Massachusetts Institute of Technology

18.721

NOTES FOR A COURSE IN
ALGEBRAIC GEOMETRY

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January 26, 2022

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PREFACE

These are notes that have been used for an algebraic geometry course at MIT. I had thought about teaching such a course for quite a while, motivated partly by the fact that MIT didn't have very many courses suitable for students who had taken the standard theoretical math classes. I got around to thinking about this seriously twelve years ago, and have now taught the subject seven times. I wanted to get to cohomology of $\mathcal{O}$-modules (aka quasicoherent sheaves) in one semester, without presupposing a knowledge of sheaf theory or of much commutative algebra, so it has been a challenge. Fortunately, MIT has many outstanding students who are interested in mathematics. The students and I have made some progress, but much remains to be done. Ideally, one would like the development to be so natural as to seem obvious. Though I haven't tried to put in anything unusual, this has yet to be achieved. And there are too many pages for my taste. To paraphrase Pascal, we haven't had the time to make it shorter.

To cut the material down, I decided to work exclusively with varieties over the complex numbers, and to use that restriction freely. Schemes are not discussed. Some people may disagree with these decisions, but I feel that absorbing schemes and general ground fields won't be too difficult for someone who is familiar with complex varieties. Also, I don't go out of my way to state and prove things in their most general form.

If one plans to teach such a course in a single semester, it is essential to keep moving. One can't linger over the topics in the first Chapter. To save time, one can replace some proofs with heuristic reasoning, or omit them. Proposition 1.8.12 on the order of vanishing of the discriminant is a candidate for some hand-waving, and Lemma 1.9.7 on flex points may be a proof to skip.

Indices can cloud the picture. When that happens, I recommend focussing attention on a low dimensional case. Schelter's neat proof of Chevalley's Finitness Theorem is an example. Schelter discovered the proof while studying $\mathbb{P}^1$. That case demonstrates the main point, and is a bit easier to follow.

In Chapter 6 on $\mathcal{O}$-modules, all technical points about sheaves are eliminated when one sticks to affine open sets and localizations. Sections over other open sets are important, mainly because one wants the global sections, but the proof that a module extends to arbitrary open sets can safely be put on a back burner, as is done in these notes.

In Chapter 7, I decided to restrict to $\mathcal{O}$-modules when defining cohomology, and to characterize the cohomology axiomatically. This was in order to minimize technical points. Simplicial operations are eliminated, though they appear in disguise in the resolution (7.4.13).

The special topics at the ends of Chapters 2,3,4 enrich the subject. I don't recommend skipping them. And, without some of the applications at the end of Chapter 8, the Riemann Roch Theorem would be pointless.

When I last taught the subject in the spring of 2020, MIT semester had 39 class hours. I followed this schedule: Chapter 1, 6 hours, Chapters 2-7, roughly 4 hours each, Chapter 8, 7 hours, in-class quizzes, 2 hours. This was a brisk pace. The topics in the notes could be covered comfortably in a one-year course, and there might be time for some extra material.

Great thanks are due to the students who have been in my classes. Many of you have contributed to these notes by commenting on the drafts or by creating figures. Though I remember you well, I'm not naming you individually because I'm sure I'd overlook someone important. I hope that you will understand. And I want to thank Edgar Costa, Sam Schiavone, and Raymond van Bommel, who taught a class using the notes, and who made many helpful comments.

A Note for the Student

The prerequisites are standard undergraduate courses in algebra, analysis, and topology, and the definitions of category and functor. We will also use the implicit function theorem for complex variables. But don't worry too much about the prerequisites. There is a review of some points in Chapter 9. You can refer to it as needed, or look on the web. And, many points are reviewed briefly in the notes as they come up.

Proofs of some lemmas and propositions are omitted. I have omitted a proof when I consider it simple enough that including it would just clutter up the text or, occasionally, when I feel that it is important for the reader to supply a proof.

As with all mathematics, working exercises and, most importantly, writing up the solutions carefully is, by far, the best way to learn the material well.

Chapter 1 PLANE CURVES

planecurves

Plane curves were the first algebraic varieties to be studied, and they provide instructive examples, so we begin with them. Chapters 2 – 7 are about varieties of arbitrary dimension. We will see in Chapter 5 how curves scontrol higher-dimensional varieties, and we come back to curves in Chapter 8.

1.1 The Affine Plane

affine-plane

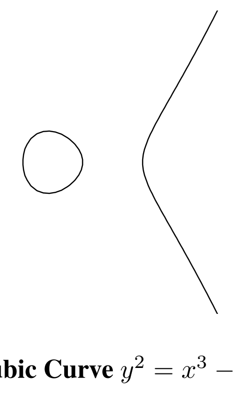
The n-dimensional *affine space* $\mathbb{A}^n$ is the space of n-tuples of complex numbers. The affine plane is the two-dimensional affine space.

Let $f(x_1, x_2)$ be an irreducible polynomial in two variables, with complex coefficients. The set of points of the affine plane at which f vanishes, the *locus of zeros* of f, is a *plane affine curve*. Let's denote that locus by X. Writing x for the vector (x_1, x_2),

affcurve (1.1.1)
$$X = \{x \mid f(x) = 0\}$$

The *degree* of the curve X is the degree of its defining polynomial f.

When it seems unlikely to cause confusion, we may abbreviate the notation for an indexed set, using a single letter, as here, where x stands for $(x_!, x_2)$.



goober8 **1.1.2.** **The Cubic Curve** $y^2 = x^3 - x$ (real locus)

About Figures. In algebraic geometry, the dimensions are too big to allow realistic figures. Even with an affine plane curve, one is dealing with a locus in the affine plane $\mathbb{A}^2$, which has dimension 4 as a real vector space. In some cases, the real locus can be helpful, but in most cases, even the real locus is too big, and one must make do without a figure, or with a schematic diagram. $\square$

We will get an understanding of the geometry of a plane curve as we go along, and we mention just one point here. A plane curve is called a *curve* because it is defined by one equation in two variables. Its *algebraic* dimension is 1. The only proper subsets of a curve X that can be defined by polynomial equations are the finite sets (see Proposition 1.3.12). But the affine plane $\mathbb{A}^2$ is a real space of dimension 4, and X will be a surface in that space.

One can see that a plane curve X has dimension 2, geometrically, by inspecting its projection to an affine line. Let $X \xrightarrow{\pi} \mathbb{A}^1$ be the projection to the x_1-line. We write the defining polynomial as a polynomial in x_2:

$$f(x_1, x_2) = c_0 x_2^d + c_1 x_2^{d-1} + \cdots + c_d$$

whose coefficients c_i are polynomials in x_1, and we suppose that d is positive, i.e., that f isn't a polynomial in x_1 alone.

The *fibre* of a map $V \to U$ over a point p of U is the inverse image of p, the set of points of V that map to p. The fibre of the projection π over the point $x_1 = a$ is the set of points (a, b), in which b is a root of the one-variable polynomial

$$f(a, x_2) = \overline{c}_0 x_2^d + \overline{c}_1 x_2^{d-1} + \cdots + \overline{c}_d$$

with $\overline{c}_i = c_i(a)$. There will be finitely many points in this fibre, and it won't be empty unless $f(a, x_2)$ is a constant. The plane curve X covers most of the x_1-line, a complex plane, finitely often.

1.1.3. Note. Plane curves are the zero loci of *irreducible* polynomials $f(x_1, x_2)$. In contrast with complex polynomials in one variable, most polynomials in two or more variables are irreducible — they cannot be factored. This can be shown by a method called *counting constants*. For instance, quadratic polynomials in x_1, x_2 depend on the six coefficients of the monomials $1, x_1, x_2, x_1^2, x_1x_2, x_2^2$ of degree at most two. Linear polynomials $ax_1 + bx_2 + c$ depend on three coefficients, but the product of two linear polynomials depends on only five parameters, because a scalar factor can be moved from one of the linear factors to the other. So the quadratic polynomials cannot all be written as products of linear polynomials. This reasoning is fairly convincing. It can be justified formally in terms of *dimension*, which will be discussed in Chapter 5. $\square$ polyirred

1.1.4. changing coordinates planecoords

We allow linear changes of variable and translations in the affine plane $\mathbb{A}^2$. When a point x is written as the column vector $(x_1, x_2)^t$, the coordinates $x' = (x_1', x_2')^t$ after such a change will be related to x by a formula

(1.1.5) $$x = Qx' + a$$ chgcoord

where Q is an invertible 2×2 matrix with complex coefficients and $a = (a_1, a_2)^t$ is a complex translation vector. This changes a polynomial equation $f(x) = 0$, to $f(Qx' + a) = 0$. One may also multiply a polynomial f by a nonzero complex scalar without changing its locus of zeros. Using these operations, all *lines*, plane curves of degree 1, become equivalent.

An *affine conic* is a plane affine curve of degree 2. Every affine conic is equivalent to one of the two loci

(1.1.6) $$x_1^2 - x_2^2 = 1 \quad \text{or} \quad x_2 = x_1^2$$ affconic

by a suitable linear change of variable, translation, and scaling. The proof of this is similar to the one used to classify real conics. These loci might be called a complex 'hyperbola' and 'parabola', respectively. The complex 'ellipse' $x_1^2 + x_2^2 = 1$ becomes the 'hyperbola' when one multiplies the coordinate x_2 by i.

On the other hand, there are infinitely many inequivalent cubic curves. Cubic polynomials in two variables depend on the coefficients of the ten monomials $1,\ x_1, x_2,\ x_1^2, x_1x_2, x_2^2,\ x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$ of degree at most 3 in x_1, x_2. Linear changes of variable, translations, and scalar multiplication, give us only seven scalars to work with, leaving three essential parameters.

1.2 The Projective Plane

projplane

Let n be a positive integer. The n-dimensional *projective space* $\mathbb{P}^n$ is the set of equivalence classes of *nonzero* vectors $x = (x_0, x_1, ..., x_n)$, the equivalence relation being

equivrel

$$(1.2.1) \qquad (x'_0, ..., x'_n) \sim (x_0, ..., x_n) \quad \text{if} \quad (x'_0, ..., x'_n) = (\lambda x_0, ..., \lambda x_n) \qquad \big(\text{if} \quad x' = \lambda x\big)$$

for some nonzero complex number λ. The equivalence classes are the *points* of $\mathbb{P}^n$. One often refers to a point by giving a particular vector in its class.

When x is a nonzero vector, the one-dimensional subspace of $\mathbb{C}^{n+1}$ spanned by x consists of the vectors λx, together with the zero vector. So points of $\mathbb{P}^n$ correspond to one-dimensional subspaces of the complex vector space $\mathbb{C}^{n+1}$.

projline

1.2.2. the projective line

Points of the *projective line* $\mathbb{P}^1$ are equivalence classes of nonzero vectors $x = (x_0, x_1)$.

If the first coordinate x_0 of a vector $x = (x_0, x_1)$ isn't zero, we may multiply by $\lambda = x_0^{-1}$ to normalize the first entry to 1, and write the point that x represents in a unique way as $(1, u_1)$, with $u_1 = x_1/x_0$. There is one remaining point, the one represented by the vector $(0, 1)$. The projective line $\mathbb{P}^1$ can be obtained by adding that point, called the *point at infinity*, to the affine u_1-line, a complex plane. As u_1 tends to infinity in any direction, the point $(1, u_1)$ approaches the point $(0, 1)$. Topologically, $\mathbb{P}^1$ is a two-dimensional sphere.

projpl

1.2.3. lines in projective space

Let p and q be vectors that represent distinct points of the projective space $\mathbb{P}^n$. There is a unique *line* L in $\mathbb{P}^n$ that contains those points, the set of points $L = \{rp + sq\}$, with r, s in $\mathbb{C}$ not both zero. Points of the line L correspond bijectively to points of the projective line $\mathbb{P}^1$, by

pline

$$(1.2.4) \qquad rp + sq \quad \longleftrightarrow \quad (r, s)$$

The *projective plane* $\mathbb{P}^2$ is the two-dimensional projective space. A line in the projective plane $\mathbb{P}^2$ can also be described as the locus of solutions of a homogeneous linear equation

eqline

$$(1.2.5) \qquad s_0 x_0 + s_1 x_1 + s_2 x_2 = 0$$

linesmeet

1.2.6. Lemma. *In the projective plane, two distinct lines have exactly one point in common, and in a projective space of any dimension, a pair of distinct points is contained in exactly one line.* $\square$

standcov

1.2.7. the standard covering of the projective plane

Points of the projective plane are equivalence classes of nonzero vectors (x_0, x_1, x_2). If the first entry x_0 of a point $p = (x_0, x_1, x_2)$ isn't zero, we may normalize it to 1 without changing the point: $(x_0, x_1, x_2) \sim (1, u_1, u_2)$, where $u_i = x_i/x_0$. We did the analogous thing for $\mathbb{P}^1$ above. The representative vector $(1, u_1, u_2)$ is uniquely determined by p, so points with $x_0 \neq 0$ correspond bijectively to points of an affine plane $\mathbb{A}^2$ with coordinates (u_1, u_2):

$$(x_0, x_1, x_2) \sim (1, u_1, u_2) \quad \longleftrightarrow \quad (u_1, u_2)$$

We regard the affine u_1, u_2-plane as a subset of $\mathbb{P}^2$ by this correspondence, and we denote that subset by $\mathbb{U}^0$. The points of $\mathbb{U}^0$, those with $x_0 \neq 0$, are the *points at finite distance*. The *points at infinity* of $\mathbb{P}^2$ are those of the form $(0, x_1, x_2)$. They are on the *line at infinity* L^0, the locus $\{x_0 = 0\}$ in $\mathbb{P}^2$. The projective plane is the union of the two sets $\mathbb{U}^0$ and L^0. When a point is given by a coordinate vector, one can assume that the first coordinate is either 1 or 0.

We may write a point (x_0, x_1, x_2) that is in $\mathbb{U}^0$ as $(1, u_1, u_2)$, with $u_i = x_i/x_0$ as above. The notation $u_i = x_i/x_0$ is important when the coordinate vector (x_0, x_1, x_2) has been given. When no coordinate vector of a point p has been given, one may simply assume that the first coordinate is 1 and write $p = (1, x_1, x_2)$.

There is an analogous correspondence between points $(x_0, 1, x_2)$ and points of an affine plane $\mathbb{A}^2$, and between points $(x_0, x_1, 1)$ and points of an affine plane. We denote the subsets $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$ by $\mathbb{U}^1$ and $\mathbb{U}^2$, respectively. The three sets $\mathbb{U}^0, \mathbb{U}^1, \mathbb{U}^2$ form the *standard covering* of $\mathbb{P}^2$ by three *standard open sets*.

Since the vector $(0,0,0)$ has been ruled out, every point of $\mathbb{P}^2$ lies in at least one of the three standard open sets. Points whose three coordinates are nonzero lie in all of them.

We use similar notation for a projective space of arbitrary dimension, denoting by $\mathbb{U}^i$ the set of points of $\mathbb{P}^n$ at which the coordinate x_i is nonzero, and we call it a *standard open set*. The sets $\mathbb{U}^0, ..., \mathbb{U}^n$ form the *standard covering* of $\mathbb{P}^n$. The standard open set $\mathbb{U}^i$ of $\mathbb{P}^n$ correponds bijectively to an affine space $\mathbb{A}^n$.

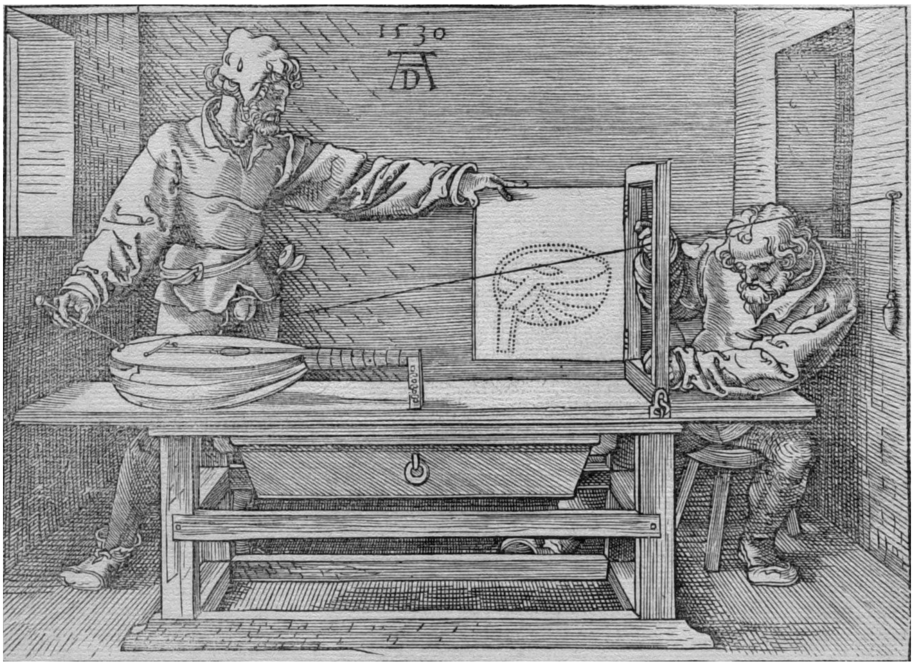
1.2.8. Note. Which points of $\mathbb{P}^2$ are at infinity depends on which of the standard open sets is taken to be the one at finite distance. When the coordinates are (x_0, x_1, x_2), I like to normalize x_0 to 1, as above. Then the points at infinity are those of the form $(0, x_1, x_2)$. But when coordinates are (x, y, z), I may normalize z to 1. I hope this won't cause too much confusion. $\square$ pointatinfinity

1.2.9. digression: the real projective plane realprojplane

Points of the *real projective plane* $\mathbb{RP}^2$ are equivalence classes of nonzero real vectors $x = (x_0, x_1, x_2)$, the equivalence relation being $x' \sim x$ if $x' = \lambda x$ for some nonzero real number λ. The real projective plane can also be thought of as the set of one-dimensional subspaces of the real vector space $\mathbb{R}^3$.

Let's denote $\mathbb{R}^3$ by V. The plane U in V defined by the equation $x_0 = 1$ is analogous to the standard open subset $\mathbb{U}^0$ of the complex projective plane. One can project V from the origin $p_0 = (0,0,0)$ to U, sending a point $x = (x_0, x_1, x_2)$ of V to the point $(1, u_1, u_2)$, with $u_i = x_i/x_0$. The fibres of this projection are the lines through p_0 and x, with p_0 omitted. Looking from the origin, U becomes a "picture plane".

The projection to U is undefined at the points $(0, x_1, x_2)$, which are orthogonal to the x_0-axis. The line connecting such a point to p_0 doesn't meet U. Those points are the points at infinity of $\mathbb{RP}^2$.

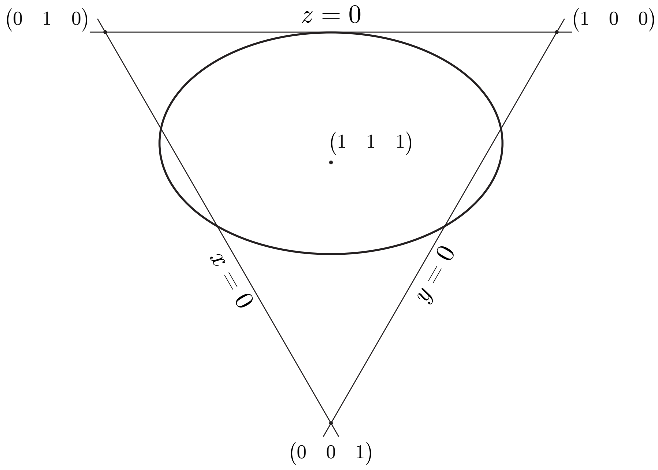


1.2.10. durerfig

This is an illustration from a book on perspective by Albrecht Dürer

The projection from three-space to a picture plane goes back to the the 16th century, the time of Desargues and Dürer. Projective coordinates were introduced by Möbius, 200 years later.

The figure below shows the plane W: $x+y+z = 1$ in the real vector space $\mathbb{R}^3$, together with coordinate lines and a conic. The one-dimensional subspace spanned by a nonzero vector (x_0, y_0, z_0) in $\mathbb{R}^3$ will meet W in a single point unless that vector is on the line $L: \; x+y+z = 0$. So W represents all points of $\mathbb{RP}^2$. except those on L. (The vector that represents the midpoint needs to be multiplied by $\lambda = 1/3$ to satisfy the defining equation of W.)



goober4 **1.2.11. The Real Projective Plane**

chgcoordssec **1.2.12. changing coordinates in the projective plane**

An invertible 3×3 matrix P determines a linear change of coordinates in $\mathbb{P}^2$. With $x=(x_0,x_1,x_2)^t$ and $x'=(x_0',x_1',x_2')^t$ represented as column vectors, that coordinate change is given by

chg
$$x = Px' \tag{1.2.13}$$

The next proposition shows that four special points, the points

$$e_0=(1,0,0)^t,\ e_1=(0,1,0)^t,\ e_2=(0,0,1)^t \quad \text{and} \quad \epsilon=(1,1,1)^t$$

determine the coordinates.

fourpoints **1.2.14. Proposition.** *Let p_0,p_1,p_2,q be four points of $\mathbb{P}^2$, no three of which lie on a line. There is a unique linear coordinate change $Px'=x$ such that $Pp_i=e_i$ and $Pq=\epsilon$.*

proof. The hypothesis that the points p_0,p_1,p_2 don't lie on a line tells us that the vectors that represent those points are independent. They span $\mathbb{C}^3$. So q will be a combination $q=c_0p_0+c_1p_1+c_2p_2$, and because no three of the four points lie on a line, the coefficients c_i will be nonzero. We can *scale* the vectors p_i (multiply them by nonzero scalars) to make $q=p_0+p_1+p_2$ without changing the points. Next, the columns of P can be an arbitrary set of independent vectors. We let them be p_0,p_1,p_2. Then $Pe_i=p_i$, and $P\epsilon=q$. The matrix P is unique up to scalar factor. □

projconics **1.2.15. conics**

A polynomial $f(x_0,...,x_n)$ is *homogeneous, of degree* d, if all monomials that appear with nonzero coefficient have (total) degree d. For example, $x_0^3+x_1^3-x_0x_1x_2$ is a homogeneous cubic polynomial. A homogeneous quadratic polynomal is a combination of the six monomials

$$x_0^2,\ x_1^2,\ x_2^2,\ x_0x_1,\ x_1x_2,\ x_0x_2$$

The locus of zeros of an irreducible homogeneous quadratic polynomial is a *conic*.

classifyconic **1.2.16. Proposition.** *For any conic C, there is a choice of coordinates so that it becomes the locus*

$$x_0x_1+x_0x_2+x_1x_2=0$$

proof. A conic will contain three points that aren't colinear. Let's leave the verification of this fact as an exercise. We choose three non-colinear points on the conic C, and adjust coordinates so that they become the points e_0, e_1, e_2. Let f be the irreducible homogeneous quadratic polynomial in those coordinates, whose zero locus is C. Because e_0 is a point of C, $f(1,0,0) = 0$, and therefore the coefficient of x_0^2 in f is zero. Similarly, the coefficients of x_1^2 and x_2^2 are zero. So f has the form

$$f = ax_0x_1 + bx_0x_2 + cx_1x_2$$

Since f is irreducible, a, b, c aren't zero. By *scaling* appropriately (adjusting f, x_0, x_1, x_2 by scalar factors), we can make $a = b = c = 1$. We will be left with the polynomial $x_0x_1 + x_0x_2 + x_1x_2$. $\square$

1.3 Plane Projective Curves

projcurve

The loci in projective space that are studied in algebraic geometry are those that can be defined by systems of *homogeneous* polynomial equations.

The reason that we use homogeneous equations is this: To say that a polynomial $f(x_0, ..., x_n)$ vanishes at a point of projective space $\mathbb{P}^n$ means that, if the vector $a = (a_0, ..., a_n)$ represents a point p, then $f(a) = 0$. Perhaps this is obvious. Now, if a represents p, the other vectors that represent p are the vectors λa $(\lambda \neq 0)$. When f vanishes at p, $f(\lambda a)$ must also be zero. The polynomial $f(x)$ vanishes at p if and only if $f(\lambda a) = 0$ for every λ.

We write a polynomial $f(x_0, ..., x_n)$ as a sum of its *homogeneous parts*:

(1.3.1) $$f = f_0 + f_1 + \cdots + f_d$$ homparts

where f_0 is the constant term, f_1 is the linear part, etc., and d is the degree of f.

1.3.2. Lemma. *Let $f = f_0 + \cdots + f_d$ be a polynomial in $x_0, ..., x_n$, and let $a = (a_0, ..., a_n)$ be a nonzero vector. Then $f(\lambda a) = 0$ for every nonzero complex number λ if and only if $f_i(a) = 0$ for every $i = 0, ..., d$.* *hompart-szero*

This lemma shows that we may as well work with homogeneous equations.

proof of the lemma. We substitute into (1.3.1): $f(\lambda x) = f_0 + \lambda f_1(x) + \lambda^2 f_2(x) + \cdot + \lambda^d f_d(x)$. Evaluating at $x = a$, $f(\lambda a) = f_0 + \lambda f_1(a) + \lambda^2 f_2(a) + \cdot + \lambda^d f_d(a)$, and $f_i(a)$ are scalars (complex numbers). The right side of this equation is a polynomial in λ, with coefficients $f_i(a)$. Since a nonzero polynomial has finitely many roots, $f(\lambda a)$ won't be zero for *every* λ unless that polynomial is zero — unless $f_i(a)$ is zero for every i. $\square$

1.3.3. Lemma. (i) *If the product $f = gh$ of two polynomials is homogeneous, then g and h are homogeneous.* *fequalsgh*
(ii) *The zero locus in projective space of a product gh of homogeneous polynomials is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.*
(iii) *The zero locus in affine space of a product gh of polynomials, not necessarily homogeneous, is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.* $\square$

1.3.4. loci in the projective line

locipone

Before going to plane projective curves, we describe the zero locus in $\mathbb{P}^1$ of a homogeneous polynomial in two variables.

1.3.5. Lemma. *A homogeneous polynomial $f(x,y) = a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d$ of positive degree, and with complex coefficients, is a product of homogeneous linear polynomials that are unique up to scalar factor.* *fac-torhom-poly*

To prove this, one uses the fact that the field of complex numbers is algebraically closed. A one-variable complex polynomial factors into linear factors. To factor $f(x,y)$, one can factor the one-variable polynomial $f(1,y)$ into linear factors, substitute y/x for y, and multiply the result by x^d. When one adjusts scalar factors, one will obtain the expected factorization of $f(x,y)$. For instance, to factor $f(x,y) = x^2 - 3xy + 2y^2$, we substitute $x = 1$: $2y^2 - 3y + 1 = 2(y-1)(y-\frac{1}{2})$. Substituting $y = y/x$ and multiplying by x^2, $f(x,y) = 2(y-x)(y-\frac{1}{2}x)$. The scalar 2 can be put into one of the linear factors. $\square$

When a homogeneous polynomial f is a product of linear factors, one can adjust the factors by scalars, to put it into the form

(1.3.6) $$f(x,y) = c(v_1x - u_1y)^{r_1} \cdots (v_kx - u_ky)^{r_k}$$ factor-polytwo where no factor $v_ix - u_iy$ is a constant multiple of another, c is a nonzero scalar, and $r_1 + \cdots + r_k$ is the degree of f. The exponent r_i is the *multiplicity* of the linear factor $v_ix - u_iy$.

A linear polynomial $vx - uy$ determines a point (u, v) in the projective line $\mathbb{P}^1$, the unique *zero* of that polynomial, and changing the polynomial by a scalar factor doesn't change its zero. Thus the linear factors of the homogeneous polynomial (1.3.6) determine points of $\mathbb{P}^1$, the zeros of f. The points (u_i, v_i) are zeros of *multiplicity* r_i. The total number of those points, counted with multiplicity, will be the degree of f.

zerosand-roots **1.3.7.** The zero (u_i, v_i) of f corresponds to a root $x = u_i/v_i$ of multiplicity r_i of the one-variable polynomial $f(x, 1)$, except when the zero is the point $(1, 0)$. This happens when the coefficient a_0 of f is zero, and y is a factor of f. One could say that $f(x, y)$ has a zero at infinity in that case.

This sums up the information contained in the zero locus of a homogeneous polynomial in the projective line. It will be a finite set of points with multiplicities.

intersect-line **1.3.8. intersections with a line**

Let Z be the zero locus of a homogeneous polynomial $f(x_0, ..., x_n)$ of degree d in projective space $\mathbb{P}^n$, and let L be a line in $\mathbb{P}^n$ (1.2.4). Say that L is the set of points $rp+sq$, where the points p and q are represented by specific vectors $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$, respectively. So L corresponds to the projective line $\mathbb{P}^1$, by $rp + sq \leftrightarrow (r, s)$. Let's assume that L isn't entirely contained in the zero locus Z. The intersection $Z \cap L$ corresponds to the zero locus in $\mathbb{P}^1$ of the polynomial $\overline{f}$ in r, s obtained by substituting $rp + sq$ into f. This substitution yields a homogeneous polynomial $\overline{f}(r, s)$ of degree d, and the zeros of $\overline{f}$ in $\mathbb{P}^1$ correspond to the points of $Z \cap L$. If f has degree d, there will be d zeros, counted with multiplicity.

For instance, let f be the polynomial $x_0x_1+x_0x_2+x_1x_2$. Then with $p = (a_0, a_1, a_2)$ and $q = (b_0, b_1, b_2)$, $\overline{f}$ is the following quadratic polynomial in r, s:

$$\begin{aligned}\overline{f}(r, s) = f(rp + sq) &= (ra_0 + sb_0)(ra_1 + sb_1) + (ra_0 + sb_0)(ra_2 + sb_2) + (ra_1 + sb_1)(ra_2 + sb_2) \\ &= (a_0a_1+a_0a_2+a_1a_2)r^2 + \big(\textstyle\sum_{i\neq j} a_ib_j\big)rs + (b_0b_1+b_0b_2+b_1b_2)s^2\end{aligned}$$

intersect-linetwo **1.3.9. Definition.** With notation as above, the *intersection multiplicity* of the zero locus Z and the line L at a point p is the multiplicity of zero of the polynomial $\overline{f}$. □

XcapL **1.3.10. Corollary.** *Let Z be the zero locus of a homogeneous polynomial f in projective space $\mathbb{P}^n$, and let L be a line in $\mathbb{P}^n$ that isn't contained in Z. The number of intersections of Z and L, counted with multiplicity, is equal to the degree of f.* □

lociptwo **1.3.11. loci in the projective plane**

The locus of zeros in $\mathbb{P}^2$ of a single irreducible homogeneous polynomial $f(x, y, z)$ is called a *plane projective curve*. The *degree* of a plane projective curve is the degree of its irreducible defining polynomial.

The next proposition shows that plane projective curves are the most interesting loci in the projective plane.

fgzerofi-nite **1.3.12. Proposition.** *Homogeneous polynomials $f_1, ..., f_r$ in three variables with no common factor and with $r > 1$ have finitely many common zeros in $\mathbb{P}^2$.*

The proof of this proposition is below.

redcurve **1.3.13. Note.** Suppose that a homogeneous polynomial $f(x, y, z)$ is reducible, say $f = g_1 \cdots g_k$, that g_i are irreducible, and that no two of them are scalar multiples of one another. Then the zero locus C of f is the union of the zero loci V_i of the factors g_i. In this case, C may be called a *reducible curve*.

When there are multiple factors, say $f = g_1^{e_1} \cdots g_k^{e_k}$, and some of the exponents are greater than 1, it is still true that the locus $C : \{f = 0\}$ is the union of the zero loci V_i of g_i, but the connection between the geometry of C and the algebra is weakened. In this situation, the structure of a *scheme* becomes useful, but we won't discuss schemes. The only situation in which we may need to keep track of multiple factors is when counting intersections with another curve D. For this purpose, one can use the *divisor* of f, which is defined to be the integer combination $e_1V_1 + \cdots + e_kV_k$. □

A *rational function* is a fraction of polynomials. The polynomial ring $\mathbb{C}[x,y]$ embeds into its field of fractions, the field of rational functions in x,y, which is often denoted by $\mathbb{C}(x,y)$. Let's denote it by F here. The polynomial ring $\mathbb{C}[x,y,z]$ in three variables becomes a subring of the one-variable polynomial ring $F[z]$. When one is presented with a problem about the ring $\mathbb{C}[x,y,z]$, it can be useful to begin by studying it in the ring $F[z]$ because it is a principal ideal domain. The polynomial rings $\mathbb{C}[x,y]$ and $\mathbb{C}[x,y,z]$ are unique factorization domains, but not principal ideal domains.

1.3.14. Lemma. *Let F be the field of rational functions in x,y.* relprime
(i) *If $f_1,...,f_k$ are homogeneous polynomials in x,y,z with no common factor, their greatest common divisor in $F[z]$ is 1, and therefore they generate the unit ideal of $F[z]$. So there is an equation of the form $\sum g_i' f_i = 1$, with g_i' in $F[z]$.*
(ii) *An irreducible element of $\mathbb{C}[x,y,z]$ that has positive degree in z is also an irreducible element of $F[z]$.*

The *unit ideal* of a ring R is the ring R itself.

proof. **(i)** This is a proof by contradiction. Let h' be a nonzero element of $F[z]$ that isn't a unit of that ring, i.e., isn't an element of F, and suppose that h' divides f_i in $F[z]$ for every i. Say that $f_i = u_i'h'$ with u_i' in $F[x]$. The coefficients of h' and u_i' are rational functions, whose denominators are polynomials in x,y. We multiply by a polynomial in x,y to clear the denominators from the coefficients of all of the elements h' and u_i'. This will give us equations of the form $d_if_i = u_ih$, where d_i are polynomials in x,y, and h and u_i are polynomials in x,y,z. Since h' isn't in F, neither is h. So h will have positive degree in z. Let g be an irreducible factor of h of positive degree in z. Then g divides d_if_i, but it doesn't divide d_i, which has degree zero in z. So g divides f_i, and this is true for every i. This contradicts the hypothesis that $f_1,...,f_k$ have no common factor.

(ii) Say that a polynomial $f(x,y,z)$ factors in $F[z]$, $f = g'h'$, where g' and h' are polynomials of positive degree in z with coefficients in F. When we clear denominators from g' and h', we obtain an equation of the form $df = gh$, where g and h are polynomials in x,y,z of positive degree in z and d is a polynomial in x,y. Since neither g nor h divides d, f must be reducible. □

proof of Proposition 1.3.12. We are to show that homogeneous polynomials $f_1,...,f_r$ in x,y,z with no common factor have finitely many common zeros. Lemma 1.3.14 tells us that we may write $\sum g_i'f_i = 1$, with g_i' in $F[z]$. Clearing denominators from the elements g_i' gives us an equation of the form

$$\sum g_if_i = d$$

where g_i are polynomials in x,y,z and d is a polynomial in x,y. Taking suitable homogeneous parts of g_i and d produces an equation $\sum g_if_i = d$ in which all terms are homogeneous.

Lemma 1.3.5 asserts that $d(x,y)$ is a product of linear polynomials, say $d = \ell_1 \cdots \ell_r$. Since $f_1,...,f_k$ have no common factor, they aren't all divisible by any one of the linear factors ℓ_j. Corollary 1.3.10 shows that if ℓ_j doesn't divide f_i, then ℓ_j and f_i have finitely many common zeros. A common zero of $f_1,...,f_k$ is also a zero of d, and therefore it is a zero of ℓ_j for some j. So there are finitely many common zeros of $f_1,...,f_k$ in all. □

1.3.15. Corollary. *Every locus in the projective plane $\mathbb{P}^2$ that is defined by a system of homogeneous polynomial equations is a finite union of points and curves.* □ pointscurves

The next corollary is a special case of the Strong Nullstellensatz, which will be proved in the next chapter.

1.3.16. Corollary. *Let $f(x,y,z)$ be an irreducible homogeneous polynomial that vanishes on an infinite set S of points of $\mathbb{P}^2$. If another homogeneous polynomial $g(x,y,z)$ vanishes on S, then f divides g. Therefore, if an irreducible polynomial vanishes on S, that polynomial is unique up to scalar factor.* idealprincipal

proof. If the irreducible polynomial f doesn't divide g, then f and g have no common factor, and therefore they have finitely many common zeros. □

1.3.17. the classical topology classicaltopology

The usual topology on the affine space $\mathbb{A}^n$ will be called the *classical topology*. A subset U of $\mathbb{A}^n$ is open in the classical topology if, whenever U contains a point p, it contains all points sufficiently near to p. We call this the classical topology to distinguish it from another topology, the *Zariski topology*, which will be discussed in the next chapter.

The projective space $\mathbb{P}^n$ also has a classical topology. A subset U of $\mathbb{P}^n$ is open if, whenever a point p of U is represented by a vector $(x_0, ..., x_n)$, all vectors $x' = (x'_0, ..., x'_n)$ sufficiently near to x represent points of U.

isopts **1.3.18. isolated points**

A point p of a topological space X is *isolated* if the set $\{p\}$ is both open and closed, or if both $\{p\}$ and its complement $X - \{p\}$ are closed. If X is a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$, a point p of X is isolated in the classical topology if X contains no sequence of points p_i distinct from p, and with limit p.

noisolatedpoint **1.3.19. Proposition** *Let n be an integer greater than one. In the classical topology, the zero locus of a polynomial in $\mathbb{A}^n$, or of a homogeneous polyomial in $\mathbb{P}^n$, contains no isolated points.*

polymonic **1.3.20. Lemma.** *Let f be a polynomial of degree d in $x_1, ..., x_n$. After a suitable coordinate change and scaling, $f(x)$ will be a monic polynomial of degree d in the variable x_n.*

proof. We write $f = f_0 + f_1 + \cdots + f_d$, where f_i is the homogeneous part of f of degree i. We choose a point p of $\mathbb{A}^n$ at which f_d isn't zero, and change variables so that p becomes the point $(0, ..., 0, 1)$. We relabel, calling the new variables $x_1, .., .x_n$ and the new polynomial f. Then $f_d(0, ..., 0, x_n)$ will be equal to cx_n^d for some nonzero constant c, and f/c will be monic. □

proof of Proposition 1.3.19. We look at the affine case. Let $f(x_1, ..., x_n)$ be a polynomial, and let Z be its zero locus. If f is a product, say $f = gh$, then Z will be the union of the zero loci $Z_1 : \{g = 0\}$ and $Z_2 : \{h = 0\}$. A point p of Z will be in one of those two sets, say in Z_1. If p is an isolated point of Z, then its complement $U = Z - \{p\}$ in Z is closed. If so, then its complement $Z_1 - \{p\}$ in Z_1, which is the intersection $U \cap Z_1$, will be closed in Z_1, and therefore p will be an isolated point of Z_1. So it suffices to prove the proposition in the case that f is irreducible. Let p be a point of Z. We adjust coordinates and scale, so that p becomes the origin $(0, ..., 0)$ and f becomes monic in x_n. We relabel x_n as y, and write f as a polynomial in y:

$$\widetilde{f}(y) = f(x_1, ..., x_{n-1}, y) = y^d + c_{d-1}y^{d-1} + \cdots + c_0$$

where c_i are polynomials in $x_1, ..., x_{n-1}$. Since f is irreducible, $c_0(x) \neq 0$. Since p is the origin and $f(p) = 0$, $c_0(0) = 0$. So $c_0(x)$, which is the product of the roots of $\widetilde{f}(y)$, will tend to zero with x. When $c_0(x)$ is small, at least one root of $\widetilde{f}$ will be small. So there are points of Z distinct from p, but arbitrarily close to p. □

functioniszero **1.3.21. Corollary.** *Let C' be the complement of a finite set of points in a plane curve C. In the classical topology, a continuous function g on C that is zero at every point of C' is identically zero.* □

1.4 Tangent Lines

tanlines

standnot **1.4.1. notation for working locally**

We will often want to inspect a plane projective curve $C : \{f(x_0, x_1, x_2) = 0\}$ in a neighborhood of a particular point p. To do this we may adjust coordinates so that p becomes the point $(1, 0, 0)$, and work with points $(1, x_1, x_2)$ in the standard open set $\mathbb{U}^0 : \{x_0 \neq 0\}$. When we identify $\mathbb{U}^0$ with the affine x_1, x_2-plane, p becomes the origin $(0, 0)$ and C becomes the zero locus of the nonhomogeneous polynomial $f(1, x_1, x_2)$. On the subset $\mathbb{U}^0$, the loci $f(x_0, x_1, x_2) = 0$ and $f(1, x_1, x_2) = 0$ are equal.

This will be a standard notation for working locally. Of course, it doesn't matter which variable we set to 1. If the variables are x, y, z, we may prefer to take for p the point $(0, 0, 1)$ and work with the polynomial $f(x, y, 1)$.

homdehomone **1.4.2. homogenizing and dehomogenizing**

Let $f(x_0, x_1, ..., x_n)$ be a homogeneous polynomial. The polynomial $f(1, x_1, ..., x_n)$ is called the *dehomogenization* of f, with respect to the variable x_0. A simple procedure, *homogenization*, inverts this dehomogenization. Suppose given a nonhomogeneous polynomial $F(x_1, ..., x_n)$ of degree d. To homogenize

F, we replace the variables x_i, $i = 1, ..., n$, by $u_i = x_i/x_0$. Then since u_i have degree zero in x, so does $F(u_1, ..., u_n)$. When we multiply by x_0^d, the result will be a homogeneous polynomial, of degree d in $x_0, ..., x_n$, that isn't divisible by x_0.

For example, let $F(x_1, x_2) = 1 + x_1 + x_2^3$. Then $x_0^3 F[u_1, u_2] = x_0^3 + x_0^2 x_1 + x_2^3$.

1.4.3. Lemma. *A homogeneous polynomial $f(x_0, x_1, x_2)$ that isn't divisible by x_0 is irreducible if and only if $f(1, x_1, x_2)$ is irreducible.* □ *foneirred*

1.4.4. smooth points and singular points smsingpts

Let C be the plane curve defined by an irreducible homogeneous polynomial $f(x_0, x_1, x_2)$, and let f_i denote the partial derivative $\frac{\partial f}{\partial x_i}$, computed by the usual calculus formula. A point of C at which the partial derivatives f_i aren't all zero is a *smooth point* of C. A point at which all partial derivatives are zero is a *singular point*. A curve is *smooth*, or *nonsingular*, if it contains no singular point. A curve that contains a singular point is a *singular curve*.

The *Fermat Curve* fermatcurve

$$x_0^d + x_1^d + x_2^d = 0 \tag{1.4.5}$$

is smooth because the only common zero of the partial derivatives $dx_0^{d-1}, dx_1^{d-1}, dx_2^{d-1}$, which is $(0, 0, 0)$, doesn't represent a point of $\mathbb{P}^2$. The cubic curve $x_0^3 + x_1^3 - x_0 x_1 x_2 = 0$ is singular at the point $(0, 0, 1)$.

The Implicit Function Theorem, which is reviewed in Section 9.2, explains the meaning of smoothness. Suppose that $p = (1, 0, 0)$ is a point of C. We set $x_0 = 1$ and inspect the locus $f(1, x_1, x_2) = 0$ in the standard open set $\mathbb{U}^0$. If $f_2 = \frac{\partial f}{\partial x_2}$ isn't zero at p, the Implicit Function Theorem tells us that we can solve the equation $f(1, x_1, x_2) = 0$ for x_2 locally, i.e., for small x_1, as an analytic function φ with $\varphi(0) = 0$, and then $f(1, x_1, \varphi(x_1))$ will be zero. Sending x_1 to $(1, x_1, \varphi(x_1))$ inverts the projection from C to the affine x_1-line locally.

At a smooth point, a plane curve C is locally homeomorphic to the affine line.

1.4.6. Euler's Formula. If $f(x_0, ..., x_n)$ is a homogeneous polynomial of degree d, then eulerformula

$$\sum_i x_i \frac{\partial f}{\partial x_i} = d\, f$$

It suffices to check the formula when f is a monomial. You will be able to do this. For instance, if the variables are x, y, z, and $f = x^2 y^3 z$, then

$$x f_x + y f_y + z f_z = x(2xy^3 z) + y(3x^2 y^2 z) + z(x^2 y^3) = 6x^2 y^3 z = 6\, f$$ □

1.4.7. Corollary. **(i)** *If all partial derivatives of an irreducible homogeneous polynomial $f(x_0, x_1, x_2)$ are zero at a point p of $\mathbb{P}^2$, then f is zero at p, and therefore p is a singular point of the curve $\{f = 0\}$.* *singpointoncurve*

(ii) *At a smooth point of the plane curve defined by an irreducible homogeneous polynomial f, at least two partial derivatives of f will be nonzero.*

(iii) *At a smooth point of the curve $\{f = 0\}$, the dehomogenization $f(1, u_1, u_2)$ will have a nonvanishing partial derivative.*

(iv) *The partial derivatives of an irreducible polynomial have no common (nonconstant) factor.*

(v) *A plane curve has finitely many singular points.* □

1.4.8. tangent lines and flex points tangent

Let C be the plane projective curve defined by an irreducible homogeneous polynomial $f(x_0, x_1, x_2)$. A line L is *tangent* to C at a smooth point p if the intersection multiplicity of C and L at p is at least 2. (See (1.3.9).) A smooth point p of C is a *flex point* if the intersection multiplicity of C and its tangent line at p is at least 3, and p is an *ordinary* flex point if the intersection multiplicity is equal to 3.

