψ is the composed map $Y \xrightarrow{\pi} X \xrightarrow{u} \mathbb{P}^{g-1}$:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle u} \\ \mathbb{P}^{g-1} & = & \mathbb{P}^{g-1} \end{array}$$

proof. Let x be an affine coordinate in X, so that the standard affine open subset $\mathbb{U}^0$ of X is $\operatorname{Spec}\mathbb{C}[x]$. We suppose that the point of X at infinity isn't a branch point of the covering π. The open set $Y^0 = \pi^{-1}\mathbb{U}^0$ will be described by an equation of the form $y^2 = f(x)$, where f is a polynomial of degree $n = 2g + 2$ with simple roots, and there will be two points of Y above the point of X at infinity, that are interchanged by the automorphism $y \to -y$. Let's call those points q_1 and q_2.

We start with the differential dx, which we view as a rational differential on Y. Then $2y\,dy = f'(x)dx$. Since f has simple roots, f' doesn't vanish at any of those roots. Solving for dx, we see that it has simple zeros on Y above the roots of f, which are the points at which $y = 0$. We also have a regular function on Y^0 with simple roots at those points, namely the function y. Therefore the differential $\omega = \frac{dx}{y}$ is regular and nowhere zero on Y^0. Because the degree of a differential on Y is $2g-2$, ω has a total of $2g-2$ zeros at infinity. By symmetry, ω has zeros of order $g-1$ at the points q_1 and q_2. Then $K = (g-1)q_1 + (g-1)q_2$ is a canonical divisor on Y.

Since K has zeros of order $g-1$ at infinity, the rational functions $1, x, x^2, ..., x^{g-1}$, viewed as functions on Y, are among the global sections of $\mathcal{O}_Y(K)$. They are independent, and there are g of them. Since $\mathbf{h}^0\mathcal{O}_Y(K) = g$, they form a basis of $H^0(\mathcal{O}_Y(K))$. The map $Y \to \mathbb{P}^{g-1}$ defined by the global sections of $\mathcal{O}_Y(K)$ evaluates these powers of x, so it factors through X. □

oned-blcover **8.8.24. Corollary.** *A curve of genus $g \geq 2$ can be presented as a branched covering of $\mathbb{P}^1$ of degree 2 in at most one way.* □

canonemb **(8.8.25) canonical embedding**

Let Y be a smooth projective curve of genus $g \geq 2$, and let K be a canonical divisor on Y. Its global sections define a morphism $Y \to \mathbb{P}^{g-1}$. This morphism is called the *canonical map*. We denote the canonical map by ψ. Since $\mathcal{O}(K)$ has no base point, the degree of ψ is the degree $2g-2$ of the canonical divisor. Theorem 8.8.23 shows that, when Y is hyperelliptic, the image of the canonical map is isomorphic to $\mathbb{P}^1$.

canonem-btwo **8.8.26. Theorem.** *Let Y be a smooth projective curve of genus g at least two. If Y isn't hyperelliptic, the canonical map embeds Y as a closed subvariety of projective space $\mathbb{P}^{g-1}$.*

proof. We show first that, if the canonical map $Y \xrightarrow{\psi} \mathbb{P}^{g-1}$ isn't injective, then Y is hyperelliptic.

Let p and q be distinct points of Y with the same image: $\psi(p) = \psi(q)$. We choose an effective canonical divisor K whose support doesn't contain p or q, and we inspect the global sections of $\mathcal{O}(K-p-q)$. Since $\psi(p) = \psi(q)$, any global section of $\mathcal{O}(K)$ that vanishes at p vanishes at q too. Therefore $\mathcal{O}(K-p)$ and $\mathcal{O}(K-p-q)$ have the same global sections, and so q is a base point of $\mathcal{O}(K-p)$. We've computed the cohomology of $\mathcal{O}(K-p)$ before: $\mathbf{h}^0\mathcal{O}(K-p) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p) = 1$. Therefore $\mathbf{h}^0\mathcal{O}(K-p-q) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p-q) = 2$. The Serre dual of $\mathcal{O}(K-p-q)$ is $\mathcal{O}(p+q)$, so by Riemann-Roch, $\mathbf{h}^0\mathcal{O}(p+q) = 2$. For any divisor D of degree one on a curve of positive genus, $\mathbf{h}^0(\mathcal{O}(D)) \leq 1$. So $\mathcal{O}(p+q)$ has no base point, and the global sections of $\mathcal{O}(p+q)$ define a morphism $Y \to \mathbb{P}^1$ of degree 2. This shows that Y is hyperelliptic.

If Y isn't hyperelliptic, the canonical map is injective, so Y is mapped bijectively to its image Y' in $\mathbb{P}^{g-1}$. This almost proves the theorem. But: can Y' have a cusp? We must show that the bijective map $Y \xrightarrow{\psi} Y'$ is an isomorphism. We go over the computation made above for a pair of points p, q, this time taking $q = p$. The computation is the same. Since Y isn't hyperelliptic, p isn't a base point of $\mathcal{O}_Y(K-p)$. Therefore $\mathbf{h}^0\mathcal{O}_Y(K-2p) = \mathbf{h}^0\mathcal{O}_Y(K-p) - 1$. This tells us that there is a global section f of $\mathcal{O}_Y(K)$ that has a zero of order exactly 1 at p. When properly interpreted, this fact shows that ψ doesn't collapse any tangent vector at p, and that ψ is an isomorphism. Since we haven't discussed tangent vectors, we prove this directly.