Let L be a line through a point p and let q be a point of L distinct from p. We represent p and q by specific vectors (p_0, p_1, p_2) and (q_0, q_1, q_2), to write a variable point of L as $p + tq$, and we expand the restriction of f to L in a Taylor's series. (The Taylor expansion carries over to complex polynomials because it is an identity.) Let $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Taylor's formula is

taylor (1.4.9)
$$f(p+tq) \;=\; f(p) \;+\; \Big(\sum_i f_i(p)\, q_i\Big)t \;+\; \tfrac{1}{2}\Big(\sum_{i,j} q_i\, f_{ij}(p)\, q_j\Big)t^2 \;+\; O(3)$$

where the symbol $O(3)$ stands for a polynomial in which all terms have degree at least 3 in t. The point q is missing from this parametrization, but this won't be important.

The intersection multiplicity of C and L at a point p was defined in (1.3.8). It is equal to the lowest power of t that has nonzero coefficient in $f(p + tq)$. The point p lies on the curve C if $f(p) = 0$. If so, and if p is a smooth point of C, then the line L with the parametrization $p + tq$ will be a tangent line to C at p, provided that the coefficient $\sum_i f_i(p)\, q_i$ of t is zero. If p is a smooth point and L is a tangent line, then p is a flex point if, in addition, $\sum_{i,j} q_i f_{ij}(p)\, q_j$ is zero.

One can write the equation (1.4.9) in terms of the *gradient vector* $\nabla = (f_0, f_1, f_2)$ and the *Hessian matrix* H of f. The Hessian is the matrix of second partial derivatives f_{ij}:

hessian-matrix (1.4.10)
$$H \;=\; \begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix}$$

Let ∇_p and H_p denote the evaluations of ∇ and H, respectively, at p.

When we regard p and q as the column vectors $(p_0, p_1, p_2)^t$ and $(q_0, q_1, q_2)^t$, Equation 1.4.9 can be written as

texp (1.4.11)
$$f(p+tq) = \; f(p) \;+\; \big(\nabla_p\, q\big)t \;+\; \tfrac{1}{2}(q^t H_p\, q)t^2 \;+\; O(3)$$

in which q^t is the transpose of the column vector q, a row vector, and where $\nabla_p q$ and $q^t H_p q$ are computed as matrix products.

So p is a smooth point of C if $f(p) = 0$ and $\nabla_p \neq 0$. If p is a smooth point, then L is tangent to C at p if $\nabla_p\, q$ is zero, and p is a flex point if $\nabla_p\, q$ and $q^t H_p\, q$ are both zero.

The equation of the tangent line L at a smooth point p of C is $\nabla_p\, x = 0$, or

tanlineeq (1.4.12)
$$f_0(p)x_0 + f_1(p)x_1 + f_2(p)x_2 = 0$$

A line L is the tangent line at a smooth point p if it is orthogonal to the gradient ∇_p.

tangentex **1.4.13. Example.** Let C be the plane curve defined by the polynomial $f = x_0^2 x_1 + x_1^3 x_2 + x_2^2 x_0$. The point $p = (1, 0, 0)$ is on C. Then $\nabla f = \big(2x_0x_1 {+} x_2^2,\; sx_1x_2 {+} x_0^2,\; 2x_0x_2 {+} x_1^2\big)$, and $\nabla_p = (0, 1, 0)$. The tangent line at p is the line $x_1 = 0$. Also,

$$H = 2\begin{pmatrix} x_1 & x_0 & x_2 \\ x_0 & x_2 & x_1 \\ x_2 & x_1 & x_0 \end{pmatrix} \quad \text{and} \quad H_p = 2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When $q = (0, 0, 1), \;\; f(p + tq) = t^2 + O(3).$ □

Note. We will speak of tangent lines only at smooth points of the curve, though Taylor's Formula shows that the restriction of f to any line through a singular point has a multiple zero.

The next lemma is obtained by applying Euler's Formula to the entries of ∇_p and H_p.

applyeuler **1.4.14. Lemma.** $\nabla_p\, p = d\, f(p)$ *and* $p^t H_p = (d-1)\nabla_p$. □

We rewrite Equation 1.4.9 one more time, using the notation $\langle u, v\rangle$ to represent the symmetric bilinear form $u^t H_p\, v$ on the complex vector space $\mathbb{C}^3$. It makes sense to say that this form vanishes on a pair of points of $\mathbb{P}^2$, because the condition $\langle u, v\rangle = 0$ doesn't change when u or v is multiplied by a nonzero scalar.

1.4.15. Proposition. *With notation as above,* linewith-form

(i) *Equation (1.4.9) can be written as*

$$f(p+tq) \;=\; \tfrac{1}{d(d-1)}\langle p,p\rangle \;+\; \tfrac{1}{d-1}\langle p,q\rangle t \;+\; \tfrac{1}{2}\langle q,q\rangle t^2 \;+\; O(3)$$

(ii) *A point p is a smooth point of C if and only if $\langle p,p\rangle = 0$ but $\langle p,v\rangle$ is'nt identically zero.*

proof. **(i)** This is obtained by applying Lemma 1.4.14 to (1.4.11).

(ii) $\langle p,v\rangle = (d-1)^{-1}\nabla_p v$ is identically zero if and only if $\nabla_p = 0$. □

1.4.16. Corollary. *Let p be a smooth point of C, let q be a point of $\mathbb{P}^2$ distinct from p, and let L be the line through p and q. Then* bilinform

(i) *L is tangent to C at p if and only if $\langle p,p\rangle = \langle p,q\rangle = 0$, and*

(ii) *p is a flex point of C with tangent line L if and only if $\langle p,p\rangle = \langle p,q\rangle = \langle q,q\rangle = 0$.* □

1.4.17. Theorem. *A smooth point p of the curve C is a flex point if and only if the Hessian determinant $\det H_p$ at p is zero.* tangent-line

proof. Let p be a smooth point of C. So $\langle p,p\rangle = 0$. If $\det H_p = 0$, the form $\langle u,v\rangle$ is degenerate. Then there is a nonzero null vector q, and $\langle p,q\rangle = \langle q,q\rangle = 0$. But p isn't a null vector, because $\langle p,v\rangle$ isn't identically zero at a smooth point. So q is distinct from p. Therefore p is a flex point.

Conversely, suppose that p is a flex point and let q be a point on the tangent line at p and distinct from p, so that $\langle p,p\rangle = \langle p,q\rangle = \langle q,q\rangle = 0$. The restriction of the form to the two-dimensional space spanned by p and q is zero, and this implies that the form is degenerate. If (p,q,v) is a basis of V, the matrix of the form will look like this:

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$$

□

1.4.18. Proposition. hess-notzero

(i) *Let $f(x,y,z)$ be an irreducible homogeneous polynomial of degree at least two and let H be the Hessian matrix. The Hessian determinant $\det H$ isn't divisible by f. In particular, it isn't identically zero.*

(ii) *A plane curve that isn't a line has finitely many flex points.*

proof. **(i)** Let C be the plane curve defined by f. If f divides the Hessian determinant, every smooth point of C will be a flex point. We set $z = 1$ and look on the standard open set $\mathbb{U}^2$, choosing coordinates so that the origin p is a smooth point of C, and so that $\frac{\partial f}{\partial y} \neq 0$ at p. The Implicit Function Theorem tells us that we can solve the equation $f(x,y,1) = 0$ for y locally, say $y = \varphi(x)$, where φ is an analytic function. The graph $\Gamma : \{y = \varphi(x)\}$ will be equal to C in a neighborhood of p. A point of Γ is a flex point if and only if $\frac{d^2\varphi}{dx^2}$ is zero there. If this is true for all points near to p, then $\frac{d^2\varphi}{dx^2}$ is identically zero. This implies that φ is linear, and since $\varphi(0) = 0$, that $\varphi(y)$ has the form ax. Then $y = ax$ solves $f = 0$, and therefore $y - ax$ divides $f(x,y,1)$. But $f(x,y,z)$ is irreducible, and so is $f(x,y,1)$. So $f(x,y,1)$ and $f(x,y,z)$ are linear, contrary to hypothesis.

(ii) This follows from **(i)** and (1.3.12). The irreducible polynomial f and the Hessian determinant $\det H$ have finitely many common zeros. □

1.5 The Dual Curve

dualcurve

1.5.1. the dual plane

dualplane-sect

Let $\mathbb{P}$ denote the projective plane with coordinates x_0, x_1, x_2, let s_0, s_1, s_2 be scalars, not all zero, and let L be the line in $\mathbb{P}$ with the equation

(1.5.2) $$s_0x_0 + s_1x_1 + s_2x_2 = 0$$ lineequation *uptofactor* **1.5.3. Lemma.** *The solutions x of the equation 1.5.2, the points of L, determine the coefficients s up to a common nonzero factor.* □

So L determines a point (s_0, s_1, s_2) in another projective plane $\mathbb{P}^*$ called the *dual plane*. We denote that point by L^*. Moreover, a point $p = (x_0, x_1, x_2)$ in $\mathbb{P}$ determines a line in the dual plane, the line with the equation (1.5.2), when s_i are regarded as the variables and x_i as the scalar coefficients. We denote that line by p^*. The equation exhibits a duality between $\mathbb{P}$ and $\mathbb{P}^*$. A point p of $\mathbb{P}$ lies on the line L if and only if the equation is satisfied, and this means that, in $\mathbb{P}^*$, the point L^* lies on the line p^*.

The dual $\mathbb{P}^{**}$ of the dual plane $\mathbb{P}^*$ is the plane $\mathbb{P}$.

dual-curvetwo **1.5.4. the dual curve**

Let C be the plane projective curve defined by an irreducible homogeneous polyomial f of degree at least 2, and let U be the set of its smooth points. which is the complement of a finite set in C (1.4.7). We define a map

$$U \xrightarrow{t} \mathbb{P}^*$$

as follows: If p is a point of U and L is the tangent line to C at p, then $t(p)$ is the point L^* of $\mathbb{P}^*$ that corresponds to L. Thus the image $t(U)$ is the locus of the tangent lines at the smooth points of C. We assume that f has degree at least two because, if C were a line, the image $t(U)$ would be a point.

Let $\nabla f = (f_0, f_1, f_2)$ be the gradient of f. The tangent line L at a smooth point $p = (x_0, x_1, x_2)$ of C has the equation $s_0x_0 + s_1x_1 + s_2x_2 = 0$, with $s_i = f_i(x)$ (1.4.12). Therefore $L^* = t(p)$ is the point

ellstare-quation (1.5.5)
$$(s_0, s_1, s_2) \sim \big(f_0(x), f_1(x), f_2(x)\big) = \nabla f(x)$$

phize-rogzero **1.5.6. Lemma.** *Let U be the set of smooth points of a curve C, let $\varphi(s_0, s_1, s_2)$ be a homogeneous polynomial, and let $g(x_0, x_1, x_2) = \varphi(\nabla f(x))$. Then $\varphi(s)$ is identically zero on the image $t(U)$ if and only if $g(x)$ is identically zero on U, and this is true if and only if f divides g.*

proof. The point $s = (s_0, s_1, s_2)$ is in $t(U)$ if for some x in U and some $\lambda \neq 0$, $\nabla f(x) = \lambda s$. If g has degree r, $g(x) = \varphi(\nabla f(x)) = \varphi(\lambda s) = \lambda^r \varphi(s)$. So $g(x) = 0$ if and only if $\varphi(s) = 0$. If g is zero on U, then because U is the complement of a finite set, g is zero on C, and therefore f divides g. □

dual-curvethm **1.5.7. Theorem.** *Let C be the plane curve defined by an irreducible homogeneous polynomial f of degree at least two. With notation as above, the image $t(U)$ is contained in a curve C^* in the dual plane $\mathbb{P}^*$.*

The curve C^* referred to in the theorem is the *dual curve*.

proof of Theorem 1.5.7. We use transcendence degree for this proof. It you aren't familiar with that concept, read Section 9.3.

If an irreducible homogeneous polynomial $\varphi(s)$ vanishes on the image $t(U)$, it will be unique up to scalar factor (Corollary 1.3.16). We show first that there is a nonzero polynomial $\varphi(s)$, not necessarily irreducible or homogeneous, that vanishes on $t(U)$. The transcendence degree of the field $\mathbb{C}(x_0, x_1, x_2)$ over $\mathbb{C}$ is 3. Therefore the four polynomials f_0, f_1, f_2, and f are algebraically dependent. There is a nonzero polynomial $\psi(s_0, s_1, s_2, t)$ such that $\psi(f_0(x), f_1(x), f_2(x), f(x))$ is the zero polynomial. We can cancel factors of t, so we may assume that ψ isn't divisible by t. Let $\varphi(s) = \psi(s_0, s_1, s_2, 0)$. When t doesn't divide ψ, this isn't the zero polynomial. If a vector $x = (x_1, x_2, x_3)$ represents a point of U, then $f(x) = 0$, and therefore

$$\psi(f_0(x), f_1(x), f_2(x), f(x)) = \psi(\nabla f(x), 0) = \varphi(\nabla f(x))$$

Since the left side of this equation is identically zero, $\varphi(\nabla f(x)) = 0$ whenever x represents a point of U. So φ vanishes on $t(U)$

Next, if f has degree d, then the partial derivatives f_i have degree $d-1$, and $\nabla f(\lambda x) = \lambda^{d-1} \nabla f(x)$ for all λ. Because the vectors x and λx represent the same point, $\varphi(\nabla f(\lambda x)) = \varphi(\lambda^{d-1} \nabla f(x))) = 0$ for all λ, when x is in U. Writing $\nabla f(x) = s$, $\varphi(\lambda^{d-1} s) = 0$ for all λ when x is in U. Since λ^{d-1} can be any complex number, Lemma 1.3.2 tells us that the homogeneous parts of $\varphi(s)$ vanish at s, when $s = \nabla f(x)$ and x is in U. So the homogeneous parts of $\varphi(s)$ vanish on $t(U)$. This shows that there is a nonzero, homogeneous polynomial $\varphi(s)$ that vanishes on $t(U)$. We choose such a polynomial $\varphi(s)$. Let its degree be r.

Let $g(x) = \varphi(\nabla f(x))$. If f has degree d, then g will be homogeneous, of degree $r(d-1)$. It will vanish on U, and therefore on C (1.3.21). So f will divide g. If $\varphi(s)$ factors, then $g(x)$ factors accordingly, and because f is irreducible, it will divide one of the factors of g. The corresponding factor of φ will vanish on $t(U)$ (1.5.6). So we may replace the homogeneous polynomial φ by one of its irreducible factors. $\square$

In principle, the proof of Theorem 1.5.7 gives a method for finding a polynomial that vanishes on the dual curve. That method is to find a polynomial relation among f_x, f_y, f_z, f, and set $f = 0$. However, it is usually painful to determine the defining polynomial of C^* explicitly. Most often, the degrees of C and C^* will be different. Moreover, several points of the dual curve C^* may correspond to a singular point of C, and vice versa.

We give two examples in which the computation is easy.

1.5.8. Examples. exampled-ualone

(i) (*the dual of a conic*) Let $f = x_0x_1 + x_0x_2 + x_1x_2$ and let C be the conic $f = 0$. Let $(s_0, s_1, s_2) = (f_0, f_1, f_2) = (x_1{+}x_2,\ x_0{+}x_2,\ x_0{+}x_1)$. Then

(1.5.9) $$s_0^2 + s_1^2 + s_2^2 - 2(x_0^2 + x_1^2 + x_2^2) \ = 2f \qquad \text{and} \qquad s_0s_1 + s_1s_2 + s_0s_2 - (x_0^2 + x_1^2 + x_2^2) \ = 3f$$

We eliminate $(x_0^2 + x_1^2 + x_2^2)$ from the two equations:

(1.5.10) $$(s_0^2 + s_1^2 + s_2^2) - 2(s_0s_1 + s_1s_2 + s_0s_2) \ = \ -4f$$

Setting $f = 0$ gives us the equation of the dual curve. It is another conic.

(ii) (*the dual of a cuspidal cubic*) The dual of a smooth cubic is a curve of degree 6. It is too much work to compute that dual here. We compute the dual of a singular cubic instead. The curve C defined by the irreducible polynomial $f = y^2z + x^3$ has a singularity, a cusp. at the point $(0,0,1)$. The Hessian matrix of f is

$$H \ = \ \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

and the Hessian determinant $\det H$ is $h = -24xy^2$. The common zeros of f and h are the singular point $(0,0,1)$, and a single flex point $(0,1,0)$.

We scale the partial derivatives of f to simplify notation. Let $u = f_x/3 = x^2$, $v = f_y/2 = yz$, and $w = f_z = y^2$. Then

$$v^2w - u^3 = y^4z^2 - x^6 = (y^2z + x^3)(y^2z - x^3) = f(y^2z - x^3)$$

The zero locus of the irreducible polynomial $v^2w - u^3$ is the dual curve. It is another singular cubic. $\square$

1.5.11. a local equation for the dual curve equa-tionofcstar

We label the coordinates in $\mathbb{P}$ and $\mathbb{P}^*$ as x, y, z and u, v, w, respectively, and let C be the curve defined by an irreducible homogeneous polynomial $f(x,y,z)$. We work in a neighborhood of a smooth point p_0 of the curve C defined by a homogeneous polynomial $f(x,y,z)$, choosing coordinates so that $p_0 = (0,0,1)$, and that the tangent line L_0 at p_0 is the line $\{y = 0\}$. The image of p_0 in the dual curve C^* is the point L_0^* at which $(u,v,w) = (0,1,0)$.

Let $\widetilde{f}(x,y) = f(x,y,1)$. In the affine x,y-plane, the point p_0 becomes the origin $(0,0)$. So $\widetilde{f}(p_0) = 0$, and since the tangent line is L_0, $\frac{\partial \widetilde{f}}{\partial x}(p_0) = 0$, while $\frac{\partial \widetilde{f}}{\partial y}(p_0) \neq 0$. We solve the equation $\widetilde{f} = 0$ for y as an analytic function $y(x)$, with $y(0) = 0$. Let $y'(x)$ denote the derivative $\frac{dy}{dx}$. Differentiating the equation $f(x, y(x)) = 0$ shows that $y'(0) = 0$.

Let $\widetilde{p}_1 = (x_1, y_1)$ be a point of C_0 near to $\widetilde{p}_0$, so that $y_1 = y(x_1)$, and let $y_1' = y'(x_1)$. The tangent line L_1 at $\widetilde{p}_1$ has the equation

(1.5.12) $$y - y_1 = y_1'(x - x_1)$$ localtan-gent

Putting z back, the homogeneous equation of the tangent line L_1 at the point $p_1 = (x_1, y_1, 1)$ near to p_0 is

$$-y_1'x + y + (y_1'x_1 {-} y_1)z = 0$$

The point L_1^* of the dual plane that corresponds to L_1 is

projlocal-tangent (1.5.13)
$$(u_1, v_1, w_1) = (-y_1', 1, y_1'x_1 - y_1)$$

bidualone **1.5.14. the bidual**

The *bidual* C^{**} of C is the dual of the curve C^*. It is a curve in the space $\mathbb{P}^{**}$, which is $\mathbb{P}$.

bidualC **1.5.15. Theorem.** *A plane curve C of degree greater than one is equal to its bidual C^{**}.*

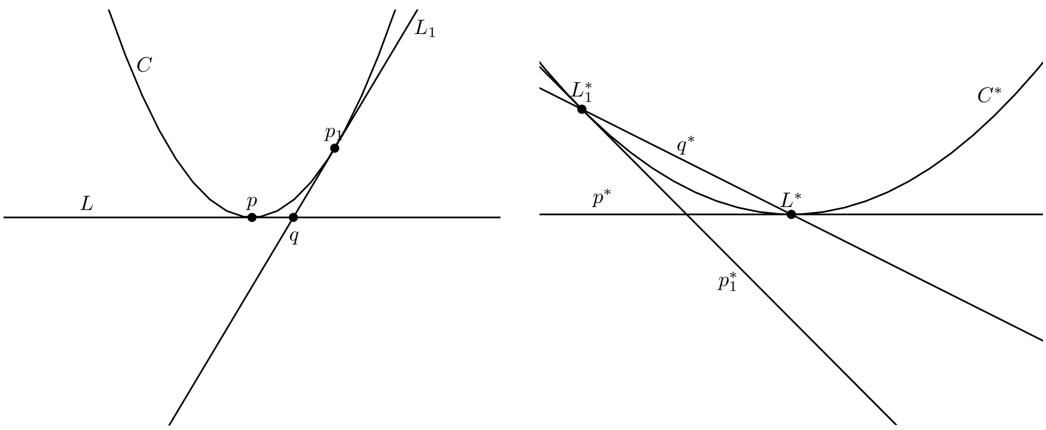
We use the following notation for the proof:

• U is the set of smooth points of C, and U^* is the set of smooth points of the dual curve C^*.
• $U^* \xrightarrow{t^*} \mathbb{P}^{**} = \mathbb{P}$ is the map analogous to the map $U \xrightarrow{t} \mathbb{P}^*$.
• V is the set of points p of C such that p is a smooth point of C and also $t(p)$ is a smooth point of C^*.

So $V \subset U \subset C$ and $t(V) \subset U^* \subset C^*$.

vopen **1.5.16. Lemma.**
(i) *V is the complement of a finite set in C, and $t(V)$ is the complement of a finite set in C^*,*
(ii) *Let p_1 be a point near to a smooth point p of a curve C, let L_1 and L be the tangent lines to C at p_1 and p, respectively, and let q be intersection point $L_1 \cap L$. Then $\lim_{p_1 \to p} q = p$.*
(iii) *If L is the tangent line to C at a point p of V, then p^* is the tangent line to C^* at the point L^*, and $t^*(L^*) = p$.*
(iv) *The map $V \xrightarrow{t} t(V)$ is bijective.*

The points and lines that appear in **(ii)** are displayed in this figure:



curveand-dual **1.5.17. A Curve and its Dual**

The curve C on the left is the parabola $y = x^2$. We used (1.5.12) to obtain a local equation $u^2 = 4w$ of the dual curve C^*.

proof of Lemma 1.5.16. **(i)** Let S and S^* denote the finite sets of singular points of C and C^*, respectively. The set V is obtained from C by deleting points of S and points in the inverse image of S^*. The fibre of the map $U \xrightarrow{t} \mathbb{P}^*$ over a point L^* of C^* is the set of smooth points of C whose tangent line is L. Since L meets C in finitely many points, the fibre is finite. So the inverse image of the finite set S^* is finite.

(ii) We work analytically in a neighborhood of p, choosing coordinates so that $p = (0, 0, 1)$ and that L is the line $\{y = 0\}$. Let $(x_q, y_q, 1)$ be the coordinates of the point q. Since q is a point of L, $y_q = 0$. The coordinate x_q can be obtained by substituting $x = x_q$ and $y = 0$ into the local equation (1.5.12) for L_1: $x_q = x_1 - y_1/y_1'$.

Now, when a function has an nth order zero at the point $x = 0$, i.e, when it has the form $y = x^n h(x)$, where $n > 0$ and $h(0) \neq 0$, the order of zero of its derivative at that point is $n-1$. This is verified by differentiating $x^n h(x)$. Since the function $y(x)$ has a zero of positive order at p, $\lim_{p_1 \to p} y_1/y_1' = 0$. We also have $\lim_{p_1 \to p} x_1 = 0$. Therefore $\lim_{p_1 \to p} x_q = 0$, and $\lim_{p_1 \to p} q = \lim_{p_1 \to p} (x_q, y_q, 1) = (0, 0, 1) = p$.

(iii) Let p_1 be a point of C near to p, and let L_1 be the tangent line to C at p_1. The image L_1^* of p_1 is the point $(f_0(p_1), f_1(p_1), f_2(p_1))$ of C^*. Because the partial derivatives f_i are continuous,

$$\lim_{p_1 \to p} L_1^* = (f_0(p), f_1(p), f_2(p)) = L^*$$

With $q = L \cap L_1$ as above, q^* is the line through the points L^* and L_1^*. As p_1 approaches p, L_1^* approaches L^*, and therefore q^* approaches the tangent line to C^* at L^*. On the other hand, it follows from **(ii)** that q^* approaches p^*. Therefore the tangent line to C^* at L^* is p^*. By definition, $t^*(L^*)$ is the point of C that corresponds to the tangent line p^* at L^*. So $t^*(L^*) = p^{**} = p$. □

proof of theorem 1.5.15. Let p be a point of V, and let L be the tangent line at p. The map t^* is defined at L^*, and $L^* = t(p)$, so $p = t^*(L^*) = t^*t(p)$. It follows that the restriction of t to V is injective, and that it defines a bijective map from V to its image $t(V)$, whose inverse function is t^*. So V is contained in the bidual C^{**}. Since V is dense in C and since C^{**} is a closed set, C is contained in C^{**}. Since C and C^{**} are curves, $C = C^{**}$. □

1.5.18. Corollary. **(i)** *Let U be the set of smooth points of a plane curve C, and let t denote the map from U to the dual curve C^*. The image $t(U)$ is the complement of a finite subset of C^*.* *cstarisimage*

(ii) *If C is a smooth curve, the map $C \xrightarrow{t} C^*$ is defined at all points of C, and it is a surjective map.*

proof. **(i)** With U, U^*, and V as above, $V = t^*t(V) \subset t^*(U^*) \subset C^{**} = C$. Since V is the complement of a finite subset of C, so is $t^*(U^*)$. The assertion to be proved follows when we interchange C and C^*.

(ii) The map t is continuous, so its image $t(C)$ is a compact subset of C^*, and by **(i)**, its complement Z is a finite set. Therefore Z is both open and closed. It consists of isolated points of C^*. Since a plane curve has no isolated point, Z is empty. □

1.5.19. Corollary. *Let C be a smooth curve. If the tangent line L at a point p of C isn't tangent to C at another point, i.e., L isn't a bitangent, then the path defined by the local equation (1.5.13) traces out all points of the dual curve C^* that are sufficently near to $L^* = (0, 1, 0)$.* *notbitangent*

proof. Let D be an open neighborhood of p in C in the classical topology, such that the equation (1.5.13) describes the point L_1^* when p_1 is in D. The complement of D in C is compact, and so is its image tZ. If $L^* = t(p)$ isn't in tZ, then p has a neighborhood U whose image is disjoint from tZ. In that neighborhood, the local equation traces out the dual curve. □

The conclusion of this corollary may be false when C is singular. The reasoning breaks down because D won't be compact.

1.6 Resultants and Discriminants

resultant

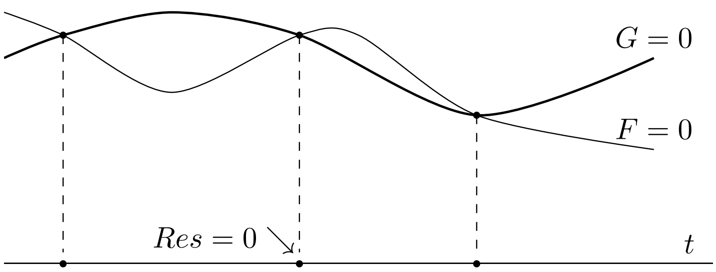
Let

$$F(x) = x^m + a_1 x^{m-1} + \cdots + a_m \quad \text{and} \quad G(x) = x^n + b_1 x^{n-1} + \cdots + b_n \qquad (1.6.1)$$ polys

be monic polynomials with variable coefficients a_i, b_j. The *resultant* $\mathrm{Res}(F, G)$ of F and G is a certain polynomial in the coefficients. Its important property is that, when the coefficients are given values in a field, the resultant is zero if and only if F and G have a common factor.

For instance, suppose that $F(x) = x + a_1$ and $G(x) = x^2 + b_1 x + b_2$. The root $-a_1$ of F is also a root of G if $G(a_1) = a_1^2 - b_1 a_1 + b_2$ is zero. The resultant of F and G is $a_1^2 - b_1 a_1 + b_2$.

restwovar **1.6.2. Example.** Suppose that the coefficients a_i and b_j in (1.6.1) are polynomials in t, so that F and G become polynomials in two variables. Let C and D be the loci $F = 0$ and $G = 0$ in the affine plane $\mathbb{A}^2_{t,x}$. The resultant $\text{Res}_x(F, G)$, computed regarding x as the variable, will be a polynomial in t whose roots are the t-coordinates of the intersections of C and D.



The analogous statement is true when there are more variables. If F and G are relatively prime polynomials in x, y, z, the loci $C : \{F = 0\}$ and $D : \{G = 0\}$ in $\mathbb{A}^3$ will be surfaces, and their intersection Z will be a curve. The resultant $\text{Res}_z(F, G)$, computed regarding z as the variable, is a polynomial in x, y. Its zero locus is the projection of the curve Z to the x, y-plane. □

The formula for the resultant is nicest when one allows leading coefficients different from 1. We work with homogeneous polynomials in two variables to prevent the degrees from dropping when a leading coefficient happens to be zero. The common zeros of two homogeneous polynomials $f(x, y)$ and $g(x, y)$ with complex coefficients correspond to the common roots of the polynomials $f(x, 1)$ and $g(x, 1)$, except when the common zero is the point $(0, 1)$ at infinity.

Let

hompolys (1.6.3) $$f(x, y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m \quad \text{and} \quad g(x, y) = b_0x^n + b_1x^{n-1}y + \cdots + b_ny^n$$

be homogeneous polynomials in x and y, of degrees m and n, respectively, and with complex coefficients. If these polynomials have a common zero $(x, y) = (u, v)$ in $\mathbb{P}^1_{xy}$, then $vx-uy$ divides both f and g (see (1.3.6)). Then the polynomial $h = fg/(vx-uy)$, which has degree $m+n-1$, will be divisible by f and also by g. Suppose that this is so, and that $h = pf = qg$, where p and q are homogeneous polynomials of degrees $n-1$ and $m-1$, respectively. Then p will be a linear combination of the polynomials x^iy^j, with $i+j = n-1$ and q will be a linear combination of the polynomials x^ky^ℓ, with $k+\ell = m-1$. The fact that the two combinations pf and qg are equal tells us that the $m+n$ polynomials

mplus-npolys (1.6.4) $$x^{n-1}f,\ x^{n-2}yf, ...,\ y^{n-1}f\ \ ;\ \ x^{m-1}g,\ x^{m-2}yg, ...,\ y^{m-1}g$$

of degree $m+n--1$ are (linearly) dependent. For example, if f has degree 3 and g has degree 2, and if f and g have a common zero, then the polynomials

$$\begin{array}{ll}
xf = & a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 \\
yf = & \qquad a_0x^3y + a_1x^2y^2 + a_2xy^3 + a_3y^4 \\
x^2g = & b_0x^4 + b_1x^3y + b_2x^2y^2 \\
xyg = & \qquad b_0x^3y + b_1x^2y^2 + b_2xy^3 \\
y^2g = & \qquad\qquad bx^2y^2 + b_1xy^3 + b_2y^4
\end{array}$$

will be dependent. Conversely, if the polynomials (1.6.4) are dependent, there will be an equation of the form $pf - qg = 0$, with p of degree $n-1$ and q of degree $m-1$. Then since g has degree n while p has degree $n - 1$, at least one zero of g must be a zero of f.

The polynomials (1.6.4) have degree $r = m+n-1$. We form a square $(m+n)\times(m+n)$ matrix $\mathcal{R}$, the *resultant matrix*, whose columns are indexed by the monomials $x^r, x^{r-1}y, ..., y^r$ of degree r, and whose rows list the coefficients of those monomials in the polynomials (1.6.4). The matrix is illustrated below for the cases $m,n=3,2$ and $m,n=1,2$, with dots representing entries that are zero:

(1.6.5)
$$\mathcal{R}=\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{pmatrix} \quad \text{or} \quad \mathcal{R}=\begin{pmatrix} a_0 & a_1 & \cdot \\ \cdot & a_0 & a_1 \\ b_0 & b_1 & b_2 \end{pmatrix}$$
resmatrix

The *resultant* of f and g is defined to be the determinant of $\mathcal{R}$:

(1.6.6)
$$\mathrm{Res}(f,g)=\det\mathcal{R}$$
resequals-det

In this definition, the coefficients of f and g can be in any ring.

The resultant $\mathrm{Res}(F,G)$ of the monic, one-variable polynomials $F(x)=x^m+a_1x^{m-1}+\cdots+a_m$ and $G(x)=x^n+b_1x^{n-1}+\cdots+b_n$ is the determinant of the matrix obtained from $\mathcal{R}$ by setting $a_0=b_0=1$.

1.6.7. Corollary. *Let f and g be homogeneous polynomials in two variables, or monic polynomials in one variable, of degrees m and n, respectively, and with coefficients in a field. The resultant $\mathrm{Res}(f,g)$ is zero if and only if f and g have a common factor. If so, there will be polynomials p and q of degrees $n-1$ and $m-1$ respectively, such that $pf=qg$. When the coefficients are complex numbers, the resultant is zero if and only if f and g have a common zero.* □ *homogre-sult*

When the leading coefficients a_0 and b_0 of f and g are both zero, the point $(1,0)$ of $\mathbb{P}^1$ will be a zero of f and of g. One might say that f and g have a common zero at infinity in that case.

1.6.8. Aside. *(the entries of the resultant matrix)* Define $a_i=0$ when i isn't in the range $0,...,m$, and $b_i=0$ when i isn't in the range $0,...,n$. The resultant matrix has two parts. For rows 1 to n, the (i,j)-entry of $\mathcal{R}$ is the coefficient a_{j-i} of f. For the bottom part of $\mathcal{R}$, one needs to adjust the indices. For $i=n+1,..,n+m$, the (i,j)-entry of $\mathcal{R}$ is the coefficient b_{j-i+n} of g: resentries

$$R_{ij}=a_{j-i} \text{ when } i=1,...,n\,, \text{ and } R_{ij}=b_{j-i+n} \text{ when } i=n+1,...,n+m.$$
□

1.6.9. weighted degree weights

When defining the degree of a polynomial, one may assign an integer called a *weight* to each variable. If one assigns weight w_i to the variable x_i, the monomial $x_1^{e_1}\cdots x_n^{e_n}$ gets the *weighted degree*

$$e_1w_1+\cdots+e_nw_n$$

For instance, one may assign weight k to the coefficient a_k of the polynomial $f(x)=x^n-a_1x^{n-1}+\cdots\pm a_n$. This is natural because, if f factors into linear factors, $f(x)=(x-\alpha_1)\cdots(x-\alpha_n)$, then a_k will be the kth elementary symmetric function in the roots $\alpha_1,...,\alpha_n$. When a_k written as a polynomial in the roots, its degree will be k.

1.6.10. Lemma. *Let $f(x,y)$ and $g(x,y)$ be homogeneous polynomials of degrees m and n, with variable coefficients a_i and b_j, as in (1.6.3). When one assigns weight k to a_k and to b_k, the resultant $\mathrm{Res}(f,g)$ becomes a weighted homogeneous polynomial of degree mn in the variables $\{a_i,b_j\}$.* □ *degresult*

For example, when the degrees of f and g are 1 and 2, the resultant $\mathrm{Res}(f,g)$ is the determinant of the 3×3 matrix depicted in (1.6.5), which is $a_0^2b_2+a_1^2b_0-a_0a_1b_1$. Its weighted degree is 2.

1.6.11. Proposition. *Let F and G be products of monic linear polynomials, say $F=\prod_{i=1}^m(x-\alpha_i)$ and $G=\prod_{j=1}^n(x-\beta_j)$. Then* *resroots*

$$\mathrm{Res}(F,G) \quad = \quad \prod_{i,j}(\alpha_i-\beta_j) \quad = \quad \prod_i G(\alpha_i)$$

proof. The equality of the second and third terms is obtained by substituting α_i for x into the formula $G = \prod(x-\beta_j)$. We prove the first equality. Let the polynomials F and G have variable roots α_i and β_j, and let $R = \mathrm{Res}(F,G)$, and $\Pi = \prod_{i.j}(\alpha_i - \beta_j)$. Lemma 1.6.10 tells us that, when we write the coefficients of F and G as symmetric functions in the roots, α_i and β_j, the resultant R will be homogeneous. Its (unweighted) degree in $\{\alpha_i, \beta_j\}$ will be mn. This is also the degree of Π. To show that $R = \Pi$, we choose i and j. We view R as a polynomial in the variable α_i, and divide by $\alpha_i - \beta_j$, which is a monic polynomial in α_i:

$$R = (\alpha_i - \beta_j)q + r$$

where r has degree zero in α_i. Corollary 1.6.7 tells us that the resultant R vanishes when we make the substitution $\alpha_i = \beta_j$. The coefficients of F and G are in the field of rational functions in $\{\alpha_i, \beta_j\}$. Looking at the equation above, we see that the remainder r also vanishes when $\alpha_i = \beta_j$. On the other hand, the remainder is independent of α_i. It doesn't change when we make that substitution. Therefore the remainder is zero, and $\alpha_i - \beta_j$ divides R. This is true for all i and j, so Π divides R, and since these two polynomials have the same degree, $R = c\Pi$ for some scalar c. One can show that $c = 1$ by computing R and Π for some particular polynomials. We suggest making the computation with $F = x^m$ and $G = x^n - 1$. $\square$

restrivialities

1.6.12. Corollary. *Let F, G, and H be monic polynomials and let c be a scalar. Then*
(i) $\mathrm{Res}(F, GH) = \mathrm{Res}(F,G)\,\mathrm{Res}(F,H)$, *and*
(ii) $\mathrm{Res}(F(x-c), G(x-c)) = \mathrm{Res}(F(x), G(x))$. $\square$

discrim-sect

1.6.13. the discriminant

The *discriminant* $\mathrm{Discr}(F)$ of a polynomial $F = a_0x^m + a_1x^{n-1} + \cdots a_m$ is the resultant of F and its derivative F':

discrdef

$$\mathrm{Discr}(F) = \mathrm{Res}(F, F') \tag{1.6.14}$$

It is computed using the formula for the resultant of a polynomial of degree m, and it will be a weighted polynomial of degree $m(m-1)$. The definition makes sense when the leading coefficient a_0 is zero, but the discriminant will be zero in that case. When F is a polynomial of degree m with complex coefficients, the discriminant is zero if and only if F and F' have a common factor, which happens when F has a multiple root.

Note. The formula for the discriminant is often normalized by a scalar factor that depends on the degee. We won't bother with this normalization, so our formula is slightly different from the usual one.

The discriminant of the quadratic polynomial $F(x) = ax^2 + bx + c$ is

discrquadr

$$\det\begin{pmatrix} a & b & c \\ 2a & b & \cdot \\ \cdot & 2a & b \end{pmatrix} = -a(b^2 - 4ac) \tag{1.6.15}$$

and the discriminant of a monic cubic $x^3 + px + q$ whose quadratic coefficient is zero is

discrcubic

$$\det\begin{pmatrix} 1 & \cdot & p & q & \cdot \\ \cdot & 1 & \cdot & p & q \\ 3 & \cdot & p & \cdot & \cdot \\ \cdot & 3 & \cdot & p & \cdot \\ \cdot & \cdot & 3 & \cdot & p \end{pmatrix} = 4p^3 + 27q^2 \tag{1.6.16}$$

As mentioned, these formulas differ from the usual ones by a scalar factor. The usual formula for the discriminant of the quadratic is $b^2 - 4ac$, and the discriminant of the cubic $x^3 + px + q$ is usually written as $-4p^3 - 27q^2$. Though it conflicts with our definition, we'll follow tradition and continue writing the discriminant of a quadratic as $b^2 - 4ac$.

discrtwovar

1.6.17. Example. Suppose that the coefficients a_i of F are polynomials in t, so that $F = F(t,x)$ becomes a polynomial in two variables. Let's suppose that it is an irreducible polynomial. Let C be the curve $F = 0$ in the t,x-plane. The discriminant $\mathrm{Discr}_x(F)$, computed regarding x as the variable, will be a polynomial in t. At a root t_0 of the discriminant, $F(t_0, x)$ will have a multiple root. Therefore the vertical line $\{t = t_0\}$ will be tangent to C, or pass though a singular point of C. $\square$

1.6.18. Proposition. *Let K be a field of characteristic zero. The discriminant of an irreducible polynomial F with coefficients in K isn't zero. Therefore F has no multiple root.* *dis-crnotzero*

proof. When F is irreducible, it cannot have a factor in common with its derivative, which has lower degree. $\square$

This proposition is false when the characteristic of K isn't zero. In characteristic p, the derivative F' might be the zero polynomial.

1.6.19. Proposition. *Let $F = \prod(x - \alpha_i)$ be a polynomial that is a product of monic linear polynomials $x - \alpha_i$. Then* *discrfor-mulas*

$$\mathrm{Discr}(F) = \prod_i F'(\alpha_i) = \prod_{i \neq j}(\alpha_i - \alpha_j) = \pm \prod_{i<j}(\alpha_i - \alpha_j)^2$$

proof. The fact that $\mathrm{Discr}(F) = \prod F'(\alpha_i)$ follows from (1.6.11). We prove the second equality by showing that $F'(\alpha_i) = \prod_{j,j \neq i}(\alpha_i - \alpha_j)$. By the product rule for differentiation,

$$F'(x) = \sum_k (x - \alpha_1) \cdots \widehat{(x - \alpha_k)} \cdots (x - \alpha_n)$$

where the hat $\widehat{\ }$ indicates that that term is deleted. When we substitute $x = \alpha_i$, all terms in this sum, except the one with $k = i$, become zero. $\square$

1.6.20. Corollary. $\mathrm{Discr}(F(x)) = \mathrm{Discr}(F(x - c))$. $\square$ *translate-discr*

1.6.21. Proposition. *Let $F(x)$ and $G(x)$ be monic polynomials. Then* *discrprop*

$$\mathrm{Discr}(FG) = \pm \mathrm{Discr}(F)\, \mathrm{Discr}(G) \mathrm{Res}(F, G)^2$$

proof. This proposition follows from Propositions 1.6.11 and 1.6.19 for polynomials with complex coefficients. It is true for polynomials with coefficients in any ring because it is an identity. For the same reason, Corollary 1.6.12 is true when the coefficients of the polynomials F, G, H are in any ring. $\square$

When f and g are polynomials in several variables, one of which is z, $\mathrm{Res}_z(f, g)$ and $\mathrm{Discr}_z(f)$ will denote the resultant and the discriminant, computed regarding f and g as polynomials in z. They will be polynomials in the other variables.

The next lemma follows from Lemma 1.3.14 **(ii)** and Proposition 1.6.18.

1.6.22. Lemma. *Let f be an irreducible polynomial in $\mathbb{C}[x, y, z]$ of positive degree in z, and not a scalar multiple of z. The discriminant $\mathrm{Discr}_z(f)$ is a nonzero polynomial in x, y.* $\square$ *firroverK*

1.7 Nodes and Cusps

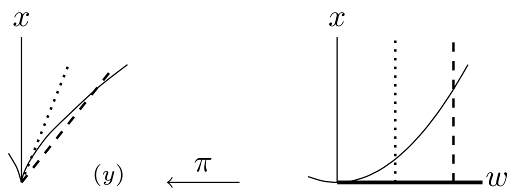
nodes

1.7.1. blowing up the plane

blup

The map $W \xrightarrow{\pi} X$ from the (x, w)-plane to the (x, y)-plane defined by $\pi(x, w) = (x, xw)$ is called an *affine blowup*. It is a useful tool for studying a singularity of a plane curve.

The fibre of the blowup map over the origin in X is the line $\{x = 0\}$ in W : $\pi(0, w) = (0, 0)$ for all w. The blowup is bijective at points (x, y) of X at which $x \neq 0$, and points $(x, 0)$ of X with $x \neq 0$ aren't in its image. It might seem more appropriate to call the inverse of π the blowup, but the inverse isn't a map.



blowupfig **1.7.2.** **Blowing up the plane**

This figure depicts the first quadrant in the two real planes. The direction of the arrow had been reversed and the axes are interchanged. The curves shown are the cusp $y^2 = x^3$ and its blowup $w^2 = x$. There is a picture on the web, in which the x, w-plane is depicted as a surface in the three-dimensional x, y, w-space.

singmult **1.7.3. the multiplicity of a singular point**

Let C be the projective curve defined by an irreducible homogeneous polynomial $f(x, y, z)$ of degree d, and let p be a point of C. We choose coordinates so that $p = (0, 0, 1)$, and we set $z = 1$. This gives us an affine curve C_0 in $\mathbb{A}^2_{x,y}$, the zero set of the polynomial $\widetilde{f}(x, y) = f(x, y, 1)$, and p becomes the origin. We write

seriesf (1.7.4) $$\widetilde{f}(x, y) = f_0 + f_1 + f_2 + \cdots + f_d$$

where f_i is the homogeneous part of $\widetilde{f}$ of degree i, which is also the coefficient of z^{d-i} in $f(x, y, z)$. If the origin p is a point of C_0, the constant term f_0 will be zero, and the linear term f_1 will define the tangent direction to C_0 at p, If f_0 and f_1 are both zero, p will be a singular point of C. It seems permissible to drop the tilde from $\widetilde{f}$ and the subscript 0 from C_0 in what follows, denoting $f(x, y, 1)$ by $f(x, y)$, and C_0 by C.

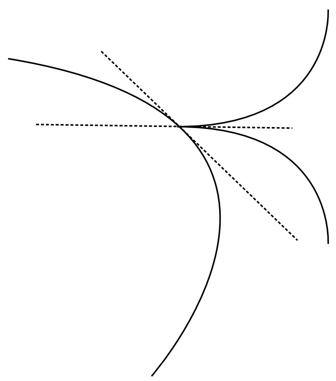
We use analogous notation for an analytic function $f(x, y)$ (9.2), denoting the homogeneous part of degree i of the series f by f_i:

multr (1.7.5) $$f(x, y) = f_0 + f_1 + \cdots$$

Let C denote the locus of zeros of f in a neighborhood of the origin p. To describe the singularity of C at the origin, we look at the part of f of lowest degree. The smallest integer r such that $f_r(x, y)$ isn't zero is called the *multiplicity* of C at p. When the multiplicity is r, f will have the form $f_r + f_{r+1} + \cdots$.