Since ψ is bijective, the function fields of Y and its image Y' are equal, and Y is the normalization of Y'. Moreover, ψ is an isomorphism except on a finite set. We work locally at a point p' of Y', and we denote the unique point of Y that maps to Y' by p. When we restrict the global section f of $\mathcal{O}_Y(K)$ found above to the image Y', we obtain an element of the maximal ideal $\mathfrak{m}_{p'}$ of $\mathcal{O}_{Y'}$ at p', that we denote by x. On Y, this element has a zero of order one at p, and therefore it is a local generator for the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_Y$. Let R' and R denote the local rings at p. We apply the Local Nakayama Lemma 5.1.1, regarding R as a finite R'-module. We substitute $V = R$ and $M = \mathfrak{m}'_{p'}$ into the statement of that lemma. Since x is in $\mathfrak{m}_{p'}$, $V/MV = R/\mathfrak{m}_{p'}R$ is the residue field $k(p)$ of R, which is spanned, as R'-module, by the element 1. The Local Nakayama Lemma tells us that R is spanned, as R'-module, by 1, and this shows that $R = R'$. □

lowgenus **(8.8.27) some curves of low genus**

curves of genus 2

When Y is a smooth projective curve of genus 2. The canonical map ψ is a map from Y to $\mathbb{P}^1$, of degree $2g-2=2$. Every smooth projective curve of genus 2 is hyperelliptic.

curves of genus 3

Let Y be a smooth projective curve of genus 3. The canonical map ψ is a morphism of degree 4 from Y to $\mathbb{P}^2$. If Y isn't hyperelliptic, its image will be a plane curve of degree 4 that is isomorphic to Y. The genus of a smooth projective curve of degree 4 is $\binom{3}{2}=3$ (1.8.25), which checks.

There is another way to arrive at the same result. We go through it because the method can be used for curves of genus 4 or 5. Let K be a canonical divisor. Riemann-Roch determines the dimension of the space of global sections of $\mathcal{O}(dK)$. When $d>1$,

$$\mathbf{h}^1\mathcal{O}(dK)=\mathbf{h}^0\mathcal{O}((1-d)K)=0$$

Then

(8.8.28) $$\mathbf{h}^0\mathcal{O}(dK)=deg(dK)+1-g=d(2g-2)-(g-1)=(2d-1)(g-1)$$ OdK

In our case $g=3$, so when $d>1$, $\mathbf{h}^0\mathcal{O}(dK)=4d-2$.

The number of monomials of degree d in $n+1$ variables $x_0,...,x_n$ is $\binom{n+d}{d}$. When $n=2$, that number is $\binom{d+2}{2}$.

We assemble this information into a table:

d	0	1	2	3	4	5
monos deg d	1	3	6	10	15	21
$\mathbf{h}^0\mathcal{O}(dK)$	1	3	6	10	14	18

Now let $(\alpha_0,\alpha_1,\alpha_2)$ be a basis of $H^0\mathcal{O}(K)$. The products $\alpha_{i_1}\cdots\alpha_{i_d}$ of length d of elements of the basis are global sections of $\mathcal{O}(dK)$. It is a fact that they generate the space $H^0\mathcal{O}(dK)$ of global sections. However, this isn't very important here, and the proof isn't easy. So we omit it. What we see from the table is that there is at least one nonzero homogeneous polynomial $f(x_0,...,x_2)$ of degree 4, such that $f(\alpha)=0$. This means that the curve Y lies in the zero locus of that polynomial, which is a quartic curve. The table also shows that Y isn't in the zero locus of any curve of lower degree. So Y is a quartic curve, and f is, up to scalar factor, the only homogeneous quartic that vanishes on Y. The monomials of degree 4 in α span a space of dimension 14, and therefore they span $H^0\mathcal{O}(4K)$. This is one case of the fact that was stated above.

The table also shows that there are (at least) three independent polynomials of degree 5 that vanish on Y. They are x_0f, x_1f, x_2f.

curves of genus 4

When Y is a smooth projective curve of genus 4 that isn't hyperelliptic, the canonical map embeds Y as a curve of degree 6 in $\mathbb{P}^3$. Let's leave the analysis of this case as an exercise.

curves of genus 5

With genus 5, things become more complicated.

Let Y be a smooth projective curves of genus 5 that isn't hyperelliptic. The canonical map embeds Y as a curve of degree 8 in $\mathbb{P}^4$. We make a computation analogous to what was done for genus 3. For $d>1$, the dimension of the space of global sections of $\mathcal{O}(dK)$ is

$$\mathbf{h}^0\mathcal{O}(dK)=(2d-1)(g-1)=8d-4$$

and the number of monomials of degree d in 5 variables is $\binom{d+4}{4}$.

We form a table:

d	0	1	2	3
monos deg d	1	5	15	35
$\mathbf{h}^0\mathcal{O}(dK)$	1	5	12	20

This table predicts that there are at least three independent homogeneous quadratic polynomials q_1, q_2, q_3 that vanish on the curve Y. Let Q_i be the quadric $\{q_i = 0\}$. Then Y will be contained in the zero locus $Z = Q_1 \cap Q_2 \cap Q_3$.

Bézout's Theorem has a generalization that applies here. Let Q_1, Q_2, Q_3 be hypersurfaces in $\mathbb{P}^4$, of degrees r_1, r_2, r_3, respectively. Let $Z_1, , , , , Z_k$ be the irreducible components of the zero locus $Z : \{q_1 = q_2 = q_3 = 0\}$. If Z has dimension 1, then the sum $\deg Z_1 + \cdots + \deg Z_k$ is at most equal to the product $r_1 r_2 r_3$, and is equal to that product when counted with a suitable multiplicity. We omit the proof, which is similar to the proof of the usual Bézout's Theorem.