Let L be the line $\{vx = uy\}$ through p, and suppose that $u \neq 0$. In analogy with Definition 1.3.9, the intersection multiplicity (1.3.9) of C and L at p is the order of zero of the series in x obtained by substituting $y = vx/u$ into f. The intersection multiplicity will be r unless $f_r(u, v)$ is zero. If $f_r(u, v) = 0$, the intersection multiplicity will be greater than r.

A line L through p whose intersection multiplicity with C at p is greater than the multiplicity r of C at p will be called a *special line*. The special lines correspond to the zeros of f_r in $\mathbb{P}^1$. Because f_r has degree r, there will be at most r special lines.



1.7.6. **a Singular Point, with its Special Lines** (real locus) goober14

1.7.7. double points dpt

Suppose that the origin p is a *double point* of the curve C defined by a polynomial or analytic function $f(x,y)$, a point of multiplicity 2, and let the quadratic part of f be

$$f_2 = ax^2 + bxy + cy^2$$ quadraticterm

We blow up the plane. We adjust coordinates so that c isn't zero, and we normalize c to 1. Writing

$$f(x,y) = ax^2 + bxy + y^2 + dx^3 + \cdots$$

we make the substitution $y = xw$ and cancel x^2. This gives us a polynomial

$$g(x,w) \;=\; f(x,xw)/x^2 \;=\; a + bw + w^2 + dx + \cdots$$

in which all terms represented by $\cdots$ are divisible by x. Let D be the locus $\{g = 0\}$ in W. The blowup map π restricts to a map $D \xrightarrow{\overline{\pi}} C$. Since π is bijective at points at which $x \neq 0$, so is $\overline{\pi}$.

Suppose first that the quadratic polynomial $y^2 + by + a$ has distinct roots α, β, so that $ax^2 + bxy + y^2 = (y - \alpha x)(y - \beta x)$. Then $g(x,w) = (w - \alpha)(w - \beta) + dx + \cdots$. The fibre of D over the origin $p = (0,0)$ in X is obtained by substituting $x = 0$ into g. It consists of the two points $(x,w) = (0,\alpha)$ and $(x,w) = (0,\beta)$. The partial derivative $\frac{\partial g}{\partial w}$ isn't zero at either of those points, so they are smooth points of D. At each of those points, we can solve $g(x,w) = 0$ for w as analytic functions of x, say $w = u(x)$ and $w = v(x)$, with $u(0) = \alpha$ and $v(0) = \beta$. So the curve C has two analytic branches $y = xu(x)$ and $y = xv(x)$ through the origin, with distinct tangent directions α and β. This singularity is called a *node*. A node is the simplest singularity that a curve can have.

When the discriminant $b^2 - 4ac$ is zero, f_2 will be a square, and f will have the form

$$f(x,y) = (y - \alpha x)^2 + dx^3 + \cdots$$

Let's change coordinates, substituting $y + \alpha x$ for y, so that

$$f(x,y) = y^2 + dx^3 + \cdots \tag{1.7.8}$$ cuspeq

The blowup substitution $y = xw$ gives

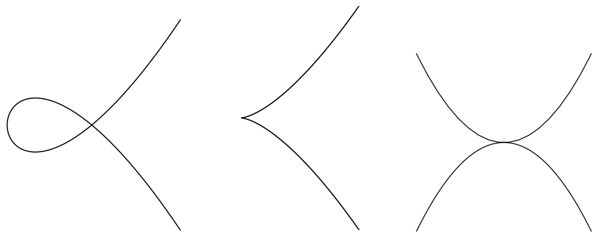
$$g(x,w) = w^2 + dx + \cdots$$

Here the fibre over $(x,y) = (0,0)$ is the point $(x,w) = (0,0)$, and $\frac{\partial g}{\partial w}(0,0) = 0$. However, if $d \neq 0$, then $\frac{\partial g}{\partial x}(0,0) \neq 0$, and if so, the zero locus of g will be smooth at $(0,0)$, and the equation of C will have the form $y^2 + dx^3 + \cdots$. This singularity is called a *cusp*.

The *standard cusp* is the locus $y^2 = x^3$. All cusps are analytically equivalent with the standard cusp.

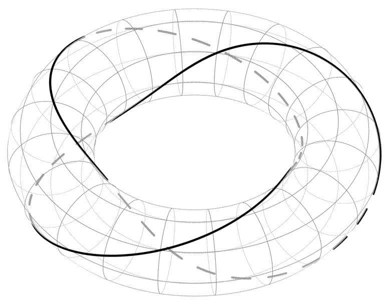
nodeor-cusp **1.7.9. Corollary.** *A double point p of a curve C is a node or a cusp if and only if the blowup of C is smooth at the points that lie over p.* □

The simplest example of a double point that isn't a node or cusp is a *tacnode*, a point at which two smooth branches of a curve intersect with the same tangent direction.



goober23 **1.7.10.** **a Node, a Cusp, and a Tacnode** (real locus)

Cusps have an interesting geometry. The intersection of the standard cusp $X : \{y^2 = x^3\}$, with a small 3-sphere $S : \{\overline{x}x + \overline{y}y = \epsilon\}$ in $\mathbb{C}^2$ is a *trefoil knot*, as is illustrated below.



trefoil **1.7.11.** **Intersection of a Cusp Curve with a Three-Sphere**

This very nice figure was made by Jason Chen and Andrew Lin.

The standard cusp X, the locus $y^2 = x^3$, can be parametrized as $(x, y) = (t^2, t^3)$. The trefoil knot is the locus of points $(x, y) = (e^{2i\theta}, e^{3i\theta})$, the set of points of X of absolute value $\sqrt{2}$. It embeds into the product of a unit x-circle and a unit y-circle in $\mathbb{C}^2$, a torus that we denote by T. The figure depicts T as a torus in $\mathbb{R}^3$, though the map to $\mathbb{R}^3$ distorts T. The circumference of T_0 represents the x-coordinate, and a loop through the hole represents y. As θ runs from 0 to 2π, the point (x, y) goes around the circumference twice, and it loops through the hole three times, as is illustrated. □

tracecusp **1.7.12. Proposition.** *Let $x(t) = t^2 + \cdots$ and $y(t) = t^3 + \cdots$ be analytic functions of t, whose orders of vanishing are 2 and 3, as indicated. For small t, the path $(x, y) = (x(t), y(t))$ in the x, y-plane traces out a curve with a cusp at the origin.*

proof. We show that there are analytic functions $b(x) = b_2x^2 + \cdots$ and $c(x) = x^3 + \cdots$ that vanish to orders 2 and 3 at $x = 0$, such that $x(t)$ and $y(t)$ solve the equation $y^2 + b(x)y + c(x) = 0$. The locus of such an equation has a cusp at the origin.

We solve for b and c. The function $x(t)$ can be written as $t^2(1 + \cdots)$. It has an analytic square root of the form $z = t(1 + \cdots)$. This follows from the Implicit Function Theorem, which also tells us that t can be written as an analytic function of z. So z is a coordinate equivalent to t, and we may replace t by z, so that $x = t^2$. The function y will still have a zero of order 3, $y = t^3 + \cdots$, though the series for y is changed.

Let's call the *even part* of a series $\sum a_n t^n$ the sum of the terms $a_n t^n$ with n even, and the *odd part* the sum of the terms with n odd. We write $y(t) = u(t) + v(t)$, where u an v are the even and the odd parts of $y(t)$, respectively. Inside its circle of convergence, the series $y(t)$ is absolutely convergent. Therefore $u(t)$ and $v(t)$

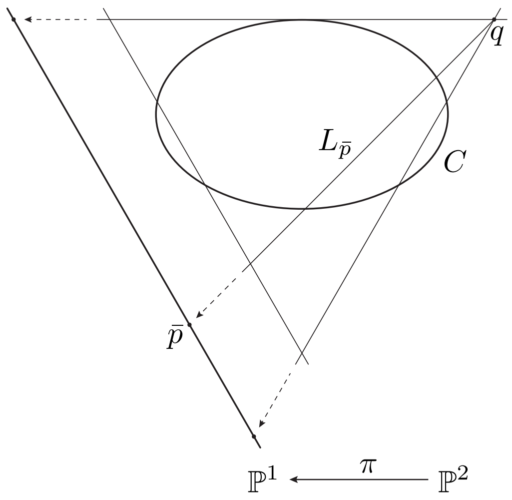
are convergent series too. Since y has a zero of order 3, v has a zero of order 3 and u has a zero of order at least 4.

Now $y^2 = (u^2+v^2) + 2uv,\ uy = u^2+uv$, and $y^2-2uy + (v^2-u^2) = 0$. The series $-2u$ and $v^2 - u^2$ that appear in this last equation are even. They can be written as convergent series in $x = t^2$, say $-2u = b(x)$ and $v^2 - u^2 = c(x)$. Since $y(t)$ has a zero of order 3, $u(t)$ has a zero of order at least 4. Then $b(x)$ will have a zero of order at least 2, c will have a zero of order equal to 3, and $y^2 + b(x)y + c(x) = 0$. The locus of such an equation has a cusp at the origin. □

1.7.13. projection to a line coverline

Let π denote the *projection* $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ that sends a point (x, y, z) to (x, y), dropping the last coordinate. This projection is defined at all points of $\mathbb{P}^2$ except at the *center of projection*, the point $q = (0, 0, 1)$.

The fibre of π over a point $\overline{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the line through $p = (x_0, y_0, 0)$ and $q = (0, 0, 1)$, with the point q omitted. We may denote that line by L_{pq} or by $L_{\overline{p}}$.



1.7.14. Projection from the Plane to a Line projtoline

Let $f(x, y, z)$ be an irreducible homogeneous polynomial whose zero locus C is a plane curve that doesn't contain the center of projection q, and let d be the degree of f. The projection π will be defined at all points of the curve. We write f as a polynomial in z,

$$f = c_0 z^d + c_1 z^{d-1} + \cdots + c_d \tag{1.7.15}$$
poly-inztwo

with c_i homogeneous, of degree i in x, y. When q isn't in C, the scalar $c_0 = f(0, 0, 1)$ won't be zero, and we can normalize c_0 to 1, so that f becomes a monic polynomial of degree d in z.

Let's assume that C is a smooth curve. The fibre of C over a point $\overline{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the intersection of C with the line L_{pq} described above. Its points are (x_0, y_0, α), where α is a root of the one-variable polynomial

$$\widetilde{f}(z) = f(x_0, y_0, z) \tag{1.7.16}$$
effp

We call the smooth curve C a *branched covering* of $\mathbb{P}^1$, of degree d. All but finitely many fibres of C over $\mathbb{P}^1$ consist of d points. The fibres of π with fewer than d points are those above the zeros of the discriminant (see Lemma 1.6.22). Those zeros are the *branch points* of the covering. We use the same term for certain points of C, calling a point p a *branch point* if the tangent line there is L_{pq}, in which case its image will be a branch point of $\mathbb{P}^1$.

1.7.17. Proposition. *Let $C \xrightarrow{\pi} \mathbb{P}^1$ be be the projection of a smooth plane curve from a generic point q of the plane, and let p be a branch point of C, so that the tangent line L at p contains q. The intersection multiplicity of L and C at p is 2, and L and C have $d-2$ other intersections of multiplicity* 1. *flexbitan*

The proof is below, but we explain the word *generic* first.

genpos **1.7.18. generic and general position**

In algebraic geometry, the word *generic* is used for an object, such as a point, that has no special 'bad' properties. Typically, the object will be parametrized somehow, and the adjective 'generic' indicates that the parameter representing that particular object avoids a proper closed subset of the parameter space, which may be described explicitly or not. The phrase *general position* has a similar meaning. It indicates that an object isn't in a special 'bad' position. In Proposition 1.7.17, what is required of the generic point q is that it shall not lie on a flex tangent line or on a *bitangent* — a line that is tangent to C at two or more points. We have seen that a smooth curve C has finitely many flex points (1.4.18), and Lemma 1.7.19 below states that it has finitely many bitangents. So q must avoid a finite set of lines. Most points of the plane will be generic in this sense. □

proof of Proposition 1.7.17. The intersection multiplicity of the tangent line L with C at p is at least 2 because L is a tangent line. It will be equal to 2 unless p is a flex point. The generic point q won't lie on any of the finitely many flex tangent lines, so the intersection multiplicity at p is 2. Next, the intersection multiplicity at another point p' of $L \cap C$ will be 1 unless L is tangent to C at p' as well as at p, i.e., unless L is a bitangent. The generic point q won't lie on a bitangent. □

finbitan **1.7.19. Lemma.** *A plane curve has finitely many bitangent lines.*

proof. This isn't especially easy to prove directly. We use the map $U \xrightarrow{t} C^*$ from the set U of smooth points of C to the dual curve C^*. If a line L is tangent to C at distinct smooth points p and p', then t will be defined at those points, and $t(p) = t(p') = L^*$. Therefore L^* will be a singular point of C^*. Since C^* has finitely many singular points (1.4.7), C has finitely many bitangents. □

genus **1.7.20. the genus of a plane curve**

The topological structure of a smooth plane curve in the classical topology is described by the next theorem.

curveshome-omorphic **1.7.21. Theorem.** *A smooth projective plane curve of degree d is a compact, orientable, and connected two-dimensional manifold.*

The fact that a smooth curve is a two-dimensional manifold follows from the Implicit Function Theorem. (See the discussion in (1.4.4)).

orientability: A two-dimensional manifold is orientable if one can choose one of its two sides (as in front and back of a sheet of paper) in a continuous, consistent way. A smooth curve C is orientable because its tangent space at a point, the affine line with the equation (1.4.11), is a one-dimensional complex vector space. Multiplication by i orients the tangent space by defining the counterclockwise rotation. Then the right-hand rule tells us which side of C is "up".

compactness: A plane projective curve is compact because it is a closed subset of the compact space $\mathbb{P}^2$.

connectedness: The fact that a plane curve is connected mixes topology and algebra, and it is subtle. Unfortunately, I don't know a nice proof that fits the discussion here. It will be proved later (see Theorem 8.2.11).

The topological *Euler characteristic* of a compact, orientable two-dimensional manifold M is the alternating sum $b^0 - b^1 + b^2$ of its Betti numbers, the dimensions of its homology groups. The Euler characteristic, which we denote by e, can also be computed using a *topological triangulation*, a subdivision of M into topological triangles, called *faces*, by the formula

vef (1.7.22)
$$e = |\text{vertices}| - |\text{edges}| + |\text{faces}|$$

For example, a sphere is homeomorphic to a tetrahedron, which has four vertices, six edges, and four faces. Its Euler characteristic is $4 - 6 + 4 = 2$. Any other topological triangulation of a sphere, such as the one given by the icosahedron, yields the same Euler characteristic.

Every compact, connected, orientable two-dimensional manifold is homeomorphic to a sphere with a finite number of "holes" (also called "handles"). Its *genus* is the number of holes. A torus has one hole. Its genus is one. The projective line $\mathbb{P}^1$, a two-dimensional sphere, has genus zero.

The Euler characteristic and the genus are related by the formula

$$e = 2 - 2g \tag{1.7.23}$$ genuseuler

The Euler characteristic of a torus is zero, and the Euler characteristic of $\mathbb{P}^1$ is two.

To compute the Euler characteristic of a smooth curve C, we analyze a generic projection (a projection from a generic point q of the plane), to represent C as a branched covering of the projective line: $C \xrightarrow{\pi} \mathbb{P}^1$. We choose generic coordinates x, y, z in $\mathbb{P}^2$ and project from the point $q = (0, 0, 1)$. Say that C has degree d. When the defining equation of C is written as a monic polynomial in z: $f = z^d + c_1 z^{d-1} + \cdots + c_d$ where c_i is a homogeneous polynomial of degree i in the variables x, y, the discriminant $\mathrm{Discr}_z(f)$ with respect to z will be a homogeneous polynomial of degree $d(d-1) = d^2 - d$ in x, y.

Let $\widetilde{p}$ be the image in $\mathbb{P}^1$ of a point p of C. The covering $C \xrightarrow{\pi} \mathbb{P}^1$ will be branched at $\widetilde{p}$ when the tangent line at p is the line L_{pq} through p and q. Proposition 1.7.17 tells us that if L_{pq} is a tangent line, there will be one intersection of multiplicity 2 and $d-1$ intersections of multiplicity 1. The discriminant will have a simple zero at such a point $\widetilde{p}$. This is proved in Proposition 1.8.13 below. Let's assume it for now.

Since the discriminant has degree $d^2 - d$, there will be $d^2 - d$ points $\widetilde{p}$ of $\mathbb{P}^1$ at which the discriminant vanishes, and the fibre over such a point will contain $d-1$ points. They are the *branch points* of the covering. All other fibres consist of d points.

We triangulate the sphere $\mathbb{P}^1$ in such a way that the branch points are among the vertices, and we use the inverse images of the vertices, edges, and faces to triangulate C. Then C will have d faces and d edges lying over each face and each edge of $\mathbb{P}^1$, respectively. There will also be d vertices of C lying over a vertex of $\mathbb{P}^1$ except when the vertex is one of the branch points, in which case the the fibre will contain only $d-1$ vertices. So the Euler characteristic of C can be obtained by multiplying the Euler characteristic of $\mathbb{P}^1$ by d and subtracting the number $d^2 - d$ of branch points:

$$e(C) \;=\; d\,e(\mathbb{P}^1) - (d^2 - d) \;=\; 2d - (d^2 - d) \;=\; 3d - d^2 \tag{1.7.24}$$ eulercover

This is the Euler characteristic of any smooth curve of degree d, so we denote it by e_d :

$$e_d = 3d - d^2 \tag{1.7.25}$$ equatione

Formula (1.7.23) shows that the genus g_d of a smooth curve of degree d is

$$g_d \;=\; \tfrac{1}{2}(d^2 - 3d + 2) = \binom{d-1}{2} \tag{1.7.26}$$ equationg

Thus smooth curves of degrees $1, 2, 3, 4, 5, 6, ...$ have genus $0, 0, 1, 3, 6, 10, ...$, respectively. A smooth plane curve cannot have genus 2.

The generic projection to $\mathbb{P}^1$ with center q can also be used to compute the degree of the dual curve C^* of a smooth curve C of degree d. The degree of C^* is the number of its intersections with the generic line q^* in $\mathbb{P}^*$. The intersections of C^* and q^* are the points L^*, where L is a tangent line that contains q. As we saw above, there are $d^2 - d$ such lines.

1.7.27. Corollary. *Let C be a smooth plane projective curve of degree d. The degree of the dual curve C^* is $d^* = d^2 - d$. It is equal to the number of tangent lines to C that pass through a generic point q of the plane.* □ *degdual*

When C is a singular curve, the degree of C^* will be less than $d^2 - d$.

If $d = 2$, so that C is a conic, $d^* = d$. We have seen before that the dual curve of a conic is also a conic. But when $d > 2$, $d^* = d^2 - d$ will be greater than d. Then the dual curve C^* will be singular. If it were smooth, the degree of its dual curve C^{**} would be ${d^*}^2 - d^*$, which would be greater than d too. This would contradict the fact that $C^{**} = C$. For instance, when $d = 3$, $d^* = 3^2 - 3 = 6$, and ${d^*}^2 - d^* = 30$. The dual curve C^* has enough singularity to account for the discrepancy between 30 and 3.

1.8 Hensel's Lemma

The resultant matrix (1.6.5) arises in a second context that we explain here. hensel

Suppose given a product $P = FG$ of two polynomials, say

multiply-polys (1.8.1) $$\left(c_0x^{m+n} + c_1x^{m+n-1} + \cdots + c_{m+n}\right) = \left(a_0x^m + a_1x^{m-1} + \cdots + a_m\right)\left(b_0x^n + b_1x^{n-1} + \cdots + b_n\right)$$

We call the relations among the coefficients that are implied by this polynomial equation the *product equations*. The product equations are

$$c_i = a_ib_0 + a_{i-1}b_1 + \cdots + a_0b_i \quad = \sum_{j=0}^{i} a_{i-j}b_j$$

for $i = 0, ..., m+n$. When $m = 3$ and $n = 2$, they are

prodeqns **1.8.2.**

$$\begin{aligned} c_0 &= a_0b_0 \\ c_1 &= a_1b_0 + a_0b_1 \\ c_2 &= a_2b_0 + a_1b_1 + a_0b_2 \\ c_3 &= a_3b_0 + a_2b_1 + a_1b_2 \\ c_4 &= \qquad\quad\ \ a_3b_1 + a_2b_2 \\ c_5 &= \qquad\qquad\qquad\quad a_3b_2 \end{aligned}$$

Let J denote the Jacobian matrix of partial derivatives of $c_1, ..., c_{m+n}$ with respect to the variables $b_1, ..., b_n$ and $a_1, ..., a_m$, treating a_0, b_0 and c_0 as constants. Here $\frac{\partial c_i}{\partial b_j} = a_{i-j}$ and $\frac{\partial c_i}{\partial a_j} = b_{i-j}$. So the i, j-entry of J is

jentries (1.8.3) $$J_{ij} = a_{i-j} \text{ when } j = 1, ..., n \text{ and } J_{ij} = b_{i-j+n} \text{ when } j = n+1, ..., n+m$$

entries with negative subscripts being set to 0.

When $m, n = 3, 2$,

prodjacob (1.8.4) $$J = \frac{\partial(c_i)}{\partial(b_j, a_k)} = \begin{pmatrix} a_0 & . & b_0 & . & . \\ a_1 & a_0 & b_1 & b_0 & . \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & . & b_2 & b_1 \\ . & a_3 & . & . & b_2 \end{pmatrix}$$

jacres **1.8.5. Lemma.** *The Jacobian matrix J is the transpose of the resultant matrix $\mathcal{R}$ (1.6.5).*

The lemma follows when one compares (1.8.5) with (1.6.8), but this may be an occasion to quote Cayley. While discussing the Cayley-Hamilton Theorem, Cayley wrote: 'I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case.' □

jacobian-notzero **1.8.6. Corollary.** *Let F and G be polynomials with complex coefficients. The Jacobian matrix is singular if and only if, either F and G have a common root, or $a_0 = b_0 = 0$.* □

This corollary has an application to polynomials with analytic coefficients. Let

poly-forhensel (1.8.7) $$P(t, x) = c_0(t)x^d + c_1(t)x^{d-1} + \cdots + c_d(t)$$

be a polynomial in x whose coefficients c_i are analytic functions of t, and let $\overline{P} = P(0, x) = \overline{c}_0x^d + \overline{c}_1x^{d-1} + \cdots + \overline{c}_d$ be the evaluation of P at $t = 0$, so that $\overline{c}_i = c_i(0)$. Suppose given a factorization $\overline{P} = \overline{F}\,\overline{G}$, where $\overline{F} = x^m + \overline{a}_1x^{m-1} + \cdots + \overline{a}_m$ and $\overline{G} = \overline{b}_0x^n + \overline{b}_1x^{n-1} + \cdots + \overline{b}_n$ are polynomials with complex coefficients, and $\overline{F}$ is monic. Are there polynomials $F(t, x) = x^m + a_1x^{m-1} + \cdots + a_m$ and $G(t, x) = b_0x^n + b_1x^{n-1} + \cdots + b_n$, with F monic, whose coefficients a_i and b_j are analytic functions of t, such that $F(0, x) = \overline{F}$, $G(0, x) = \overline{G}$, and $P = FG$?

hensellemma **1.8.8. Hensel's Lemma.** *With notation as above, suppose that $\overline{F}$ and $\overline{G}$ have no common root. Then P factors: $P = FG$, where F and G are polynomials in x, whose coefficients are analytic functions of t, and F is monic.*

proof. We look at the product equations. Since F is supposed to be monic, we set $a_0(t) = 1$. The first product equation tells us that $b_0(t) = c_0(t)$. Corollary 1.8.6 tells us that the Jacobian matrix for the remaining product equations is nonsingular at $t = 0$, so according to the Implicit Function Theorem, the product equations have a unique solution in analytic functions $a_i(t), b_j(t)$. $\square$

Note that P isn't assumed to be monic. If $\overline{c}_0 = 0$, the degree of $\overline{P}$ will be less than the degree of P. In that case, $\overline{G}$ will have lower degree than G.

1.8.9. Example. Let $P = c_0(t)x^2 + c_1(t)x + c_2(t)$. The product equations $P = FG$ with $F = x + a_1$ monic and $G = b_0x + b_1$, are henselex

(1.8.10) $$c_0 = b_0\,, \quad c_1 = a_1b_0 + b_1\,, \quad c_2 = a_1b_1$$ quadfactor

and the Jacobian matrix is
$$\frac{\partial(c_1, c_2)}{\partial(b_1, a_1)} = \begin{pmatrix} 1 & b_0 \\ a_1 & b_1 \end{pmatrix}$$

Suppose that $\overline{P} = P(0, x)$ factors: $\overline{c}_0x^2 + \overline{c}_1x + \overline{c}_2 = (x + \overline{a}_1)(\overline{b}_0x + \overline{b}_1) = \overline{F}\,\overline{G}$. The determinant of the Jacobian matrix at $t = 0$ is $\overline{b}_1 - \overline{a}_1\overline{b}_0$. It is nonzero if and only if the factors $\overline{F}$ and $\overline{G}$ are relatively prime, in which case P factors too.

On the other hand, the one-variable Jacobian criterion allows us to solve the equation $P(t, x) = 0$ for x as function of t with $x(0) = -\overline{a}_1$, provided that $\frac{\partial P}{\partial x} = 2c_0x + c_1$ isn't zero at the point $(t, x) = (0, -\overline{a}_1)$. If $\overline{P}$ factors as above, then when we substitute (1.8.10) into $\overline{P}$, we find that $\frac{\partial P}{\partial x}(0, -\overline{a}_1) = -2\overline{c}_0\overline{a}_1 + \overline{c}_1 = \overline{b}_1 - \overline{a}_1\overline{b}_0$. Not surprisingly, $\frac{\partial P}{\partial x}(0, -\overline{a}_1)$ is equal to the determinant of the Jacobian matrix at $t = 0$. $\square$

1.8.11. order of vanishing of the discriminant vandisc

We introduce some terminology for use in the next proposition. Let X be the affine x-line, let Y be the affine x, y-plane and let p be the origin in Y. Two curves are said to intersect *transversally* at a point p if they are smooth at p and their tangent lines there are distinct.

Let C be the plane affine curve defined by a polynomial $f(x, y)$ with no multiple factors, and suppose that C contains the origin p. Let L be the line $\{x = 0\}$ in Y. Suppose that all intersections of C with L are transversal, except at the point p.

1.8.12. Proposition. *With notation as above,* **(a)** *Let p be a smooth point of C with tangent line L. If p isn't a flex point of C, the discriminant* $\mathrm{Discr}_y(f)$ *has a simple zero at the origin.* *affdvan*

(b) *If p is a node of C and L is not a special line,* $\mathrm{Discr}_y(f)$ *has a double zero at the origin.*

(c) *If p is a cusp of C and L is not its special line,* $\mathrm{Discr}_y(f)$ *has a triple zero at the origin.*

(d) *If p is an ordinary flex point of C and L is its tangent line, then* $\mathrm{Discr}_y(f)$ *has a double zero at the origin.*

proof. Let $\overline{f}(y) = f(0, y)$. In cases **(a,b,c)**, $\overline{f}(y)$ will have a double zero at $y = 0$. We will have $\overline{f}(y) = y^2\overline{h}(y)$, with $\overline{h}(0) \neq 0$. So y^2 and $\overline{h}(y)$ have no common root. We apply Hensel's Lemma: $f(x, y) = g(x, y)h(x, y)$, where g and h are polynomials in y whose coefficients are analytic functions of x, g is monic, $g(0, y) = y^2$, and $h(0, y) = \overline{h}$. Then $\mathrm{Discr}_y(f) = \pm\,\mathrm{Discr}_y(g)\,\mathrm{Discr}_y(h)\,\mathrm{Res}_y(g, h)^2$ (1.6.21).

Since C is tranversal to L except at q, $\overline{h}$ has simple zeros (1.7.17). Then $\mathrm{Discr}_y(h)$ doesn't vanish at $y = 0$. Neither does $\mathrm{Res}_y(g, h)$. So the orders of vanishing of $\mathrm{Discr}_y(f)$ and $\mathrm{Discr}_y(g)$ at p are equal.

We replace f by g, so that f becomes a monic quadratic polynomial in y, of the form
$$f(x, y) = y^2 + b(x)y + c(x)$$
where the coefficients b and c are analytic functions of x, and $f(0, y) = y^2$. The discriminant $\mathrm{Discr}_y(f) = b^2 - 4c$ is unchanged when we complete the square by the substitution of $y - \frac{1}{2}b$ for y, and if p is a smooth point, a node or a cusp, that property isn't affected by this operation. So we may assume that f has the form $y^2 + c(x)$. The discriminant is then $D = 4c(x)$.

We write $c(x)$ as a series:
$$c(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

Since C contains p, the constant coefficient c_0 is zero. If $c_1 \neq 0$, p is a smooth point with tangent line $\widetilde{L}$, and D has a simple zero. If p is a node, $c_0 = c_1 = 0$ and $c_2 \neq 0$. Then D has a double zero. If p is a cusp, $c_0 = c_1 = c_2 = 0$, and $c_3 \neq 0$. Then D has a triple zero at p.

(d) In this case, the polynomial $\widetilde{f}(y) = f(0, y)$ will have a triple zero at $y = 0$. Proceding as above, we may factor: $f = gh$ where g and h are polynomials in y whose coefficients are analytic functions of x, $g(x, y) = y^3 + a(x)y^2 + b(x)y + c(x)$, and $g(0, y) = y^3$. We eliminate the quadratic coefficient a by substituting $y - \frac{1}{3a}$ for y. With $g = y^3 + by + c$ in the new coordinates, the discriminant $\mathrm{Discr}_y(g)$ is $4b^3 + 27c^2$ (1.6.16). We write $c(x) = c_0 + c_1x + \cdots$ and $b(x) = b_0 + b_1x + \cdots$. Since p is a smooth point of C with tangent line $\{y{=}0\}$, $c_0 = 0$ and $c_1 \neq 0$. Since the intersection multiplicity of C with the line $\{y{=}0\}$ at p is three, $b_0 = 0$. The discriminant $4b^3 + 27c^2$ has a zero of order two. $\square$

Let $f(x, y, z)$ be a homogeneous polynomial with no multiple factors, and let C be the (possibly reducible) plane curve $\{f = 0\}$. We project to $X = \mathbb{P}^1$ from a point q that isn't on C, adjusting coordinates so that $q = (0, 0, 1)$. Let L_{pq} denote the line through a point $p = (x_0, y_0, 0)$ and q, the set of points (x_0, y_0, z_0), and let $\widetilde{p} = (x_0, y_0)$. Suppose that all intersections of C with L_{pq} are transversal.

discrim-vanishing

1.8.13. Corollary. *With notation as above:*
(a) *If p is a smooth point of C with tangent line L_{pq}, the discriminant* $\mathrm{Discr}_z(f)$ *has a simple zero at $\widetilde{p}$.*
(b) *If p is a node of C,* $\mathrm{Discr}_z(f)$ *has a double zero at $\widetilde{p}$.*
(c) *If p is a cusp,* $\mathrm{Discr}_z(f)$ *has a double zero at $\widetilde{p}$.*
(d) *If p is an ordinary flex point of C with tangent line L_{pq},* $\mathrm{Discr}_z(f)$ *has a double zero at $z{=}0$.* $\square$

In cases **(a,b,c)**, the hypotheses are satisfied when the center of projection q is in general position. To be precise about what is required of the generic point q in those cases, we ask that q not lie on C or on any of the following lines:

generic-cond

(1.8.14)
flex tangent lines and bitangent lines,
lines that contain more than one singular point,
special lines through singular points,
tangent lines that contain a singular point.

This is a list of finitely many lines that q must avoid.

transres

1.8.15. Corollary. *Let $C : \{g = 0\}$ and $D : \{h = 0\}$ be plane curves that intersect transversally at a point $p = (z_0, y_0, z_0)$. With coordinates in general position,* $\mathrm{Res}_z(g, h)$ *has a simple zero at (x_0, y_0).*

proof. Proposition 1.8.13 **(b)** applies to the product gh, whose zero locus is the union $C \cup D$. It shows that the discriminant $\mathrm{Discr}_z(gh)$ has a double zero at p. We also have the formula

$$\mathrm{Discr}_z(gh) = \mathrm{Discr}_z(g)\,\mathrm{Discr}_z(h)\,\mathrm{Res}_z(g, h)^2$$

(1.6.21). When coordinates are in general position, $\mathrm{Discr}_z(g)$ and $\mathrm{Discr}_z(h)$ will not be zero at p. Then $\mathrm{Res}_z(g, h)$ has a simple zero there. $\square$

1.9 Bézout's Theorem

bezoutthm

Bézout's Theorem counts intersections of plane curves. We state it here in a form that is ambiguous because it contains a term "multiplicity" that hasn't yet been defined.

bezoutone

1.9.1. Bézout's Theorem. Let C and D be distinct curves of degrees m and n, respectively. When intersections are counted with an appropriate multiplicity, the number of intersections is mn. Moreover, the multiplicity at a transversal intersection is 1.

For example, a conic C and a line L meet in two points unless L is a tangent line, in which case they have one point of intersection of multiplicity 2.
As before, C and D intersect transversally at p if they are smooth at p and their tangent lines there are distinct.

bezoutline

1.9.2. Proposition. *Bézout's Theorem is true when one of the curves is a line.*

See Corollary 1.3.10. The multiplicity of intersection of a curve and a line is the one that was defined there. □

The proof in the general case requires some algebra that we would rather defer. It will be given later (Theorem 7.8.1), but we will use the theorem in the rest of this chapter.

It is possible to determine the intersections by counting the zeros of the resultant with respect to one of the variables. To do this, one chooses coordinates x, y, z, so that neither C nor D contains the point $(0, 0, 1)$. One writes their defining polynomials f and g as polynomials in z with coefficients in $\mathbb{C}[x, y]$. The resultant R with respect to z will be a homogeneous polynomial in x, y, of degree mn. It will have mn zeros in $\mathbb{P}^1_{x,y}$, counted with multiplicity (see 1.3.4). If $\widetilde{p} = (x_0, y_0)$ is a zero of R, the polynomials $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$ in z have a common root $z = z_0$, and $p = (x_0, y_0, z_0)$ will be a point of $C \cap D$. It is a fact that the multiplicity of the zero of the resultant R at the image $\widetilde{p}$ is the (as yet undefined) intersection multiplicity of C and D at p. Unfortunately, this won't be obvious, even when the multiplicity has been defined. However, one can prove the next proposition using this approach.

1.9.3. Proposition. *Let C and D be distinct plane curves of degrees m and n, respectively.* *nocom-monfactor*

(i) *C and D have at least one intersection, and the number of intersections is at most mn.*

(ii) *If all intersections are transversal, the number of intersections is precisely mn.*

It isn't obvious that two curves in the projective plane intersect. If two curves in the affine plane have no intersection, If they are parallel lines, for example, their closures in the projective plane meet on the line at infinity.

1.9.4. Lemma. *Let f and g be homogeneous polynomials in x, y, z of degrees m and n, respectively, and suppose that the point $(0, 0, 1)$ isn't a zero of f or g. If the resultant $\mathrm{Res}_z(f, g)$ with respect to z is identically zero, then f and g have a common factor.* *resnotzero*

proof. Let F denote the field of rational functions $\mathbb{C}(x, y)$. If the resultant is zero, f and g have a common factor in $F[z]$. There will be polynomials p and q in $F[z]$, of degrees at most $n-1$ and $m-1$ in z, respectively, such that $pf = qg$ (1.6.3). We may clear denominators, so we may assume that the coefficients of p and q are in $\mathbb{C}[x, y]$. This doesn't change their degree in z. Then $pf = qg$ is an equation in $\mathbb{C}[x, y, z]$, and since p has degree $n - 1$, it isn't divisible by g. Since $\mathbb{C}[x, y, z]$ is a unique factorization domain, f and g have a common factor. □

proof of Proposition 1.9.3. **(i)** Let C and D be distinct curves, defined by irreducible homogeneous polynomials f and g. Proposition 1.3.12 shows that there are finitely many intersections. We project to $\mathbb{P}^1$ from a point q that doesn't lie on any of the finitely many lines through pairs of intersection points. Then a line through q passes through at most one intersection, and the zeros of the resultant $\mathrm{Res}_z(f, g)$ that correspond to the intersection points will be distinct. The resultant has degree mn (1.6.10). It has at least one zero, and at most mn of them. Therefore C and D have at least one and at most mn intersections.

(ii) Every zero of the resultant will be the image of an intersection of C and D. To show that there are mn intersections if all intersections are transversal, it suffices to show that the resultant has simple zeros. This is Corollary 1.8.15. □

1.9.5. Corollary. *If the zero locus X of a homogeneous polynomial $f(x, y, z)$ is smooth, then f is irreducible, and therefore X is a smooth curve.* *smoothirred*

proof. Suppose that $f = gh$, and let p be a point of intersection of the loci $\{g = 0\}$ and $\{h = 0\}$. Proposition 1.9.3 shows that such a point exists. All partial derivatives of f vanish at p, so p is a singular point of the locus $f = 0$ (1.4.7). □

1.9.6. Proposition. **(i)** *Let d be an integer ≥ 3. A smooth plane curve of degree d has at least one flex point, and the number of flex points is at most $3d(d-2)$.* *numberof-flexes*

(ii) *If all flex points are ordinary, the number of flex points is equal to $3d(d-2)$.*

Thus smooth curves of degrees $2, 3, 4, 5, ...$ have at most $0, 9, 24, 45, ...$ flex points, respectively.

proof. **(i)** Let C be the smooth curve defind by a homogeneous polynomial f of degree d. Let H be the Hessian matrix of f, let $\det H = h_1^{e_1} \cdots h_k^{e_k}$ be the factorization of its determinant into irreducible polynomials, and let Z_i be the locus of zeros of h_i. The *Hessian divisor* is defined to be the combination $D = e_1 Z_1 + \cdots + e_k Z_k$.

The flex points of C are its intersections with its Hessian divisor D (1.4.17). The entries of the 3×3 Hessian matrix H, the second partial derivatives f_{ij} are homogeneous polynomials of degree $d-2$. So $\det H$ is homogeneous, of degree $3(d-2)$. Propositions 1.4.18 and 1.9.3 tell us that there are at most $3d(d-2)$ intersections.

(ii) A flex point is ordinary if the multiplicity of intersection of the curve and its tangent line is 3 (1.4.8). Bézout's Theorem asserts that the number of flex points is $3d(d-2)$ if the intersections of C with its Hessian divisor D are transversal, and therefore have multiplicity 1. So the next lemma completes the proof.

transver-salH **1.9.7. Lemma.** *A curve $C : \{f = 0\}$ intersects its Hessian divisor D transversally at a point p if and only p is an ordinary flex point of C.*

proof. We prove the lemma by computation. I don't know a conceptual proof.

Let D be the Hessian divisor $\{\det H = 0\}$. The Hessian determinant $\det H$ vanishes at a smooth point p of C if and only if p is a flex point (1.4.17).

Assume that p is a flex point, let L be the tangent line to C at p, and let $\overline{h}$ denote the restriction of the determinant $\det H$ to L. We adjust coordinates x, y, z so that p is the point $(0, 0, 1)$ and L is the line $\{y = 0\}$, and we set $z = 1$ to work in the affine space $\mathbb{A}^2_{x,y}$. Because p is a flex point, the coefficients of the monomials $1, x$ and x^2 in the polynomial $f(x, y, 1)$ are zero. So

$$f(x, y, 1) = ay + bxy + cy^2 + dx^3 + ex^2y + \cdots$$

To restrict to L, we set $y = 0$, keeping $z = 1$: $f(x, 0, 1) = dx^3 + O(4)$, where $O(k)$ stands for a polynomial all of whose terms have degree $\geq k$.

To compute the determinant $\det H$, we put the variable z back. If f has degree n, then

$$f(x, y, z) = ayz^{n-1} + bxyz^{n-2} + cy^2z^{n-2} + dx^3z^{n-3} + ex^2yz^{n-3} + \cdots$$

The Hessian divisor D will be transversal to C at p if and only if it is transversal to L there, which will be true if and only if the restriction of $\det H$ to L, which is $\det H(x, 0, 1)$, has a zero of order 1 at the origin.

We set $y = 0$ and $z = 1$ in the second order partial derivatives. Let $v = 6dx$ and $w = (n-1)a + (n-2)bx$. Then

$f_{xx}(x, 0, 1) = 6dx + O(2) = v + O(2)$,
$f_{xz}(x, 0, 1) = 0 + O(2)$,
$f_{yz}(x, 0, 1) = (n-1)a + (n-2)bx + O(2) = w + O(2)$,
$f_{zz}(x, 0, 1) = 0 + O(2)$,

We won't need f_{xy} or f_{yy}. The Hessian matrix at $y = 0, z = 1$ has the form

(1.9.8)
$$H(x, 0, 1) \;=\; \begin{pmatrix} v & * & 0 \\ * & * & w \\ 0 & w & 0 \end{pmatrix} \;+\; O(2)$$

Because of the zeros, the entries marked with $*$ don't affect the determinant. It is

$$\det H(x, 0, 1) \;=\; -vw^2 + O(2) \;=\; -6d(n-1)^2a^2x + O(2)$$

and it has a zero of order 1 at $x = 0$ if and only if a and d aren't zero there. Since C is smooth at p and since the coefficient of x in f is zero, the coefficient of y, which is a, can't be zero. Thus the curve C and its Hessian divisor D intersect transversally, if and only if d isn't zero. This is true if and only if p is an ordinary flex point. □

nineflexes **1.9.9. Corollary.** *A smooth cubic curve contains exactly 9 flex points.*

proof. Let C be a smooth cubic curve. The Hessian divisor D of C also has degree 3, so Bézout's Theorem predicts at most 9 intersections of D with C. To derive the corollary, we show that D intersects C transversally, and to do this, we show that D intersects the tangent line L to C at p transversally. According to Lemma 1.9.7, a nontransversal intersection of D and L would correspond to a point at which C has a flex that isn't ordinary, and at such a point, the intersection multiplicity of C and L would be greater than 3. This is impossible when the curve is a cubic. □

1.9.10. singularities of the dual curve singdual

Let C be a plane curve. As before, an *ordinary flex point* is a smooth point p such that the intersection multiplicity of the curve and its tangent line L at p is equal to 3. A bitangent, a line L that is tangent to C at distinct points p and p', is an *ordinary bitangent* if neither p nor p' is a flex point. A tangent line L at a smooth point p of C is an *ordinary tangent* if p isn't a flex point and L isn't a bitangent.

The tangent line L at a point p will have other intersections with C. Most often, those other intersections will be transversal, unless L is a bitangent, in which case it will be tangent to C at a second point. However, it may also happen that one of the other intersections of L with C is a singular point of C. Or, L may be a *tritangent*, tangent to C at three points. Let's call such occurences *accidents*.

1.9.11. Definition. A plane curve C is *ordinary* if it is smooth, all of its bitangents and flex points are ordinary, and there are no accidents. ordcurve

1.9.12. Lemma. *A generic curve C is ordinary.* *genisord*

We verify this by counting constants. The reasoning is fairly convincing, though not completely precise.

There are three ways in which a curve C might fail to be ordinary:

- C may be singular.
- C may have a flex point that isn't ordinary.
- A bitangent to C may be a flex tangent or a tritangent.

The curve will be ordinary if none of these occurs.

Let the coordinates in the plane be x, y, z. The homogeneous polynomials of degree d form a vector space whose dimension is equal to the number N of monomials $x^iy^jz^k$ of degree $i+j+k = d$. Let f be a homogeneous polynomial of degree d, and let $f = \prod f_i^{e_i}$ be its factorization into irreducible polynomials. If Z_i denotes the zero locus of f_i, the divisor associated to f is $\sum e_iZ_i$. The divisors of degree d are parametrized by points of a projective space of dimension $n = N-1$, and curves correspond to points in a subset of that space.

singular points. We look at the point $p_0 = (0, 0, 1)$, and we set $z = 1$. If p_0 is a singular point of the curve C defined by a polynomial f, the coefficients of $1, x, y$ in the polynomial $f(x, y, 1)$ will be zero. This is three conditions. The curves that are singular at p_0 are parametrized by a linear subspace of dimension $n-3$ in the projective space $\mathbb{P}^n$, and the same will be true when p_0 is replaced by any other point of $\mathbb{P}^2$. The points of $\mathbb{P}^2$ depend on 2 parameters. Therefore, in the space of divisors, the singular curves form a subset of dimension at most $n-1$. Most curves are smooth.

flex points. Let's look at curves that have a four-fold tangency with the line $L : \{y = 0\}$ at p_0. Setting $z = 1$ as before, we see that the coefficients of $1, y, y^2, y^3$ in f must be zero. This is four conditions. The lines through p_0 depend on one parameter, and the points of $\mathbb{P}^2$ depend on two parameters, giving us three parameters to vary. We can't get all curves this way. Most curves have no four-fold tangencies, and therefore they have only ordinary flexes.

bitangents. To be tangent to the line $L : \{y = 0\}$ at the point p_0, the coefficients of 1 and y in f must be zero. This is two conditions. Then to be tangent to L at three given points p_0, p_1, p_2 imposes 6 conditions. A set of three points of L depends on three parameters, and a line depends on two parameters, giving us 5 parameters in all. Most curves don't have a tritangent. Similar reasoning takes care of bitangents in which one tangency is a flex. □

1.9.13. Proposition. *Let p be a point of an ordinary curve C, and let L be the tangent line at p.* *dualcusp*
If L is an ordinary tangent at p, then L^* is a smooth point of C^*.
If L is a bitangent, then L^* is a node of C^*.
If p is a flex, then L^* is a cusp of C^*.

proof. We refer to the map $C \xrightarrow{t} C^*$ from C to the dual curve (1.5.4). Because C is smooth, t is defined at all points of C.

We dehomogenize the defining polynomial f by setting $z = 1$, and choose affine coordinates, so that p is the point $(x, y, z) = (0, 0, 1)$, and the tangent line L at p is the line $\{y = 0\}$. Then L^* is the point

$(u,v,w) = (0,1,0)$. Let $\widetilde{f}(x,y) = f(x,y,1)$. We solve $\widetilde{f} = 0$ for $y = y(x)$ as an analytic function of x, as before. The tangent line L_1 to C at a nearby point $p_1 = (x_1, y_1)$ has the equation (1.5.12), and L_1^* is the point $(u,v,w) = (-y_1', 1, y_1'x_1 - y_1)$ of $\mathbb{P}^*$ (1.5.13). Since there are no accidents, this path traces out all points of C^* near to L^* (Corollary 1.5.19).

If L is an ordinary tangent line, $y(x)$ will have a zero of order 2 at $x = 0$. Then $u = -y'$ will have a simple zero. So the path $(-y', 1, y'x - y)$ is smooth at $x = 0$, and therefore C^* is smooth at the origin.

If L is an ordinary bitangent, tangent to C at two points p and p', the reasoning given for an ordinary tangent shows that the images in C^* of small neighborhoods of p and p' in C will be smooth at L^*. Their tangent lines p^* and p'^* will be distinct, so p is a node.

Suppose that p is an ordinary flex point. Then, in the power series $y(x) = c_0 + c_1 x + c_2 x^2 + \cdots$, the coefficients c_0, c_1, c_2 are zero and since the flex is ordinary, $c_3 \neq 0$. We may assume that $c_3 = 1$ and that $y(x) = x^3 + \cdots$. Then, in the local equation $(u,v,w) = (-y', 1, y'x - y)$ for the dual curve, $u = -3x^2 + \cdots$ and $w = 2x^3 + \cdots$. Proposition 1.7.12 tells us that the singularity at the origin is a cusp. $\square$

1.10 The Plücker Formulas

plucker

The *Plücker Formulas* compute the number of flexes and bitangents of an ordinary plane curve. Since the bitangents aren't easy to count directly, the fact that there is a formula for bitangents is particularly interesting.

plform

1.10.1. Theorem: Plücker Formulas. *Let C be an ordinary curve of degree d at least two, and let C^* be its dual curve. Let f and b denote the numbers of flex points and bitangents of C, and let d^*, δ^* and κ^* denote the degree, the numbers of nodes, and the number of cusps of C^*, respectively. Then:*

(i) *The dual curve C^* has no flexes or bitangents. Its singularities are nodes and cusps.*

(ii) $d^* = d^2 - d, \quad f = \kappa^* = 3d(d-2), \quad$ *and* $\quad b = \delta^* = \frac{1}{2}d(d-2)(d^2-9).$

proof. First, a bitangent or a flex on C^* would produce a singularity on the bidual C^{**}, which is the smooth curve C. Next, because C is ordinary, Proposition 1.9.13 shows that its singularities are nodes and cusps, and also that $f = \kappa^*$ and $b = \delta^*$. The degree d^* was computed in Corollary 1.7.27. Bézout's Theorem counts the flex points, so $f = 3d(d-2)$ (1.9.6). Thus $\kappa^* = 3d(d-2)$.

To count the bitangents, we project C^* to $\mathbb{P}^1$ from a generic point s of $\mathbb{P}^*$. The number of branch points that correspond to tangent lines at smooth points that contain s is the degree d of the bidual C (1.7.27).

Let $F(u,v,w)$ be the defining polynomial for C^*. The discriminant $\mathrm{Discr}_w(F)$ has degree $d^{*2} - d^*$. Corollary 1.8.13 tells us that the order of vanishing of the discriminant at the images of the d tangent lines through s, the δ nodes of C^*, and the κ cusps of C^*, are $1, 2, 3$, respectively. Then $d^{*2} - d^* = d + 2\delta^* + 3\kappa^*$. Substituting the known values $d^* = d^2 - d$, and $\kappa^* = 3d(d-2)$ into this formula gives us $(d^2-d)^2 - (d^2-d) = d + 2\delta^* + 9d(d-2)$, or

$$2\delta^* = d^4 - 2d^3 - 9d^2 + 18d = d(d-2)(d^2-9)$$

$\square$

some-plucker-formulas

1.10.2. Examples.

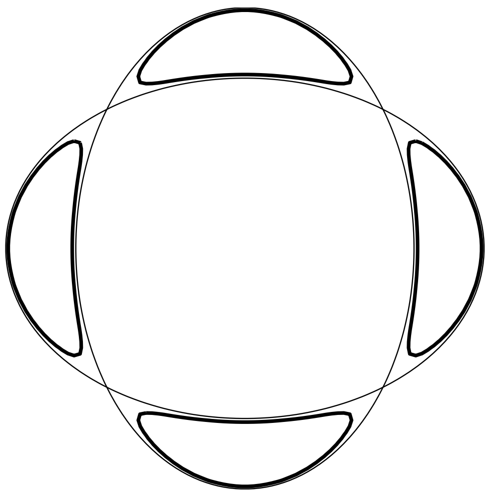
(i) All curves of degree 2 and all smooth curves of degree 3 are ordinary.

(ii) A curve of degree 2 has no flexes and no bitangents. Its dual curve has degree 2.

(iii) A smooth curve of degree 3 has 9 flexes and no bitangents. Its dual curve has degree 6.

(iv) An ordinary curve C of degree 4 has 24 flexes and 28 bitangents. Its dual curve has degree 12. $\square$

We will make use of the fact that a quartic curve has 28 bitangents in Chapter 4 (see (4.7.18)). The Plücker Formulas are rarely used for curves of larger degree, but the fact that there is such a formula is interesting.



A Quartic Curve whose 28 **Bitangents are Real** (real locus)

1.10.3.

To obtain this quartic, we added a small constant ϵ to the product of the quadratic equations of the two ellipses that are shown. The equation of the quartic is $(2x^2 + y^2 - 1)(x^2 + 2y^2 - 1) + \epsilon = 0$.

1.11 Exercises

chaponeex xsin **1.11.1.** Prove that the path $x(t) = t,\ y(t) = \sin t$ in $\mathbb{A}^2$ doesn't lie on any plane curve. (Zariski assigned this problem in an algebraic geometry class in the 1950's.)

xinfpoints **1.11.2.** Prove that a plane curve contains infinitely many points.

xtwovarirred **1.11.3.** Use counting constants to show that most (nonhomogeneous) polynomials in two or more variables are irreducible.

xnotonline **1.11.4.** Let $f(x,y,z)$ be an irreducible homogeneous polynomial of degree > 1. Prove that the locus $f = 0$ in $\mathbb{P}^2$ contains three points that do not lie on a line.

xstaysirreducible **1.11.5.** Let $f(x,y,z)$ be a homogeneous polynomial not divisible by z. Prove that f is irreducible if and only if $f(x,y,1)$ is irreducible.

xdiagform **1.11.6.** **(i)** Classify conics in $\mathbb{P}^2$ by writing an irreducible quadratic polynomial in three variables in the form X^tAX where A is symmetric, and diagonalizing this quadratic form.

(ii) Quadrics in projective space $\mathbb{P}^n$ are zero sets of irreducible homogeneous quadratic polynomials in $x_0, ..., x_n$. Classify quadrics in $\mathbb{P}^3$.

xlociequal **1.11.7.** Let f and g be irreducible homogeneous polynomials in x, y, z. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ are equal, then $g = cf$.

xcubicsing **1.11.8.** Prove that a plane projective cubic curve can have at most one singular point.

xtanconic **1.11.9.** Let C be the plane projective curve defined by the equation $x_0x_1 + x_1x_2 + x_2x_0 = 0$, and let p be the point $(-1, 2, 2)$. What is the equation of the tangent line to C at p?

xeqforcubic **1.11.10.** Let C be a smooth cubic curve in $\mathbb{P}^2$, and let p be a flex point of C. Choose coordinates so that p is the point $(0,1,0)$ and the tangent line to C at p is the line $\{z = 0\}$. Prove the following::

(i) With a suitable choice of coordinates, one can reduce the defining polynomial to the form $y^2z + x^3 + axz^2 + bz^3$, and $x^3 + ax + b$ will be a polynomial with distinct roots.

(ii) One of the coefficients a or b can be eliminated, and therefore smooth cubic curves in $\mathbb{P}^2$ depend on just one parameter.

xhessian **1.11.11.** Using Euler's formula together with row and column operations, show that the Hessian determinant is equal to $a \det H'$, where

$$H' = \begin{pmatrix} cf & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}, \quad a = \left(\tfrac{d-1}{x_0}\right)^2, \quad \text{and} \quad c = \tfrac{d}{d-1}$$

xhessianZero **1.11.12.** Prove that a smooth point of a curve is a flex point if and only if the Hessian determinant is zero, in this way: Given a smooth point p of X, choose coordinates so that $p = (0,0,1)$ and the tangent line ℓ is the line $\{x_1 = 0\}$. Then compute the Hessian.

xsymmfnindep **1.11.13.** Prove that the elementary symmetric functions in n variables are algebaically independent.

xelementtransc **1.11.14.** Let K be a field extension of a field F, and let α be an element of K that is transcendental over F. Prove that every element of the field $F(\alpha)$ that isn't in F is transcendental over F.

xtdadds **1.11.15.** Let $L \supset K \supset F$ be fields. Prove that the transcendence degree of the extension L/F is the sum of the transcendence degrees of L/K and K/F.

xdualiscurve **1.11.16.** Let $f(x_0, x_1, x_2)$ be a homogeneous polynomial of degree d, let $f_i = \frac{\partial f}{\partial x_i}$, and let C be the plane curve $\{f = 0\}$. Use the following method to prove that the image in the dual plane of the set of smooth points of C is contained in a curve C^*: Let $N_r(k)$ be the dimension of the space of polynomials of degree $\le k$ in r variables. Determine $N_r(k)$ for $r = 3$ and $r = 4$. Show that $N_4(k) > N_3(kd)$ if k is sufficiently large. Conclude that there is a nonzero polynomial $G(x_0, x_1, x_2)$ such that $G(f_0, f_1, f_2) = 0$.

xtangentq **1.11.17.** Let C be a smooth cubic curve in the plane $\mathbb{P}^2$, and let q be a generic point of $\mathbb{P}^2$. How many lines through q are tangent lines to C?

1.11.18. Let X and Y be the surfaces in $\mathbb{A}^3_{x,y,z}$ defined by the equations $z^3 = x^2$ and $yz^2 + z + y = 0$, respectively. The intersection $C = X \cap Y$ is a curve. Determine the equation of the projection of C to the x, y-plane. xprojcurve

1.11.19. Comput the resultant of the polynomials x^m and $x^n - 1$. xercres-formula

1.11.20. Let f, g, and h be polynomials in one variable. Prove that xresfgh

(i) $\mathrm{Res}(f, gh) = \mathrm{Res}(f, g)\,\mathrm{Res}(f, h)$.

(ii) If the degree of gh is less than or equal to the degree of f, then $\mathrm{Res}(f, g) = \mathrm{Res}(f + gh, g)$.

1.11.21. Prove that a generic line meets a plane projective curve of degree d in d distinct points. gener-alline

1.11.22. Let $f = x^2 + xz + yz$ and $g = x^2 + y^2$. Compute the resultant $\mathrm{Res}_x(f, g)$ with respect to the variable x. xrest-wopol

1.11.23. If $F(x) = \prod(x - \alpha_i)$, then $\mathrm{Discr}(F) = \pm \prod_{i<j}(\alpha_i - \alpha_j)^2$. Determine the sign. xsign

1.11.24. Let $f = a_0x^m + a_1x^{m-1} + \cdots a_m$ and $g = b_0x^n + b_1x^{n-1} + \cdots b_n$, and let $R = \mathrm{Res}(f, g)$ be the resultant of these polynomials. Prove that etdegres

(i) R is a polynomial that is homogeneous in each of the sets of variables a and b, and determine its degree.

(ii) If one assigns weighted degree i to the coefficients a_i and b_i, then R is homogeneous, of weighted degree mn.

(iii) Let α_i and β_j be the roots of $f(x, 1)$ and $g(x, 1)$, respectively. Show that if a_0 and b_0 are not zero, then $\mathrm{Res}(f, g) = a_0^n b_0^m \prod(\alpha_i - \beta_j)$.

1.11.25. Let coordinates in $\mathbb{A}^4$ be x, y, z, w, let Y be the variety defined by $z^2 = x^2 - y^2$ and $w(z - x) = 1$, and let π denote projection from Y to (x, y)-space. Describe the fibres of π and the image of π. xamap

1.11.26. Let p be a cusp of the curve C defined by a homogeneous polynomial f. Prove that there is just one line L through p such that the restriction of f to L has as zero of order > 2 at p, and that the order of zero for that line is precisely 3. xcusptan

1.11.27. Describe the intersection of the node $xy = 0$ at the origin with the unit 3-sphere in $\mathbb{A}^2$. xgeo-mofnode

1.11.28. Prove that the Fermat curve $C : \{x^d + y^d + z^d = 0\}$ is connected by studying its projection to $\mathbb{P}^1$ from the point $(0, 0, 1)$. xfermat-conn

1.11.29. Prove that every cusp is analytically equivalent with the standard cusp. xcuspstan-dard

1.11.30. Analyze the singularities of the plane curve $x^3y^2 - x^3z^2 + y^3z^2 = 0$. xsing-sofcurve

1.11.31. Exhibit an irreducible homogeneous polynomial $f(x, y, z)$ of degree 4 whose locus of zeros is a curve with three cusps. xthree-cusps

1.11.32. Let $p(t, x) = x^3 + x^2 + t$. Then $p(0, x) = x^2(x + 1)$. Since x^2 and $x + 1$ are relatively prime, Hensel's Lemma predicts that p factors: $p = fg$, where g and g are polynomials in x whose coefficients are analytic functions in t, and f is monic, $f(0, x) = x^2$, and $g(0, x) = x + 1$. Determine this factorization up to degree 3 in t. Do the same for the polynomial $tx^4 + x^3 + x^2 + t$. xhensel

1.11.33. Let $f(t, y) = ty^2 - 4y + t$.

(i) Solve $f = 0$ for y by the quadratic formula, and sketch the real locus $f = 0$ in the t, y plane. xxhensellemma

(ii) What does Hensel's Lemma tell us about f?

(iii) Factor f, modulo t^4.

(iv) Factor $g(t, x) = x^3 + 2tx^2 + t^2x + x + t$, modulo t^2.

1.11.34. Using a generic projection to $\mathbb{P}^1$, determine the degree of the dual C^* of a plane curve C of degree 4 with three nodes. cus-pcurved-ual

1.11.35. Let C be a cubic curve with a node. Determine the degree of the dual curve C^*, and the numbers of flexes, bitangents, nodes, and cusps of C and of C^*. xdualnode xdefsing **1.11.36.** About the order of vanishing of the discriminant: With notation as in (1.8.11), when one perturbs the equation of C, the line L that meets C at p will be replaced by a finite set of nearby tangent lines. Choose particular examples for C in parts **(b,c,d)** of (1.8.12) and compare the number of nearby tangents with the order of vanishing of the discriminant.

xthreepts **1.11.37.** **(i)** Show that there is a conic C that passes through any five points of $\mathbb{P}^2$.

(ii) Prove that a plane curve X of degree 4 can have at most three singular points.

xintconic **1.11.38.** Prove Bézout's Theorem when one of the curves is a conic. Do this by parametrizing the conic.

Chapter 2 AFFINE ALGEBRAIC GEOMETRY

affine

The next chapters are about varieties of arbitrary dimension. We will use some of the terminology that was introduced in Chapter 1, discriminant and transcendence degree for instance, but many of the results in Chapter 1 won't be used again until we come back to curves in Chapter 8.

There is a review of some commutative algebra, the theory of commutative rings, in Section 9.1. Take a look at that section, but don't spend much time on it. You can refer back as needed, and look up information on concepts that aren't familiar.

Except when the contrary is stated explicitly the word 'ring' will mean commutative ring, $ab = ba$, with identity element 1. A *domain* is a ring without zero divisors, that isn't isn't the zero ring. A ring that contains the field $\mathbb{C}$ of complex numbers as a subring is an *algebra*.

A set of elements $\{\alpha_1, ..., \alpha_n\}$ *generates* an algebra A if every element of A can be expressed (usually not uniquely) as a polynomial in those elements, with complex coefficients. A *finite-type algebra* is an algebra that are generated by a finite set of elements.

2.1 The Zariski Topology

zartop

Affine algebraic geometry is a study of subsets of affine space that can be defined by systems of polynomial equations. Those subsets are the closed sets in the Zariski topology on $\mathbb{A}^n$, the *Zariski closed* sets. A *Zariski open* set U is a set whose complement, the set of points not in U, is Zariski closed.

If $f_1, ..., f_k$ are polynomials in $x_1, ..., x_n$, the set of points of $\mathbb{A}^n$ that solve the system of equations

fequat (2.1.1)
$$f_1 = 0\,,\ \dots,\ f_k = 0$$

the *locus of zeros* of f, may be denoted by $V(f_1, ..., f_k)$ or by $V(f)$. Thus $V(f)$ is a Zariski closed set.

We use analogous notation for infinite sets. If $\mathcal{F}$ is any set of polynomials, $V(\mathcal{F})$ denotes the set of points of affine space at which all elements of $\mathcal{F}$ are zero. In particular, if I is an ideal of the polynomial ring, $V(I)$ denotes the set of points at which all elements of I vanish.

As usual, the ideal I of $\mathbb{C}[x_1, ..., x_n]$ generated by polynomials $f_1, ..., f_k$ is the set of combinations $r_1f_1 + \cdots + r_kf_k$ with polynomial coefficients r_i. Some notations for this ideal are $(f_1, ..., f_k)$ and (f). All elements of I vanish on the zero set $V(f)$, so $V(f) = V(I)$. The Zariski closed subsets of $\mathbb{A}^n$ can also be described as the sets $V(I)$, where I is an ideal.

An ideal isn't determined by its zero locus. For one thing, all powers f^k of a polynomial f have the same zeros as f.

2.1.2. Lemma. *Let I and J be ideals of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$.* *IinJ*
(i) *If $I \subset J$, then $V(I) \supset V(J)$.*
(ii) $V(I^k) = V(I)$.
(iii) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.
(iv) *If I_ν are ideals, then $V(\sum I_\nu)$ is the intersection $\bigcap V(I_\nu)$.*

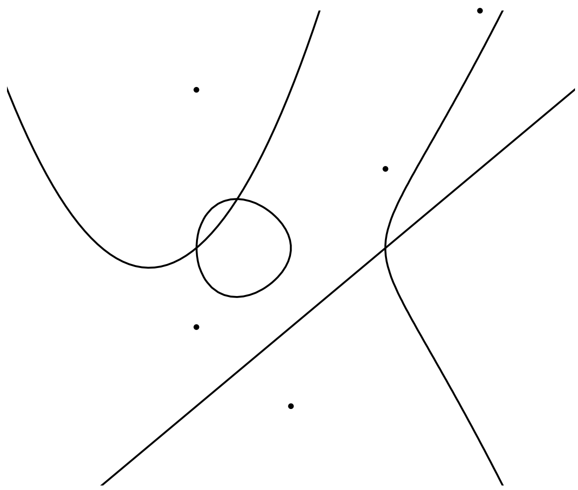
proof. **(iii)** Since $(I \cap J)^2 \subset IJ \subset I \cap J$, **(ii)** tells us that $V(I \cap J) = V(IJ)$. Because I and J contain IJ, $V(IJ) \supset V(I) \cup V(J)$. To prove that $V(IJ) \subset V(I) \cup V(J)$, we note that $V(IJ)$ is the locus of common zeros of the products fg with f in I and g in J. Suppose that a point p is a common zero: $f(p)g(p) = 0$ for all f in I and all g in J. If there is an element f in I such that $f(p) \neq 0$, we must have $g(p) = 0$ for every g in J, and then p is a point of $V(J)$. If $f(p) = 0$ for all f in I, then p is a point of $V(I)$. In either case, p is a point of $V(I) \cup V(J)$. □

2.1.3. To verify that the Zariski closed sets are the closed sets of a topology, one must show that ztop

- the empty set and the whole space are Zariski closed,
- the intersection $\bigcap C_\nu$ of an arbitrary family of Zariski closed sets is Zariski closed, and
- the union $C \cup D$ of two Zariski closed sets is Zariski closed.

The empty set and the whole space are the zero sets of the elements 1 and 0, respectively. The other conditions follow from Lemma 2.1.2. □

2.1.4. Example. The proper Zariski closed subsets of the affine line, or of a plane affine curve, are the nonempty finite sets. The proper Zariski closed subsets of the affine plane are finite unions of points and curves. We omit the proofs of these facts. The corresponding facts for loci in the projective line and the projective plane have been noted before. (See (1.3.4) and (1.3.15).) □ ztopdimone



2.1.5. **A Zariski closed subset of the affine plane** (real locus) zclosed

2.1.6. A subset S of a topological space X becomes a topological space with its *induced topology*. The closed (or open) subsets of S in the induced topology are intersections $S \cap Y$, where Y is closed (or open) in X. In these notes, a subset of X will always be given the induced topology, and to emphasize this, we may speak of it as a *subspace* of X. induced

The topology induced on a subset S from the Zariski topology on $\mathbb{A}^n$ will be called the *Zariski topology* on S too. A subset of S is closed in its Zariski topology if it has the form $S \cap Z$ for some Zariski closed subset Z of $\mathbb{A}^n$. When S is itself a Zariski closed subset of $\mathbb{A}^n$, a closed subset of S can also be described as a closed subset of $\mathbb{A}^n$ that is contained in S.

closedinaff **2.1.7. Lemma.** *Let $\{X^i\}$ be a covering of a topological space X by open sets. A subset V of X is open if and only if $V \cap X^i$ is open in X^i for every i, and V is closed if and only if $V \cap X^i$ is closed in X^i for every i. In particular, if $\{\mathbb{U}^i\}$ is the standard open covering of $\mathbb{P}^n$, a subset V of $\mathbb{P}^n$ is open (or closed) if and only if $V \cap \mathbb{U}^i$ is open (or closed) in $\mathbb{U}^i$ for every i.* $\square$

When two topologies T and T' on a set X are given, T' is said to be *coarser* than T if every closed set in T' is closed in T i.e., if T' contains fewer closed sets (or fewer open sets) than T, and T' is *finer* than T if it contains more closed sets (or more open sets) than T. The Zariski topology is coarser than the classical topology, and as the next proposition shows, it is much coarser.

opendense **2.1.8. Proposition.** *Every nonempty Zariski open subset of $\mathbb{A}^n$ is dense and path connected in the classical topology.*

proof. The (complex) *line* L through distinct points p and q of $\mathbb{A}^n$ is a Zariski closed subset of $\mathbb{A}^n$. Its points can be written as $p+t(q-p)$, with t in $\mathbb{C}$. A line corresponds bijectively to the affine t-line $\mathbb{A}^1$, and the Zariski closed subsets of L correspond to Zariski closed subsets of $\mathbb{A}^1$. They are the finite subsets, and L itself.

Let U be a nonempty Zariski open subset of $\mathbb{A}^n$, and let C be the Zariski closed complement of U. To show that U is dense in the classical topology, we choose distinct points p and q of $\mathbb{A}^n$, with p in U. If L is the line through p and q, $C \cap L$ will be a Zariski closed subset of L, a finite set that doesn't contain p. The complement of this finite set in L is $U \cap L$. In the classical topology, the closure of $U \cap L$ will be the whole line L. The closure of U contains the closure of $U \cap L$, which is L, so it contains q. Since q was arbitrary, the closure of U is $\mathbb{A}^n$.

Next, let L be the line through two points p and q of U. As before, $C \cap L$ will be a finite set of points. In the classical topology, L is the plane of complex numbers. The points p and q can be joined by a path in that plane that avoids a finite set. $\square$

Though the Zariski topology is very different from the classical topology, it is very useful in algebraic geometry. We will use the classical topology from time to time, but the Zariski topology will appear more often. Because of this, we refer to a Zariski closed subset simply as a *closed set*. Similarly, by an *open set* we mean a Zariski open set. We will mention the adjective "Zariski" only for emphasis.

irrclosed **2.1.9. irreducible closed sets**

The fact that the polynomial algebra is a noetherian ring has an important consequence for the Zariski topology that we discuss here.

A topological space X has the *descending chain condition* on closed subsets if there is no infinite, strictly descending chain $C_1 > C_2 > \cdots$ of closed subsets of X. (See (9.1.14).) The descending chain condition on closed subsets is equivalent with the *ascending chain condition* on open sets.

A *noetherian space* is a topological space that has the descending chain condition on closed sets. In a noetherian space, every nonempty family $\mathcal{S}$ of closed subsets has a minimal member, one that doesn't contain any other member of $\mathcal{S}$, and every nonempty family of open sets has a maximal member (see (9.1.12)).

noethqcomp **2.1.10. Lemma.** *A noetherian topological space is quasicompact: Every open covering has a finite subcovering.* $\square$

deschain **2.1.11. Proposition.** *With its Zariski topology, $\mathbb{A}^n$ is a noetherian space.*

proof. Suppose that a strictly descending chain $C_1 > C_2 > \cdots$ of closed subsets of $\mathbb{A}^n$ is given. Let I_j be the ideal of elements of the polynomial ring $\mathbb{C}[x_1,...,x_n]$ that are identically zero on C_j. Then $C_j = V(I_j)$. The fact that $C_j > C_{j+1}$ implies that $I_j < I_{j+1}$. The ideals I_j form a strictly increasing chain. Since $\mathbb{C}[x_1,...,x_n]$ is noetherian, that chain is finite. Therefore the chain C_j is finite. $\square$

defirrspace **2.1.12. Definition.** A topological space X is *irreducible* if it isn't the union of two proper closed subsets.

The concept of irreducibility is useful primarily for noetherian spaces. The only irreducible subsets of a Hausdorff space are its points. So, in the classical topology, the only irreducible subsets of affine space are points.

Irreducibility may seem analogous to connectedness. A topological space is *connected* if it isn't the union $C \cup D$ of two proper *disjoint* closed subsets. However, the condition that a space be irreducible is much more restrictive because, in Definition 2.1.12, the closed sets C and D aren't required to be disjoint. In the Zariski topology on the affine plane, lines are irreducible closed sets. The union of two intersecting lines is connected, but not irreducible.

2.1.13. Proposition. *The following conditions on a topological space X are equivalent.* *irredlemma*

- *X is irreducible.*
  - *If C and D are closed subsets of X, and if $X = C \cup D$, then $X = C$ or $X = D$.*
- *The intersection $U \cap V$ of nonempty open subsets is nonempty.*
- *Every nonempty open subset U of X is dense — its closure is X.* $\square$

The *closure* of a subset U of a topological space X is the smallest closed subset of X that contains U. It is the intersection of all closed subsets that contain S.

2.1.14. Lemma. *Let Y be a subspace of a topological space X, let S be a subset of Y, and let C be the closure of S in X. The closure of S in Y is $C \cap Y$.* *closureinters*

proof. The closure $\overline{S}$ of S in Y is the intersection of the closed subsets of Y that contain S. A subset W is closed in Y if and only if $W = V \cap Y$ for some closed subset V of X, and if W contains S, so does V. The intersection of the closed subsets V of X that contain S is C. Then $\overline{S} = \bigcap W = \bigcap (V \cap Y) = \big(\bigcap V\big) \cap Y = C \cap Y$. $\square$

2.1.15. Lemma. (i) *The closure $\overline{Z}$ of a subspace Z of a topological space X is irreducible if and only if Z is irreducible.* *quasicompact*

(ii) *A nonempty open subspace W of an irreducible space X is irreducible.*

(iii) *Let $Y \to X$ be a continuous map of topological spaces. The image of an irreducible subset D of Y is an irreducible subset of X.*

proof. **(i)** Let Z be an irreducible subset of X, and suppose that its closure $\overline{Z}$ is the union $\overline{C} \cup \overline{D}$ of two closed sets $\overline{C}$ and $\overline{D}$. Then Z is the union of the sets $C = \overline{C} \cap Z$ and $D = \overline{D} \cap Z$, and they are closed in Z. Therefore Z is one of those two sets, say $Z = C$. Then $Z \subset \overline{C}$, and since $\overline{C}$ is closed, $\overline{Z} \subset \overline{C}$. Because $\overline{C} \subset \overline{Z}$ as well, $\overline{C} = \overline{Z}$. Conversely, suppose that the closure $\overline{Z}$ of a subset Z of X is irreducible, and that Z is a union $C \cup D$ of closed subsets. Then $\overline{Z} = \overline{C} \cup \overline{D}$, and therefore $\overline{Z} = \overline{C}$ or $\overline{Z} = \overline{D}$. If $\overline{Z} = \overline{C}$, then $Z = \overline{C} \cap Z = C$ (2.1.14). So Z is irreducible.

(ii) The closure of W is the irreducible space X (see (2.1.13)).

(iii) Let D be an irreducible subspace of Y, whose image C is the union $C_1 \cup C_2$ of closed subsets. The inverse image D_i of C_i is closed in D, and $D = D_1 \cup D_2$. Therefore either $D_1 = D$ or $D_2 = D$. Say that $D_1 = D$. Then D is the inverse image of C_1, the image of D is C_1, and $C_1 = C$. $\square$

2.1.16. Theorem. *In a noetherian topological space, every closed subset is the union of finitely many irreducible closed sets.* *unionirred*

proof. If a closed subset C_0 of a topological space X isn't a union of finitely many irreducible closed sets, then it isn't irreducible, so it is a union $C_1 \cup D_1$, where C_1 and D_1 are proper closed subsets of C_0, and therefore closed subsets of X. Since C_0 isn't a finite union of irreducible closed sets, C_1 and D_1 cannot both be finite unions of irreducible closed sets. Say that C_1 isn't such a union. We have the beginning $C_0 > C_1$ of a chain. We repeat the argument, replacing C_0 by C_1, and we continue in this way, to construct an infinite, strictly descending chain $C_0 > C_1 > C_2 > \cdots$ of closed subsets. So X isn't a noetherian space. $\square$

2.1.17. Definition. An *affine variety* is an irreducible closed subset of affine space $\mathbb{A}^n$. defadffvar

Theorem 2.1.16 tells us that every closed subset of $\mathbb{A}^n$ is a finite union of affine varieties. Since an affine variety is irreducible, it is connected in the Zariski topology. An affine variety is also connected in the classical topology, but this isn't easy to prove. We may not get to the proof.

2.1.18. noetherian induction noethind

In a noetherian space Z one can use *noetherian induction* in proofs. Suppose that a statement Σ is to be proved for every closed subvariety X of Z. It suffices to prove Σ for X under the assumption that Σ is true for every closed subvariety that is a proper subset of X. Or, to prove a statement Σ for every closed subset X of Z, it suffices to prove it for X under the assumption that Σ is true for every proper closed subset of X.

The justification of noetherian induction is similar to the justification of complete induction. Let $\mathcal{S}$ be the family of closed subvarieties for which Σ is false. If $\mathcal{S}$ isn't empty, it will contain a minimal member X. Then Σ will be true for every proper closed subvariety of X, etc.

coordi-nateagebra

2.1.19. the coordinate algebra of a variety

Let I be an ideal of R. The *radical* of I of is the set of elements α of R such that some power α^r is in I. The radical is an ideal that contains I. It will be denoted by $\operatorname{rad} I$:

raddef **2.1.20.** $\operatorname{rad} I = \{\alpha \in R \mid \alpha^r \in I \text{ for some } r > 0\}$

An ideal that is equal to its radical is a *radical ideal*. A prime ideal is a radical ideal.

Recall that $V(P)$ denotes the set of points of affine space at which all elements of P vanish.

radpower **2.1.21. Lemma. (i)** *An ideal I of a noetherian ring R contains a power of its radical.*
(ii) *If I is an ideal of the polynomial ring $\mathbb{C}[x]$, then $V(I) = V(\operatorname{rad} I)$.*

proof. **(i)** Since R is noetherian, $\operatorname{rad} I$ is generated by a finite set of elements $\alpha = \{\alpha_1, ..., \alpha_k\}$, and for large r, α_i^r is in I. We can use the same large integer r for every i. A monomial $\beta = \alpha_1^{e_1} \cdots \alpha_k^{e_k}$ of sufficiently large degree n in α will be divisible α_i^r for at least one i, and therefore it will be in I. The monomials of degree n generate $(\operatorname{rad} I)^n$, so $(\operatorname{rad} I)^n \subset I$. $\square$

If I and J are ideals and if $\operatorname{rad} I = \operatorname{rad} J$, then $V(I) = V(J)$. The converse of this statement is also true: If $V(I) = V(J)$, then $\operatorname{rad} I = \operatorname{rad} J$. This is a consequence of the *Strong Nullstellensatz*. (See (2.3.10) below).

Because $(I \cap J)^2 \subset IJ \subset I \cap J$, $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J)$. Also, $\operatorname{rad}(I \cap J) = (\operatorname{rad} I) \cap (\operatorname{rad} J)$. Therefore $V(\operatorname{rad}(I \cap J)) = V(I) \cup V(J)$.

irredprime **2.1.22. Proposition.** *The affine varieties in $\mathbb{A}^n$ are the sets $V(P)$, where P is a prime ideal of the polynomial algebra $\mathbb{C}[x_1, ..., x_n]$. If P is a radical ideal of $\mathbb{C}[x]$, then $V(P)$ is an affine variety if and only if P is a prime ideal.*

We will use Proposition 2.1.22 in the next section, where we give a few examples of varieties, but we defer the proof to Section 2.4, where the proposition will be proved in a more general form. (See Proposition 2.4.12).)

defco-ordalg **2.1.23. Definition.** Let P be a prime ideal of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, and let V be the affine variety $V(P)$ in $\mathbb{A}^n$. The *coordinate algebra* of V is the quotient algebra $A = \mathbb{C}[x]/P$.

Geometric properties of the variety are reflected in algebraic properties of its coordinate algebra and vice versa. In a primitive sense, one can regard the geometry of an affine variety V as given by closed subsets and incidence relations — the inclusion of one closed set into another, as when a point lies on a line. A finer study of the geometry takes into account other things, such as tangency, but it is reasonable to begin by studying incidences $C' \subset C$ among closed subvarieties. Such incidences translate into inclusions $P' \supset P$ in the opposite direction among prime ideals.

2.2 Some affine varieties

somevari-eties ptvar **2.2.1.** A *point* of affine space $\mathbb{A}^n$ is a variety. The point $p = (a_1, \ldots, a_n)$ is the set of solutions of the n equations $x_i - a_i = 0$, $i = 1, \ldots, n$. The polynomials $x_i - a_i$ generate a maximal ideal in the polynomial algebra $\mathbb{C}[x]$, and a maximal ideal is a prime ideal. We denote the maximal ideal that corresponds to the point p by $\mathfrak{m}_p$. It is the kernel of the substitution homomorphism $\pi_p : \mathbb{C}[x] \to \mathbb{C}$ that evaluates a polynomial $g(x_1, ..., x_n)$ at p: $\pi_p(g) = g(a_1, ..., a_n) = g(p)$.

The coordinate algebra of the point p is the quotient $\mathbb{C}[x]/\mathfrak{m}_p$, a field. It is called the *residue field* at p, and it will be denoted by $k(p)$. The residue field $k(p)$ is isomorphic to the field of complex numbers, but it is a particular quotient of the polynomial ring.

varinline **2.2.2.** The varieties in the affine line $\mathbb{A}^1$ are its points and the whole line. The varieties in the affine plane $\mathbb{A}^2$ are points, plane affine curves, and the whole plane.

This is true because the varieties correspond to the prime ideals of the polynomial ring. The prime ideals of $\mathbb{C}[x_1, x_2]$ are the maximal ideals, the principal ideals generated by irreducible polynomials, and the zero ideal. The proof is an exercise.

2.2.3. The set X of solutions of a single irreducible polynomial equation $f_1(x_1, ..., x_n) = 0$ in $\mathbb{A}^n$ is a variety called an *affine hypersurface*. hsurf

A hypersurface in the affine plane $\mathbb{A}^2$ is a plane affine curve. The *special linear group SL_2*, the group of complex 2×2 matrices with determinant 1, is a hypersurface in $\mathbb{A}^4$, the locus of zeros of the irreducible polynomial $x_{11}x_{22} - x_{12}x_{21} - 1$.

The reason that an affine hypersurface is a variety is that an irreducible element of a unique factorization domain is a prime element, and a prime element generates a prime ideal. The polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is a unique factorization domain.

2.2.4. A *line* in the plane, the locus of a linear equation $ax + by = c$, is a plane affine curve. Its coordinate algebra, which is $\mathbb{C}[x, y]/(ax + by - c)$, is isomorphic to a polynomial ring in one variable. Every line is isomorphic to the affine line $\mathbb{A}^1$. apc

2.2.5. Let $p = (a_1, \dots, a_n)$ and $q = (b_1, \dots, b_n)$ be distinct points of $\mathbb{A}^n$. The *point pair* (p, q) isn't irreducible, so it isn't a variety. It is the closed set defined by the system of n^2 equations $(x_i - a_i)(x_j - b_j) = 0$, $1 \le i, j \le n$, and the ideal I generated by the polynomials $(x_i - a_i)(x_j - b_j)$ isn't a prime ideal. The next corollary, which follows from the Chinese Remainder Theorem 9.1.7, describes that ideal: pointpr

2.2.6. Corollary. *The ideal I of polynomials that vanish on a point pair p, q is the product $\mathfrak{m}_p\mathfrak{m}_q$ of the maximal ideals at those points, and the quotient algebra $\mathbb{C}[x]/I$ is isomorphic to the product algebra $\mathbb{C}\times\mathbb{C}$.* *pointpair*
□

2.3 Hilbert's Nullstellensatz

Hilbert's Nullstellesatz establishes the fundamental relation between affine algebraic geometry and algebra. It identifies the points of an affine variety with maximal ideals. null

2.3.1. Nullstellensatz (version 1). *Let $\mathbb{C}[x]$ be the polynomial algebra in the variables $x_1, \dots, x_n$. There are bijective correspondences between the following sets:* *nullone*

- *points p of the affine space $\mathbb{A}^n$,*
- *algebra homomorphisms $\pi_p : \mathbb{C}[x] \to \mathbb{C}$,*
- *maximal ideals $\mathfrak{m}_p$ of $\mathbb{C}[x]$.*

The homomorphism π_p evaluates a polynomial at a point p of $\mathbb{A}^n$. If $p = (a_1, ..., a_n)$, then $\pi_p(g) = g(p) = g(a_1, ..., a_n)$. The maximal ideal $\mathfrak{m}_p$ is the kernel of π_p. It is the ideal generated by the linear polynomials $x_1 - a_1, \dots, x_n - a_n$. □

It is obvious that every algebra homomorphism $\mathbb{C}[x] \to \mathbb{C}$ is surjective, so its kernel is a maximal ideal. It isn't obvious that every maximal ideal of $\mathbb{C}[x]$ is the kernel of such a homomorphism. The proof can be found manywhere. (I learned this unusual word from the mathematician Nagata. While writing a paper, he decided that the English language needed this word, and then he found it in a dictionary.)

The Nullstellensatz gives a way to describe the set $V(I)$ of zeros of an ideal I in affine space in terms of maximal ideals. The points of $V(I)$ are those at which all elements of I vanish — the points p such that I is contained in $\mathfrak{m}_p$.

(2.3.2) $$V(I) = \{p \in \mathbb{A}^n \mid I \subset \mathfrak{m}_p\}$$ maxide-alscheme

2.3.3. Proposition. *Let I be an ideal of the polynomial ring $\mathbb{C}[x]$. If the zero locus $V(I)$ is empty, then I is the unit ideal.* *Vempty*

proof. Every ideal I except the unit ideal is contained in a maximal ideal. □

nulltwo **2.3.4. Nullstellensatz (version 2).** *Let A be a finite-type algebra. There are bijective correspondences between the following sets:*

- *algebra homomorphisms $\overline{\pi} : A \to \mathbb{C}$,*
- *maximal ideals $\overline{\mathfrak{m}}$ of A.*

The maximal ideal $\overline{\mathfrak{m}}$ that corresponds to a homomorphism $\overline{\pi}$ is the kernel of $\overline{\pi}$.

If A is presented as a quotient of a polynomial ring, say $A \approx \mathbb{C}[x_1, ..., x_n]/I$, then these sets also correspond bijectively to points of the zero locus $V(I)$ of I in $\mathbb{A}^n$.

The symbol $\approx$, as it is used here, stands for an isomorphism.

As before, a finite-type algebra is an algebra that can be generated by a finite set of elements. A *presentation* of a finite-type algebra A is an isomorphism of A with a quotient $\mathbb{C}[x_1, ..., x_n]]/I$ of a polynomial ring. (This is not the same as a presentation of a module.)

proof of version 2 *of the Nullstellensatz.* We choose a presentation of A as a quotient of a polynomial ring, to identify A with a quotient $\mathbb{C}[x]/I$. The Correspondence Theorem (9.1.6) tells us that maximal ideals of A correspond to maximal ideals of $\mathbb{C}[x]$ that contain I. Those maximal ideals correspond to points of $V(I)$.

Let τ denote the canonical homomorphism $\mathbb{C}[x] \to A$. The Mapping Property 9.1.5, applied to τ, tells us that homomorphisms $A \xrightarrow{\overline{\pi}} \mathbb{C}$ correspond to homomorphisms $\mathbb{C}[x] \xrightarrow{\pi} \mathbb{C}$ whose kernels contain I. Those homomorphisms also correspond to points of $V(I)$.

polyring-toA (2.3.5)

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\pi"] \arrow[d, "\tau"'] & \mathbb{C} \arrow[d, equal] \\ A \arrow[r, "\overline{\pi}"] & \mathbb{C} \end{tikzcd}$$

□

commdiag **2.3.6. commutative diagrams**

In the diagram displayed above, the maps $\overline{\pi}\tau$ and π from $\mathbb{C}[x]$ to $\mathbb{C}$ are equal. This is referred to by saying that the diagram is *commutative*. A commutative diagram is one in which every map that can be obtained by composing its arrows depends only on its domain and range. In these notes, almost all diagrams of maps are commutative. We won't mention commutativity most of the time. □

strongnull **2.3.7. Strong Nullstellensatz.** *Let I be an ideal of the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, and let V denote the locus of zeros $V(I)$ in affine space. If a polynomial $g(x)$ vanishes at every point of V, then I contains a power of g.*

proof. This is Rainich's beautiful proof. Let $g(x)$ be a polynomial that is identically zero on V. We are to show that I contains a power of g. The zero polynomial is in I, so we may assume that g isn't zero.

The Hilbert Basis Theorem tells us that I is a finitely generated ideal. Let $f = (f_1, \dots, f_k)$ be a set of generators. We introduce a new variable y. Let W be the locus of solutions of the $k+1$ equations

fgy (2.3.8)
$$f_1(x) = 0, \dots, f_k(x) = 0 \quad \text{and} \quad g(x)y - 1 = 0$$

in the $n+1$–dimensional affine space with coordinates $(x_1, \dots, x_n, y)$. Suppose that we have a solution of the equations $f(x) = 0$, say $(x_1, ..., x_n) = (a_1, ..., a_n)$. Then a is a point of V, and our hypothesis tells us that $g(a) = 0$. So there can be no b such that $g(a)b = 1$. There is no point $(a_1, ..., a_n, b)$ that solves the equations (2.3.8): The locus W is empty. Proposition 2.3.3 tells us that the polynomials $f_1, ..., f_k$ and $gy - 1$ generate the unit ideal of $\mathbb{C}[x_1, ..., x_n, y]$. There are polynomials $p_1(x, y), \dots, p_k(x, y)$ and $q(x, y)$ such that

rabinowitz (2.3.9)
$$p_1 f_1 + \cdots + p_k f_k + q(gy - 1) = 1$$

The ring $R = \mathbb{C}[x, y]/(gy - 1)$ can be described as the one obtained by adjoining an inverse of g to the polynomial ring $\mathbb{C}[x]$. The residue of y is the inverse. Since g isn't zero, $gy - 1$ doesn't divide any polynomial in x. So $\mathbb{C}[x]$ is a subring of R. In R, $gy-1=0$, and the equation (2.3.9) becomes $p_1f_1+\cdots+p_kf_k=1$. When we multiply both sides of this equation by a large power of g, we can use the equation $gy=1$, which is true in R, to eliminate all occurences of y in the polynomials $p_i(x,y)$. Let $h_i(x)$ denote the polynomial in x that is obtained by eliminating y from g^Np_i. Then

$$h_1(x)f_1(x)+\cdots+h_k(x)f_k(x) \;=\; g^N(x)$$

is a polynomial equation that is true in R and in its subring $\mathbb{C}[x]$. Since $f_1,...,f_k$ are in I, this equation shows that g^N is in I. □

2.3.10. Corollary. *Let $\mathbb{C}[x]$ denote the polynomial ring in the variables $x_1,...,x_n$.* *VIsupVJ*

(i) *Let P be a prime ideal of $\mathbb{C}[x]$, and let $V=V(P)$ be the variety of zeros of P in $\mathbb{A}^n$. If a polynomial g vanishes at every point of V, then g is an element of P.*

(ii) *Let f be an irreducible polynomial in $\mathbb{C}[x]$. If a polynomial g vanishes at every point of $V(f)$, then f divides g.*

(iii) *Let I and J be ideals of $\mathbb{C}[x]$. Then $V(I)\supset V(J)$ if and only if $\operatorname{rad} I\subset \operatorname{rad} J$, and $V(I)>V(J)$ if and only if $\operatorname{rad} I<\operatorname{rad} J$ (see (2.1.20)).* □

2.3.11. Examples.

(i) Let I be the ideal of the polynomial algebra $\mathbb{C}[x,y]$ generated by y^5 and y^2-x^3. In the affine plane, the origin $(0,0)$, is the only common zero of these polynomials, and the polynomial x also vanishes at the origin. The Strong Nullstellensatz predicts that I contains a power of x. This is verified by the following equation: strongnullex

$$yy^5-(y^4+y^2x^3+x^6)(y^2-x^3) \;=\; x^9$$

(ii) We may regard pairs A,B of $n\times n$ matrices as points of the affine space $\mathbb{A}$ of dimension $2n^2$ that has coordinates a_{ij},b_{ij}, $1\le i,j\le n$. The pairs of *commuting matrices* ($AB=BA$) form a closed subset of $\mathbb{A}$, the locus of common zeros of the n^2 polynomials c_{ij} that compute the matrix entries of $AB-BA$:

$$c_{ij}(a,b)=\sum_\nu a_{i\nu}b_{\nu j}-b_{i\nu}a_{\nu j}$$

If I is the ideal of the polynomial algebra $\mathbb{C}[a,b]$ generated by the set of polynomials $\{c_{ij}\}$, then $V(I)$ is the set of pairs of commuting matrices. The Strong Nullstellensatz asserts that, if a polynomial $g(a,b)$ vanishes on every pair of commuting matrices, some power of g is in I. Is g itself in I? It is a famous conjecture that I is a prime ideal. If so, g would be in I. Proving the conjecture would establish your reputation as a mathematician, but I don't recommend spending much time on it right now. □

2.4 The Spectrum

When a finite-type domain A is presented as a quotient of a polynomial ring $\mathbb{C}[x]/P$, where P is a prime ideal, A becomes the coordinate algebra of the variety $V(P)$ in affine space. The points of $V(P)$ correspond to maximal ideals of A and also to homomorphisms $A\to\mathbb{C}$. spectrumalg

The Nullstellensatz allows us to associate a set of points to a finite-type domain A without reference to a presentation. We can do this because the maximal ideals of A and the homomorphisms $A\to\mathbb{C}$ don't depend on a presentation.