When Q_i are the quadrics $\{q_i = 0\}$, $i = 1, 2, 3$, the intersection $Z = Q_1 \cap Q_2 \cap Q_3$ will contain Y. If Z has dimension one, the generalized Bézout's Theorem shows that its degree is 8, the same as the degree of the embedded curve Y. In this case, $Y = Z$, and Y is called a *complete intersection* of the three quadrics. However, it is possible that the intersection Z has dimension 2.

A curve Y that can be represented as a three-sheeted covering of $\mathbb{P}^1$ is called a *trigonal curve* (another peculiar term).

trigonal **8.8.29. Proposition.** *A trigonal curve of genus* 5 *is not isomorphic to an intersection of three quadrics in* $\mathbb{P}^4$.

proof. A trigonal curve Y has a morphism of degree 3 to the projective line: $Y \to X = \mathbb{P}^1$. Let's suppose that the point at infinity of X isn't a branch point. Let the fibre over the point at infinity be $\{p_1, p_2, p_3\}$. With coordinates (x_0, x_1) on X, the rational function $u = x_1/x_0$ on X has poles $D = p_1 + p_2 + p_3$ on Y, so $H^0(Y, \mathcal{O}(D))$ contains 1 and u, and therefore $\mathbf{h}^0\mathcal{O}(D) \geq 2$. By Riemann-Roch, $\chi\mathcal{O}(D) = 3 + 1 - g = -1$. Therefore $\mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(K - D) \geq 3$. There are (at least) three independent global sections of $\mathcal{O}(K)$ that vanish on D. Let them be $\alpha_0, \alpha_1, \alpha_2$. We extend this set to a basis $(\alpha_0, ..., \alpha_4)$ of $\mathcal{O}(K)$. When Y is embedded into $\mathbb{P}^4$ by that basis, the three planes $\{x_i = 0\}$, $i = 0, 1, 2$ contain the points p_1, p_2, p_3. The intersection of those planes is a line L that contains the three points.

We go back to the three quadrics Q_1, Q_2, Q_3 that contain Y. Since they contain Y, they contain D. A quadric Q intersects the line L in at most two points unless it contains L. Therefore each of the quadrics Q_i contains L, and then $Q_1 \cap Q_2 \cap Q_3$ contains L as well as Y. Suppose that $Z = Q_1 \cap Q_2 \cap Q_3$ has dimension 1. Then, according to Bézout, the sum $1 + 8$ of the degrees of L and Y, must be at most $2 \cdot 2 \cdot 2 = 8$. Nope: $Z = Q_1 \cap Q_2 \cap Q_3$ cannot have dimension 1. $\square$

In fact, this is the only exceptional case. A curve of genus 5 is either hyperelliptic, or trigonal, or else it is a complete intersection of three quadrics in $\mathbb{P}^4$. But we omit the proof. We have done enough.

8.9 Exercises

8.9.1. Let D be a divisor on a smooth projective curve Y, and suppose that $\mathbf{h}^0\mathcal{O}(D) = k > 1$. When Y is mapped to projective space using a basis for $H^0(\mathcal{O}_Y(D))$, what is the inverse image in Y of a hyperplane? chapeightex

8.9.2. **(i)** Prove that every projective curve of degree 2 is a plane conic. degtwoisconic
(ii) Classify projective curves of degree 3.

8.9.3. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules. Prove that ${}_{\mathcal{O}}(\mathcal{M},\mathcal{N}) = \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ is an $\mathcal{O}$-module. xhomcoherent

8.9.4. Let D be a divisor of degree d on a smooth projective curve Y, such that $\mathbf{h}^0\mathcal{O}(D) = k > 0$. genericnotbp
(i) Prove that if p is a generic point of Y, then $\mathbf{h}^0\mathcal{O}(D-p) = k-1$.
(ii) Prove that $\mathbf{h}^0\mathcal{O}(D) \le d+1$, and that if $\mathbf{h}^0\mathcal{O}(D) = d+1$, then X is isomorphic to $\mathbb{P}^1$.

8.9.5. Prove that every nonempty open subset of a smooth affine curve is affine. openaffine

8.9.6. Let D be a divisor of degree d on a smooth projective curve Y. Show that $\mathbf{h}^0(\mathcal{O}(D)) \le d+1$, and if $\mathbf{h}^0(\mathcal{O}(D)) = d+1$, then Y is a smooth rational curve, isomorphic to $\mathbb{P}^1$. xsmrat

8.9.7. Prove that a projective curve Y such that $\mathbf{h}^1(\mathcal{O}_Y) = 0$, smooth or not, is isomorphic to the projective line $\mathbb{P}^1$. xHonezero

8.9.8. Use version 1 of the Riemann-Roch Theorem to compute $\mathbf{h}^q(\mathcal{O}(rp))$ for a smooth projective curve of genus 1 xcohgenusone

8.9.9. Let C be a plane projective curve of degree d, with δ nodes and κ cusps. Determine the genus of the normalization $C^\#$ of C. xnodesandcusps

8.9.10. Let Y be a smooth projective curve of genus 2. Determine the possible dimensions of $H^q(Y, \mathcal{O}(D))$, when D is an effective divisor of degree n. xgenustwo