We replace the variety $V(P)$ by an abstract set of points, the *spectrum* of A, that we denote by $\operatorname{Spec} A$ and call an *affine variety*. We put one point p into the spectrum for every maximal ideal of A, and then we turn around and denote the maximal ideal that corresponds to a point p by $\overline{\mathfrak{m}}_p$. The Nullstellensatz tells us that p also corresponds to a homomorphism $A\to\mathbb{C}$, whose kernel is $\overline{\mathfrak{m}}_p$. We denote that homomorphism by $\overline{\pi}_p$. In analogy with (2.1.23), A is called the *coordinate algebra* of the affine variety $\operatorname{Spec} A$. To work with $\operatorname{Spec} A$, we may interpret its points as maximal ideals or as homomorphisms to $\mathbb{C}$, whichever is convenient.

When defined in this way, the variety $\operatorname{Spec} A$ isn't embedded into any affine space, but because A is a finite-type domain, it can be presented as a quotient $\mathbb{C}[x]/P$, where P is a prime ideal. When this is done, points of $\operatorname{Spec} A$ correspond to points of the variety $V(P)$ in $\mathbb{A}^n$. Even when the coordinate ring A of an affine variety is presented as $\mathbb{C}[x]/P$, we may denote it by $\operatorname{Spec} A$ rather than by $V(P)$.

primespec **2.4.1. Note.** In modern terminology, the word "spectrum" is usually used to denote the set of prime ideals of a ring. This becomes important when one studies rings that aren't finite-type algebras. When working with finite-type domains, there are enough maximal ideals. The other prime ideals aren't needed to fill out $\operatorname{Spec} A$, and it is simpler not to include them here. $\square$

Let $X = \operatorname{Spec} A$. An element α of A defines a (complex-valued) function on X that we denote by the same letter α. The definition of the function α is as follows: A point p of X corresponds to a homomorphism $A \xrightarrow{\overline{\pi}_p} \mathbb{C}$. By definition The *value* $\alpha(p)$ of the function α at p is $\overline{\pi}_p(\alpha)$:

defalphap (2.4.2)
$$\alpha(p) \overset{def}{=} \overline{\pi}_p(\alpha)$$

Thus the kernel of $\overline{\pi}_p$, which is $\overline{\mathfrak{m}}_p$, is the set of elements α of the coordinate algebra A at which the value of α is 0:

$$\overline{\mathfrak{m}}_p = \{\alpha \in A \,|\, \alpha(p) = 0\}$$

The functions defined by the elements of A are called the *regular functions* on X. (See (2.5.2) below.)

When A is a polynomial algebra $\mathbb{C}[x_1, ..., x_n]$, the function defined by a polynomal $g(x)$ is simply the usual polynomial function, because π_p is defined by evaluating a polynomial at p: $\pi_p(g) = g(p)$.

gpequal-salphap **2.4.3. Lemma.** *Let A be a quotient $\mathbb{C}[x]/P$ of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, modulo a prime ideal P, so that $\operatorname{Spec} A$ identifies with the closed subset $V(P)$ of $\mathbb{A}^n$, and a point p of $\operatorname{Spec} A$ becomes a point of $\mathbb{A}^n$. Say that $p = (a_1, ..., a_n)$. When an element α of A is represented by a polynomial $g(x)$, the value of α at p can be obtained by evaluating g: $\alpha(p) = g(p) = g(a_1, ..., a_n)$.*

So when $A = \mathbb{C}[x]/P$, the value of an element α at a point p of $\operatorname{Spec} A$ can be obtained by evaluating a suitable polynomial g. However, unless P is the zero ideal, that polynomial won't be unique.

proof of Lemma 2.4.3. The point p of $\operatorname{Spec} A$ gives us a diagram (2.3.5), with $\pi = \pi_p$ and $\overline{\pi} = \overline{\pi}_p$, and where τ is the canonical map $\mathbb{C}[x] \to A$. Then $\alpha = \tau(g)$, and

redefinefn (2.4.4)
$$g(p) \overset{def}{=} \pi_p(g) = \overline{\pi}_p \tau(g) = \overline{\pi}_p(\alpha) \overset{def}{=} \alpha(p).$$
$\square$

regfndetelt **2.4.5. Lemma.** *The regular functions determined by distinct elements α and β of a finite type domain A are distinct. In particular, the only element α of A that is zero at all points of $\operatorname{Spec} A$ is the zero element.*

proof. We replace α by $\alpha - \beta$. Then what is to be shown is that, if the function determined by an element α is the zero function, then α is the zero element.

We present A as $\mathbb{C}[x]/P$, $x = (x_1, ..., x_n)$, where P is a prime ideal. Let X be the locus of zeros of P in $\mathbb{A}^n$. Corollary 2.3.10 **(i)** tells us that P is the ideal of all elements that are zero on X. Let $g(x)$ be a polynomial that represents α. If p is a point of X, and if $\alpha(p) = 0$, then $g(p) = 0$. So if α is the zero function, then g is in P, and therefore $\alpha = 0$. $\square$

ztoponvar

2.4.6. the Zariski topology on an affine variety

Let $X = \operatorname{Spec} A$ be an affine variety with coordinate algebra A. An ideal $\overline{J}$ of A defines a locus in X, a *closed subset*, that we denote by $V(\overline{J})$, using the same notation as for loci in affine space. The points of $V(\overline{J})$ are the points of X at which all elements of $\overline{J}$ vanish. This is analogous to (2.3.2):

locusin-spec (2.4.7)
$$V(\overline{J}) \;=\; \{p \in \operatorname{Spec} A \,|\, \overline{J} \subset \overline{\mathfrak{m}}_p\}$$

Say that A is presented as $A = \mathbb{C}[x]/P$. An ideal $\overline{J}$ of A corresponds to an ideal J of $\mathbb{C}[x]$ that contains P, and $\overline{J} = J/P$.

zerolocus-inX **2.4.8. Lemma.** **(i)** *Let $V(J)$ denote the zero locus of J in $\mathbb{A}^n$. When we regard $\operatorname{Spec} A$ as a subvariety of $\mathbb{A}^n$, the loci $V(\overline{J})$ in $\operatorname{Spec} A$ and $V(J)$ in $\mathbb{A}^n$ are equal.*
(ii) *Let $\overline{J}$ be an ideal of a finite-type domain A. The zero set $V(\overline{J})$ in $X = \operatorname{Spec} A$ is empty if and only if $\overline{J}$ is the unit ideal of A. If X is empty, then A is the zero ring.* $\square$

2.4.9. Note. We have put bars on the symbols $\overline{\mathfrak{m}}$, $\overline{\pi}$, and $\overline{J}$ here, in order to distinguish ideals of A from ideals of $\mathbb{C}[x]$ and homomorphisms $A \to \mathbb{C}$ from homomorphisms $\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$. From now on we will put bars over the letters only when there is a danger of confusion. Most of the time, we will drop the bars, and write $\mathfrak{m}$, π, and J instead of $\overline{\mathfrak{m}}$, $\overline{\pi}$, and $\overline{J}$. $\square$ nobar

2.4.10. Proposition. *Let I be an ideal of noetherian ring R. The radical of I is the intersection of the prime ideals of R that contain I.* *intersprimes*

proof. Let x be an element of $\operatorname{rad} I$. Some power x^k is in I. If P is a prime ideal that contains I, then $x^k \in P$, and since P is a prime ideal, $x \in P$. So $\operatorname{rad} I \subset P$. Conversely, let x be an element not in $\operatorname{rad} I$. So no power of x is in I. We show that there is a prime ideal that contains I but not x. Let $\mathcal{S}$ be the set of ideals that contain I, but don't contain any power of x. The ideal I is one such ideal, so $\mathcal{S}$ isn't empty. Since R is noetherian, $\mathcal{S}$ contains a maximal member P. We show that P is a prime ideal by showing that, if two ideals A and B are strictly larger than P, their product AB isn't contained in P (9.1.2)**(iii')**. Since P is a maximal member of $\mathcal{S}$, A and B aren't in $\mathcal{S}$. They contain I and they contain powers of x, say $x^k \in A$ and $x^\ell \in B$. Then $x^{k+\ell}$ is in AB but not in P. Therefore $AB \not\subset P$. $\square$

The closed subsets of an affine variety have the properties of the closed sets in affine space given in Lemmas 2.1.2 and 2.1.21. In particular, $V(J) = V(\operatorname{rad} J)$, and $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.

2.4.11. Corollary. *Let I and J be ideals of a finite-type domain A, and let $X = \operatorname{Spec} A$. Then $V(I) \supset V(J)$ if and only if $\operatorname{rad} I \subset \operatorname{rad} J$.* *subsofspecA*

This follows from the case of a polynomial ring, Corollary 2.3.10 **(iii)**, and Lemma 2.4.8. $\square$

The next proposition includes Proposition 2.1.22 as a special case.

2.4.12. Proposition. *Let A be a finite-type domain, let $X = \operatorname{Spec} A$, and let P be a radical ideal of A. The closed set $V(P)$ of zeros of P is irreducible if and only if P is a prime ideal.* *proofirredprime*

proof. Let $Y = V(P)$, and let C and D be closed subsets of X such that $Y = C \cup D$. Say $C = V(I)$ and $D = V(J)$. We may suppose that I and J are radical ideals. Then the inclusion $C \subset Y$ implies that $I \supset P$, and similarly, $J \supset P$ (2.4.11). Because $Y = C \cup D$, we also have $V(P) = V(I) \cup V(J) = V(IJ)$. Therefore $\operatorname{rad}(IJ) = P$. If P is a prime ideal, then $P = I$ or $P = J$ (9.1.2) **(iii')**, and therefore $C = Y$ or $D = Y$. So Y is irreducible. Conversely, if P isn't a prime ideal, there are ideals I, J strictly larger than the radical ideal P, such that $IJ \subset P$. Then Y will be the union of the two proper closed subsets $V(I)$ and $V(J)$ (2.4.11), so Y isn't irreducible. $\square$

2.4.13. the nilradical

thenilradical

The *nilradical* of a ring is the set of its nilpotent elements. It is the radical of the zero ideal. If a ring R is noetherian, its nilradical N will be *nilpotent*: some power will be zero. The nilradical of a domain is the zero ideal.

The next corollary follows from Proposition 2.4.10.

2.4.14. Corollary. *The nilradical of a noetherian ring R is the intersection of the prime ideals of R.* $\square$ *intersectprimes*

Note. The conclusion of this corollary is true whether or not R is noetherian.

2.4.15. Corollary.

(i) *Let A be a finite-type algebra. An element of A that is in every maximal ideal of A is nilpotent.* *strongnullA*

(ii) *Let A be a finite-type domain. The intersection of the maximal ideals of A is the zero ideal.*

proof. **(i)** Say that A is presented as $\mathbb{C}[x_1, ..., x_n]/I$. Maximal ideals of A correspond to the maximal ideals of $\mathbb{C}[x]$ that contain I, and to points of the closed subset $V(I)$ of $\mathbb{A}^n$. Let α be the element of A that is represented by a polynomial $g(x)$ in $\mathbb{C}[x]$. Then α is in every maximal ideal of A if and only if $g = 0$ at all points of $V(I)$. If so, the Strong Nullstellensatz asserts that some power g^N is in I, and then $\alpha^N = 0$. $\square$

2.4.16. Corollary. *An element α of a finite-type domain A is determined by the function that α defines on $\operatorname{Spec} A$.* *fndetermineselt*

proof. It is enough to show that an element α that defines the zero function is the zero element. Such an element is in every maximal ideal, so it is nilpotent, and since A is a domain, it is zero. $\square$

2.5 Morphisms of Affine Varieties

morphism

The maps between varieties that are allowed are called morphisms. The morphisms between affine varieties, which will be defined in this section, correspond to algebra homomorphisms in the opposite direction between their coordinate algebras. Morphisms of projective varieties will be defined in the next chapter.

regfnone

2.5.1. regular functions

The *function field* K of an affine variety $X = \operatorname{Spec} A$ is the field of fractions of A. A *rational function* on X is a nonzero element of its function field.

A rational function f is *regular* at a point p of X if it can be written as a fraction $f = a/s$ with $s(p) \neq 0$, and f is regular on a subset U of X if it is regular at every point of U.

We have seen that an element of the coordinate algebra A defines a function on X. The value $a(p)$ of a function a at a point p is $\pi_p(a)$, where π_p is the homomorphism $A \to \mathbb{C}$ that corresponds to p. So a rational function $f = a/s$ defines a function wherever $s \neq 0$.

deloca

2.5.2. Proposition. *The regular functions on an affine variety $X = \operatorname{Spec} A$ are the elements of the coordinate algebra A.*

proof. The subset of points of X at which an element s of A isn't zero is called a localization of X, and is denoted by X_s. Localizations are discussed in the next section Section. Here we need only know that X_s is an open subset of X.

Let f be a rational function that is regular on X. For every point p of X, f can be written as a fraction a/s such that $s(p) \neq 0$ — such that the localization X_s contains p. Since p is arbitrary, those localizations cover X, and because X is quasicompact, a finite set $X_{s_1}, \dots, X_{s_k}$ of them covers X. Then for $i = 1, ..., k$, f can be written as a fraction a_i/s_i, and at least one s_i is nonzero at any point p. So $s_1, ..., s_k$ have no common zeros on X. They generate the unit ideal of A. Since f is in A_{s_i}, we can write $f = s_i^{-n} b_i$, or $s_i^n f = b_i$, with b_i in A, and we can use the same exponent n for each i. Since the elements s_i generate the unit ideal of A, so do the powers s_i^n. Writing $1 = \sum s_i^n a_i$ with a_i in A, $\quad f = \sum s_i^n a_i f = \sum a_i b_i$. So f is an element of A.

This reasoning, in which the identity element is written as a sum, occurs often. $\square$

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties, and let $A \xrightarrow{\varphi} B$ be an algebra homomorphism. A point q of Y corresponds to an algebra homomorphism $B \xrightarrow{\pi_q} \mathbb{C}$. When we compose π_q with φ, we obtain a homomorphism $A \xrightarrow{\pi_q\varphi} \mathbb{C}$. By definition, points p of $\operatorname{Spec} A$ correspond to homomorphisms $A \xrightarrow{\pi_p} \mathbb{C}$. So there is a unique point p of X such that $\pi_q\varphi = \pi_p$.

defmorphaff

2.5.3. Definition. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. A *morphism* $Y \xrightarrow{u} X$ is a map that is defined by an algebra homomorphism $A \xrightarrow{\varphi} B$ in this way: If q is a point of Y, then uq is the point p of X such that $\pi_p = \pi_q\varphi$:

piqphi

(2.5.4)
$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \pi_p}\downarrow & & \downarrow{\scriptstyle \pi_q} \\ \mathbb{C} & = & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} X & \xleftarrow{u} & Y \\ \uparrow & & \uparrow \\ p & \longleftarrow & q \end{array}$$

So $p = uq$ means that $\pi_q\varphi = \pi_p$.

betaqalphap

2.5.5. Lemma. *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $Y \xrightarrow{u} X$ be the morphism defined by a homomorphism $A \xrightarrow{\varphi} B$. Let q be a point of Y, and let $p = uq$ be its image in X.*
(i) *If α is an element of A and if $\beta = \varphi(\alpha)$, then $\beta(q) = \alpha(p)$.*
(ii) *Let $\mathfrak{m}_p$ and $\mathfrak{m}_q$ be the maximal ideals of A and B at p and q, respectively. Then $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$.*

proof. **(i)** $\beta(q) = \pi_q(\beta) = \pi_q(\varphi\alpha) = \pi_p(\alpha) = \alpha(p)$.
(ii) $\alpha(p) = 0$ if and only if $[\varphi\alpha](q) = 0$. $\square$

Thus the homomorphism φ is determined by the morphism u, and vice-versa. But a map $A \to B$ needn't be a homomorphism, and a map $Y \to X$ needn't be a morphism.

Notation. Parentheses tend to accumulate, and this can make expressions hard to read. When we want to denote the value of a complicated function such as $\varphi(\alpha)$ on q we may, for clarity, drop some parentheses and enclose the function in square brackets, writing $[\varphi\alpha](q)$ instead of $(\varphi(\alpha))(q)$. When a square bracket is used in this way, there is no logical difference between it and a parenthesis. $\square$ brackets

A morphism $Y \xrightarrow{u} X$ is an *isomorphism* if it is bijective and its inverse function is a morphism. This will be true if and only if $A \xrightarrow{\varphi} B$ is an isomorphism of algebras. The morphism u is the identity if and only if φ is the identity.

The definition of a morphism can be confusing because the direction of the arrow is reversed. It will become clearer as we expand the discussion, but the reversal of arrows always remains a potential source of confusion.

morphisms to affine space.

A morphism $Y \xrightarrow{u} \mathbb{A}^1$ from a variety $Y = \operatorname{Spec} B$ to the affine line $\operatorname{Spec} \mathbb{C}[x]$ is defined by an algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} B$, and such a homomorphism substitutes an element β of B for x. The corresponding morphism u sends a point q of Y to the point $x = \beta(q)$ of the x-line.

For example, let Y be the space of 2×2 matrices, $Y = \operatorname{Spec} \mathbb{C}[y_{ij}]$, where y_{ij} are variable matrix entries, $1 \le i, j \le 2$. The determinant defines a morphism $Y \to \mathbb{A}^1$ that sends a matrix to its determinant. The corresponding algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} \mathbb{C}[y_{ij}]$ substitutes $y_{11}y_{22} - y_{12}y_{21}$ for x. It sends a polynomial $f(x)$ to $f(y_{11}y_{22} - y_{12}y_{21})$.

A morphism in the other direction, from the affine line $\mathbb{A}^1$ to a variety Y may be called a (complex) polynomial path in Y. When Y is the space of matrices, a morphism $\mathbb{A}^1 \to Y$ corresponds to a homomorphism $\mathbb{C}[y_{ij}] \to \mathbb{C}[x]$. It substitutes a polynomial in x for each variable y_{ij}.

A morphism from an affine variety $Y = \operatorname{Spec} B$ to affine space $\mathbb{A}^n$, defined by a homomorphism $\mathbb{C}[x_1, ..., x_n] \xrightarrow{\Phi} B$, substitutes elements β_i of B for x_i: $\Phi(f(x)) = f(\beta)$. (We use an upper case Φ here, keeping φ in reserve.) The corresponding morphism $Y \xrightarrow{u} \mathbb{A}^n$ sends a point q of Y to the point $(\beta_1(q), ..., \beta_n(q))$ of $\mathbb{A}^n$.

morphisms to affine varieties.

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties. Say that we have chosen a presentation $A = \mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$ of A, so that X becomes the closed subvariety $V(f)$ of affine space $\mathbb{A}^m$. There is no need to choose a presentation of B. A natural way to define a morphism from a variety Y to X is as a morphism $Y \xrightarrow{u} \mathbb{A}^m$ to affine space whose image is contained in X. We check that this agrees with Definition 2.5.3.

A morphism $Y \xrightarrow{u} \mathbb{A}^m$ corresponds to the homomorphism $\mathbb{C}[x_1, ..., x_m] \xrightarrow{\Phi} B$, and Φ is determined by a set $(\beta_1, ..., \beta_m)$ of elements of B, with the rule that $\Phi(x_i) = \beta_i$. Since X is the locus of zeros of the polynomials f, the image of Y will be contained in X if and only if $f_i(\beta_1(q), ..., \beta_m(q)) = 0$ for every point q of Y and every i, i.e., if and only if $f_i(\beta)$ is in every maximal ideal of B, in which case $f_i(\beta) = 0$ (2.4.15)**(ii)**. A better way to say this is: The image of Y is contained in X if and only if $\beta = (\beta_1, ..., \beta_m)$ solves the equations $f(x) = 0$. And, if β is a solution, the homomorphism Φ defines a homomorphism $A \xrightarrow{\varphi} B$.

$$\begin{tikzcd}
\mathbb{C}[x] \arrow[r, "\Phi"] \arrow[d] & B \arrow[d, equal] \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

There is an elementary, but important, principle here:

Homomorphisms from the algebra $A = \mathbb{C}[x]/(f)$ to an algebra B correspond to solutions of the equations $f = 0$ in B.

2.5.6. Corollary. *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties, and say that $A = \mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$. There are bijective correspondences between the following sets:* *morphandhomom*

- *algebra homomorphisms $A \to B$,*
- *morphisms $Y \to X$,*
- *morphisms $Y \to \mathbb{A}^m$ whose images are contained in X,*
- *solutions of the equations $f_i(x) = 0$ in B.* $\square$

The first set doesn't refer to an embedding of the variety X into affine space. It shows that a morphism depends only on the varieties X and Y, not on their embeddings.

maptocusp **2.5.7. Example.** Let $B = \mathbb{C}[x]$ be the polynomial ring in one variable, and let A be the coordinate algebra $\mathbb{C}[u,v]/(v^2 - u^3)$ of a cubic curve with a cusp. A homomorphism $A \to B$ is determined by a solution of the equation $v^2 = u^3$ in $\mathbb{C}[x]$. The solutions have the form $u = g^2$, $v = g^3$ with g in $\mathbb{C}[x]$. One solution is $u = x^2$, $v = x^3$. $\square$

We note a few more facts about morphisms here. Their geometry will be analyzed further in Chapter 4.

phisurj **2.5.8. Proposition.** *Let $Y \xrightarrow{u} X$ be the morphism of affine varieties that corresponds to a homomorphism of finite-type domains $A \xrightarrow{\varphi} B$.*
(i) *Suppose that $B = A/P$, where P is a prime ideal of A, and that φ is the canonical homomorphism $A \to A/P$. Then u is the inclusion of the zero set of P into $X : Y = V(P)$.*
(ii) *The homomorphism φ is surjective if and only if u maps Y isomorphically to a closed subvariety of X.*
(iii) *Let $Z \xrightarrow{v} Y$ be another morphism, that corresponds to a homomorphism $B \xrightarrow{\psi} R$ of finite-type domains, the composed map $Z \xrightarrow{uv} X$ corresponds to the composed homomorphism $A \xrightarrow{\psi\varphi} R$.* $\square$

It can be useful to phrase the definition of the morphism $Y \xrightarrow{u} X$ that corresponds to a homomorphism $A \xrightarrow{\varphi} B$ in terms of maximal ideals. Let $\mathfrak{m}_q$ be the maximal ideal of B at a point q of Y. Its inverse image in A is the kernel of the composed homomorphism $A \xrightarrow{\varphi} B \xrightarrow{\pi_q} \mathbb{C}$, so it is a maximal ideal of A: $\varphi^{-1}\mathfrak{m}_q = \mathfrak{m}_p$, for some p in X. That point p is the image of q: If $p = uq$, then $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$.

The *fibre* over a point p of the morphism $Y \xrightarrow{u} X$ defined by a homomorphism $A \xrightarrow{\varphi} B$ is described as follows: let $\mathfrak{m}_p$ be the maximal ideal at a point p of X, and let J be the extended ideal $\mathfrak{m}_p B$, the ideal generated by the image of $\mathfrak{m}_p$ in B. Its elements are finite sums $\sum \varphi(z_i)b_i$ with z_i in $\mathfrak{m}_p$ and b_i in B. If q is is a point of Y, then $uq = p$ if and only if $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$. This will be true if and only $J \subset \mathfrak{m}_q$. The fibre is the locus of zeros of the ideal J in Y (2.4.7). The fibre is empty when J is the unit ideal.

cusp-normx **2.5.9. Example.** *(blowing up the plane)*

Let W and X be planes with coordinates (x, w) and (x, y), respectively. The affine blowup $W \xrightarrow{\pi} X$ has been described before (1.7.7). It is defined by $\pi(x, w) = (x, xw)$, and it corresponds to the algebra homomorphism $\mathbb{C}[x, y] \xrightarrow{\varphi} \mathbb{C}[x, w]$ defined by $\varphi(x) = x$ and $\varphi(y) = xw$. The image of the point $(x, w) = (a, c)$ of W is the point $(x, y) = (a, ac)$ of X.

As was explained in (1.7.7), the blowup π is bijective at points (x, y) at which $x \neq 0$. The fibre of Z over a point of Y of the form $(0, y)$ is empty unless $y = 0$, and the fibre over the origin $(0, 0)$ is the w-axis, the line $x = 0$ in the plane W. $\square$

mcont **2.5.10. Proposition.** *A morphism $Y \xrightarrow{u} X$ of affine varieties is a continuous map in the Zariski topology and also in the classical topology.*

proof. the Zariski topology. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, so that u corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. A closed subset C of X will be the zero locus of a set $\alpha = \{\alpha_1, ..., \alpha_k\}$ of elements of A. Let $\beta_i = \varphi\alpha_i$. The inverse image $u^{-1}C$ is the set of points q such that $p = uq$ is in C, i.e., such that $\alpha_i(uq) = \beta_i(q) = 0$ (2.5.5). So $u^{-1}C$ is the zero locus in Y of the elements $\beta_i = \varphi(\alpha_i)$. It is a closed set.

the classical topology. We use the fact that polynomials are continuous functions. First, a morphism of affine spaces $\mathbb{A}^n_y \xrightarrow{U} \mathbb{A}^m_x$ is defined by an algebra homomorphism $\mathbb{C}[x_1, ..., x_m] \xrightarrow{\Phi} \mathbb{C}[y_1, ..., y_n]$, and that homomorphism is determined by the polynomials $h_i(y)$ that are the images $\Phi(x_i)$. The morphism U sends the point $(y_1, ..., y_n)$ of $\mathbb{A}^n$ to the point $(h_1(y), ..., h_m(y))$ of $\mathbb{A}^m$. It is continuous because polynomials are continuous functions.

Next, say that a morphism $Y \xrightarrow{u} X$ is defined by a homomorphism $A \xrightarrow{\varphi} B$ of algebras that are presented as $A = \mathbb{C}[x]/I$ and $B = \mathbb{C}[y]/J$. We form a diagram of homomorphisms and the associated diagram of morphisms:

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & \mathbb{C}[y] \arrow[d, "\beta"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^n_x & \mathbb{A}^m_y \arrow[l, "U"'] \\ X \arrow[u] & Y \arrow[l, "u"'] \arrow[u] \end{tikzcd}$$

Here the map α sends x_i to α_i, $i = 1, ..., n$, and β sends y_j to $\beta_j = \varphi(\alpha_j)$. Then Φ is obtained by choosing elements h_j of $\mathbb{C}[y]$, such that $\beta(h_j) = \beta_j$.

In the diagram on the right, U is a continuous map, and the vertical arrows are the embeddings of X and Y into their affine spaces. Since the topologies on X and Y are induced from their embeddings into affine spaces, u is continuous. □

2.5.11. Thus every morphism of affine varieties can be obtained by restriction from a morphism of affine spaces. However, in the diagram above, the morphism U depends on the choice of the polynomials h_i and on the presentations of A and B. It isn't unique. extend-toaffsp

2.6 Localization

In these notes, the word "localization" refers to the process of adjoining inverses to an algebra, and to the effect of that process on the spectrum. boldloc

Let s be a nonzero element of a domain A. The ring $A_s = A[s^{-1}]$ obtained by adjoining an inverse of s to A is called a *localization* of A. If $A[z]$ denotes the ring of polynomials in one variable z with coefficients in A, the localization is isomorphic to the quotient $A[z]/(sz-1)$ of $A[z]$ modulo the principal ideal generated by $sz - 1$. The residue of z becomes the inverse of s.

If $X = \operatorname{Spec} A$, the variety $\operatorname{Spec} A_s$ will be called a localization of X, and it will be denoted by X_s.

The reason for the term "localization' is explained by the next proposition.

2.6.1. Proposition. *The localization $X_s = \operatorname{Spec} A_s$ is homeomorphic to the open subspace U of X of points at which the function defined by s isn't zero.* *toponloc*

proof. Let p be a point of X, and let $A \xrightarrow{\pi_p} \mathbb{C}$ be the corresponding homomorphism. If p is a point of U, i.e., if $s(p) \neq 0$, then π_p extends uniquely to a homomorphism $A_s \to \mathbb{C}$ that sends s^{-1} to $s(p)^{-1}$. This gives us a unique point p' of X_s that maps to p. If $c = 0$, then π_p doesn't extend to A_s. So the morphism $X_s \xrightarrow{f} X$ defined by the map $A \to A_s$, maps X_s bijectively to U. The map f is continuous because it is a morphism (2.5.10), and since the topology on U is induced from X, the restriction of f to U is also continuous.

We show that the inverse map $U \to X_s$ is continuous. Let D be a closed subset of X_s, say the zero set in X_s of some elements $\beta_1, ..., \beta_k$ of A_s. Then $\beta_i = b_i s^{-n_i}$ with b_i in A. Since s^{-1} doesn't vanish on U, the elements β_i and b_i have the same zeros there. The zero set of $b_1, ..., b_k$ is the image of D. So the image is closed. □

Thus we may identify a localization X_s with the open subset of X of points at which the value of s isn't zero. Then the effect of adjoining the inverse is to throw out the points of X at which s vanishes. For example, the spectrum of the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ becomes the complement of the origin in the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$.

Most varieties contain open sets that aren't localizations. The complement X' of the origin in the affine plane $X = \operatorname{Spec} \mathbb{C}[x_1, x_2]$ is a simple example. Every polynomial that vanishes at the origin vanishes on an affine curve, which has points distinct from the origin. Its inverse doesn't define a function on X'. So X' isn't a localization of X. This is rather obvious, but it is often hard to tell whether or not a given open set is a localization.

Localizations are important for two reasons:

2.6.2. basistop

- The relation between an algebra A and a localization A_s is easy to understand.
- The localizations X_s of an affine variety X form a basis for the Zariski topology on X.

A *basis* for the topology on a topological space X is a family $\mathcal{B}$ of open sets with the property that every open subset of X is a union of open sets that are members of $\mathcal{B}$. To show that the localizations X_s of an affine variety X form a basis for the topology on X, we must show that every open subset U of $X = \operatorname{Spec} A$ can be covered by sets of the form X_s. Let C be the complement of U in X. Then C is closed, so it is the set of common zeros of some nonzero elements $s_1, ..., s_k$ of A. The zero set $V(s_i)$ of s_i is the complement of the locus X_{s_i} in X. So C is the intersection of the sets $V(s_i)$, and U is the union of the sets X_{s_i}. □

coverby-locs **2.6.3. Corollary.** *Let $X = \operatorname{Spec} A$ be an affine variety.*

(i) *Let $s_1, ..., s_k$ be elements of A. If the localizations $X_{s_1}, ..., X_{s_k}$ cover X, then $s_1, ..., s_k$ generate the unit ideal of A.*

(ii) *If $\{U_\nu\}$ is an open covering of X (a covering by open sets), there are elements $s_1, ..., s_k$ of A such that the sets $X_{s_1}, ..., X_{s_k}$ cover X, and each localization X_{s_i} is contained in some U_ν.* □

locloc **2.6.4. Lemma.** *Let $X = \operatorname{Spec} A$ be an affine variety.*

(i) *If A_s and A_t are localizations of A, and if $A_s \supset A_t$, then A_s is a localization of A_t. Or, if X_s and X_t are localizations of X, and if $X_s \subset X_t$, then X_s is a localization of X_t.*

(ii) *If u is an element of a localization A_s of A, then the localization $(A_s)_u$ of A_s is also a localization of A.* □

extcontr **2.6.5. extension and contraction of ideals**

Let $A \subset B$ be the inclusion of a ring A as a subring of a ring B. The *extension* of an ideal I of A is the ideal IB of B generated by I. Its elements are finite sums $\sum_i z_i b_i$ with z_i in I and b_i in B. The *contraction* of an ideal J of B is the intersection $J \cap A$. The contraction is an ideal of A.

When A_s is a localization of A and I is an ideal of A, the elements of the extended ideal IA_s are fractions of the form zs^{-k}, with z in I. We denote this extended ideal by I_s.

xtcontr-prop **2.6.6. Lemma.** *Let s be a nonzero element of a domain A.*

(i) *Let J be an ideal of the localization A_s and let $I = J \cap A$. Then $J = I_s$. Every ideal of A_s is the extension of an ideal of A.*

(ii) *Let P be a prime ideal of A. If s isn't in P, the extended ideal P_s is a prime ideal of A_s. If I is an ideal of A that contains s, the extended ideal I_s is the unit ideal.* □

multsys **2.6.7. multiplicative systems**

To work with the inverses of finitely many nonzero elements, one may simply adjoin the inverse of their product. The concept of a multiplicative system is useful for working with an infinite set of inverses. A *multiplicative system* S in a domain A is a set of nonzero elements of A that is closed under multiplication and contains 1. If S is a multiplicative system, the *ring of fractions* AS^{-1} is the domain obtained by adjoining inverses of all elements of S. Its elements are equivalence classes of fractions as^{-1} with a in A and s in S, the equivalence relation and the laws of composition being the usual ones for fractions.

inverseex-amples **2.6.8. Examples.** **(i)** The set consisting of the powers of a nonzero element s of a domain A is a multiplicative system. Its ring of fractions is the localization A_s.

(ii) The set S of all nonzero elements of a domain A is a multiplicative system. Its ring of fractions is the field of fractions of A.

(iii) An ideal P of a domain A is a prime ideal if and only if its complement, the set of elements of A not in P, is a multiplicative system. □

extendide-altoloc **2.6.9. Proposition.** *Let S be a multiplicative system in a domain A, and let A' be the ring of fractions AS^{-1}.*

(i) *Let I be an ideal of A. The extended ideal IA' is the set IS^{-1} whose elements are classes of fractions xs^{-1}, with x in I and s in S. The extended ideal is the unit ideal if and only if I contains an element of S.*

(ii) *Let J be an ideal of the ring of fractions A' and let $I = J \cap A$. Then $IA' = J$.*

(iii) *If P is a prime ideal of A and if $P \cap S$ is empty, the extended ideal $P' = PA'$ is a prime ideal of A', and the contraction $P' \cap A$ is equal to P. If $P \cap S$ isn't empty, the extended ideal is the unit ideal. Thus prime ideals of AS^{-1} correspond bijectively to prime ideals of A that don't meet S.* □

2.6.10. Corollary. *Every ring of fractions AS^{-1} of a noetherian domain A is a noetherian domain.* $\square$ *locfintype*

2.6.11. When S is a multiplicative system in a domain A, the *module of fractions* MS^{-1} of an A-module M is defined in a way analogous to the one used for localizations: It is the AS^{-1}-module whose elements are equivalence classes of fractions ms^{-1} with m in M and s in S. To take care of torsion, two fractions $m_1s_1^{-1}$ and $m_2s_2^{-1}$ are defined to be equivalent if there is an element s in S such that $m_1s_2s = m_2s_1s$. Then $ms_1^{-1} = 0$ if and only if $ms = 0$ for some s in S. As with simple localizations, there will be a homomorphism $M \to MS^{-1}$ that sends an element m to the fraction $m/1$. loc-modtwo

2.6.12. Proposition. *Let S be a multiplicative system in a domain A.* *localexact*

(i) *The process of making fractions is exact: A homomorphism $M \xrightarrow{\varphi} N$ of A-modules induces a homomorphism $MS^{-1} \xrightarrow{\varphi'} NS^{-1}$ of AS^{-1}-modules. If $M \xrightarrow{\varphi} N \xrightarrow{\psi} P$ is an exact sequence of A-modules, the localized sequence $MS^{-1} \xrightarrow{\varphi'} NS^{-1} \xrightarrow{\psi'} PS^{-1}$ is exact.*

(ii) *Let M be an A-module and let N be an AS^{-1}-module. When N is made into an A-module ${}_AN$ by restriction of scalars, homomorphisms of A-modules $M \to {}_AN$ correspond bijectively to homomorphisms of AS^{-1}-modules $MS^{-1} \to N$.*

(iii) *If multiplication by s is an injective map $M \to M$ for every s in S, then $M \subset MS^{-1}$. If multiplication by every s is a bijective map $M \to M$, then $M \approx MS^{-1}$.* $\square$

2.6.13. a general principle import-princ

An elementary principle for working with fractions is that any finite sequence of computations in a ring of fractions AS^{-1} will involve finitely many denominators, and can therefore be done in a simple localization A_s, where s is a common denominator for the fractions that occur.

2.7 Finite Group Actions

grpi

Let G be a finite group of automorphisms of a finite-type domain B. An *invariant element* β of B is an element such that $\sigma\beta = \beta$ for every element σ of G. For example, if b is an element of B, the product and the sum

(2.7.1) $$\prod_{\sigma\in G} \sigma b \quad , \quad \sum_{\sigma\in G} \sigma b$$ invarelts

are invariant elements. The invariant elements form a subalgebra of B that is often denoted by B^G. Theorem 2.7.5 below asserts that B^G is a finite-type domain, and that points of the variety $\operatorname{Spec} B^G$ correspond bijectively to G-orbits in $\operatorname{Spec} B$.

2.7.2. Examples. actonpla-neex

(i) The symmetric group $G = S_n$ operates on the polynomial algebra $R = \mathbb{C}[y_1, ..., y_n]$ by permuting the variables. The Symmetric Functions Theorem asserts that the elementary symmetric functions

$$s_1(y) = \sum_i y_i\,, \quad s_2(y) = \sum_{i<j} y_iy_j \quad , \ldots, \quad s_n(y) = y_1y_2\cdots y_n$$

generate the algebra R^G of invariant polynomials. Moreover, $s_1, ..., s_n$ are algebraically independent, so R^G is the polynomial algebra $\mathbb{C}[s_1, ..., s_n]$. The inclusion of R^G into R gives us a morphism $Y \to S$, from affine y-space $Y = \mathbb{A}^n_y$ to affine s-space $S = \mathbb{A}^n_s$. The symmetric group G operates on Y. If $a = (a_1, ..., a_n)$, one can evaluate the variables $s_1, ..., s_n$ at $y = a$, to obtain a point $c = (c_1, ..., c_n) = (s_1(a), ..., s_n(a))$ of S. The points a of Y with image c in S are those such that $s_j(a) = c_j$. They are the roots of the polynomial $y^n - c_1y^{n-1} + \cdots \pm c_n$. The roots form a G-orbit, so the set of G-orbits in Y maps bijectively to S.

(ii) Let σ be the automorphism of the polynomial ring $B = \mathbb{C}[y_1, y_2]$ defined by $\sigma y_1 = \zeta y_1$ and $\sigma y_2 = \zeta^{-1}y_2$, where $\zeta = e^{2\pi i/n}$. Let G be the cyclic group of order n generated by σ, and let A denote the algebra B^G of invariant elements. A monomial $m = y_1^iy_2^j$ is invariant if and only if n divides $i-j$, and an invariant polynomial is a linear combination of invariant monomials. You will be able to show that the three monomials

(2.7.3) $$u_1 = y_1^n\,, \quad u_2 = y_2^n\,, \quad \text{and} \quad w = y_1y_2$$ zetainvar generate the invariant algebra A. Let's use the same symbols u_1, u_2, w to denote variables in a polynomial ring $\mathbb{C}[u_1, u_2, w]$. Let J be the kernel of the canonical homomorphism $\mathbb{C}[u_1, u_2, w] \xrightarrow{\tau} A$ that sends u_1, u_2 and w to y_1^n, y_2^n and y_1y_2, respectively.

slaction **2.7.4. Lemma.** *With notation as above, the kernel J is the principal ideal of $\mathbb{C}[u_1, u_2, w]$, generated by the polynomial $f = w^n - u_1u_2$. Thus $A \approx \mathbb{C}[u_1, u_2, w]/(w^n - u_1u_2)$.*

proof. Clearly, f is in J. Let $g(u_1, u_2, w)$ be any element of J. So $g(y_1^n, y_2^n, y_1y_2) = 0$. We divide g by f, considered as a monic polynomial in w, say $g = fq + r$, where the remainder $r(u_1, u_2, w)$ has degree $< n$ in w. The remainder will be in J too: $r(y_1^n, y_2^n, y_1y_2) = 0$. We write r as a polynomial in w: $r = r_{n-1}w^{n-1} + \cdots + r_1w + r_0$, where r_i are polynomials in u_1, u_2. When we substitute y_1^n, y_2^n, y_1y_2 for u_1, u_2, w, the term $r_i(u_1, u_2)w^i$ becomes $r_i(y_1^n, y_2^n)(y_1y_2)^i$. The degree in y_1 of every monomial that appears there will be congruent to i modulo n, and the same is true for the degree in y_2. Since the indices i are distinct, and since $r(y_1^n, y_2^n, y_1y_2) = 0$, $r_i(u_1, u_2) = 0$ must be zero for every i. So $r = 0$, which means that f divides g. □

We go back to the operation of the cyclic group G on $B = \mathbb{C}[y_1, y_2]$ and the algebra of invariants A. Let Y denote the affine plane $\operatorname{Spec} B$, and let $X = \operatorname{Spec} A$. The group G operates on Y, and except for the origin, which is a fixed point, the orbit of a point (y_1, y_2) consists of the n points $(\zeta^i y_1, \zeta^{-i} y_2)$, $i = 0, \ldots, n-1$. To show that G-orbits in Y correspond bijectively to points of X, we fix complex numbers u_1, u_2, w with $w^n = u_1u_2$, and look for solutions of the equations (2.7.3). When $u_1 \neq 0$, the equation $u_1 = y_1^n$ has n solutions for y_1, and when a soluion is given, y_2 is determined by the equation $y_1y_2 = w$. So the fibre has order n. Similarly, there are n points in the fibre if $u_2 \neq 0$. If $u_1 = u_2 = 0$, then $y_1 = y_2 = w = 0$, and the fibre contains just one point. In all cases, the fibres are the G-orbits. □

groupoper-one **2.7.5. Theorem.** *Let G be a finite group of automorphisms of a finite-type domain B, and let A denote the algebra B^G of invariant elements. Let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.*
(i) *A is a finite-type domain and B is a finite A-module.*
(ii) *G operates by automorphisms on Y.*
(iii) *The morphism $Y \to X$ defined by the inclusion $A \subset B$ is surjective. Its fibres are the G-orbits of points of Y.*

When a group G operates on a set Y, one often denotes the set of G-orbits of Y by Y/G ('Y mod G'). With that notation, part **(iii)** of the theorem asserts that there is a bijective map

$$Y/G \to X$$

proof of Theorem 2.7.5 **(i)**. *The invariant algebra $A = B^G$ is a finite-type algebra, and B is a finite A-module.*

This is an interesting indirect proof. To show that A is a finite-type algebra, one constructs a finite-type subalgebra R of A such that B is a submodule of a finite R-module.

Let $\{z_1, \ldots, z_k\}$ be the G-orbit of an element z_1 of B. The orbit is the set of roots of the polynomial

spoly
$$f(t) = (t - z_1) \cdots (t - z_k) = t^k - s_1t^{k-1} + \cdots \pm s_k$$

Its coefficients s_i are the elementary symmetric functions in $\{z_1, ..., z_k\}$. Let R_1 denote the algebra generated by those coefficients. Because the symmetric functions are invariant, $R_1 \subset A$. Using the equation $f(z_1) = 0$, we can write any power of z_1 as a polynomial in z_1 of degree less than k, with coefficients in R_1.

We choose a finite set of generators $\{y_1, \ldots, y_r\}$ for the algebra B. If the order of the orbit of y_j is k_j, then y_j will be the root of a monic polynomial f_j of degree k_j with coefficients in A. Let R denote the finite-type algebra generated by all of the coefficients of all of the polynomials $f_1, ..., f_r$. We can write any power of y_j as a polynomial in y_j with coefficients in R, and of degree less than k_j. Using such expressions, we can write every monomial in $y_1 \cdots y_r$ as a polynomial with coefficients in R, whose degree in the variable y_j is less than k_j. Since $y_1, ..., y_r$ generate B, we can write every element of B as such a polynomial. Then the finite set of monomials $y_1^{e_1} \cdots y_r^{e_r}$ with $e_j < k_j$ spans B as an R-module. Therefore B is a finite R-module.

The algebra A of invariants is a subalgebra of B that contains R. Since R is a finite-type algebra, it is noetherian. When regarded as an R-module, A is a submodule of the finite R-module B. Therefore A is also a finite R-module. When we put a finite set of algebra generators for R together with a finite set of R-module generators for A, we obtain a finite set of algebra generators for A, so A is a finite-type algebra. And, since B is a finite R-module, it is also a finite module over the larger ring A.

proof of Theorem 2.7.5 **(ii)**. *The group G operates on Y.*

A group element σ is a homomorphism $B \xrightarrow{\sigma} B$. It defines a morphism $Y \xleftarrow{u_\sigma} Y$, as in Definition 2.5.3. Since σ is an invertible homomorphism, i.e., an automorphism of B, u_σ is an automorphism of Y. Thus G operates on Y. However, there is a point that should be mentioned.

We write the operation of G on B on the left as usual, so that a group element σ maps an element β of B to σb. Then if σ and τ are two group elements, the product $\sigma\tau$ acts as *first do τ, then σ*: $(\sigma\tau)\beta = \sigma(\tau\beta)$.

(2.7.6) $$B \xrightarrow{\tau} B \xrightarrow{\sigma} B$$ BBB

We substitute $u = u_\sigma$ into Definition 2.5.3: If q is a point of Y, the morphism $Y \xleftarrow{u_\sigma} Y$ sends q to the point p such that $\pi_p = \pi_q\sigma$. It seems permissible to drop the symbol u, and to write the morphism simply as $Y \xleftarrow{\sigma} Y$. But since arrows are reversed when going from homomorphisms of algebras to morphisms of their spectra (2.5.4), the maps displayed in (2.7.6) above, give us morphisms

(2.7.7) $$Y \xleftarrow{\tau} Y \xleftarrow{\sigma} Y$$ YYY

On $Y = \operatorname{Spec} B$, the product $\sigma\tau$ acts as *first do σ, then τ*. This is a problem, but we can get around it by putting the symbol σ on the right when it operates on Y, so that σ sends a point q to $q\sigma$. Then we will have $q(\sigma\tau) = (q\sigma)\tau$, as required of the operation.

- *If G operates on the left on B, then it operates on the right on* $\operatorname{Spec} B$.

This is important only when one wants to compose morphisms. In Definition 2.5.3, we followed custom and wrote the morphism u that corresponds to an algebra homomorphism φ on the left. We will continue to write morphisms on the left where possible, but not here.

Let β be an element of B and let q be a point of Y. The value of the function $\sigma\beta$ at a point q is the same as the value of β at the point $q\sigma$ (see (2.5.5) **(i)**):

(2.7.8) $$[\sigma\beta](q) = \beta(q\sigma)$$ $\square$ bsigmay

proof of Theorem 2.7.5 **(iii)**: *The fibres of the morphism $Y \to X$ are the G-orbits in Y.*

We go back to the subalgebra $A = B^G$. For σ in G, we have a diagram of algebra homomorphisms and the corresponding diagram of morphisms of varieties

(2.7.9) $$\begin{array}{ccccccc} B & \xrightarrow{\sigma} & B & \qquad & Y & \xleftarrow{\sigma} & Y \\ \uparrow & & \uparrow & & \downarrow & & \downarrow \\ A & = & A & & X & = & X \end{array}$$ actonB

The diagram of morphisms shows that points of Y that are in a G-orbit have the same image in X, and therefore that the set of G-orbits in Y, which we may denote by Y/G, maps to X. We show that the map $Y/G \to X$ is bijective.

2.7.10. Lemma. **(i)** *Let $p_1, \dots, p_k$ be distinct points of affine space $\mathbb{A}^n$, and let $c_1, \dots, c_k$ be complex numbers. There is a polynomial $f(x_1, \dots, x_n)$ such that $f(p_i) = c_i$ for $i = 1, \dots, n$.* *arbvals*
(ii) *Let B be a finite-type algebra, let $q_1, \dots, q_k$ be distinct points of* $\operatorname{Spec} B$, *and let $c_1, \dots, c_k$ be distinct complex numbers. There is an element β in B such that $\beta(q_i) = c_i$ for $i = 1, \dots, k$.* $\square$

injectivity of the map $Y/G \to X$. Let O_1 and O_2 be distinct G-orbits in Y. Lemma 2.7.10 tells us that there is an element β in B whose value is 0 at all points of O_1, and 1 at all points of O_2. Since G permutes the orbits, $\sigma\beta$ will also be 0 at all points of O_1 and 1 at all points of O_2. Then the product $\gamma = \prod_\sigma \sigma\beta$ will be 0 at points of O_1 and 1 at points of O_2, and the product γ is invariant. If p_i denotes the image in X of the orbit O_i, the maximal ideal $\mathfrak{m}_{p_i}$ of A is the intersection $A \cap \mathfrak{m}_q$, where q is any point in the orbit O_i. Therefore γ is in the maximal ideal $\mathfrak{m}_{p_1}$, but not in $\mathfrak{m}_{p_2}$. The images of the two orbits are distinct.

surjectivity of the map $Y/G \to X$. It suffices to show that the map $Y \to X$ is surjective.

extideal **2.7.11. Lemma.** *If I is an ideal of the invariant algebra A, and if the extended ideal IB is the unit ideal of B, then I is the unit ideal of A.*

As before, the extended ideal IB is the ideal of B generated by I.

Let's assume the lemma for the moment, and use it to prove surjectivity of the map $Y \to X$. Let p be a point of X. The lemma tells us that the extended ideal $\mathfrak{m}_p B$ isn't the unit ideal. So it is contained in a maximal ideal $\mathfrak{m}_q$ of B, where q is a point of Y. Then $\mathfrak{m}_p \subset (\mathfrak{m}_p B) \cap A \ \subset \mathfrak{m}_q \cap A$. The contraction $\mathfrak{m}_q \cap A$ is an ideal of A, and it isn't the unit ideal because it doesn't contain 1, which isn't in $\mathfrak{m}_q$. Since $\mathfrak{m}_p \subset \mathfrak{m}_q \cap A$ and $\mathfrak{m}_p$ is a maximal ideal, $\mathfrak{m}_p = \mathfrak{m}_q \cap A$. This means that q maps to p in X. □

proof of the lemma. If $IB = B$, there will be an equation $\sum_i z_i b_i = 1$, with z_i in I and b_i in B. The sums $\alpha_i = \sum_\sigma \sigma b_i$ are invariant, so they are elements of A, and the elements z_i are invariant because they are in A. Therefore $\sum_\sigma \sigma(z_i b_i) = z_i \sum_\sigma \sigma b_i = z_i \alpha_i$ is in I. Then

$$\sum_\sigma 1 = \sum_\sigma \sigma(1) = \sum_{\sigma,i} \sigma(z_i b_i) = \sum_i z_i \alpha_i$$

The right side is in I, and the left side is the order of the group which is an invertible element of A, because A contains the complex numbers. So I is the unit ideal. □

2.8 Exercises

2.8.1. Prove that relatively prime polynomials in F, G two variables x, y, not necessarily homogeneous, have finitely many common zeros in $\mathbb{A}^2$. chaptwoex xfinmanyzeros

2.8.2. Prove that if A, B are finite-type domains, defining $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2 \otimes b_1b_2)$ makes the tensor product $A \otimes B$ into a finite-type domain. xtensftyp

2.8.3. Prove that if a noetherian ring contains just one prime ideal, then that ideal is nilpotent. xidealnilp

2.8.4. Prove that an algebra A that is a complex vector space of dimension d contains at most d maximal ideals. xatmostmax

2.8.5. Let T denote the ring $\mathbb{C}[\epsilon]$, with $\epsilon^2 = 0$. If A is the coordinate ring of an affine variety X, an (infinitesimal) tangent vector to X is, by definition, given by an algebra homomorphism $\varphi : A \to T$. xtanvect

(i) Show that when such a homomorphism is written in the form $\varphi(a) = f(a) + d(a)\epsilon$, where f and d are functions $A \to \mathbb{C}$, f is an algebra homomorphism, and d is an f*-derivation*, a linear map that satisfies the identity $d(ab) = f(a)d(b) + d(a)f(b)$.

(ii) Let $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_r)$. Show that the tangent vectors to $X = \operatorname{Spec} A$ are defined by the equations $\nabla f_i(p)x = 0$. The tangent vectors are the vectors orthogonal to the gradients.

2.8.6. Let $i = (i_1, ..., i_n)$ be a set of non-negative integers, and let $x^{(i)}$ denote the monomial $x_1^{i_1} \cdots x_n^{i_n}$. A *formal power series* is a sum $\sum_{(i)} a_{(i)}x^{(i)}$, where $a_{(i)}$ are arbitrary complex numbers. There is no condition of convergence. Prove that the set of formal power series forms a domain $\mathbb{C}[[x_1, ..., x_n]]$, and that a power series whose constant term is nonzero is invertible. xinvertseries

2.8.7. Prove that that the varieties in the affine plane $\mathbb{A}^2$ are points, curves, and the affine plane $\mathbb{A}^2$ itself. xvarinplane

2.8.8. Derive version 1 of the Nullstellensatz from the Strong Nulletellensatz. xstrongimpliesN

2.8.9. Find generators for the ideal of $\mathbb{C}[x, y]$ of polynomials that vanish at the three points $(0, 0), (0, 1), (1, 0)$. xptsideal

2.8.10. Let A be a noetherian ring. Prove that a radical ideal I of A is the intersection of finitely many prime ideals. xintprime

2.8.11. A minimal prime ideal is an ideal that doesn't properly contain any other prime ideal. Prove that a nonzero, finite-type algebra A (not necessarily a domain) contains at least one and only finitely many minimal prime ideals. Try to find a proof that doesn't require much work. xminprime

2.8.12. Let K be a field and let R be the polynomial ring $K[x]$. Prove that the field of fractions of R is not a finitely generated K-algebra. franotfg

2.8.13. Prove that the algebra $A = \mathbb{C}[x, y]/(x^2 + y^2 + 1)$ is isomorphic to the Laurent Polynomial Ring $\mathbb{C}[t, t^{-1}]$, but that $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ is not isomorphic to $\mathbb{R}[t, t^{-1}]$. xcircle

2.8.14. Let C and D be closed subsets of an affine variety $X = \operatorname{Spec} A$. Suppose that no component of D is contained in C. Prove that there is a regular function f that vanishes on C and isn't identically zero on any component of D. xnotvanish

2.8.15. Classify algebras that are complex vector spaces of dimensions two or three. xdimthree

2.8.16. Let B be a finite-type domain, let p and q be points of the affine variety $Y = \operatorname{Spec} B$, and let A be the subset of B of elements f such that $f(p) = f(q)$. Prove xgluepts

(i) A is a finite type domain.

(ii) B is a finite A-module.

(iii) Let $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the morphism obtained from the inclusion $A \subset B$. Show that $\varphi(p) = \varphi(q)$, and that φ is bijective everywhere else.

2.8.17. The equation $y^2 = x^3$ defines a plane curve X with a cusp at the origin, the spectrum of the algebra $A = \mathbb{C}[x, y]/(y^2 - x^3)$. There is a homomorphism $A \xrightarrow{\varphi} \mathbb{C}[t]$, such that $\varphi(x) = t^2$ and $\varphi(y) = t^3$, and the associated morphism $\mathbb{A}^1_t \xrightarrow{u} X$ sends a point t of $\mathbb{A}^1$ to the point $(x, y) = (t^2, t^3)$ of X. Prove that u is a homeomorphism in the Zariski topology and in the classical topology. xcusp xspellout-morph **2.8.18.** Explain what a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ means in terms of polynomials, when $A = \mathbb{C}[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ and $B = \mathbb{C}[y_1, \ldots, y_n]/(g_1, \ldots, g_k)$.

xadjoin-frac **2.8.19.** Let $A = \mathbb{C}[x_1, ..., x_n]$, and let $B = A[\alpha]$, where α is an element of the fraction field $\mathbb{C}(x)$ of A. Describe the fibres of the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$.

xparam-curve **2.8.20.** Let X be the plane curve $y^2 = x(x-1)^2$, let $A = \mathbb{C}[x, y]/(y^2 - x(x-1)^2)$ be its coordinate algebra, and let x, y denote the residues of those elements in A too.

(i) Points of X can be parametrized. Use the lines $y = t(x-1)$ to determine such a parametrization.

(ii) Let $B = \mathbb{C}[t]$ and let T be the affine line $\operatorname{Spec} \mathbb{C}[t]$. The parametrization **(i)** gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.

xgr-plawtwo **2.8.21.** Let X be the affine line $\operatorname{Spec} \mathbb{C}[x]$. When we view $\operatorname{Spec} \mathbb{C}[x_1, x_2]$ as the product $X \times X$, a homomorphism $\mathbb{C}[x] \to \mathbb{C}[x_1, x_2]$ defines a law of composition on X, a morphism $X \times X \to X$. Determine the homomorphisms that define group laws on X with the point $x = 0$ as the identity.

xre-mainirred **2.8.22.** Let A be a finite-type domain, and let f be an irreducible element of $\mathbb{C}[x_1, ..., x_n]$ of positive degree. Prove that when f is viewed as an element of $A[x_1, ..., x_n]$, it is irreducible.

xzetaxy **2.8.23.** The cyclic group $G = \langle \sigma \rangle$ of order n operates on the polynomial algebra $A = \mathbb{C}[x, y]$ by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, where $\zeta = e^{2\pi i/n}$.

(i) Describe the invariant ring A^G by exhibiting generators and defining relations.

(ii) Prove that the there is a $2 \times n$ matrix whose 2×2-minors are defining relations for A^G.

(iii) Prove that the morphism $\operatorname{Spec} A = \mathbb{A}^2 \to \operatorname{Spec} B$ defined by the inclusion $B \subset A$ is surjective, and that its fibres are the G-orbits. Don't use Theorem 2.7.5.

Chapter 3 PROJECTIVE ALGEBRAIC GEOMETRY

projgeom

As before, points of projective space $\mathbb{P}^n$ are equivalence classes of nonzero vectors $(x_0, ..., x_n)$, the equivalence relation being that$(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n)$ for any nonzero complex number λ.

The irreducible closed subsets of projective space are called projective varieties. Though affine varieties are important, most of algebraic geometry concerns projective varieties. It won't be immediately clear why this is so, but one property of projective space gives a hint of its importance: With its classical topology, projective space is *compact*. Therefore projective varieties are compact.

We use this definition: A topological space is *compact* if:

It is a *Hausdorff space*: distinct points p and q of X have disjoint open neighborhoods, and
it is *quasicompact*: if a family $\{U^i\}$ of open sets covers X, then a finite subfamily covers X.

A space that is quasicompact is often called a compact space, but in these notes the word compact implies Hausdorff as well as quasicompact.

By the way, when we say that the sets $\{U^i\}$ *cover* a topological space X, we mean that X is the union $\bigcup U^i$. We don't allow U^i to contain elements that aren't in X, though that would be a customary usage in English.

Affine space isn't quasicompact in the classical topology, and therefore it isn't compact. The *Heine-Borel Theorem* asserts that a subset of $\mathbb{A}^n$ is compact in the classical topology if and only if it is closed and bounded.

We show that $\mathbb{P}^n$ is compact, assuming that the Hausdorff property has been verified. The $2n+1$-dimensional sphere $\mathbb{S}$ of unit length vectors in $\mathbb{A}^{n+1}$ is a bounded set, and because it is the zero locus of the equation $\overline{x}_0 x_0 + \cdots + \overline{x}_n x_n = 1$, it is closed. The Heine-Borel Theorem tells us that $\mathbb{S}$ is compact. The map $\mathbb{S} \to \mathbb{P}^n$ that sends a vector $(x_0, ..., x_n)$ to the point of projective space with that coordinate vector is continuous and surjective. The next lemma of topology shows that $\mathbb{P}^n$ is compact.

image-compact

3.0.1. Lemma. *Let $Y \xrightarrow{f} X$ be a continuous map. Suppose that Y is compact and that X is a Hausdorff space. Then the image $f(Y)$ is a closed, compact subset of X.* □

3.1 Projective Varieties

pvariety

A subset of $\mathbb{P}^n$ is *Zariski closed* if it is the set of common zeros of a family of *homogeneous* polynomials $f_1, ..., f_k$ in the coordinate variables $x_0, ..., x_n$, or if it is the set of zeros of the ideal $\mathcal{I}$ generated by such a family. As was explained in (1.3.1), $f(\lambda x) = 0$ for all λ if and only if all of the homogeneous parts of f vanish at x.

The Zariski closed sets are the closed sets in the *Zariski topology* on $\mathbb{P}^n$. We usually refer to Zariski closed sets simply as *closed sets*.

Because the polynomial ring $\mathbb{C}[x_0, ..., x_n]$ is noetherian, projective space $\mathbb{P}^n$ is a noetherian space: Every strictly increasing family of ideals of $\mathbb{C}[x]$ is finite, and every strictly decreasing family of closed subsets of $\mathbb{P}^n$ is finite. Therefore every closed subset of $\mathbb{P}^n$ is a finite union of irreducible closed sets (2.1.16).

3.1.1. Definition. A *projective variety* is an irreducible closed subspace of a projective space $\mathbb{P}^n$. defprojvar

We will want to know when two projective varieties are isomorphic. This will be explained in Section 3.5, when morphisms are defined.

The Zariski topology on a projective variety X is induced from the topology on the projective space that contains it (2.1.6). Since a projective variety X is closed in $\mathbb{P}^n$, a subset of X is closed in X if and only if it is closed in $\mathbb{P}^n$.

3.1.2. Lemma. *The one-point sets in projective space are closed.* *point-closed*

proof. Let p be the point $(a_0, ..., a_n)$. The first guess might be that the one-point set $\{p\}$ is defined by the equations $x_i = a_i$, but the polynomials $x_i - a_i$ aren't homogeneous in x. This is reflected in the fact that, for any $\lambda \neq 0$, the vector $(\lambda a_0, ..., \lambda a_n)$ represents the same point, but it doesn't satisfy those equations. The equations that define the set $\{p\}$ are

$$a_i x_j = a_j x_i, \tag{3.1.3}$$ define-point

for $i, j = 0, ..., n$, which imply that the ratios a_i/a_j and x_i/x_j are equal. □

3.1.4. Lemma. *The proper closed subsets of the projective line are the nonempty finite subsets, and the proper closed subsets of the projective plane are finite unions of points and curves.* □ *closedin-line*

The rest of this section contains a few examples of projective varieties.

3.1.5. linear subspaces linsubsp

If W is a subspace of dimension $r+1$ of the vector space $\mathbb{C}^{n+1}$, the points of $\mathbb{P}^n$ that are represented by the nonzero vectors in W form a *linear subspace* L of $\mathbb{P}^n$, of dimension r. If $(w_0, ..., w_r)$ is a basis of W, the linear subspace L corresponds bijectively to a projective space of dimension r, by

$$c_0 w_0 + \cdots + c_r w_r \quad \longleftrightarrow \quad (c_0, ..., c_r)$$

For example, the set of points $(x_0, ..., x_r, 0, ..., 0)$ is a linear subspace. A *line* is a linear subspace of dimension 1.

3.1.6. a quadric surface quadric-surface

A *quadric* in projective space $\mathbb{P}^n$ is the locus of zeros of an irreducible homogeneous quadratic polynomial in $n + 1$ variables. We describe a bijective map from the product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines to a quadric in projective three-space $\mathbb{P}^3$.

Let coordinates in the two copies of $\mathbb{P}^1$ be (x_0, x_1) and (y_0, y_1), respectively, and let the four coordinates in $\mathbb{P}^3$ be z_{ij}, with $0 \leq i, j \leq 1$. The map is defined by $z_{ij} = x_i y_j$. Its image is the quadric Q whose equation is

$$z_{00} z_{11} = z_{01} z_{10} \tag{3.1.7}$$ ponepone

To check that the map $\mathbb{P}^1 \times \mathbb{P}^1 \to Q$ defined by the equation $z_{ij} = x_i y_j$ is bijective, we choose a point w of Q. One of its coordinates, say z_{00}, will be nonzero. Then if (x, y) is a point of $\mathbb{P}^1 \times \mathbb{P}^1$ whose image is w, so that $z_{ij} = x_i y_j$, the coordinates x_0 and y_0 must be nonzero too. When we normalize z_{00}, x_0, and y_0 to 1, the equation of the quadric becomes $z_{11} = z_{01} z_{10}$. This equation has a unique solution for x_1 and y_1 such that $z_{ij} = x_i y_j$, namely $x_1 = z_{10}$ and $y_1 = z_{01}$.

The quadric Q contains two families of lines, the images of the subsets $x \times \mathbb{P}^1$ and $\mathbb{P}^1 \times y$ of $\mathbb{P} \times \mathbb{P}$. Its equation (3.1.7) can be diagonalized by substituting $z_{00} = s+t$, $z_{11} = s - t$, $z_{01} = u+v$, $z_{10} = u - v$. This changes (3.1.7) to $s^2 - t^2 = u^2 - v^2$. When we look at the affine open set $\{u = 1\}$, the equation becomes $s^2 + v^2 - t^2 = 1$. The real locus of this equation is a one-sheeted hyerboloid in $\mathbb{R}^3$, and the two families of complex lines in the quadric correspond to the familiar rulings of that hyperboloid by real lines.

hsurftwo

3.1.8. hypersurfaces

A *hypersurface* in projective space $\mathbb{P}^n$ is the locus of zeros of an irreducible homogeneous polynomial $f(x_0,...,x_n)$. Its *degree* is the degree of the polynomial f. Plane projective curves and quadric surfaces are hypersurfaces.

segreemb

3.1.9. the Segre embedding of a product

The product $\mathbb{P}^m_x \times \mathbb{P}^n_y$ of projective spaces can be embedded by its *Segre embedding* into a projective space $\mathbb{P}^N_z$ that has coordinates z_{ij}, with $i = 0,...,m$ and $j = 0,...,n$. So $N = (m+1)(n+1)-1$. The Segre embedding is defined by

segreco-ords

$$z_{ij} = x_i y_j \tag{3.1.10}$$

We call the coordinates z_{ij} the *Segre variables*. The map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ that was described in (3.1.6) is the simplest case of a Segre embedding.

segreeq

3.1.11. Proposition. *The Segre embedding maps the product $\mathbb{P}^m \times \mathbb{P}^n$ bijectively to the locus S of the* ***Segre equations***

segree-quations

$$z_{ij}z_{k\ell} - z_{i\ell}z_{kj} = 0 \tag{3.1.12}$$

The proof is analogous to the one given in (3.1.6). □

The Segre embedding is important because it makes the product of projective spaces into a projective variety, the closed subvariety of $\mathbb{P}^N$ defined by the Segre equations. However, to show that the product is a variety, we need to show that the locus of the Segre equations is irreducible, and this isn't obvious. We defer the proof to Section 3.3. (See Proposition 3.3.4.)

verone-seemb

3.1.13. the Veronese embedding of projective space

Let the coordinates in $\mathbb{P}^n$ be x_i, and let those in $\mathbb{P}^N$ be v_{ij}, with $0 \le i \le j \le n$. So $N = \binom{n+2}{2} - 1$. The *Veronese embedding* is the map $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^N$ defined by $v_{ij} = x_i x_j$. The Veronese embedding resembles the Segre embedding, but in the Segre embedding, there are distinct sets of coordinates x and y, and $i \le j$ isn't required.

The proof of the next proposition is similar to the proof of (3.1.11), once one has untangled the inequalities.

veroneq

3.1.14. Proposition. *The Veronese embedding f maps $\mathbb{P}^n$ bijectively to the locus X in $\mathbb{P}^N$ of the equations*

$$v_{ij}v_{k\ell} = v_{i\ell}v_{kj} \quad \textit{for} \quad 0 \le i \le k \le j \le \ell \le n$$ □

For example, the Veronese embedding maps $\mathbb{P}^1$ bijectively to the conic $v_{01}v_{01} = v_{00}v_{11}$ in $\mathbb{P}^2$.

twistcubic

3.1.15. the twisted cubic

Higher order Veronese embeddings can be defined by evaluating monomials of some degree $d > 2$. The first example is the embedding of $\mathbb{P}^1$ by the cubic monomials in two variables, which maps $\mathbb{P}^1$ to $\mathbb{P}^3$. Let the coordinates in $\mathbb{P}^3$ be $v_0,...,v_3$. The cubic Veronese embedding is defined by

$$v_0 = x_0^3, \quad v_1 = x_0^2 x_1, \quad v_2 = x_0 x_1^2, \quad v_3 = x_1^3$$

Its image, the locus $(v_0, v_1, v_2, v_3) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$, is called a *twisted cubic* in $\mathbb{P}^3$. It is the set of common zeros of the three 2×2 minors of the 2×3 matrix

$$\begin{pmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

which are

twcubic

$$v_0v_2 - v_1^2 \ , \quad v_1v_2 - v_0v_3 \ , \quad v_1v_3 - v_2^2 \tag{3.1.16}$$

A 2×3 matrix has rank ≤ 1 if and only if its 2×2 minors are zero. So a point (v_0, v_1, v_2, v_3) lies on the twisted cubic if that matrix has rank 1, which means that the vectors (v_0, v_1, v_2) and (v_1, v_2, v_3), if both are nonzero, represent the same point of $\mathbb{P}^2$.

Setting $x_0 = 1$ and $x_1 = t$, the twisted cubic becomes the locus of points $(1, t, t^2, t^3)$. There is one more point on the twisted cubic, the point $(0, 0, 0, 1)$. □

3.2 Homogeneous Ideals

Let R denote the polynomial algebra $\mathbb{C}[x_0,...,x_n]$. homogen

3.2.1. Lemma. *Let $\mathcal{I}$ be an ideal of R. The following conditions are equivalent.* *homogideal*
(i) *$\mathcal{I}$ can be generated by homogeneous polynomials.*
(ii) *A polynomial is in $\mathcal{I}$ if and only if its homogeneous parts are in $\mathcal{I}$.* □

An ideal that satisfies these conditions is a *homogeneous ideal*.

3.2.2. Corollary. *Let S be a subset of projective space $\mathbb{P}^n$. The set of elements of R that vanish at all points of S is a homogeneous ideal.* *idealishom*

This follows from Lemma 1.3.2. □

3.2.3. Lemma. *The radical of a homogeneous ideal is homogeneous.* *radicalhomogeneous*

proof. Let $\mathcal{I}$ be a homogeneous ideal, and let f be an element of its radical $\operatorname{rad}\mathcal{I}$ (2.1.20). Some power f^r is in $\mathcal{I}$. When f is written as the sum $f_0+\cdots+f_d$ of its homogeneous parts, the highest degree part of f^r is $(f_d)^r$. Since $\mathcal{I}$ is homogeneous, $(f_d)^r$ is in $\mathcal{I}$ and f_d is in $\operatorname{rad}\mathcal{I}$. Then $f_0+\cdots+f_{d-1}$ is also in $\operatorname{rad}\mathcal{I}$. By induction on d, all of the homogeneous parts $f_0,...,f_d$ are in $\operatorname{rad}\mathcal{I}$. □

3.2.4. The locus of zeros of a set f of homogeneous polynomials in $\mathbb{P}^n$ may be denoted by $V(f)$, and the locus of zeros of a homogeneous ideal $\mathcal{I}$ by $V(\mathcal{I})$. This is the same notation as the one we use for closed subsets of affine space. defVXI

The complement of the origin in the affine space $\mathbb{A}^{n+1}$ is mapped to the projective space $\mathbb{P}^n$ by sending a vector $(x_0,...,x_n)$ to the point of $\mathbb{P}^n$ defined by that vector. A homogeneous ideal $\mathcal{I}$ has a zero locus W in affine space and a locus V in projective space. Unless $\mathcal{I}$ is the unit ideal, the origin $x=0$ will be a point of W, and the complement of the origin in W will map surjectively to V.

If W contains a point x other than the origin, then every point of the one-dimensional subspace of $\mathbb{A}^{n+1}$ spanned by x is in W, because a homogeneous polynomial f vanishes at x if and only if it vanishes at λx. An affine variety that is the union of lines through the origin is called an *affine cone*. If the locus W contains a point x other than the origin, it is an affine cone.

The loci $\{x_0^2+x_1^2-x_2^2=0\}$ and $\{x_0^3+x_1^3+x_2^3+x_0x_1x_2=0\}$ are affine cones in $\mathbb{A}^3$.

Note. The real locus $x_0^2+x_1^2-x_2^2=0$ in $\mathbb{R}^3$ decomposes into two parts when the origin is removed. Because of this, it is sometimes called a "double cone". The complex locus doesn't decompose.

3.2.5. the irrelevant ideal irrel

In the polynomial algebra $R=\mathbb{C}[x_0,...,x_n]$, the maximal ideal $\mathcal{M}=(x_0,...,x_n)$ generated by the variables is called the *irrelevant ideal* because its zero locus in projective space is empty.

3.2.6. Proposition. *The zero locus of a homogeneous ideal $\mathcal{I}$ of R in projective space is empty if and only if $\mathcal{I}$ contains a power of the irrelevant ideal $\mathcal{M}$.* *nozeros*

Another way to say this is: The zero locus of a proper homogeneous ideal $\mathcal{I}$ is empty if and only if its radical is the irrelevant ideal.

proof of Proposition 3.2.6. Let V be the zero locus of $\mathcal{I}$ in $\mathbb{P}^n$. If $\mathcal{I}$ contains a power of $\mathcal{M}$, it contains a power of each variable. Powers of the variables have no common zeros in projective space, so V is empty.

Suppose that V is empty, and let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0,...,x_n$. Since the complement of the origin in W maps to the empty locus V, it is empty. The origin is the only point that might be in W. If W is the one point space consisting of the origin, then $\operatorname{rad}\mathcal{I}=\mathcal{M}$. If W is empty, $\mathcal{I}$ is the unit ideal. □

3.2.7. Strong Nullstellensatz, projective version. *Let g be a nonconstant homogeneous polynomial in $x_0,...,x_n$, and let $\mathcal{I}$ be a homogeneous ideal of $\mathbb{C}[x]$, not the unit ideal. If g vanishes at every point of the zero locus $V(\mathcal{I})$ in $\mathbb{P}^n$, then $\mathcal{I}$ contains a power of g.* *homstrnull* *proof.* Let W be the locus of zeros of $\mathcal{I}$ in the affine space with coordinates $x_0, ..., x_n$. A homogeneous polynomial g that vanishes on $V(\mathcal{I})$ vanishes at every point of W different from the origin, and since g isn't a constant, it vanishes at the origin too. So the affine Strong Nullstellensatz 2.3.7 applies. If a nonconstant homogeneous polynomial g vanishes on W, then $\mathcal{I}$ contains a power of g. $\square$

hstrongnull-cor

3.2.8. Corollary. **(i)** *Let f and g be homogeneous polynomials. If f is irreducible and $V(f) \subset V(g)$, then f divides g.*

(ii) *Let $\mathcal{I}$ and $\mathcal{J}$ be homogeneous radical ideals, neither of which is the unit ideal. If $V(\mathcal{I}) = V(\mathcal{J})$, then $\mathcal{I} = \mathcal{J}$.*

proof. **(ii)** Suppose that $V(\mathcal{I}) = V(\mathcal{J})$. Let g be a homogeneous element of $\mathcal{J}$. Then g vanishes on $V(\mathcal{J})$ and therefore on $V(\mathcal{I})$. Since $\mathcal{I}$ is a radical ideal, the Strong Nullstellensatz tells us that $\mathcal{I}$ contains g. This shows that $\mathcal{J} \subset \mathcal{I}$. Similarly, $\mathcal{I} \subset \mathcal{J}$. $\square$

homprime

3.2.9. Lemma. *Let $\mathcal{P}$ be a homogeneous ideal in the polynomial algebra $\mathbb{C}[x_0, ..., x_n]$, not the unit ideal. The following conditions are equivalent:*

(i) *$\mathcal{P}$ is a prime ideal.*

(ii) *If f and g are homogeneous polynomials, and if $fg \in \mathcal{P}$, then $f \in \mathcal{P}$ or $g \in \mathcal{P}$.*

(iii) *If $\mathcal{A}$ and $\mathcal{B}$ are homogeneous ideals, and if $\mathcal{AB} \subset \mathcal{P}$, then $\mathcal{A} \subset \mathcal{P}$ or $\mathcal{B} \subset \mathcal{P}$. Or, if $\mathcal{A} \supset \mathcal{P}$ and $\mathcal{B} \supset \mathcal{P}$, and if $\mathcal{AB} \subset \mathcal{P}$, then $\mathcal{A} = \mathcal{P}$ or $\mathcal{B} = \mathcal{P}$.*

In other words, a homogeneous ideal is a prime ideal if the usual conditions for a prime ideal are satisfied when the polynomials or ideals are homogeneous.

proof of the lemma. When the word homogeneous is omitted, **(ii)** and **(iii)** become the definition of a prime ideal. So **(i)** implies **(ii)** and **(iii)**. The fact that **(iii)** $\Rightarrow$ **(ii)** is proved by considering the principal ideals generated by f and g.

(ii) $\Rightarrow$ **(i)** Suppose that a homogeneous ideal $\mathcal{P}$ satisfies condition **(ii)**. We show that if a product fg of two polynomials, not necessarily homogeneous, is in $\mathcal{P}$, then f or g is in $\mathcal{P}$. If f has degree d and g has degree e, the highest degree part of fg is the product of the homogeneous parts f_d and g_e. Since $\mathcal{P}$ is a homogeneous ideal that contains fg, it contains $f_d g_e$. Therefore one of the factors, say f_d, is in $\mathcal{P}$. Let $h = f - f_d$. Then $hg = fg - f_d g$ is in $\mathcal{P}$, and it has lower degree than fg. By induction on the degree of fg, h or g is in $\mathcal{P}$, and if h is in $\mathcal{P}$, so is f. $\square$

closedirred

3.2.10. Proposition. *Let V be the zero locus in $\mathbb{P}^n$ of a homogeneous radical ideal $\mathcal{I}$ that isn't the irrelevant ideal or the unit ideal. Then V is a projective variety, an irreducible closed subset of $\mathbb{P}^n$, if and only if $\mathcal{I}$ is a prime ideal. Thus a subset V of $\mathbb{P}^n$ is a projective variety if and only if it is the zero locus of a homogeneous prime ideal other than the irrelevant ideal.*

proof. The closed set V isn't empty, so the locus W of zeros of the radical ideal $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ contains points other than the origin. Let W' be the complement of the origin in W. Then W' maps surjectively to V. If V is irreducible, then W' is irreducible and therefore W is irreducible (2.1.15) **(ii)**. Proposition 2.1.22 tells us that $\mathcal{I}$ is a prime ideal.

Conversely, suppose that $\mathcal{I}$ isn't a prime ideal. Then there are homogeneous ideals $\mathcal{A} > \mathcal{I}$ and $\mathcal{B} > \mathcal{I}$, such that $\mathcal{AB} \subset \mathcal{I}$. Since $\mathcal{I}$ is a radical ideal, $\operatorname{rad}(\mathcal{AB}) \subset \mathcal{I}$, and since $\operatorname{rad}\mathcal{A} \operatorname{rad}\mathcal{B} \subset \operatorname{rad}(\mathcal{AB})$, $\operatorname{rad}\mathcal{A} \operatorname{rad}\mathcal{B} \subset \mathcal{I}$. Therefore we may suppose that $\mathcal{A}$ and $\mathcal{B}$ are radical ideals. If α is an element of $\mathcal{A}$ that isn't in $\mathcal{I}$, the Strong Nullstellensatz asserts that α doesn't vanish on $V(\mathcal{I})$. So $V(\mathcal{A}) < V(\mathcal{I})$ and similarly, $V(\mathcal{B}) < V(\mathcal{I})$. But $V(\mathcal{A}) \cup V(\mathcal{B}) = V(\mathcal{AB}) \supset V(\mathcal{I})$. Therefore $V(\mathcal{I})$ isn't an irreducible space. $\square$

var-quasiproj

3.2.11. quasiprojective varieties

We will sometimes want to study a nonempty open subset of a projective variety in addition to the projective variety itself. We call such an open subset a *variety* too. The topology on such a variety is induced from the topology on projective space. It will be an irreducible topological space (Lemma 2.1.15). For example, the complement of a point in a projective variety is variety.

An affine variety $X = \operatorname{Spec} A$ may be embedded as a closed subvariety into the standard open set $\mathbb{U}^0$, which is an affine space. It becomes an open subset of its closure in $\mathbb{P}^n$, a projective variety (Lemma 2.1.15).

So it is a variety. And of course, a projective variety is a variety. Most of the the varieties we encounter will be either affine or projective.

Elsewhere, what we are calling a variety is called a *quasiprojective variety*. We drop the adjective 'quasiprojective'. There are abstract varieties that aren't quasiprojective. They cannot be embedded into any projective space. But such varieties aren't very important and we won't study them. In fact, it is hard enough to find examples that we won't try to give one here. For us, the adjective 'quasiprojective' is superfluous as well as ugly.

3.2.12. Lemma. *The topology on the standard open subset $\mathbb{U}^0$ that isinduced from the Zariski topology on* *topon-*
$\mathbb{P}^n$ is same as the Zariski topology that is obtained by viewing $\mathbb{U}^0$ as the affine space $\operatorname{Spec}\mathbb{C}[u_1,...,u_n]$, *standaff*
$u_i = x_i/x_0$.

This follows from the fact that a homogeneous polynomial $f(x_0,...,x_n)$ and its dehomogenization $f(1,u_1,...,u_n)$ have the same zeros on $\mathbb{U}^0$. $\square$

3.3 Product Varieties

The properties of products of varieties are intuitively plausible, but one must be careful, because the Zariski prodvar
topology on a product isn't the product topology.

The *product topology* on the product $X\times Y$ of topological spaces is the coarsest topology such that the projection maps $X\times Y \to X$ and $X\times Y \to Y$ are continuous. If C and D are closed subsets of X and Y respectively, then $C\times D$ is a closed subset of $X\times Y$ in the product topology, and every closed set in the product topology is a finite union of such subsets.

The product topology is much coarser than the Zariski topology. For example, the proper Zariski closed subsets of $\mathbb{P}^1$ are the nonempty finite subsets. In the product topology, the proper closed subsets of $\mathbb{P}^1\times\mathbb{P}^1$ are finite unions of sets of the form $p\times\mathbb{P}^1$, $\mathbb{P}^1\times q$, and $p\times q$ ('vertical' lines, 'horizontal' lines, and points). Most Zariski closed subsets of $\mathbb{P}^1\times\mathbb{P}^1$, the diagonal $\Delta=\{(p,p)\,|\,p\in\mathbb{P}^1\}$ for instance, aren't of this form.

3.3.1. the Zariski topology on $\mathbb{P}^m\times\mathbb{P}^n$ zartopprod

As has been mentioned, the product of projective spaces $\mathbb{P}^m\times\mathbb{P}^n$ can be embedded into a projective space by the Segre map (3.1.9), which identifies the product as a closed subset of $\mathbb{P}^N$, with $N = mn+m+n$. It is the locus of the Segre equations $z_{ij}z_{k\ell} = z_{i\ell}z_{kj}$, Since $\mathbb{P}^m\times\mathbb{P}^n$, with its Segre embedding, becomes a closed subset of $\mathbb{P}^N$, we don't really need a separate definition of its Zariski topology. Its closed subsets are the zero sets of families of homogeneous polynomials in the Segre variables z_{ij}, families that include the Segre equations. However, it is important to know that the Segre embedding maps the product $\mathbb{P}^m\times\mathbb{P}^n$ to an *irreducible* closed subset of $\mathbb{P}^N$, so that the product becomes a projective variety. This will be proved below, in Corollary 3.3.5.

One can describe the closed subsets of $\mathbb{P}^m\times\mathbb{P}^n$ directly, in terms of bihomogeneous polynomials. A polynomial $f(x,y)$ in $x = (x_0,...,x_m)$ and $y = (y_0,...,y_n)$ is *bihomogeneous* if it is homogeneous in the variables x and homogeneous in the variables y. For example, $x_0^2y_0 + x_0x_1y_1$ is a bihomogeneous polynomial, of degree 2 in x and degree 1 in y.

The bihomogeneous part of bidegree i,j of a polynomial $f(x,y)$ is the sum of terms whose degrees in x and y are i and j, respectively. Because (x,y) and $(\lambda x,\mu y)$ represent the same point of $\mathbb{P}^m\times\mathbb{P}^n$ for all nonzero scalars λ and μ, we want the locus $f = 0$ to have the property that $f(x,y) = 0$ if and only if $f(\lambda x,\mu y) = 0$ for all nonzero λ and μ. This will be true if and only if all bihomogeneous parts of f are zero.

3.3.2. Proposition. **(i)** *The Segre image of a subset Z of $\mathbb{P}^m\times\mathbb{P}^n$ is closed if and only if Z is the locus of* *closedin-*
zeros of a family of bihomogeneous polynomials. *pxp*

(ii) *If X and Y are closed subsets of $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively, then $X\times Y$ is a closed subset of $\mathbb{P}^m\times\mathbb{P}^n$.*

(iii) *The projection maps $\mathbb{P}^m\times\mathbb{P}^n \xrightarrow{\pi_1} \mathbb{P}^m$ and $\mathbb{P}^m\times\mathbb{P}^n \xrightarrow{\pi_2} \mathbb{P}^n$ are continuous.*

(iv) *For all x in $\mathbb{P}^m$ the fibre $x\times\mathbb{P}^n$ of π_1 is homeomorphic to $\mathbb{P}^n$, and for all y in $\mathbb{P}^n$, the fibre $\mathbb{P}^m\times y$ is homeomorphic to $\mathbb{P}^m$.*

proof. **(i)** Let Π denote the Segre image of $\mathbb{P}^m\times\mathbb{P}^n$, and let $f(z)$ be a homogeneous polynomial in the Segre variables z_{ij}. When we substitute $z_{ij} = x_iy_j$ into f, we obtain a bihomogeneous polynomial $\widetilde{f}(x,y)$ whose degree in x and in y is the same as the degree of f. The inverse image of the zero set of f in Π is the zero set of $\widetilde{f}$ in $\mathbb{P}^m \times \mathbb{P}^n$. Therefore the inverse image of a closed subset of Π is the zero set of a family of bihomogeneous polynomials in $\mathbb{P}^m \times \mathbb{P}^n$.

Conversely, let $\widetilde{g}(x,y)$ be a bihomogeneous polynomial, say of degrees r in x and degree s in y. If $r=s$, we may collect variables that appear in $\widetilde{g}$ in pairs x_iy_j and replace each pair x_iy_j by z_{ij}. We will obtain a homogeneous polynomial g in z such that $g(z)=\widetilde{g}(x,y)$ when $z_{ij}=x_iy_j$. The zero set of g in Π is the image of the zero set of $\widetilde{g}$ in $\mathbb{P}^m \times \mathbb{P}^n$.

Suppose that $r \geq s$, and let $k=r-s$. Because the variables y cannot all be zero at any point of $\mathbb{P}^n$, the equation $\widetilde{g}=0$ on $\mathbb{P}^m \times \mathbb{P}^n$ is equivalent with the system of equations $\widetilde{g}y_0^k = \widetilde{g}y_1^k = \cdots = \widetilde{g}y_n^k = 0$. The polynomials $\widetilde{g}y_i^k$ are bihomogeneous, and of same degree in x as in y. This puts us back in the first case.

(ii) A homogeneous polynomial $f(x)$ is a bihomogeneous polynomial of degree zero in y, and a homogeneous polynomial $g(y)$ as a bihomogeneous polynomial of degree zero in x. So $X\times Y$, which is a locus of the form $f(x)=g(y)=0$ in $\mathbb{P}^m \times \mathbb{P}^n$, is closed in $\mathbb{P}^m \times \mathbb{P}^n$.

(iii) For the projection π_1, we must show that if X is a closed subset of $\mathbb{P}^m$, its inverse image $X\times\mathbb{P}^n$ is closed. This is a special case of **(ii)**.

(iv) It will be best to denote the chosen point of $\mathbb{P}^m$ by a symbol other than x here. We'll denote it by x^0. The bijective map $x^0\times\mathbb{P}^n \to \mathbb{P}^n$ is continuous because π_2 is continuous. To show that the inverse map is continuous, we must show that a closed subset Z of $x^0\times\mathbb{P}^n$ is the inverse image of a closed subset of $\mathbb{P}^n$. Say that Z is the zero locus of a set of bihomogeneous polynomials $f(x,y)$. The polynomials $\overline{f}(y)=f(x^0,y)$ are homogeneous in y, and the inverse image of their zero locus is Z. □

projcont **3.3.3. Corollary.** *Let X and Y be projective varieties, and let Π denote the product $X\times Y$, regarded as a closed subspace of $\mathbb{P}^m\times\mathbb{P}^n$.*

- *The projections $\Pi \to X$ and $\Pi \to Y$ are continuous.*
- *For all x in X and all y in Y, the fibres $x\times Y$ and $X\times y$ are homeomorphic to Y and X, respectively.* □

Pirred **3.3.4. Lemma.** *Let X and Y be irreducible topological spaces, and suppose that a topology on the product $\Pi=X\times Y$ is given. Then Π is an irreducible topological space if it has these properties:*

- *The projections $\Pi \xrightarrow{\pi_1} X$ and $\Pi \xrightarrow{\pi_2} Y$ are continuous.*
- *For all x in X and all y in Y, the fibres $x\times Y$ and $X\times y$ are homeomorphic to Y and X, respectively.*

The first condition tells us that the topology on $X\times Y$ is at least as fine as the product topology, and the second one tells us that the topology isn't too fine. (It is unlikely that we want to give Π the discrete topology.)