8.9.11. Let Y be a curve of genus 2, and let p be a point of Y. Suppose that $\mathbf{h}^1\mathcal{O}(2p) = 0$. Show that there is a basis of global sections of $\mathcal{O}(4p)$ of the form $(1, x, y)$, where x and y have poles of orders 3 and 4 at p. Prove that this basis defines a morphism $Y \to \mathbb{P}^2$ whose image is a singular curve Y' of degree 4. xgenustwotwo

8.9.12. The projective line $X = \mathbb{P}^1$ with coordinates x_0, x_1 is covered by the two standard affine open sets $U^0 = \operatorname{Spec} R_0$ and $U^1 = \operatorname{Spec} R_1$, $R_0 = \mathbb{C}[u]$ with $u = x_1/x_0$, and $R_1 = \mathbb{C}[v]$ with $v = x_0/x_1 = u^{-1}$. The intersection U^{01} is the spectrum of the Laurent polynomial ring $R_{01} = \mathbb{C}[u, v] = \mathbb{C}[u, u^{-1}]$. The units of R_{01} are the monomials cu^k, where k can be any integer. xproveBG

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible R_{01}-matrix. Prove that there is an invertible R_0-matrix Q, and there is an invertible R_1-matrix P, such that $Q^{-1}AP$ is diagonal.
(ii) Use part **(i)** to prove the Birkhoff-Grothendieck Theorem for torsion-free $\mathcal{O}_X$-modules of rank 2.

8.9.13. On $\mathbb{P}^1$, when is $\mathcal{O}(m) \oplus \mathcal{O}(n)$ isomorphic to $\mathcal{O}(r) \oplus \mathcal{O}(s)$? xOmplusOn

8.9.14. Let Y be an elliptic curve. xellgrplaw
(i) Prove that, with the law of composition $\oplus$ defined in (**8.8.15**), Y is an abelian group.
(ii) Let p be a point of Y. Describe the sum $p \oplus p \cdots \oplus p$ of k copies of p.
(iii) Determine the number of points of order 2 on Y.
(iv) Suppose that Y is a plane curve. Show that, if origin is a flex point, the other the flexes of Y are the points of order 3, and determine the number of points of Y of order 3.

8.9.15. How many real flex points can a real cubic curve have? xrealflex

8.9.16. Prove that a finite $\mathcal{O}$-module on a smooth curve is a direct sum of a torsion module and a locally free module. xmoddirsum

8.9.17. Let A be a finite-type domain. xdescomega
(i) Let $B = A[x]$ be the ring of polynomials in one variable with coefficients in A. Deascribe the module Ω_B in terms of Ω_A.
(ii) Let s be a nonzero element of A and let A' be the localization $A[x]/(sx-1)$. Describe the module $\Omega_{A'}$.

xgenomega **8.9.18.** Let $Y = \operatorname{Spec} B$ a smooth affine curve, and let y be an element of B. At what points does dy generate Ω_Y locally?

xtraced-eriv **8.9.19.** Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth affine curves, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $B \xrightarrow{\delta} {}_A(B, \Omega_A)$ be the composition of the derivation $B \xrightarrow{d} \Omega_B$ with the trace map $\Omega_B \approx_B (B, \Omega_B) \xrightarrow{\tau\otimes}_A (B, \Omega_A)$. Prove that δ is a derivation from B to the B-module ${}_A(B, \Omega_A)$.

xprove-trace **8.9.20.** Let $Y \to X$ be a branched covering, and let p be a point of X whose inverse image in Y consists of one point q. Prove a local analytic version of the main theorem on the trace map for differentials by computation.

xdegfive **8.9.21.** **(i)** Let Y be plane curve of degree 5 with a node. Show that the projection of the plane to $X = \mathbb{P}^1$ with the double point as center of projection represents Y as a trigonal curve of genus 5.

(ii) The canonical embedding of a trigonal genus 5 curve Y will have three colinear points $D = p_1 + p_2 + p_3$, Show that $\mathbf{h}^0\mathcal{O}(K-D) = 3$ and that $\mathcal{O}(K-D)$ has no base point. Show that a basis of $H^0\mathcal{O}(K-D)$ maps Y to a curve of degree 5 in $\mathbb{P}^2$ with a double point.

bptfre **8.9.22.** *the basepoint-free trick.* Let D be an effective divisor on a smooth projective curve Y, and suppose that $\mathcal{O}(D)$ has no base point, and that $\mathbf{h}^1\mathcal{O}(D) = 0$. Choose global sections α, β of $\mathcal{O}(D)$ with no common zeros. Prove the following:

(i) The sections α, β generate the $\mathcal{O}$-module $\mathcal{O}(D)$, and there is an exact sequence

$$0 \to \mathcal{O}(-D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}(D) \to 0$$

.

(ii) The tensor product of this sequence with $\mathcal{O}(kD)$ is an exact sequence

$$0 \to \mathcal{O}((k-1)D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}(kD)^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}((k+1)D) \to 0$$

(iii) If $H^1\mathcal{O}((k-1)D) = 0$, every global section of $\mathcal{O}((k+1)D)$ can be obtained as a combination $\alpha u + \beta v$ with $u, v \in H^0\mathcal{O}(kD)$

xponcelet **8.9.23.** ???kill this??? Let C and D be conics that meet in four distinct points in the projective plane $\mathbb{P}$, and let D^* be the dual conic of tangent lines to D. Let E be the locus of points (p, ℓ^*) in $\mathbb{P}\times\mathbb{P}^*$ such that $\ell^* \in D^*$ and $p \in \ell$.