Some notation for use in the proof: Let x be a point of X. If W is a subset of $X\times Y$, we denote the intersection $W\cap(x\times Y)$ by ${}_xW$, and similarly, if y is a point of Y, we denote $W\cap(X\times y)$ by W_y. By analogy with the x,y-plane, we call ${}_xW$ and W_y a *vertical slice* and a *horizontal slice* of W, respectively.

proof of Lemma 3.3.4. We prove irreducibility by showing that the intersection of two nonempty open subsets W and W' of $X\times Y$ isn't empty (2.1.13).

We show first that, if W is an open subset of $X\times Y$, then its projection $U=\pi_2 W$ is an open subset of Y. We are given that, for every x, the fibre $x\times Y$ is homeomorphic to Y. Since W is open in $X\times Y$, the vertical slice ${}_xW$ is open in $x\times Y$, and its image $\pi_2({}_xW)$ is open in the homeomorphic space Y. Since W is the union of the sets ${}_xW$, U is the union of the open sets $\pi_2({}_xW)$. So U is open.

Now let W and W' be nonempty open subsets of $X\times Y$, and let U and U' be their images via projection to Y. So U and U' are nonempty open subsets of Y. Since Y is irreducible, $U\cap U'$ isn't empty. Let y be a point of $U\cap U'$. Since $U=\pi_2W$ and y is a point of U, the horizontal slice W_y, which is an open subset of the fibre $X\times y$, isn't empty. Similarly, W'_y isn't empty. Since $X\times y$ is homeomorphic to the irreducible space X, it is irreducible. So $W_y\cap W'_y$ isn't empty. Therefore $W\cap W'$ isn't empty, as was to be shown. □

prodirred **3.3.5. Corollary.** *The product $X\times Y$ of projective varieties X and Y is a projective variety.* □

3.3.6. products of affine varieties

Let $X=\operatorname{Spec} A$ and $Y=\operatorname{Spec} B$ be affine varieties. Say that X is embedded as a closed subvariety of $\mathbb{A}^m$, so that $A=\mathbb{C}[x_1,...,x_m]/P$ for some prime ideal P, and that Y is embedded similarly into $\mathbb{A}^n$,

$B = \mathbb{C}[y_1, ..., y_n]/Q$ for some prime ideal Q. Then in affine x, y-space $\mathbb{A}^{m+n}$, $X \times Y$ is the locus of the equations $f(x) = 0$ and $g(y) = 0$, with f in P and g in Q. Let I be the ideal of $\mathbb{C}[x, y]$ generated by P and Q. Its locus of zeros in $\mathbb{A}^{m+n}$ is $X \times Y$. Lemma 3.3.4 shows that $X \times Y$ is irreducible, so it is a variety. The radical of I is a prime ideal (3.2.10).

3.3.7. Proposition. *The ideal I contains every polynomial that vanishes on the variety $X \times Y$. Therefore I is a prime ideal.* *fgprime*

proof of Proposition 3.3.7. Let $R = \mathbb{C}[x, y]/I$. The projection $X \times Y \to X$ is surjective. Therefore the map $A \to R$ is injective, and similarly, $B \to R$ is injective. We identify A and B with their images in R.

Any polynomial $f(x, y)$ can the written, in many ways, as a sum, each of whose terms is a product of a polynomial in x with a polynomial in y: $f(x, y) = \sum a_i(x)b_i(y)$. Therefore any element ρ of R can be written as a finite sum of products

$$\rho = \sum_{i=1}^{k} a_i b_i \tag{3.3.8}$$

with a_i in A and b_i in B. To show that 0 is the only element of R that vanishes identically on $X \times Y$, we show that a sum ρ of k products $a_i b_i$ that vanishes on $X \times Y$ can also be written as a sum of $k-1$ such products.

Say that $\rho = \sum_1^k a_i b_i$. If $a_k = 0$, then ρ is the sum $\sum_{i=1}^{k-1} a_i b_i$ of $k-1$ products. If $a_k \neq 0$, the function on X defined by a_k isn't identically zero. We choose a point x^0 of X such that $a_k(x^0) \neq 0$. Let $\overline{a}_i = a_i(x^0)$ and $\overline{\rho}(y) = \rho(x^0, y)$. Then $\overline{\rho}(y) = \sum_{i=1}^{k} \overline{a}_i b_i$ is an element of B. Since ρ vanishes on $X \times Y$, $\overline{\rho}$ vanishes on $Y = \operatorname{Spec} B$. Therefore $\overline{\rho} = 0$. Let $c_i = \overline{a}_i/\overline{a}_k$. Then $b_k = -\sum_{i=1}^{k-1} c_i b_i$. Substituting for b_k into ρ and collecting coefficients of $b_1, ..., b_{k-1}$ gives us an expression for ρ as a sum of $k-1$ terms. When $k = 1$, $b_1 = 0$, and therefore $\rho = 0$. □

3.4 Rational Functions

3.4.1. the function field ratfnsec fnfld

Let X be a projective variety, a closed subvariety of $\mathbb{P}^n$, and let $\mathbb{U}^i : \{x_i \neq 0\}$ be one of the standard open subsets of $\mathbb{P}^n$. The intersection $X^i = X \cap \mathbb{U}^i$, if it isn't empty, will be a closed subvariety of the affine space $\mathbb{U}^i$ and a dense open subset of X. It will be an affine variety. We will call the sets $X^i = X \cap \mathbb{U}^i$ that aren't empty the *standard open subsets* of X.

3.4.2. Lemma. *The localizations of the standard open sets $X^i = X \cap \mathbb{U}^i$ are affine varieties, and they form a basis for the topology on X.* *enufaff*

This follows from (2.6.2). □

There are affine open sets that aren't localizations of the standard open sets, but we don't yet have a definition of a general affine open set. Rather than defining those sets here, we postpone discussion to Section 3.6.

Let X be a closed subvariety of $\mathbb{P}^n$, and let $x_0, ..., x_n$ be coordinates in $\mathbb{P}^n$. For each $i = 0, ..., n$, let $X^i = X \cap \mathbb{U}^i$. We omit the indices for which X^i is empty. Then X^i will be affine, and the intersection $X^{ij} = X^i \cap X^j$ will be a localization of X^i and also localization of X^j. The coordinate algebra A_i of X^i is generated by the images of the elements $u_{ij} = x_j/x_i$ in A_i, and if we denote those images by u_{ij} too, then $X^{ij} = \operatorname{Spec} A_{ij}$, where $A_{ij} = A_i[u_{ij}^{-1}] = A_j[u_{ji}^{-1}]$.

3.4.3. Definition. The *function field* K of a projective variety X is the function field of any one of the standard open subsets X^i, and the function field of an open subvariety X' of a projective variety X is the function field of X. All open subvarieties of variety have the same function field. deffnfld

For example, let x_0, x_1, x_2 be coordinates in $\mathbb{P}^2$. To describe the function field of $\mathbb{P}^2$, we can use the standard open set $\mathbb{U}^0$, which is an affine plane $\operatorname{Spec} \mathbb{C}[u_1, u_2]$ with $u_i = x_i/x_0$. The function field of $\mathbb{P}^2$ is the field of rational functions: $K = \mathbb{C}(u_1, u_2)$.

We must use u_1, u_2 as coordinates here. We shouldn't normalize x_0 to 1 and use coordinates x_1, x_2, because we may want to change to another standard open set such as $\mathbb{U}^1$. The coordinates in $\mathbb{U}^1$ are $v_0 = x_0/x_1$ and $v_2 = x_2/x_1$, and the function field K is also the field of rational functions $\mathbb{C}(v_0, v_2)$. The two fields $\mathbb{C}(u_1, u_2)$ and $\mathbb{C}(v_0, v_2)$ are the same.

A *rational function* on a variety X is an element of its function field. When a point p of X lies in a standard open set $X^i = \operatorname{Spec} A_i$, a rational function α is *regular* at p if it can be written as a fraction a/s of elements of A_i with $s(p) \neq 0$. If so, its *value* at p is $\alpha(p) = a(p)/s(p)$. If X' is an open subvariety of a projective variety X, a rational function on X' is *regular* at a point p of X' if it is a regular function on X at p.

When we regard an affine variety $X = \operatorname{Spec} A$ as a closed subvariety of $\mathbb{U}^0$, its function field will be the field of fractions of A. Proposition 2.5.2 shows that the regular functions on $\operatorname{Spec} A$ are the elements of A.

Thus a rational function on a projective variety X defines a function on a nonempty open subset of X. It can be evaluated at some points of X, not at all points.

regindep **3.4.4. Lemma. (i)** *Let p be a point of a projective variety X. The regularity of a rational function at p doesn't depend on the choice of a standard open set that contains p.*

(ii) *A rational function that is regular on a nonempty open subset X' is determined by the function it defines on X'.*

Part **(ii)** follows from Corollary 2.4.16. □

ptvlfld **3.4.5. points with values in a field**

Let K be a field that contains the complex numbers. A *point* of projective space $\mathbb{P}^n$ *with values in* K is an equivalence class of nonzero vectors $\alpha = (\alpha_0, ..., \alpha_n)$ with α_i in K, the equivalence relation being analogous to the one for ordinary points: $\alpha \sim \alpha'$ if $\alpha' = \lambda\alpha$ for some λ in K. If X is the subvariety of $\mathbb{P}^n$ defined by a homogeneous prime ideal $\mathcal{P}$ of $\mathbb{C}[x]$, a *point α of X with values in K* is a point of $\mathbb{P}^n$ with values in K such that $f(\alpha) = 0$ for all f in $\mathcal{P}$.

Let K be the function field of a projective variety X. The embedding of X into $\mathbb{P}^n$ defines a point of X with values in K. To get this point, we choose a standard affine open set $\mathbb{U}^i$ of $\mathbb{P}^m$ such that $X^i = X \cap \mathbb{U}^i$ isn't empty. Say that $i = 0$. Then X^0 will be affine, $X^0 = \operatorname{Spec} A_0$. The embedding of X^0 into the affine space $\mathbb{U}^0$ is defined by a homomorphism $\mathbb{C}[u_1, ..., u_n] \to A_0$, with $u_i = x_i/x_0$. If α_i denotes the image of u_i in A_0, for $i = 1, ..., n$ and $\alpha_0 = 1$, then $(\alpha_0, ..., \alpha_n)$ is the point of $\mathbb{P}^n$ with values in K defined by the projective embedding of X.

ffprod **3.4.6. Note.** *(the function field of a product)* The function field of the product $X\times Y$ of varieties isn't generated by the function fields K_X and K_Y of X and Y. For example, let $X = \operatorname{Spec} \mathbb{C}[x]$ and $Y = \operatorname{Spec} \mathbb{C}[y]$ (one x and one y). Then $X \times Y = \operatorname{Spec} \mathbb{C}[x, y]$. The function field of $X \times Y$ is the field of rational functions $\mathbb{C}(x, y)$ in two variables. The algebra generated by the fraction fields $\mathbb{C}(x)$ and $\mathbb{C}(y)$ consists of the rational functions $p(x, y)/q(x, y)$ in which $q(x, y)$ is a product of a polynomial in x and a polynomial in y. Most rational functions, $1/(x + y)$ for instance, aren't of that type. □

ratfnspspace **3.4.7. interlude: rational functions on projective space**

A *homogeneous fraction* of polynomials in $x_0, ..., x_n$ is a fraction g/h of homogeneous polynomials. The *degree* of such a fraction is the difference of degrees: $\deg\, g/h = \deg\, g - \deg\, h$. A homogeneous fraction f is *regular* at a point p of $\mathbb{P}^n$ if, when it is written as a fraction g/h of relatively prime homogeneous polynomials, the denominator h isn't zero at p, and f is *regular on a subset* U if it is regular at every point of U.

A homogeneous fraction f of degree $d \neq 0$ won't define a function anywhere on projective space, because $f(\lambda x) = \lambda^d f(x)$. In particular, a nonconstant homogeneous polynomial g of won't define a function, though it makes sense to say that such a polynomial vanishes at a point of $\mathbb{P}^n$. On the other hand, a homogeneous fraction g/h of degree zero, so that g and h have the same degree r, defines a function wherever h isn't zero, because $g(\lambda x)/h(\lambda x) = \lambda^r g(x)/\lambda^r h(x) = g(x)/h(x)$.

homogfractsfnfld **3.4.8. Lemma. (i)** *Let h be a homogeneous polynomial of positive degree d, and let V be the open subset of $\mathbb{P}^n$ of points at which h isn't zero. The rational functions that are regular on V are those of the form g/h^k, where $k \geq 0$ and g is a homogeneous polynomial of degree dk.*

(ii) *The only rational functions that are regular at every point of $\mathbb{P}^n$ are the constant functions.*

For example, the homogeneous polynomials that don't vanish at any point of the standard open set $\mathbb{U}^0$ are scalar multiples of powers of x_0. So the rational functions that are regular on $\mathbb{U}^0$ are those of the form g/x_0^k, with g homogeneous of degree k. Because $g(x_0,...,x_m)/x_0^k = g(u_1,...,u_n)$, this agrees with the fact that the coordinate algebra of $\mathbb{U}^0$ is the polynomial ring $\mathbb{C}[u_1,...,u_n]$, with $u_i = x_i/x_0$.

proof of Lemma 3.4.8. **(i)** Let α be a rational function that is regular on the open set V, say $\alpha = g_1/h_1$, where g_1 and h_1 are relatively prime homogeneous polynomials. Then h_1 doesn't vanish on V, so its zero locus in $\mathbb{P}^n$ is contained in the zero locus of h. According to the Strong Nullstellensatz, h_1 divides a power of h. Say that $h^k = fh_1$. Then $g_1/h_1 = fg_1/fh_1 = fg_1/h^k$.

(ii) If a rational function f is regular at every point of $\mathbb{P}^n$, then it is regular on $\mathbb{U}^0$, so it will have the form g/x_0^k, where g is a homogeneous polynomial of degree k not divisible by x_0. Since f is also regular on $\mathbb{U}^1$, it will have the form h/x_1^ℓ, where h is homogeneous and not divisible by x_1. Then $gx_1^\ell = hx_0^k$. Since x_0 doesn't divide g, $k = 0$. Therefore g is a constant. □

It is also true that the only rational functions that are regular at every point of a projective variety are the constants. The proof of this will be given later (Corollary 8.2.9). When studying projective varieties, the constant functions are useless, so one has to look at at regular functions on open subsets. Affine varieties appear in projective algebraic geometry as open subsets on which there are enough regular functions.

3.5 Morphisms

As with affine varieties, morphisms are the allowed maps between varieties. Some morphisms, such as the projection from the product $X \times Y$ to X, are sufficiently obvious that they don't require much discussion, but many morphisms aren't obvious. sectmorph

Let X and Y be subvarieties of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. A morphism $Y \to X$, as defined below, will be determined by a morphism from Y to $\mathbb{P}^m$ whose image is contained in X. But in most cases, such a morphism won't be the restriction of a morphism from $\mathbb{P}^n$ to $\mathbb{P}^m$. This is an important point: Though every morphisms of affine varieties can be obtained by restricting a morphism between affine spaces, it is usually impossible to define f using polynomials in the coordinate variables of $\mathbb{P}^n$.

3.5.1. Example. Let the coordinates in $\mathbb{P}^2$ be y_0, y_1, y_2. The Veronese map from the projective line $\mathbb{P}^1$ to $\mathbb{P}^2$, defined by $(x_0, x_1) \rightsquigarrow (x_0^2, x_0x_1, x_1^2)$, is an obvious morphism. Its image is the conic C in $\mathbb{P}^2$ defined by the polynomial $\{y_0y_2 - y_1^2\}$. The Veronese map defines a bijective morphism $\mathbb{P}^1 \xrightarrow{f} C$, whose inverse function π sends a point (y_0, y_1, y_2) of C with $y_0 \neq 0$ to the point $(x_0, x_1) = (y_1, y_2)$, and sends the remaining point, which is $(0, 0, 1)$, to $(0, 1)$. Though π is a morphism $C \to \mathbb{P}^1$, there is no way to extend it to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. In fact, the only morphisms from $\mathbb{P}^2$ to $\mathbb{P}^1$ are the constant morphisms, whose images are points. □ veronex

It is somewhat artificial, but convenient, to define morphisms using points with values in a field.

3.5.2. morphisms to projective space morphtoP

It will be helpful here, to have a separate notation for the point with values in a field K determined by a nonzero vector $\alpha = (\alpha_0, ..., \alpha_n)$, with entries in K. We'll denote that point by $\underline{\alpha}$. If α and α' are points with values in K, then $\underline{\alpha} = \underline{\alpha}'$ if $\alpha' = \lambda\alpha$ for some nonzero λ in K. We'll drop this notation later.

Let Y be a variety with function field K. A morphism from Y to projective space $\mathbb{P}^n$ will be defined by a point of $\mathbb{P}^n$ with values in K. The fact that points of projective space are equivalence classes of vectors, not the vectors themselves, is useful.

Suppose given a vector $\alpha = (\alpha_0, ..., \alpha_n)$ with entries in the function field K of a variety Y. We try to use the point $\underline{\alpha}$ with values in K to define a morphism from Y to projective space $\mathbb{P}^n$. To define the image $\underline{\alpha}(q)$ of a point q of Y (an ordinary point), we look for a vector $\alpha' = (\alpha'_0, ..., \alpha'_n)$, such that $\underline{\alpha}' = \underline{\alpha}$, i.e., $\alpha' = \lambda\alpha$, with $\lambda \in K$, and such that the rational functions α'_i are regular at q and not all zero there. Such a vector may exist or not. If α' exists, we define

(3.5.3) $$\underline{\alpha}(q) = (\alpha'_0(q), ..., \alpha'_n(q)) \quad \big(= \underline{\alpha}'(q)\big)$$ defmorphP

We call $\underline{\alpha}$ a *good point* if such a vector α' exists for every point q of Y.

twoconds **3.5.4. Lemma.** *A point $\underline{\alpha}$ of $\mathbb{P}^n$ with values in the function field K of a variety Y is a good point if either one of the following conditions holds for every point q of Y:*

- *There is an element λ of K such that the rational functions $\alpha'_i = \lambda\alpha_i,\ i = 0, ..., n$, are regular and not all zero at q, for $i = 0, ..., n$.*
- *There is an index j such that $\alpha_j \neq 0$, and the rational functions α_i/α_j are regular at q, for $i = 0, ..., n$.*

proof. The first condition simply restates the definition. We show that it is equivalent with the second one. Suppose that α_i/α_j is regular at q for every i. Let $\lambda = \alpha_j^{-1}$, and let $\alpha'_i = \lambda\alpha_i = \alpha_i/\alpha_j$. The rational functions α'_i are regular at q, and they aren't all zero there because $\alpha'_j = 1$. Conversely, suppose that for some nonzero λ in K, $\alpha'_i = \lambda\alpha_i$ are all regular at q, and that α'_j isn't zero there. Then α'^{-1}_j is a regular function at q, so the rational functions α'_i/α'_j, which are equal to α_i/α_j, are regular at q for all i. □

doesntdepend **3.5.5. Lemma.** *Let $\underline{\alpha}$ be a good point of $\mathbb{P}^n$ with values in the function field K of a variety Y. The image $\underline{\alpha}(q)$ in $\mathbb{P}^n$ of a point q of Y is independent of the choice of the vector that represents $\underline{\alpha}$.*

This follows from Lemma 3.5.4 because the second condition doesn't involve λ. □

defmorphtoP **3.5.6. Definition.** Let Y be a variety with function field K. A *morphism* from Y to projective space $\mathbb{P}^n$ is a map that can be defined, as in (3.5.3), by a good point $\underline{\alpha}$ of $\mathbb{P}^n$ with values in K.

We'll denote the morphism defined by a good point $\underline{\alpha}$ by the same symbol $\underline{\alpha}$.

maptoui **3.5.7. Proposition.** *Let α be a vector with values in the function field K of a variety Y, and suppose that $\underline{\alpha}$ is a good point that defines a morphism $Y \to \mathbb{P}^n$. Suppose that the inverse image in Y of the standard open set $\mathbb{U}^0$ of $\mathbb{P}^n$ is nonempty. Then $\alpha_0 \neq 0$, and the inverse image of $\mathbb{U}^0$ the set of points $q \in Y$ at which the functions α_i/α_0 are regular, for all $j = 1, ..., n$.*

proof. If α_0 were zero, $\underline{\alpha}$ would map Y to the hyperplane $\{x_0 = 0\}$. So $\alpha_0 \neq 0$. Let q be a point of Y. Since α is a good point, there is a λ such that $\alpha'_i = \lambda\alpha_i$ are all regular at q and not all zero, and then $\underline{\alpha}(q) = (\alpha'_0(q), ..., \alpha'_n(q))$. The image will be in $\mathbb{U}^0$ if $\alpha'_0(q) \neq 0$. If so, we let $\alpha'' = \alpha'^{-1}_0\alpha' = \alpha_0^{-1}\alpha$. Then α''_i are all regular at q and $\alpha''_0(q) = 1$. □

identmap **3.5.8. Examples.**

(i) The identity map $\mathbb{P}^1 \to \mathbb{P}^1$. Let $X = \mathbb{P}^1$, and let (x_0, x_1) be coordinates in X. The function field of X is the field $K = \mathbb{C}(u)$ of rational functions in the variable $u = x_1/x_0$. The identity map $X \to X$ is defined by the point $\alpha = (1, u)$ with values in K. For every point p of X except the point $(0, 1)$, $\underline{\alpha}(p) = (1, u(p))$. For the point $q = (0, 1)$, we let $\alpha' = u^{-1}\alpha = (u^{-1}, 1)$. Then $\underline{\alpha}(q) = \alpha'(q) = (x_0(q)/x_1(q), 1) = (0, 1)$. So $\underline{\alpha}$ is a good point.

(ii) We go back to Example 3.5.1, in which C is the conic $y_0y_2 = y_1^2$ and f is the morphism $\mathbb{P}^1 \to C$ defined by $f(x_0, x_1) = (x_0^2, x_0x_1, x_1^2)$. The inverse morphism π can be described as the projection from C to the line $L_0 : \{y_0 = 0\}$, $\pi(y_0, y_1, y_2) = (y_1, y_2)$. This formula for π is undefined at the point $q = (1, 0, 0)$, though the map extends to the whole conic C. Let's write this projection using a point with values in the function field K of C. The standard affine open set $\{y_0 \neq 0\}$ of $\mathbb{P}^2$ is the polynomial algebra $\mathbb{C}[u_1, u_2]$, with $u_1 = y_1/y_0$ and $u_2 = y_2/y_0$. Denoting the restriction of the function u_i to $C^0 = C \cap \mathbb{U}^0$ by u_i too, the restricted functions are related by the equation $u_2 = u_1^2$ that is obtained by dehomogenizing f. The function field K is $\mathbb{C}(u_1)$.

The projection π is defined by the point $\alpha = (u_1, u_1^2)$ with values in K: $\pi(y_0, y_1, y_2) = \pi(1, u_1, u_2) = (u_1, u_1^2)$. Lemma 3.5.4 tells us that α is a good point if and only if one of the two vectors $\alpha' = (1, u_1)$ or $\alpha'' = (u_1^{-1}, 1)$ is regular at every (ordinary) point p of C. Since $u_1 = y_1/y_0$, α' is regular at all points at which $y_0 \neq 0$. This leaves only the one point $p = (0, 0, 1)$ to consider. Noting that $u_1^{-1} = y_0/y_1 = y_1/y_2$, we see that α'' is regular at p. So α is a good point. □

morphtoV **3.5.9. morphisms to projective varieties**

defmorphtoX **3.5.10. Definition.** Let Y be a variety, and let X be a subvariety of a projective space $\mathbb{P}^m$. A morphism of varieties $Y \to X$ is the restriction of a morphism to $\mathbb{P}^m$, whose image is contained in X.

ptfvalKfzero **3.5.11. Lemma.** *Let X be a projective variety that is the locus of zeros of a family f of homogeneous polynomials. A morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ defines a morphism $Y \to X$ if and only if $f(\alpha) = 0$.*

Note. A morphism $Y \xrightarrow{\alpha} X$ won't restrict to a map of function fields $K_X \to K_Y$ unless the image of Y is dense in X.

proof of Lemma 3.5.11. Let $f(x_0,...,x_m)$ be a homogeneous polynomial of degree d, with zero locus X in $\mathbb{P}^m$. We show that the image of a morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ is contained in X if and only if $f(\alpha) = 0$. Whether or not X is a variety (i.e., f is irreducible) is irrelevant. Suppose that $f(\alpha) = 0$, and let q be a point of Y. Since $\underline{\alpha}$ is a good point, the ratios $\alpha'_j = \alpha_j/\alpha_i$ are regular at q for some i, and $\underline{\alpha}(q) = \underline{\alpha}'(q)$. Then $f(\alpha') = \alpha_i^{-d} f(\alpha) = 0$. Therefore q is a point of X.

Conversely, suppose that $f(\alpha) \neq 0$. Let Y' be the open subset of Y of points at which all of the nonzero α_i are invertible regular functions. Then $f(\alpha') = \alpha_i^d f(\alpha)$ will be a nonzero rational function on Y'. It will be nonzero at some points q. □

3.5.12. Proposition. *A morphism of varieties $Y \xrightarrow{\alpha} X$ is a continuous map in the Zariski topology, and also in the classical topology.* *morphcont*

proof. Since the topology on a projective variety X is induced from the one on projective space $\mathbb{P}^m$, we may suppose that $X = \mathbb{P}^m$. Let $\mathbb{U}^i$ be a standard open subset of X whose inverse image in Y isn't empty, and let Y' be a localization of a standard open subset of that inverse image. The restriction $Y' \to \mathbb{U}^i$ of the morphism $\underline{\alpha}$ is continuous in either topology because it is a morphism of affine varieties (3.5.7). Since Y is covered by open sets such as Y', $\underline{\alpha}$ is continuous. □

3.5.13. Lemma. *firstprop-morph*

(i) *The inclusion of an open or a closed subvariety Y into a variety X is a morphism.*

(ii) *Let $Y \xrightarrow{f} X$ be a map whose image lies in an open or a closed subvariety Z of X. Then f is a morphism if and only if its restriction $Y \to Z$ is a morphism.*

(iii) *A composition of morphisms $Z \xrightarrow{\beta} Y \xrightarrow{\alpha} X$ is a morphism.*

(iv) *Let $\{Y^i\}$ be an open covering of a variety Y, and let $Y^i \xrightarrow{f^i} X$ be morphisms. If the restrictions of f^i and f^j to the intersections $Y^i \cap Y^j$ are equal for all i, j, there is a unique morphism f whose restriction to Y^i is f^i.*

proof. **(iv)** This is true because the points with values in K that define the morphisms f^i will be equal. We omit the proofs of **(i) - (iii)**. □

3.5.14. the mapping property of a product

mapprop-prod

Let X and Y be sets. The product set $X \times Y$ can be characterized by this property: Maps from a set T to the product set $X \times Y$ correspond bijectively to pairs of maps $T \xrightarrow{f} X$ and $T \xrightarrow{g} Y$. The map $T \xrightarrow{h} X \times Y$ that corresponds to a pair of maps f, g sends a point t to the point pair $(f(t), g(t))$. So $h = (f, g)$. If $T \xrightarrow{h} X \times Y$ is a map to the product, the corresponding maps to X and Y are the compositions with the projections $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$: $f = \pi_1 h$ and $g = \pi_2 h$.

The analogous statements are true for morphisms of varieties:

3.5.15. Proposition. *Let X and Y be varieties, and let $X \times Y$ be the product variety.* *mapprop-var*

(i) *The projections $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$ are morphisms.*

(ii) *Morphisms from a variety T to the product variety $X \times Y$ correspond bijectively to pairs of morphisms $T \to X$ and $T \to Y$, the correspondence being the same as for maps of sets.*

(iii) *If, $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$ are morphisms of varieties, the product map $X \times Y \xrightarrow{f \times g} U \times V$, which is defined by $[f \times g](x,y) = (f(x), g(y))$, is a morphism.*

proof. Perhaps it suffices to exhibit the points with values in the function fields that define the morphisms.

(i) The function field $K_{X \times Y}$ of $X \times Y$ contains the function field K_X of X. So a point with values in K_X also can be viewed as a point with values in $K_{X \times Y}$. The point with values in $K_{X \times Y}$ that defines the projection π_1 is the point with values in K_X defined by the embedding of X into projective space.

(ii) Let $z_{ij} = x_i y_j$ be the Segre coordinates for $X \times Y$, and let $x = \alpha$ and $y = \beta$ be the points with values in the function field K_T of T that define the morphisms $T \to X$ and $T \to Y$. The point with values in K_T the defines the map $T \to X \times Y$ is $z_{ij} = \alpha_i \beta_j$.

(iii) Let the coordinates in U, V, and $U \times V$ be u_i and v_j and $w_{ij} = u_i v_j$, respectively. Say that f is defined by the point α with values in K_X, and that g is defined by the point β with values in K_Y. The function field $K_{X \times Y}$ contains K_X and K_Y, and $w_{ij} = \alpha_i \beta_j$ defines the product morphism $X \times Y \to U \times V$. $\square$

isomorphisms

3.5.16. isomorphisms

An *isomorphism* of varieties is a bijective morphism $Y \xrightarrow{u} X$ whose inverse function is also a morphism. Isomorphisms are important because they allow us to identify different incarnations of what might be called the 'same' variety.

twistcubicinverse

3.5.17. Example. The projective line $\mathbb{P}^1$, a conic in $\mathbb{P}^2$, and a twisted cubic in $\mathbb{P}^3$ are isomorphic. Let Y denote the projective line with coordinates y_0, y_1. The function field K of Y is the field of rational functions in $t = y_1/y_0$. The degree 3 Veronese map $Y \longrightarrow \mathbb{P}^3$ (3.1.15) defines an isomorphism from Y to its image X, a twisted cubic. It is defined by the vector $\alpha = (1, t, t^2, t^3)$ of $\mathbb{P}^3$ with values in K, and $\alpha' = (t^{-3}, t^{-2}, t^{-1}, 1)$ defines the same point.

The twisted cubic X is the locus of zeros of the equations $v_0 v_2 = v_1^2$, $v_2 v_1 = v_0 v_3$, $v_1 v_3 = v_2^2$. To identify the function field of X, we let $u_i = v_i/v_o$, obtaining relations $u_2 = u_1^2, u_3 = u_1^3$. The function field is the field $F = \mathbb{C}(u_1)$. The point of $Y = \mathbb{P}^1$ with values in F that defines the inverse $X \to Y$ of the morphism $\underline{\alpha}$ is defined by the point $\beta = (1, u_1)$. $\square$

propisom

3.5.18. Lemma. *Let $Y \xrightarrow{f} X$ be a morphism of varieties, let $\{X^i\}$ and $\{Y^i\}$ be open coverings of X and Y, respectively, such that the image of Y^i in X is contained in X^i. If the restrictions $Y^i \xrightarrow{f^i} X^i$ of f are isomorphisms, then f is an isomorphism.*

proof. Let g^i denote the inverse of the morphism f^i. Then $g^i = g^j$ on $X^i \cap X^j$ because $f^i = f^j$ on $Y^i \cap Y^j$. By (3.5.13) **(iv)**, there is a unique morphism $X \xrightarrow{g} Y$ whose restriction to Y^i is g^i. That morphism is the inverse of f. $\square$

diagonal

3.5.19. the diagonal

Let X be a variety. The *diagonal* X_Δ is the set of points (p, p) in the product variety $X \times X$. It is a subset that is closed in the Zariski topology, but not in the product topology.

diagclosed

3.5.20. Proposition. *Let X be a variety. The diagonal X_Δ is a closed subvariety of the product $X \times X$, and it is isomorphic to X.*

proof. Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$, and let $x_0, ..., x_n$ and $y_0, ..., y_n$ be coordinates in the two factors of $\mathbb{P} \times \mathbb{P}$. The diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$ is the closed subvariety defined by the bilinear equations $x_i y_j = x_j y_i$, or in the Segre variables, by the equations $z_{ij} = z_{ji}$, which show that the ratios x_i/x_j and y_i/y_j are equal.

Next, let X be the closed subvariety of $\mathbb{P}$ defined by a system of homogeneous equations $f(x) = 0$. The diagonal X_Δ can be identified as the intersection of the product $X \times X$ with the diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$, so it is a closed subvariety of $X \times X$. As a closed subvariety of $\mathbb{P} \times \mathbb{P}$, the diagonal X_Δ is defined by the equations

Xdelta

$$x_i y_j = x_j y_i \quad \text{and} \quad f(x) = 0 \tag{3.5.21}$$

The morphisms $X \xrightarrow{(id,id)} X_\Delta \xrightarrow{\pi_1} X$ show that X_Δ is isomorphic to X. $\square$

The proof of the next lemma is often assigned as an exercise in topology.

hausdorffdiagonal

3.5.22. Lemma. *A topological space X is a Hausdorff space if and only if, when $X \times X$ is given the product topology, the diagonal X_Δ becomes a closed subset of $X \times X$.* $\square$

Though a variety X, with its Zariski topology, isn't a Hausdorff space unless it is a point, Lemma 3.5.22 doesn't contradict Proposition 3.5.20, because the Zariski topology on $X \times X$ is finer than the product topology.

3.5.23. the graph of a morphism ggraph

Let $Y \xrightarrow{f} X$ be a morphism of varieties. The *graph* Γ_f of f is the subset of $Y\times X$ of pairs (q,p) such that $f(q)=p$.

3.5.24. Proposition. *The graph Γ_f of a morphism $Y \xrightarrow{f} X$ is a closed subvariety of $Y\times X$, and it is isomorphic to Y.* *graph*

proof. We form a diagram of morphisms

(3.5.25) graphdiagram
$$\begin{array}{ccc} \Gamma_f & \longrightarrow & Y\times X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f\times id} \\ X_\Delta & \longrightarrow & X\times X \end{array}$$

where v sends a point (q,p) of Γ_f to the point $(f(q),p)=(p,p)$ of the diagonal X_Δ. The graph Γ_f is the inverse image of X_Δ in $Y\times X$. Since X_Δ is closed in $X\times X$, Γ_f is closed in $Y\times X$.

Let π_1 denote the projection from $Y\times X$ to Y. The composition of the morphisms $Y \xrightarrow{(id,f)} Y\times X \xrightarrow{\pi_1} Y$ is the identity map on Y, and the image of the map (id,f) is Γ_f. The two maps $Y\to\Gamma_f$ and $\Gamma_f\to Y$ are inverses, so Γ_f is isomorphic to Y. $\square$

3.5.26. projection defprojection

The map

(3.5.27) projectiontwo
$$\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^{n-1}$$

that drops the last coordinate of a point: $\pi(x_0,...,x_n)=(x_0,...,x_{n-1})$ is called a *projection.* It is defined at all points of $\mathbb{P}^n$ except at the *center of projection*, the point $q=(0,...,0,1)$, so it is a morphism from the complement $U=\mathbb{P}^n-\{q\}$ of q to $\mathbb{P}^{n-1}$:

$$U \xrightarrow{\pi} \mathbb{P}^n$$

The points of U are the ones that can be written in the form $(x_0,...,x_{n-1},1)$

Let the coordinates in $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ be $x=x_0,...,x_n$ and $y=y_0,...,y_{n-1}$, respectively. The fibre $\pi^{-1}(y)$ over a point $(y_0,...,y_{n-1})$ is the set of points $(x_0,...,x_n)$ such that $(x_0,...,x_{n-1})=\lambda(y_0,...,y_{n-1})$, while x_n is arbitrary. It is the line in $\mathbb{P}^n$ through the points $(y_1,...,y_{n-1},0)$ and $q=(0,...,0,1)$, with the center of projection q omitted.

In Segre coordinates, the graph Γ of π in $U\times\mathbb{P}^{n-1}_y$ is the locus of solutions of the equations $x_iy_j=x_jy_i$ with $0\le i,j\le n-1$, which imply that the vectors $(x_0,...,x_{n-1})$ and $(y_0,...,y_{n-1})$ are proportional.

3.5.28. Proposition. *The closure $\overline{\Gamma}$ of the graph Γ of π in $\mathbb{P}^n_x\times\mathbb{P}^{n-1}_y$ contains all points of the form (q,y). It is the union of Γ with $q\times\mathbb{P}^{n-1}$, and it is the locus of the equations $x_iy_j=x_jy_i$, with $0\le i,j\le n-1$.* *projgraph*

proof. Let W be the locus of solutions of those equations in $\mathbb{P}^n\times\mathbb{P}^{n-1}$. Then W contains the points of Γ, and all points of $\mathbb{P}^n\times\mathbb{P}^{n-1}$ of the form (q,y) are solutions. So $W=\Gamma\cup(q\times\mathbb{P}^{n-1})$. To show that $W=\overline{\Gamma}$, we show that every homogeneous polynomial $g(w)$ that vanishes on Γ also vanishes at all points of W. Given $y=(y_0,...,y_{n-1})$ in $\mathbb{P}^{n-1}$, let $x_t=(ty_0,...,ty_{n-1},1)$. For all $t\neq 0$, the point (x_t,y) is in Γ and therefore $g(x_t,y)=0$. Since g is a continuous function, $g(x_t,y)$ approaches $g(q,y)$ as $t\to 0$. So $g(q,y)=0$. $\square$

The projection $\overline{\Gamma}\to\mathbb{P}^n_x$ that sends a point (x,y) to x is bijective except when $x=q$, and the fibre over q, which is $q\times\mathbb{P}^{n-1}$, is a projective space of dimension $n-1$. Because the point q of $\mathbb{P}^n$ is replaced by a projective space in $\overline{\Gamma}$, the map $\overline{\Gamma}\to\mathbb{P}^n_x$ is called a blowup, a *projective blowup*, of the point q.

3.6 Affine Varieties

We have used the term 'affine variety' in several contexts: An irreducible closed subset of affine space $\mathbb{A}^n_x$ is an affine variety. The spectrum $\operatorname{Spec} A$ of a finite type domain A is an affine variety. A closed subvariety in $\mathbb{A}^n$ becomes an affine open subset of $\mathbb{P}^n$ when the ambient affine space $\mathbb{A}^n$ is identified with the standard open subset $\mathbb{U}^0$. We combine these definitions now, in a rather obvious way: affineV

An *affine variety* X is a variety that is isomorphic to a variety of the form $\operatorname{Spec} A$.

If X is an affine variety with coordinate algebra A, the function field K of X will be the field of fractions of A, and Proposition 2.5.2 shows that the regular functions on X are the elements of A. So A and $\operatorname{Spec} A$ are determined uniquely by X. The isomorphism $\operatorname{Spec} A \to X$ is also determined uniquely. When A is the coordinate algebra of the affine variety X, it seems permissible to identify X with $\operatorname{Spec} A$.

affopens **3.6.1. affine open sets**

Now that we have a definition of an affine variety, we can make the next definition. Though obvious, it is important: An *affine open subset* of a variety X is an open subset that is an affine variety. A nonempty open subset V is an affine open subset if and only if

- The algebra R of regular functions on V is a finite-type domain, so that $\operatorname{Spec} R$ is defined, and
- V is isomorphic to $\operatorname{Spec} R$.

Since the localizations of the standard open sets are affine, the affine open subsets on a variety form a basis for the its topology (2.6.2).

locloctwo **3.6.2. Lemma.** *Let U and V be affine open subsets of an affine variety X.*
(i) *If U is a localization of X and V is a localization of U, then V is a localization of X.*
(ii) *If $V \subset U$ and V is a localization of X, then V is a localization of U.*
(iii) *Let p be a point of $U \cap V$. There is an open set Z containing p that is a localization of U and also a localization of V.* □

pullback-reg **3.6.3. Proposition.** *Let $Y \xrightarrow{f} X$ be a morphism of varieties, let q be a point of X, and let $p = f(q)$. If a rational function g on X is regular at p, then gf is a regular function on Y at q.*

proof. We choose an affine open neighborhood U of p in X on which g is a regular function and an affine open neighborhood V of q in Y that is contained in the inverse image $f^{-1}U$. The morphism f restricts to a morphism $V \to U$ that we denote by the same letter f. Let A and B be the coordinate algebras of U and V, respectively. The morphism $V \xrightarrow{f} U$ corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. On U, the function g is an element of A, and $gf = \varphi(g)$. □

Xmaps **3.6.4. Lemma.** *Let $X = \operatorname{Spec} A$ be an affine variety, and let R be the algebra of regular functions on an arbitrary variety Y. An algebra homomorphism $A \to R$ defines a morphism $Y \xrightarrow{f} X$.*

proof. **(i)** Here, Y isn't assumed to be affine. The algebra R consists of the regular functions on Y, but we don't know more about it.

Let $\{Y^i\}$ be a covering of Y by affine open sets, and let R_i be the coordinate algebra of Y^i. A rational function that is regular on Y is regular on Y^i, so $R \subset R_i$. The composition of the homomorphism $A \to R \subset R_i$ define morphisms $\operatorname{Spec} R_i = Y^i \xrightarrow{f^i} X = \operatorname{Spec} A$ for each i, and it is true that $f^i = f^j$ on the affine variety $Y^i \cap Y^j$. Lemma 3.5.13 **(iv)** shows that there is a unique morphism $Y \xrightarrow{f} \operatorname{Spec} A$ that restricts to f^i on Y^i. □

inter-sectaffine **3.6.5. Theorem.** *Let U and V be affine open subvarieties of a variety X. The intersection $U \cap V$ is an affine open subvariety. If $U \approx \operatorname{Spec} A$ and $V \approx \operatorname{Spec} B$, the coordinate algebra of $U \cap V$ is generated by the two algebras A and B.*

proof. Let $[A, B]$ denote the subalgebra generated by two subalgebras A and B of the function field K of X. Its elements are finite sums of products $\sum \alpha_\nu \beta_\nu$ with α_ν in A and β_ν in B. If $A = \mathbb{C}[a_1, ..., a_r]$ and $B = \mathbb{C}[b_1, ..., b_s]$, then $[A, B]$ will be the finite-type algebra generated by the set $\{a_i\} \cup \{b_j\}$.

The algebras A and B that appear in the statement of the theorem are subalgebras of the function field K. Let $R = [A, B]$ and let $W = \operatorname{Spec} R$. To prove the theorem, we show that W is isomorphic to $U \cap V$. The varieties U, V, W, and X have the same function field K, and the inclusions of coordinate algebras $A \to R$ and $B \to R$ give us morphisms $W \to U$ and $W \to V$. We also have inclusions $U \subset X$ and $V \subset X$, and X is a subvariety of a projective space $\mathbb{P}^n$. Restricting the projective embedding of X gives us embeddings of U and V and it gives us a morphism from W to $\mathbb{P}^n$. All of the morphisms to $\mathbb{P}^n$ are defined by the same good point α with values in K, the point that defines the projective embedding of X. Let's denote the morphisms to $\mathbb{P}^n$ by $\underline{\alpha}_X, \underline{\alpha}_U, \underline{\alpha}_V$ and $\underline{\alpha}_W$. The morphism $\underline{\alpha}_W$ can be obtained as the composition of the morphisms $W \to U \subset X \subset \mathbb{P}^n$, and also as the analogous composition, in which V replaces U. Therefore the image of W in $\mathbb{P}^n$ is contained in $U \cap V$. Thus $\underline{\alpha}_W$ restricts to a morphism $W \xrightarrow{\epsilon} U \cap V$. We show that ϵ is an isomorphism.

Let p be a point of $U \cap V$. We choose an affine open subset U_s of $U \cap V$ that is a localization of U and that contains p (3.6.2). The coordinate algebra of U_s will be the localization A_s of A, and since $U_s \subset V$, B will be a subalgebra of A_s. Then

$$R_s = [A, B]_s = [A_s, B] = A_s$$

So ϵ maps the localization $W_s = \operatorname{Spec} R_s$ of W isomorphically to the open subset $U_s = \operatorname{Spec} A_s$ of $U \cap V$. Since we can cover $U \cap V$ by open sets such as U_s, Lemma 3.5.13 **(ii)** shows that ϵ is an isomorphism. $\square$

3.7 Lines in Three-Space

linesinp

The *Grassmanian* $\mathbf{G}(m,n)$ is a variety whose points correspond to subspaces of dimension m of the vector space $\mathbb{C}^n$, and to linear subspaces of dimension $m-1$ of $\mathbb{P}^{n-1}$. One says that $\mathbf{G}(m,n)$ *parametrizes* those subspaces. For example, the Grassmanian $\mathbf{G}(1,n{+}1)$ is the projective space $\mathbb{P}^n$. The points of $\mathbb{P}^n$ parametrize one-dimensional subspaces of $\mathbb{C}^{n+1}$ as well as points of $\mathbb{P}^n$.

The Grassmanian $\mathbf{G}(2,4)$ parametrizes two-dimensional subspaces of $\mathbb{C}^4$, or lines in $\mathbb{P}^3$. We denote that Grassmanian by $\mathbb{G}$, and we describe it in this section. The point of $\mathbb{G}$ that corresponds to a line ℓ in $\mathbb{P}^3$ will be denoted by $[\ell]$.