(i) Prove that E is a smooth elliptic curve.

(ii) Show that, for most $p \in C$, there will be two tangent lines ℓ to D such that (p, ℓ^*) is in E, and that, for most $\ell^* \in D^*$, there will be two points p such that (p, ℓ^*) is in E. Identify the exceptional points.

(iii) If (p_1, ℓ_1) is given, let p_2 denote the second intersection of C with ℓ_1, and let ℓ_2 denote the second tangent to D that contains p_2. Define a map, where possible, by sending $(p_1, \ell_1^*) \to (p_2, \ell_2^*)$. Show that this map extends to a morphism $E \xrightarrow{\gamma} E$ on E, and that this morphism is a translation $p \to p \oplus a$, for some point a of E.

(iv) It might happen that for some point p of C and some n, $\gamma^n(p) = p$. Show that if this occurs, the same is true for every point of C. For example, if $\gamma^3(p) = p$, the lines ℓ_1, ℓ_2, ℓ_3 will form a triangle whose vertices are on C, and this will be true for all points p of C. This is *Poncelet's Theorem.*

xOmn **8.9.24.** Let x_0, x_1 and y_0, y_1 be the coordinates in the two factors of the product $X = \mathbb{P}^1 \times \mathbb{P}^1$.A homogeneous fraction of bidegree m, n on X is a fraction g/h of bihomogeneous polynomials in x, y such that, if the bidegree of g is i, j an the bidegree of h is k, ℓ, then $m = i - k$ and $n = j - \ell$. Rational functions on X can be represented as bihomogeneous fractions of bidegree $0, 0$. A curve C in X of bidegree m, n is the zero locus of a bihomogeneous polynomial $f(x, y)$ of bidegree m, n.

Let $\mathcal{O}_X(m, n)$ denote the $\mathcal{O}$-module whose sections on an open subset W of X are the rational functions f on X such that $f x_0^m y_0^n$ is a regular function on W. We say that such a function f has poles of orders $\le m$ on V and $\le n$ on H, where H is the 'horizontal' line $y_0 = 0$, and V is the 'vertical' line $x_0 = 0$.

(i) Determine the cohomology of $\mathcal{O}_X(m, n)$.

(ii) Determine the genus of a smooth curve of bidegree m, n.

8.9.25. Let Y be a smooth projective curve Y of genus g, and let d be an integer. Prove that xgminu-
(i) If $d < g-1$, then $\mathbf{h}^1\mathcal{O}(D) > 0$ for every divisor D of degree d on Y. sone
(ii) If $d \le 2g-2$, there exist divisors D of degree d on Y such that $\mathbf{h}^1\mathcal{O}(D) > 0$.
(iii) If $d \ge g-1$, there exist divisors D of degree d on Y such that $\mathbf{h}^1\mathcal{O}(D) = 0$.
(iv) If $d > 2g-2$, then $\mathbf{h}^1\mathcal{O}(D) = 0$ for every divisor D of degree d on Y.

8.9.26. Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves. Use the trace from $\mathcal{O}_Y$ to $\mathcal{O}_X$ to prove that its direct image is isomorphic to the direct sum $\mathcal{O}_X \oplus \mathcal{M}$ for some locally free $\mathcal{O}_X$-module $\mathcal{M}$. xOYdi-
rectsum

8.9.27. Let Y be a smooth projective curve of genus $g > 1$, and let D be an effective divisor of degree $g+1$ on Y, such that $\mathbf{h}^1\mathcal{O}(D) = 0$ and $\mathbf{h}^0\mathcal{O}(D) = 2$. Let $Y \xrightarrow{\pi} X$ be the morphism to the projective line X defined by a basis $(1, f)$ of $H^0\mathcal{O}(D)$. The $\mathcal{O}_X$-module $\mathcal{O}_Y$ is isomorphic to a direct sum $\mathcal{O}_X \oplus \mathcal{M}$, where $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of rank g (Exercise **??**). decom-
poseOY
(i) Let p be the point at infinity of X. Prove that $\mathcal{O}_Y(D)$ is isomorphic to $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(p)$.
(ii) Determine the dimensions of cohomology of $\mathcal{M}$ and of $\mathcal{M}(p)$.
(iii) According to the Birkhoff-Grothendieck Theorem, $\mathcal{M}$ is isomorphic to a sum of twisting modules $\sum_{i=1}^{g} \mathcal{O}_X(r_i)$. Determine the twists r_i.

8.9.28. **(a)** Let C be a plane curve of degree d with a node as its only singularity. Determine the genus of its normalization $C^{\#}$. Do the same for a curve with a cusp. xs-
inggenus
(b) Let $f(x_0, x_1, x_2)$ be a homogeneous polynomial of degree d. Suppose that, when $f(x, y, 1)$ is written as a sum of its homogeneous parts $f_0 + f_1 + f_2 + \cdots$, $f_0 = f_1 = f_2 = 0$, and that f_3 has three distinct zeros, so that the plane curve $C : \{f = 0\}$ has an ordinary triple point at $p = (0, 0, 1)$, and suppose that there are no other singularities. Determine the genus of the normalization $C^{\#}$.
(c) A point p of curve C in $\mathbb{P}^3$ may be a triple point, in which three smooth points p_1, p_3, p_3 of the normalization $C^{\#}$ lie over p. ####etcetc#####

8.9.29. Use the results of Exercise 8.9.28 as an aid to factor the polynomial $x^3y^2 - x^3z^2 + y^3z^2$. xpolirred

2. *## There is an error in the statement of this problem. See web page.##*

Let f and g be homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2, x_3]$, of degrees d and e respectively, and with no common factor. Let X be the locus of common zeros of f and g in the projective space $\mathbb{P}^3$ with coordinates x, and let i be the inclusion $X \to \mathbb{P}$.

(a) Construct an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-d-e) \to \mathcal{O}_{\mathbb{P}}(-d) \oplus \mathcal{O}_{\mathbb{P}}(-e) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_X \to 0$$

(b) Prove that X is connected, i.e., that it is not the union of two proper disjoint Zariski-closed subsets of $\mathbb{P}$.

(c) Determine the cohomology of $\mathcal{O}_X$.

1. Let

$$N = \begin{matrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{matrix}$$

be a 3×2 matrix whose entries are homogeneous polynomials of degree d in $R = \mathbb{C}[x_0, x_1, x_2]$, and let $M = (m_1, m_2, m_3)$ be the 1×3 matrix of minors

$$m_1 = y_{21}y_{32} - y_{22}y_{31}, \quad m_2 = -y_{11}y_{32} + y_{12}y_{31}, \quad m_3 = y_{11}y_{22} - y_{12}y_{21}.$$

Let I be the ideal of R generated by the minors m_1, m_2, m_3.

(a) Prove that if I is the unit ideal of R, the sequence

$$0@<<< R@<M<< R^3@<N<< R^2@<<< 0$$

is exact.

(We've written the arrows from right to left here so that matrix multiplication by M on R^3 and N on R^2 are defined, when elements of R^3 and R^2 are represented as column vectors.)

(b) Let $X = \mathbb{P}^2$, and suppose that the locus Y of zeros of I in X has dimension zero. Prove that the sequence

$$0@<<< R/I@<<< R@<M<< R^3@<N<< R^2@<<< 0$$

is exact.

(c) The sequence in **(b)** corresponds to the following sequence, in which the terms R are replaced by twisting modules:

$$0@<<< \mathcal{O}_Y@<<< \mathcal{O}_X@<M<< \mathcal{O}_X(-2d)^3@<N<< \mathcal{O}_X(-3d)^2@<<< 0$$

Use this sequence to determine $h^0(Y, \mathcal{O}_Y)$. Check your work in some example in which y_{ij} are homogeneous linear polynomials.

2. There are 10 monomials of degree 3 in x_0, x_1, x_2, so the homogeneous polynomials of degree 3 form a vector space of dimension 10. Let Z be the corresponding projective space of dimension 9, whose points are classes of nonzero homogeneous cubic polynomials up to scalar factor. Prove that the subset of Z of classes of reducible polynomials is (Zariski) closed.

2. With coordinates x_0, x_1, x_2 in the plane $\mathbb{P}$ and s_0, s_1, s_2 in the dual plane $\mathbb{P}^*$, let C be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where f is an irreducible homogeneous polynomial in x. Let Γ be the locus of pairs (x, s) of $\mathbb{P} \times \mathbb{P}^*$ such that $x \in C$ and the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$ is the tangent line to C at x. Prove that Γ is a (Zariski) closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

GLOSSARY

algebra: a ring that contains the complex numbers **??**).

analytic function: A function that can be represented by a convergent power series (1.4.18).

annihilator: The annihilator of an element m of an R-module is the ideal of elements a of R such that $am = 0$ (6.7).

arithmetic genus: The arithmetic genus of a smooth projective curve is $p_a = 1 - \mathbf{C}^1\mathcal{O}$ (7.6.2).

basis for a topology: A basis $\mathcal{B}$ for a topology is a set of open subsets such that every open subset if a union of members of $\mathcal{B}$ (2.7.2).

bitangent: a line that is tangent to a curve at two points (**1.8.17**).

branch point: a point at which the ramification index of a branched covering is greater than 1. (8.5).

branched covering: a finite morphism of curves (1.8.15), (8.5).

canonical: A mathematical construction is called canonical if it is the natural one in the context.

canonical divisor: a divisor K on a smooth projective curve X such that $\mathcal{O}(K)$ is isomorphic to Ω_X (**8.8.7**).

canonical map: the map fom a curve to projective space defined by the regular differentials (8.8.26).

classical topology: the usual topology (**1.3.17**).

closure: The closure of a subset S of a topological space is the smallest closed subset that contains S.

coarser topology: A topology T' on a set X is coarser than another topology T if T' contains fewer closed subsets than T.

cohomological functor: A sequence of functors $H^0, H^1, H^2, ...$ to vector spaces such that a short exact sequence produces a long cohomology sequence (7.1.4).

cokernel: The cokernel of a homomorphism $M \to N$ is the quotient $N/\operatorname{im} M$ (**2.1.17**).

commutative diagram: A diagram of maps is commutative if all maps from A to B that can be obtaind by composition of the ones in the diagram are equal (**2.1.5**).

complement: The complement of a subset S of a set X is the set of elements of X that are not in S.

complex: A complex of vector spaces is a sequence $\cdots \to V^{n-1} \xrightarrow{d^n} V^n \to \cdots$ of vector spaces such that $\ker\, d^n \subset \operatorname{im}\, d^{n-1}$ (7.2).

constructible set: a finite union of locally closed sets (5.3).

cusp: a certain type of singular point of a curve (1.8).

dimension: The dimension of a variety is the transcendence degree of its field of rational functions, or the length of a maximal chain of closed subvarieties (4.5).

discriminant: a polynomial in the coefficients of a polynomial f that vanishes if and only if f has a double root (**1.7.13**).

divisor: A divisor on a smooth curve is an integer combination $a_1q_1 + \cdots + a_kq_k$ of points (8.1).

domain: A nonzero ring with no zero divisors.

dual curve: The dual curve of a smooth plane curve is the locus of tangent lines as smooth points (**1.6.3**).

dual plane: the projective plane whose points correspond to lines in the given plane (**1.6.1**).

elliptic curve: a smooth projective curve of genus 1 (**8.8.12**).

Euler characteristic: The Euler characteristic of an $\mathcal{O}$-module is the alternating sum of the dimensions of its cohomology (**7.7.7**).

exact sequence: a sequence $\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \to \cdots$ is exact if $\ker\, d_n = \operatorname{im}\, d^{n-1}$ (**2.1.17**).

exterior algebra: the graded algebra generated by the elements of a vector space, with the relations $vv = 0$ (**3.7.2**).

Fermat Curve: one of the plane curves $x_o^k + x_1^k + x_2^k = 0$.

fibre of a map: The fibre of a map $Y \xrightarrow{\pi} X$ over a point x is the number of points in its inverse image.

finer topology: A topology T' on a set X is finer than another topology T if T' contains more closed subsets than T.

finite module: a module that can be generated by finitiely many of its elements (2.1).

finite-type algebra: an algebra that can be generated, as algbera, by finitely many elements (2.1).

generic, general position: not in a special "bad" position (**1.8.17**).

genus: the genus of a compact two-dimensional manifold is the number of its handles (1.8.22).

Grassmanian: a variety that parametrizes subspaces of a given dimension of a vector space (3.7).

Hessian matrix: the matrix of second partial derivatives (1.4.10).

homogeneous parts: the homogeneous part of degree k of a polynomial is the sum of terms of degree k (1.3.1).

hyperelliptic curve: a curve of genus at least two, that can be represented as a double cover of $\mathbb{P}^1$ (**8.8.22**).

hypersurface: a subvariety of projective space that is defined by one equation (2.3.3).

increasing sequence: a sequence S_n of sets is increasing if $S_n \subset S_{n+1}$ for all n, and it is strictly increasing if if $S_n < S_{n+1}$ for all n (**2.1.12**).

integral morphism: a morphism $Y \to X$ such that $\mathcal{O}_Y$ becomes a finite $\mathcal{O}_X$-module (4.2.4).

invertible module: a locally free module of rank 1 (**8.1.16**).

irreducible polynomial: a polynomial of positive degree that isn't the product of two polynomials of positive degree.

irreducible space: a topological space that isn't the union of two proper closed subsets (2.2.11).

isolated point: a point p of topological space such that both p and its complement are closed (**1.3.18**).

line at infinity: The line at infinity in the projective plane $\mathbb{P}^2$ is the locus $\{x_0 = 0\}$ (**1.2.7**).

local property: a property that is true in an open neighborhood of any point (5.1.3).

localization: the process of adjoining inverses (**2.1.23**).

locally closed set: the intersection of a closed set and an open set (5.3).

member: When a set is made up of subsets of another set, we call an element of that set a member to avoid confusion (**2.1.12**).

module homomorphism: a homomorphism from an R-module M to an R'-module M' is defined in Section (6.2).

morphism: one of the allowed maps between varieties (2.6),(3.5).

nilradical: the radical of the zero ideal (**2.5.14**).

node: a point at which two branches of a curve met transverslly (1.8).

noetherian space: a space that satisfies the descending chain condition on closed sets (**2.2.8**).

normal domain: an integrally closed domain (4.3).

Nullstellensatz: the theorem that identifies points with maximal ideals (2.4).

ordinary: a plane curve is ordinary if all flexes and bitangents are ordinary, and there are no accidents (1.10.11).

Plücker formulas: the formulas that count flexes, bitangents, nodes and cusps of an ordinary curve (1.11).

quadric: the locus of an irreducible homogeneous quadratic equation in projective space (**3.1.6**).

quasicompact: A topological space is quasicompact if every open covering has a finite subcovering.

radical of an ideal: the radical of an ideal I is the set of elements such that some power is in I (**2.2.20**).

reducible curve: a union of finitely many irreducible curves.

resultant: a polynomial in the coefficients of two polynomials f and g that vanishes if and only if they have a common root (1.7).

scalar: a complex number.

scaling: adjusting by scalar factors.

Segre embedding. a map that embeds a product of projective varieties into projective space (**3.1.9**).

Serre dual. The Serre dual $\mathcal{M}^S$ of a locally free $\mathcal{O}$-module $\mathcal{M}$ is the module $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \Omega)$ (**8.7.1**).

smooth point, singular point: a point p of a plane curve $\{f = 0\}$ is a smooth point if at least one partial derivative is nonzero at p. Otherwise it is a singular point (**1.4.4**).

Special line: A line L through a singular point of a curve whose intersection multiplicity with C is greater than the multiplicity of C. (1.8.4).

spectrum: the spectrum of a finite-tpe domain is the set of its maximal ideals (2.5).

structure sheaf: its sections on an open set are the regular functions on that set (6.1).

tensor algebra: the graded algebra T such that T^n is the nth tensor power of a vector space V (3.7.21).

torsion: an element m of am R-module is a torsion element if there is a nonzero element r in R such that $rm = 0$ (**2.1.24**).

transcendence basis: a maximal algebraically independent set of elements (1.5).

transcendence degree: the number of elements in a transcendnce basis (1.5).

transversal intersection: two curves intersect transvesally at a point p if they are smooth at p and their tangent lines there are distinct (1.9.11).

trigonal curve: a curve that can be represented as a overing of $\mathbb{P}^1$ of degree 3 (8.8.29).

twisted cubic: the locus of points $(x_0^3, x_0^2x_1, x_1^2x_2, x_1^2)$ in $\mathbb{P}^3$ (**3.1.15**).

unit ideal: the unit ideal of a ring R is R.

valuation: a surjective homomorphism from the multiplicative group $K^\times$ of nonzero elements of a field K to the additive group $\mathbb{Z}^+$ of integers (**5.1.4**).

valuation ring: the set of elements with value greater than zero, together with the identity element.

variety: an irreducible subspace of affine or projective space (2.2.17),(3.0.1),(**3.2.11**).

Veronese embedding: the embeding of a projective space using the monomials of given degree (**3.1.13**).

weight: a variable may be assigned an integer called a *weight* (4.6.20).

weighted projective space: projective space when the variables have weights (4.6.20).

Zariski topology: the closed sets are the zero sets of families of polynomial equations (2.2).

INDEX OF NOTATION

$\mathbb{A}^n$ affine space (1.1).

(affines) the category of affine open sets, morphism being localizations (6.1).

ann annihilator (6.7).

C^* the dual of the curve C (1.6.6).

$\mathbf{C}^q$ the cohomology of a complex (7.2).

$\mathrm{Discr}(F)$ the discriminant of f (1.7.14).

Δ the diagonal, or the branch locus (**3.5.19**), (**4.6.14**).

e the Euler characteristic, or the ramification index (1.8.21),(**7.7.7**),(8.5).

g often, the genus of a curve. (**1.8.19**)

H ; the Hessian matrix of second partial derivatives. (1.4.10)

H_p the evaluation of H at the point p.

H^q cohomology (7.1).

$\mathbf{h}^q$ the dimension of H^q.

$K^\times$ the multiplicative group of nonzero elements of the field K.

$k(p)$ the residue field at a point (2.3.1),(2.5).

L^* the point of $\mathbb{P}^*$ that corresponds to the line L in the plane $\mathbb{P}$ (**1.6.1**).

$\mathcal{M}$ an $\mathcal{O}$-module (6.2.1).

$\mathfrak{m}$ a maximal ideal (2.3.1),(2.5).

$\mathcal{O}$ the structure sheaf on a variety (6.1.1).

${}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$, ${}_X(\mathcal{M},\mathcal{N})$ abbreviated notations for the $\mathcal{O}_X$-module of homomorphisms $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$.

(opens) the category whose objects are open subsets (6.1).

p^* the line in the dual plane $\mathbb{P}^*$ that corresponds to the point p of the plane $\mathbb{P}$ (**1.6.1**).

p_a the arithmetic genus (8.7.4),(8.8.2).

$\mathbb{P}$, $\mathbb{P}^n$ projective space (1.2).

$\mathbb{P}^*$ the dual of the plane $\mathbb{P}$ (**1.6.1**).

π_p the homomorphism to the residue field $k(p)$ (2.3.1),(2.5).

$\mathrm{rad}\, I$ the radical of the ideal I (**2.2.20**).

$\mathrm{Res}(f,g)$ the resultant of f and g (1.7).

^{S} $\mathcal{M}^S$ is the Serre dual of $\mathcal{M}$ (**8.7.1**).

$\mathrm{Spec}\, A$ the set of maximal ideals of a finite-type algebra A (2.5).

$\mathbb{U}^i$ the standard affine open subset $\{x_i \neq 0\}$ of projective space (**1.2.7**).

$V(f)$ the locus of zeros of f (2.2),(3.2.4).

$\bigwedge V$ the exterior algebra (**3.7.2**).

∇ the gradient vector of partial derivatives (1.4.10.

∇_p the evaluation of ∇ at the point p.

$\approx$ an isomorphism.

$\otimes$ tensor product (**2.1.25**).

$\bigcap$ intersection.

$<$ $S < T$ means that the set S is a subset of T and is not equal to T (**2.1.12**).

$A^\#$ denotes the normalization of the algebra A.

[ ] square brackets are sometimes used in place of parentheses for clarity.

BIBLIOGRAPHY

Two classic works:

Fulton, William, *Algebraic Curves*, 2008, available on the web www.math.lsa.umich.edu/ wfulton/CurveBook.pdf

Mumford, David *Algebraic Geometry I, Complex Projectie Varieties* Grundlehren, Springer, 1976.

An introduction based on many specific examples:

Harris, J. *Algebraic Geometry, A First Course* Grad. Texts in Math. Springer, 1992.

Books for a more formal introduction:

Cutkosky, D. *Introduction to Algebraic Geometry* Grad. Studies in Math., AMS, 2018.

Hartshorne, Robin *Algebraic Geometry* Grad. Texts in Math. Springer, 1977.

Books with an analytic point of view:

Griffiths, P., Harris, J *Principles of Algebraic Geometry* Wiley, 2014.

Miranda, R. *Algebraic Curves and Riemann Surfaces* Grad. Studies in Math., AMS, 1979.

A computational approach to algebraic geometry:

Cox, D., Little, J., O'Shea, D. *Using Algebraic Geometry* 2nd ed Grad. Texts in Math., Springer, 2004.

Books for the background on algebra:

Cohn, P. M. *Algebra* vol, 2 Wiley,, 1977.

Lang, Serge *Algebra* 3rd ed Grad. Texts in Math., Springer, 2002.

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