One can get some insight into the structure of $\mathbb{G}$ using row reduction. Let $V = \mathbb{C}^4$, let u_1, u_2 be a basis of a two-dimensional subspace U of V, and let M be the 2×4 matrix whose rows are u_1, u_2. The rows of the matrix M' obtained from M by row reduction span the same space U, and the row-reduced matrix M' is uniquely determined by U. Provided that the left hand 2×2 submatrix of M is invertible, M' will have the form

(3.7.1)
$$M' = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$
rowre-duced

The Grassmanian $\mathbb{G}$ contains, as an open subset, a four-dimensional affine space whose coordinates are the variable entries of M'.

In any 2×4 matrix M with independent rows, some pair of columns will be independent, and the corresponding 2×2 submatrix will be invertible. That pair of columns can be used in place of the first two in a row reduction. So $\mathbb{G}$ is covered by six four-dimensional affine spaces that we denote by $\mathbb{W}^{ij}$, $1 \le i < j \le 4$, $\mathbb{W}^{ij}$ being the space of 2×4 matrices such that column i is $(1,0)^t$ and column j is $(0,1)^t$.

The fact that $\mathbb{P}^4$ and $\mathbb{G}$ are both covered by affine spaces of dimension four might lead one to guess that they are similar, but they are quite different.

3.7.2. the exterior algebra

extalgone

Let V be a complex vector space. The *exterior algebra* $\bigwedge V$ ('wedge V') is a noncommutative algebra — an algebra whose multiplication law isn't commutative. It is generated by the elements of V, with the relations

(3.7.3)
$$vw = -wv \quad \text{for all } v, w \text{ in } V.$$
extalg

3.7.4. Lemma. *The condition (3.7.3) is equivalent with:* $vv = 0$ *for all* v *in* V. *vvzero*

proof. To get $vv = 0$ from (3.7.3), one sets $w = v$. Suppose that $vv = 0$ for all v in V. Then vv, ww, and $(v{+}w)(v{+}w)$ are all zero. Since $(v{+}w)(v{+}w) = vv + vw + wv + ww$, it follows that $vw + wv = 0$. $\square$

To familiarize yourself with computation in $\bigwedge V$, verify that $v_3v_2v_1 = -v_1v_2v_3$ and that $v_4v_3v_2v_1 = v_1v_2v_3v_4$.

Let $\bigwedge^r V$ denote the subspace of $\bigwedge V$ spanned by products of length r of elements of V. The exterior algebra $\bigwedge V$ is the direct sum of those subspaces. An algebra A that is a direct sum of subspaces A^i, and such that multiplication maps $A^i \times A^j$ to A^{i+j}, is called a *graded algebra*. The exterior algebra is a *noncommutative graded algebra*.

dependentproduct **3.7.5. Proposition.** *Let $(v_1,...,v_n)$ be a basis for V. The products $v_{i_1}\cdots v_{i_r}$ of length r, with increasing indices $i_1<i_2<\cdots<i_r$, form a basis for $\bigwedge^r V$.*

The proof of this proposition is at the end of the section.

wedgezero **3.7.6. Corollary.** *Let $v_1,...,v_r$ be elements of V. In $\bigwedge^r V$, the product $v_1\cdots v_r$ is zero if and only if the elements are dependent.* □

For the rest of the section, we let V be a vector space of dimension four, with basis $(v_1,...,v_4)$. Proposition 3.7.5 tells us that

extbasistwo (3.7.7)

$\bigwedge^0 V=\mathbb{C}$ is a space of dimension 1, with basis $\{1\}$
$\bigwedge^1 V=V$ is a space of dimension 4, with basis $\{v_1,v_2,v_3,v_4\}$
$\bigwedge^2 V$ is a space of dimension 6, with basis $\{v_iv_j\,|\,i<j\}=\{v_1v_2,v_1v_3,v_1v_4,v_2v_3,v_2v_4,v_3v_4\}$
$\bigwedge^3 V$ is a space of dimension 4, with basis $\{v_iv_jv_k\,|\,i<j<k\}=\{v_1v_2v_3,v_1v_2v_4,v_1v_3v_4,v_2v_3v_4\}$
$\bigwedge^4 V$ is a space of dimension 1, with basis $\{v_1v_2v_3v_4\}$
$\bigwedge^q V=0$ when $q>4$.

The elements of $\bigwedge^2 V$ are combinations

wedgetwo (3.7.8)
$$w=\sum_{i<j}a_{ij}v_iv_j$$

We regard $\bigwedge^2 V$ as an affine space of dimension 6, identifying the combination w with the vector $(a_{12},a_{13},a_{14},a_{23},a_{24},a_{34})$, and we use the same symbol w to represent the corresponding point of the projective space $\mathbb{P}^5$.

An element of $\bigwedge^2 V$ is *decomposable* if it is the product of two elements of V.

describedecomp **3.7.9. Proposition.** *The decomposable elements w of $\bigwedge^2 V$ are those such that $ww=0$, and the relation $ww=0$ is equivalent with the following equation in its coefficients a_{ij}:*

eqgrass (3.7.10)
$$a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}=0$$

proof. If w is decomposable, say $w=u_1u_2$ with u_i in V, then $w^2=u_1u_2u_1u_2=-u_1^2u_2^2$ is zero because $u_1^2=0$. For the converse, we compute w^2 with $w=\sum_{i<j}a_{ij}v_iv_j$. The result is

$$ww=2\big(a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}\big)v_1v_2v_3v_4$$

To show that w is decomposable if $w^2=0$, it seems simplest to factor w explictly. Since the assertion is trivial when $w=0$, we may suppose that some coefficient of w is nonzero. Say that $a_{12}\neq 0$. Then if $w^2=0$,

factorw (3.7.11)
$$w\;=\;\frac{1}{a_{12}}\big(a_{12}v_2+a_{13}v_3+a_{14}v_4\big)\big(-a_{12}v_1+a_{23}v_3+a_{24}v_4\big)$$

The computation for another pair of indices is similar. □

decomptwo **3.7.12. Corollary.** **(i)** *Let w be a nonzero decomposable element of $\bigwedge^2 V$, say $w=u_1u_2$, with u_i in V. Then (u_1,u_2) is a basis for a two-dimensional subspace of V.*
(ii) *Let (u_1,u_2) and (u_1',u_2') be bases for two subspaces U and U' of V, and let $w=u_1u_2$ and $w'=u_1'u_2'$. Then $U=U'$, if and only if w and w' differ by a scalar factor — if and only if they represent the same point of $\mathbb{P}^5$.*
(iii) *The Grassmanian $\mathbb{G}$ corresponds bijectively to the quadric Q in $\mathbb{P}^5$ whose equation is (3.7.10). If U is a two-dimensional subspace of V with basis (u_1,u_2), the point of $\mathbb{G}$ that represents U is sent to the point $w=u_1u_2$ of Q.*

Thus we may identify the Grassmanian $\mathbb{G}$ with the quadric in $\mathbb{P}^5$ defined by the equation (3.7.10).

proof. **(i)** If an element w of $\bigwedge^2 V$ is decomposable, say $w = u_1u_2$, and if w isn't zero, then u_1 and u_2 must be independent. They span a two-dimensional subspace.

(ii) Suppose that $U' = U$. When we write the second basis in terms of the first one, say $(u_1', u_2') = (au_1 + bu_2, cu_1 + du_2)$, the product w' becomes the scalar multiple $(ad-bc)w$ of w, and $ad-bc \neq 0$.

If $U' \neq U$, then at least three of the vectors u_1, u_2, u_1', u_2' will be independent. Say that u_1, u_2, u_1' are independent. Then, according to Corollary 3.7.6, the product $u_1u_2u_1'$ isn't zero. Since $u_1'u_2'u_1' = 0$, $u_1'u_2'$ cannot be a scalar multiple of u_1u_2.

(iii) This follows from **(i)** and **(ii)**. □

For the rest of this section, we will use the concept of the algebraic dimension of a variety X. The algebraic dimension is the length d of the longest chain $C_0 > C_1 > \cdots > C_d$ of closed subvarieties of X. We refer to the algebraic dimension simply as the *dimension*, and we use some of its properties informally here, deferring proofs to the discussion of dimension in the next chapter.

The topological dimension of X, its dimension in the classical topology, is always twice the algebraic dimension. Because the Grassmanian $\mathbb{G}$ is covered by affine spaces of dimension 4, its algebraic dimension is 4 and its topological dimension is 8.

3.7.13. Proposition. *Let $\mathbb{P}^3$ be the projective space associated to a four dimensional vector space V. In the product $\mathbb{P}^3 \times \mathbb{G}$, the locus Γ of pairs $p,[\ell]$ such that p lies on ℓ is a closed subset of dimension* 5. *pinlclosed*

proof. Let ℓ be the line in $\mathbb{P}^3$ that corresponds to a subspace U with basis (u_1, u_2), let $w = u_1u_2$, and let p be the point represented by a vector x in V. Then $p \in \ell$ means $x \in U$, which is true if and only if (x, u_1, u_2) is a dependent set — if and only if $xw = 0$ (3.7.5). An element w of $\bigwedge^2 V$ is decomposable if $w^2 = 0$. So Γ is the closed subset of points (x, w) of $\mathbb{P}^3_x \times \mathbb{P}^5_w$ defined by the bihomogeneous equations $xw = 0$ and $w^2 = 0$. □

When we project Γ to $\mathbb{G}$, the fibre over a point $[\ell]$ of $\mathbb{G}$ is the set of pairs $p, [\ell]$ such that $p \in \ell$. The projection maps that fibre bijectively to the line ℓ. Thus Γ can be viewed as a family of lines, parametrized by $\mathbb{G}$. Its dimension is $\dim \ell + \dim \mathbb{G} = 1 + 4 = 5$.

3.7.14. lines in a surface linesina-surface

When a surface S in $\mathbb{P}^3$ is given, one may ask: Does S contain a line? One surface that contains lines is the quadric Q in $\mathbb{P}^3$ whose equation is $z_{01}z_{10} = z_{00}z_{11}$, the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3_w$ (3.1.6). It contains two families of lines, the lines that correspond to the two 'rulings' $p \times \mathbb{P}^1$ and $\mathbb{P}^1 \times q$ of $\mathbb{P}^1 \times \mathbb{P}^1$. There are surfaces of arbitrary degree that contain lines, but a generic surface of degree four or more won't contain any line.

We use coordinates x_i with $i = 1, 2, 3, 4$ for $\mathbb{P}^3$ here. There are $\binom{d+3}{3}$ monomials of degree d in four variables, so homogeneous polynomials of degree d are parametrized by an affine space of dimension $\binom{d+3}{3}$, and surfaces of degree d in $\mathbb{P}^3$ by a projective space of dimension $n = \binom{d+3}{3} - 1$. Let $\mathbb{S}$ denote that projective space, let $[S]$ denote the point of $\mathbb{S}$ that corresponds to a surface S, and let f be the irreducible polynomial whose zero locus is S. The coordinates of $[S]$ are the coefficients of f. Speaking infomally, we say that a point of $\mathbb{S}$ *is* a surface of degree d in $\mathbb{P}^3$. (When f is reducible, its zero locus isn't a variety. Let's not worry about this.)

Consider the line ℓ_0 defined by $x_3 = x_4 = 0$. Its points are those of the form $(x_1, x_2, 0, 0)$, and a surface $S : \{f = 0\}$ will contain ℓ_0 if and only if $f(x_1, x_2, 0, 0) = 0$ for all x_1, x_2. Substituting $x_3 = x_4 = 0$ into f leaves us with a polynomial in two variables:

(3.7.15) $$f(x_1, x_2, 0, 0) = c_0x_1^d + c_1x_1^{d-1}x_2 + \cdots + c_dx_2^d$$ scon-tainslzero

where c_i are some of the coefficients of the polynomial f. If $f(x_1, x_2, 0, 0)$ is identically zero, all of those coefficients will be zero. So the surfaces that contain ℓ_0 correspond to the points of the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by the equations $c_0 = \cdots = c_d = 0$. Its dimension is $n-d-1$. This is a satisfactory answer to the question of which surfaces contain ℓ_0, and we can use it to make a guess about lines in a generic surface of degree d.

scon-tainslclosed **3.7.16. Lemma.** *In the product variety $\mathbb{G}\times\mathbb{S}$, the set Σ of pairs $[\ell]$,$[S]$, such that S is a surface of degree d and ℓ is a line contained in S, is a closed set.*

proof. Let $\mathbb{W}^{ij}$, $1\le i<j\le 4$ denote the six affine spaces that cover the Grassmanian, as at the beginning of this section. It suffices to show that the intersection $\Sigma^{ij}=\Sigma\cap(\mathbb{W}^{ij}\times\mathbb{S})$ is closed in $\mathbb{W}^{ij}\times\mathbb{S}$ for all i,j (2.1.7). We inspect the case $i,j=1,2$.

A line ℓ such that $[\ell]$ is in $\mathbb{W}^{12}$ corresponds to a two-dimensional subspace of $V=\mathbb{C}^4$ that has a basis of the form $u_1=(1,0,a_2,a_3)$, $u_2=(0,1,b_2,b_3)$, and the coordinates of the points of ℓ are combinations ru_1+su_2 of u_1,u_2. Let $f(x_1,x_2,x_3,x_4)$ be the polynomial that defines a surface S of degree d. The line ℓ is contained in S if and only if $f(r,s,ra_2+sb_2,ra_3+sb_3)=\widetilde{f}(r,s)=0$ for all r and s, and $\widetilde{f}(r,s)$ is a homogeneous polynomial of degree d in r,s. If we write $\widetilde{f}(r,s)=z_0r^d+z_1r^{d-1}s+\cdots+z_ds^d$, the coefficients z_ν will be polynomials in a_i,b_i and in the coefficients of f. The locus $z_0=\cdots=z_d=0$ is the closed subset Σ^{12} of $\mathbb{W}^{12}\times\mathbb{S}$ that represents surfaces containing a line. $\square$

The set of surfaces that contain our special line ℓ_0 corresponds to the linear space $\mathbb{L}_0$ of $\mathbb{S}$ of dimension $n-d-1$, and ℓ_0 can be carried to any other line ℓ by a linear map $\mathbb{P}^3\to\mathbb{P}^3$. So the sufaces that contain another line ℓ also form a linear subspace of $\mathbb{S}$ of dimension $n-d-1$. Those subspaces are the fibres of the set Σ over $\mathbb{G}$. The dimension of the Grassmanian $\mathbb{G}$ is 4. Therefore the dimension of Σ is

$$\dim\Sigma\;=\;\dim\mathbb{L}_0+\dim\mathbb{G}\;=\;(n-d-1)+4$$

Since $\mathbb{S}$ has dimension n,

dimspace-lines (3.7.17)
$$\dim\Sigma\;=\;\dim\mathbb{S}-d+3$$

When we project the product $\mathbb{G}\times\mathbb{S}$ and its subvariety Σ to $\mathbb{S}$, the fibre of Σ over a point $[S]$ is the set of pairs $[\ell]$,$[S]$ such that ℓ is contained in S — the set of lines in S.

lines-lowdeg **3.7.18.**

When the degree d of the surfaces we are studying is 1, $\dim\Sigma=\dim\mathbb{S}+2$. Every fibre of Σ over $\mathbb{S}$ will have dimension at least 2. In fact, every fibre has dimension equal to 2. Surfaces of degree 1 are planes, and the lines in a plane form a two-dimensional family.

When $d=2$, $\dim\Sigma=\dim\mathbb{S}+1$. We can expect that most fibres of Σ over $\mathbb{S}$ will have dimension 1. This is true: A smooth quadric contains two one-dimensional families of lines. (All smooth quadrics are equivalent with the quadric (3.1.7).) But if a quadratic polynomial $f(x_1,x_2,x_3,x_4)$ is the product of linear polynomials, its locus of zeros will be a union of planes. It will contain two-dimensional families of lines. Some fibres of Σ over $\mathbb{S}$ have dimension 2.

When $d\ge 4$, $\dim\Sigma<\dim\mathbb{S}$. The projection $\Sigma\to\mathbb{S}$ cannot be surjective. Most surfaces of degree 4 or more contain no lines.

The most interesting case is that the degree d is 3. In this case, $\dim\Sigma=\dim\mathbb{S}$. Most fibres will have dimension zero. They will be finite sets. In fact, a generic cubic surface contains 27 lines. We have to wait to see why the number is precisely 27 (see Theorem 4.7.14).

Our conclusions are intuitively plausible, but to be sure about them, we need to study dimension carefully. We do this in the next chapters.

proof of Proposition 3.7.5. The proposition asserts that, if $v=(v_1,...,v_n)$ is a basis of a vector space V, then the products $v_{i_1}\cdots v_{i_r}$ of length r with increasing indices $i_1<i_2<\cdots<i_r$ form a basis for $\bigwedge^r V$. To prove this, we need to be more precise about the definition of the exterior algebra $\bigwedge V$.

We start with the algebra $T(V)$ of noncommutative polynomials in the basis v, which is also called the *tensor algebra* on V. The part $T^r(V)$ of $T(V)$ of degree r has a basis that consists of the n^r noncommutative monomials of degree r, the products $v_{i_1}\cdots v_{i_r}$ of length r of elements of the basis. Its dimension is n^r. When $n=2$, the eight-dimensional space $T^3(V)$ has basis $(x_1^3,\ x_1^2x_2,\ x_1x_2x_1,\ x_2x_1^2,\ x_1x_2^2,\ x_2x_1x_2,\ x_2^2x_1,\ x_2^3)$.

The exterior algebra $\bigwedge V$ is the quotient of $T(V)$ obtained by forcing the relations $vw+wv=0$ (3.7.3). Using the distributive law, one sees that the relations $v_iv_j+v_jv_i=0$, $1\le i,j\le n$, are sufficient to define this quotient.

We can multiply the relations $v_iv_j+v_jv_i$ on left and right by noncommutative monomials $p(v)$ and $q(v)$ in $v_1,...,v_n$. When we do this with all pairs p,q of monomials, the sum of whose degrees is $r-2$, the noncommutative polynomials

$$p(v)(v_iv_j+v_jv_i)q(v) \tag{3.7.19}$$ vivj

span the kernel of the linear map $T^r(V) \to \bigwedge^r V$. In $\bigwedge^r V$, $p(v)(v_iv_j)q(v) = -p(v)(v_jv_i)q(v)$. Using these relations, any product $v_{i_1}\cdots v_{i_r}$ in $\bigwedge^r V$ is, up to sign, equal to a product in which the indices i_ν are in increasing order. Thus the products with indices in increasing order span $\bigwedge^r V$, and because $v_iv_i = 0$, such a product will be zero unless the indices are strictly increasing.

We proceed with the proof of Proposition3.7.5. Let $v = (v_1,...,v_n)$ be a basis for V, as above. We show first that the product $w = v_1\cdots v_n$ of the basis elements in increasing order is a basis of the space $\bigwedge^n V$. As we have shown above w spans $\bigwedge^n V$, and it remains to show that $w \neq 0$, or that $\bigwedge^n V \neq 0$.

Let's use multi-index notation, writing $(i) = (i_1,...,i_r)$, and $v_{(i)} = v_{i_1}\cdots v_{i_r}$. We define a surjective linear map $T^n(V) \xrightarrow{\varphi} \mathbb{C}$. The products $v_{(i)} = (v_{i_1}\cdots v_{i_n})$ of length n form a basis of $T^n(V)$. If there is no repetition among the indices $i_1,...,i_n$, then (i) will be a permutation of the indices $1,...,n$. In that case, we set $\varphi(v_{(i)}) = \varphi(v_{i_1}\cdots v_{i_n}) = sign(i)$. If there is a repetition, we set $\varphi(v_{(i)}) = 0$.

Let p and q be noncommutative monomials whose degrees sum to $n-2$. If the product $p(v_iv_j)q$ has no repeated index, the indices in $p(v_iv_j)q$ and in $p(v_jv_i)q$ will be permutations of $1,...,n$, and those permutations will have opposite signs. So $p(v_iv_j+v_jv_i)q$ will be in the kernel of φ. Since these elements span the space of relations that define $\bigwedge^n V$ as a quotient of $T^n(V)$, the surjective map $T^n(V) \xrightarrow{\varphi} \mathbb{C}$ defines a surjective map $\bigwedge^n V \to \mathbb{C}$. Therefore $\bigwedge^n V \neq 0$.

We must now show that for $r \le n$, the products $v_{i_1}\cdots v_{i_r}$ with $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$, and we have seen that those products span $\bigwedge^r V$. We show that they are independent. Suppose that a combination $z = \sum c_{(i)}v_{(i)}$ is zero, the sum being over the sets $\{i_1,...,i_r\}$ of strictly increasing indices. We choose a particular set $(j_1,...,j_r)$ of n strictly increasing indices, and we let $(k) = (k_1,...,k_{n-r})$ be the set of indices that don't occur in (j), listed in arbitrary order. Then all terms in the sum $zv_{(k)} = \sum c_{(i)}v_{(i)}v_{(k)}$ will be zero except the term with $(i) = (j)$. On the other hand, since $z = 0$, $zv_{(k)} = 0$. Therefore $c_{(j)}v_{(j)}v_{(k)} = 0$, and since $v_{(j)}v_{(k)}$ differs by sign from $v_1\cdots v_n$, it isn't zero. It follows that $c_{(j)} = 0$. This is true for all (j), so $z = 0$. $\square$

3.8 Exercises

chapthreeex closureinP **3.8.1.** Let X be the affine surface in $\mathbb{A}^3$ defined by the equation $x_1^3 + x_1x_2x_3 + x_1x_3 + x_2^2 + x_3 = 0$, and let $\overline{X}$ be its closure in $\mathbb{P}^3$. Describe the intersection of $\overline{X}$ with the plane at infinity in $\mathbb{P}^3$.

xcubiceq **3.8.2.** Let C be a cubic curve, the locus of a homogeneous cubic polynomial $f(x, y, z)$ in $\mathbb{P}^2$. Suppose that $(0, 0, 1)$ and $(0, 1, 0)$ are flex points of C, that the tangent line to C at $(0, 0, 1)$ is the line $\{y = 0\}$, and the tangent line at $(0, 1, 0)$ is the line $\{z = 0\}$. What are the possible polynomials f? Disregard the question of whether f is irreducible.

xtendtoinf **3.8.3.** Let Y and Z be the zero sets in $\mathbb{P}$ of relatively prime homogeneous polynomials g and h of the same degree r. Prove that the rational function $\alpha = g/h$ will tend to infinity as one approaches a point of Z that isn't also a point of Y and that, at intersections of Y and Z, α is indeterminate in the sense that the limit isn't independent of the path.

homogprimeconverse **3.8.4.** Let $\mathcal{P}$ be a homogeneous ideal in $\mathbb{C}[x_0, ..., x_n]$, and suppose that its dehomogenization P is a prime ideal. Is $\mathcal{P}$ a prime ideal?

xcondpt **3.8.5.** Let U be the complement of a closed subset Z in a projective variety X. Say that X and Z are the loci of solutions of the homogeneous polynomial equations $f = 0$ and $g = 0$ in $\mathbb{P}^n$, respectively. What conditions must a point p of $\mathbb{P}^n$ satisfy in order to be a point of U?

xfnbounded **3.8.6.** Let U be a nonempty open subset of $\mathbb{P}^n$. Prove that a rational function that is bounded on U is a constant.

xidealinAxP **3.8.7.** Describe the ideals that define closed subsets of $\mathbb{A}^m \times \mathbb{P}^n$.

xlocusinPxP **3.8.8.** With coordinates x_0, x_1, x_2 in the plane $\mathbb{P}$ and s_0, s_1, s_2 in the dual plane $\mathbb{P}^*$, let C be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where f is an irreducible homogeneous polynomial in x. Let Γ be the locus of pairs (x, s) of $\mathbb{P} \times \mathbb{P}^*$ such that the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$ is the tangent line to C at x. Prove that Γ is a Zariski closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

xmapcusp **3.8.9.** Let Y be the cusp curve $\operatorname{Spec} B$, where $B = \mathbb{C}[x, y]/(y^2 - x^3)$. This algebra embeds as subring into $\mathbb{C}[t]$, by $x = t^2$. $y = t^3$. Show that the two vectors $v_0 = (x-1, y-1)$ and $v_1 = (t+1, t^2+t+1)$ define the same point of $\mathbb{P}^1$ with values in the fraction field K of B, and that they define morphisms from Y to $\mathbb{P}^1$ wherever the entries are regular functions on Y. Prove that the two morphisms they define piece together to give a morphism $Y \to \mathbb{P}^1$.

xdoesntextend **3.8.10.** Let C be a conic in $\mathbb{P}^2$, and let π be the projection $C \to \mathbb{P}^1$ from a point q of C. Prove that there is no way to extend this map to a morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$.

zautPone **3.8.11.** A pair f_0, f_1 of homogeneous polynomials in x_0, x_1 of the same degree d can be used to define a morphism $\mathbb{P}^1 \to \mathbb{P}^1$. At a point q of $\mathbb{P}^1$, the morphism evaluates $(1, f_1/f_0)$ or $(f_0/f_1, 1)$ at q.

(i) The *degree* of such a morphism is the number of points in a generic fibre. Determine the degree.

(ii) Describe the group of automorphisms of $\mathbb{P}^1$.

xmorphPtwo **3.8.12.** **(i)** What are the conditions that a triple of $f = (f_0, f_1, f_2)$ homogeneous polynomials in x_0, x_1, x_2 of the same degree d must satisfy in order to define a morphism $\mathbb{P}^2 \to \mathbb{P}^2$?

(ii) If f does define a morphism, what is its degree?

cusptopone **3.8.13.** Let C be the plane projective curve $x^3 - y^2z = 0$.

(i) Show that the function field K of C is the field $\mathbb{C}(t)$ of rational functions in $t = y/x$.

(ii) Show that the point $(t^2 - 1, t^3 - 1)$ of $\mathbb{P}^1$ with values in K defines a morphism $C \to \mathbb{P}^1$.

xPtwoPone **3.8.14.** Describe all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$.

xblowup **3.8.15.** *blowing up a point in* $\mathbb{P}^2$. Consider the Veronese embedding of $\mathbb{P}^2_{xyz} \to \mathbb{P}^5_u$ by monomials of degree 2 defined by $(u_0, u_1, u_2, u_3, u_4, u_5) = (z^2, y^2, x^2, yz, xz, xy)$. If we drop the coordinate u_0, we obtain a map $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^4$: $\varphi(x, y, z) = (y^2, x^2, yz, xz, xy)$ that is defined at all points except the point $q = (0, 0, 1)$. Find defining equations for the closure of the image X. Prove that the inverse map $X \xrightarrow{\varphi^{-1}} \mathbb{P}^2$ is everywhere defined, that the fibre of φ^{-1} over q is a projective line, and that f is bijective everywhere else.

3.8.16. Show that the conic C in $\mathbb{P}^2$ defined by the polynomial $y_0^2 + y_1^2 + y_2^2 = 0$ and the twisted cubic V in $\mathbb{P}^3$, the zero locus of the polynomials $v_0v_2 - v_1^2$, $v_0v_3 - v_1v_2$, $v_1v_3 - v_2^2$ are isomorphic by exhibiting inverse morphisms between them. xconic-cubic

3.8.17. Let X be the affine plane with coordinates (x, y). Given a pair of polynomials $u(x, y), v(x, y)$ in x, y, one may try to define a morphism $f : X \to \mathbb{P}^1$ by $f(x, y) = (u, v)$. Under what circumstances is f a morphism? xmapplan-etoPone

3.8.18. Let x_0, x_1, x_2 be the coordinate variables in the projective plane X, and for $i = 1, 2$, let $u_i = x_i/x_0$. The function field K of X is the field of rational functions in the variables u_i. Let $f(u_1, u_2)$ and $g(u_1, u_2)$ be polynomials. Under what circumstances does the point $(1, f, g)$ with values in K define a morphism $X \to \mathbb{P}^2$? xmap-toPthree

3.8.19. Let Y be an affine variety. Prove that morphisms $Y \to \mathbb{P}^n$ whose images are in $\mathbb{U}^0$ correspond bijectively to morphisms of affine varieties $Y \to \mathbb{U}^0$, as defined in (2.5.3). xmorphaff

3.8.20. Prove that every finite subset S of a projective variety X is contained in an affine open subset. easyk-leiman

3.8.21. Describe the affine open subsets of the projective plane $\mathbb{P}^2$. xaffinopen-plane

3.8.22. Prove that the complement of a hypersurface in $\mathbb{P}^n$ is an affine open subvariety xcomphy-per

3.8.23. According to (3.7.18), a generic quartic surface in $\mathbb{P}^3$ won't contain any lines. Will a generic quartic surface contain a plane conic? xconicin-quartic

3.8.24. Let V be a vector space of dimension 5, and let $\mathbb{G}$ denote the Grassmanian $\mathbf{G}(2, 5)$ of lines in $\mathbb{P}(V) = \mathbb{P}^4$. So $\mathbb{G}$ is a subvariety of the projective space $\mathbb{P}(W)$, $W = \bigwedge^2 V$, which has dimension 10. let D denote the subset of decomposable vectors in $\mathbb{P}(W)$. Prove that there is a bijective correspondence between two-dimensional subspaces of V and points of D, and that a vector w in $\bigwedge^2 V$ is decomposable if and only if $ww = 0$. Exhibit defining equations for $\mathbb{G}$ in the space $\mathbb{P}(W)$. xlinesp-five

3.8.25. *a flag variety*. Let $\mathbb{P} = \mathbb{P}^3$. The space of planes in $\mathbb{P}$ is the *dual projective space* $\mathbb{P}^*$. The variety F that parametrizes triples (p, ℓ, H) consisting of a point p, a line ℓ, and a plane H in $\mathbb{P}$, with $p \in \ell \subset H$, is called a *flag variety*. Exhibit defining equations for F in $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^{3*}$. The equations should be homogeneous in each of 3 sets of variables. xflag

Chapter 4 INTEGRAL MORPHISMS

The concept of an algebraic integer is one of the important concepts of algebraic number theory, and as the work of Noether and Zariski shows, its analogue is essential in algebraic geometry. We study that analog here.

4.1 The Nakayama Lemma

nakayama

eigen **4.1.1. eigenvectors**

It won't be a surprise that eigenvectors are important, but the way that they are used to study modules may be less familiar.

Let P be an $n\times n$ matrix with entries in a ring A. The concept of an eigenvector for P makes sense when the entries of a vector are in an A-module. A column vector $v=(v_1,...,v_n)^t$ with entries in an A-module M is an *eigenvector* of P with *eigenvalue* λ in A if $Pv=\lambda v$.

When the entries of a vector are in a module, it becomes hard to adapt the usual requirement that an eigenvector must be nonzero. So we drop it, though the zero vector tells us nothing.

eigenval **4.1.2. Lemma.** *Let P be a square matrix with entries in a ring A and let $p(t)$ be its characteristic polynomial* $\det(tI-P)$. *If v is an eigenvector of P with eigenvalue λ, then $p(\lambda)v=0$.*

The usual proof, in which one multiplies the equation $(\lambda I-P)v=0$ by the cofactor matrix of $(\lambda I-P)$, carries over. $\square$

The next lemma is a cornerstone of the theory of modules. In it, JM denotes the set of (finite) combinations $\sum_i a_im_i$ with a_i in J and m_i in M.

nakaya-malem **4.1.3. Nakayama Lemma.** Let M be a finite module over a ring A, and let J be an ideal of A. If $M=JM$, there is an element z in J such that $m=zm$ for all m in M, i.e., such that $(1-z)M=0$.

proof of the Nakayama Lemma. Let $v_1,...,v_n$ be generators for the finite A-module M. The equation $M=JM$ tells us that there are elements p_{ij} in J such that $v_i=\sum p_{ij}v_j$. We write this equation in matrix notation, as $v=Pv$, where v is the column vector $(v_1,...,v_n)^t$ and P is the matrix $P=(p_{ij})$. Then v is an eigenvector of P with eigenvalue 1, and if $p(t)$ denotes the characteristic polynomial of P, then $p(1)v=0$. Since the entries of P are in the ideal J, inspection of the determinant of the matrix $I-P$ shows that $p(1)$ has the form $1-z$, with z in J. Then $(1-z)v_i=0$ for all i. Since $v_1,....,v_n$ generate M, $(1-z)M=0$. $\square$

With notation as in the Nakayama Lemma, let $s = 1-z$, so that $sM = 0$. The localized module M_s is the zero module.

4.1.4. Corollary. *Let I and J be ideals of a noetherian domain A.* *idealzero*

(i) *If $I = JI$, then either I is the zero ideal or J is the unit ideal.*

(ii) *Let B be a domain that contains A, and that is a finite A-module. If the extended ideal JB is the unit ideal of B, then J is the unit ideal of A.*

proof. **(i)** Since A is noetherian, I is a finite A-module. If $I = JI$, the Nakayama Lemma tells us that there is an element z of J such that $zx = x$ for all x in I. Suppose that I isn't the zero ideal. We choose a nonzero element x of I. Because A is a domain, we can cancel x from the equation $zx = x$, obtaining $z = 1$. Then 1 is in J, and J is the unit ideal.

(ii) The elements of the extended ideal JB are sums $\sum u_i b_i$ with u_i in J and b_i in B. Suppose that $B = JB$. Then there is an element z of J such that $b = zb$ for all b in B. Setting $b = 1$ shows that $z = 1$. So J is the unit ideal. □

4.1.5. Corollary. *Let x be an element of a noetherian domain A and let J be the principal ideal xA. Suppose that x isn't a unit.* *xkdividesy*

(i) *The intersection $\bigcap J^n$ is the zero ideal.*

(ii) *If y is a nonzero element of A, the integers k such that x^k divides y in A are bounded.*

(iii) *For every $k > 0$, $J^k > J^{k+1}$.*

proof. Let $I = \bigcap J^n$. Since $J = xA$, $J^n = x^n A$. The elements of I are those that are divisible by x^n for every n. If y is an element of I, then for every n, there is an element a_n in A such that $y = a_n x^n$. Then $y/x = a_n x^{n-1}$, which is an element of J^{n-1}. This is true for every n, so y/x is in I, and y is in JI. Since y is an arbitrary element of I, $I = JI$. Since x isn't a unit, J isn't the unit ideal. Corollary 4.1.4**(i)** tells us that $I = 0$. This proves **(i)**, and **(ii)** follows. For **(iii)**, we note that if $J^k = J^{k+1}$. Multiplying by J^{n-k} shows that $J^n = J^{n+1}$ for every $n \geq k$. Therefore $J^k = \bigcap J^n = 0$. But $x^k \in J^k$, and since A is a domain, $x^k \neq 0$. So $J^k \neq 0$. □

4.2 Integral Extensions

An *extension* of a domain A is a domain B that contains A as a subring. int

4.2.1. Definition. Let B be an extension of a domain A. An element β of B is *integral over A* if it is a root of a monic polynomial with coefficients in A, and B is an *integral extension* if all of its elements are integral over A. deffinmorphaff

If $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, the morphism $Y \xrightarrow{u} X$ defined by an integral extension $A \subset B$ is called an *integral morphism* of affine varieties. Thus an integral morphism of affine varieties $Y \to X$ is a morphism whose associated algebra homomorphism $A \xrightarrow{\varphi} B$ is injective, and such that B is a finite A-module.

The inclusion su of a proper closed subvariety Y into X isn't called an integral morphism, though B is a finite A-module.

4.2.2. Lemma. *Let $A \subset B$ be an extension of noetherian domains.* *aboutintegral*

(i) *An element b of B is integral over A if and only if the subring $A[b]$ of B it generates is a finite A-module.*

(ii) *The set of elements of B that are integral over A is a subring of B.*

(iii) *If B is generated as A-algebra by finitely many integral elements, then B is a finite A-module.*

(iv) *Let $R \subset A \subset B$ be domains, and suppose that A is an integral extension of R. An element of B is integral over A if and only if it is integral over R. Therefore, if A is an integral extension of R and B is an integral extension of A, then B is an integral extension of R.* □

4.2.3. Corollary. *An extension $A \subset B$ of finite-type domains is an integral extension if and only if B is a finite A-module.* □ *integraliffinite*

If $f(x)$ is a monic irreducible polynomial with cofficients in A and $B = A[x]/(f)$, then every element of B is integral over A.

betainte-gral **4.2.4. Lemma.** *Let $A \subset B$ be an extension of domains, with A noetherian, let I be a nonzero ideal of A, and let b be an element of B. If $bI \subset I$, then b is integral over A.*

proof. Because A is noetherian, I is finitely generated. Let $v = (v_1, ..., v_n)^t$ be a vector whose entries generate I. The hypothesis $bI \subset I$ allows us to write $bv_i = \sum p_{ij} v_j$ with p_{ij} in A, or in matrix notation, $bv = Pv$. So v is an eigenvector of P with eigenvalue b, and if $p(t)$ is the characteristic polynomial of P, then $p(b)v = 0$. Since I isn't zero, at least one v_i is nonzero, but $p(b)v_i = 0$. Since A is a domain, $p(b) = 0$. The characteristic polynomial is a monic polynomial with coefficients in A, so b is integral over A. $\square$

integral-surj **4.2.5. Proposition.** *An integral morphism $Y \xrightarrow{u} X$ of affine varieties is a surjective map.*

proof. Let $A \subset B$ be the extension that corresponds to the morphism u, and $\mathfrak{m}_x$ be the maximal ideal at point x of X. Corollary 4.1.4 **(ii)** shows that the extended ideal $\mathfrak{m}_x B$ isn't the unit ideal of B, so $\mathfrak{m}_x B$ is contained in a maximal ideal $\mathfrak{m}_y$ of B, where y is a point of Y. Then $\mathfrak{m}_y \cap A$ contains $\mathfrak{m}_x$ and it isn't the unit ideal because it doesn't contain 1. So $\mathfrak{m}_y \cap A = \mathfrak{m}_x$. This tells us that x is the image uy. Therefore u is surjective. $\square$

grpii **4.2.6. Example.** Let G be a finite group of automorphisms of a normal, finite-type domain B, and let A be the algebra B^G of invariant elements of B. As Theorem 2.7.5 tells us, A is a finite-type domain, B is a finite integral extension of A, and points of $X = \operatorname{Spec} A$ correspond to G-orbits of points of $Y = \operatorname{Spec} B$. $\square$

The next example is helpful for an intuitive understanding of the geometric meaning of integrality.

inte-graloverc **4.2.7. Example.** Let f be an irreducible polynomial in $\mathbb{C}[x, y]$ (one x and one y), let $A = \mathbb{C}[x]$, and let $B = \mathbb{C}[x, y]/(f)$. So $X = \operatorname{Spec} A$ is an affine line and $Y = \operatorname{Spec} B$ is a plane affine curve. The canonical map $A \to B$ defines the morphism $Y \to X$ that is obtained by restricting the projection $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_x$ to Y.

We write f as a polynomial in y, whose coefficients are polynomials in x, say

fxy (4.2.8)
$$f(x, y) = a_0 y^n + a_1 y^{n-1} + \cdots + a_n$$

with $a_i = a_i(x)$. Let $x = x_0$ be a point of X. The fibre of the map $Y \to X$ over x_0 consists of the points (x_0, y_0) such that y_0 is a root of the one-variable polynomial $f(x_0, y)$.

The discriminant $\delta_y(x)$ of $f(x, y)$ as a polynomial in y isn't identically zero because f is irreducible (1.6.22). For all but finitely many values x_0 of x, both a_0 and δ will be nonzero. Then $f(x_0, y)$ will have n distinct roots, and the fibre of Y over x_0 will have order n.

When $f(x, y)$ is a monic polynomial in y, the morphism $Y \to X$ will be an integral morphism. If so, the leading term y^n of f will be the dominant term, when y is large. For x_1 near to a point x_0 of X, there will be a positive real number N such that, when $|y| > N$,

$$|y^n| > |a_1(x_1) y^{n-1} + \cdots + a_n(x_1)|$$

and therefore $f(x_1, y) \neq 0$. The roots y of $f(x_1, y)$ are bounded by N for all x_1 near to x_0.

On the other hand, when the leading coefficient $a_0(x)$ isn't a constant, B won't be integral over A. If x_0 is a root of $a_0(x)$, then $f(x_0, y)$ will have degree less than n. What happens there is that some roots of $f(x_1, y)$ are unbounded at points x_1 near to x_0. In calculus, one says that the locus $f(x, y) = 0$ has a vertical asymptote at x_0.

To see this, we divide f by its leading coefficient. Let $g(x, y) = f(x, y)/a_0 = y^n + c_1 y^{n-1} + \cdots + c_n$ with $c_i(x) = a_i(x)/a_0(x)$. For any x at which $a_0(x)$ isn't zero, the roots of g are the same as those of f. However, let x_0 be a root of a_0. Because f is irreducible. At least one coefficient $a_j(x)$ doesn't have x_0 as a root. Then $c_j(x)$ is unbounded near x_0, and because the coefficient c_j is a symmetric function in the roots, the roots aren't all bounded.

This is the general picture: The roots of a polynomial remain bounded near points at which the leading coefficient isn't zero, but some roots are unbounded near to a point at which the leading coefficient is zero. $\square$

noether-normal **4.2.9. Noether Normalization Theorem.** Let A be a finite-type domain over an infinite field k. There exist elements $x_1, \ldots, x_n$ in A that are algebraically independent over k, such that A is an integral extension of R, i.e., such that A is a finite module over the polynomial subalgebra $R = k[x_1, \ldots, x_n]$.

When $k = \mathbb{C}$, the theorem can be stated by saying that every affine variety X admits an integral morphism to an affine space.

The Noether Normalization Theorem remains true when A is a finite-type algebra over a finite field, though the proof given below needs to be modified.

4.2.10. Lemma. *Let k be an infinite field, and let $f(x)$ be a nonzero polynomial of degree d in $x_1, \dots, x_n$, with coefficients in k. After a suitable linear change of variable and scaling, f will be a monic polynomial in x_n.* *nonzero-coeff*

proof. Let f_d be the homogeneous part of f of maximal degree d. We regard f_d as a function $k^n \to k$. Since k is infinite, that function isn't identically zero. We choose coordinates $x_1, ..., x_n$ so that the point $q = (0, ..., 0, 1)$ isn't a zero of f_d. Then $f_d(0, ..., 0, x_n) = cx_n^d$, and the coefficient c, which is $f_d(0, ..., 0, 1)$, will be nonzero. We can scale f to make it monic. □

proof of the Noether Normalization Theorem. Say that the finite-type domain A is generated by elements $x_1, \dots, x_n$. If those elements are algebraically independent over k, A will be the polynomial algebra $\mathbb{C}[x]$. In that case we let $R = A$. If $x_1, ..., x_n$ aren't algebraically independent, they satisfy a polynomial relation $f(x) = 0$ of some positive degree d, with coefficients in k. The lemma tells us that, after a suitable change of variable and scaling, f will be a monic polynomial in x_n, with coefficients in the subalgebra B of A generated by $x_1, \dots, x_{n-1}$. So x_n will be integral over B, and A will be a finite B-module. By induction on n, we may assume that B is a finite module over a polynomial subalgebra R. Then A will be a finite module over R too. □

The next corollary is an example of a general principle that has been noted before. In a ring of fractions, any construction involving finitely many denominators can be done in a simple localization.

4.2.11. Corollary. *Let $A \subset B$ be finite-type domains. There is a nonzero element s in A such that B_s is a finite module over a polynomial subring $A_s[y_1, ..., y_r]$.* *localntheorem*

proof. Let S be the multiplicative system of nonzero elements of A, so that $K = AS^{-1}$ is the fraction field of A, and let $B_K = BS^{-1}$ be the ring obtained from B by adjoining inverses of all of the elements of S. If $\beta = (\beta_1, ..., \beta_k)$ is a set of algebra generators for the finite-type algebra B, then B_K is generated by β as a K-algebra . It is a finite-type K-algebra. The Noether Normalization Theorem tells us that B_K is a finite module over a polynomial subring $R_K = K[y_1, ..., y_r]$. So B_K is an integral extension of R_K. An element β of B will be in B_K. Therefore it will be the root of a monic polynomial, say

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$$

where the coefficients $c_j(y)$ are elements of R_K. Each coefficient c_j is a combination of finitely many monomials in y, with coefficients in K. If $d \in A$ is a common denominator of those coefficients, $c_j(x)$ will have coefficients in $A_d[y]$. Since the generators $\beta_1, ..., \beta_k$ of B are integral over R_K, we may choose a single denominator s so that all of the generators are integral over $A_s[y]$. The algebra B_s is generated over A_s by $\beta_1, ..., \beta_k$, so it will be an integral extension of $A_s[y]$. □

4.3 Normalization

finint

Let A be a domain with fraction field K. The *normalization* $A^\#$ of A is the set of elements of K that are integral over A. The normalization is a domain that contains A (4.2.2) **(ii)**.

A domain A is *normal* if it is equal to its normalization, and a *normal variety* X is a variety that has an *affine covering* $\{X^i = \operatorname{Spec} A_i\}$, a covering by affine open sets, in which the algebras A_i are normal domains.

To justify the definition of normal variety, we need to show that if an affine variety $X = \operatorname{Spec} A$ has an affine covering $\{X^i = \operatorname{Spec} A_i\}$, in which A_i are normal domains, then A is a normal domain. This follows from Lemma 4.3.4 **(iii)** below.

Our goal here is the next theorem, whose proof is at the end of the section.

4.3.1. Theorem. *Let A be a finite-type domain with fraction field K of characteristic zero. The normalization $A^\#$ of A is a finite A-module and a finite-type domain.* *normalfinite*

Thus there is an integral morphism $\operatorname{Spec} A^\# \to \operatorname{Spec} A$.

Asharp-modA **4.3.2. Corollary.** *With notation as above, there is a nonzero element s in A such that $sA^\# \subset A$.*

proof. We assume that the theorem has been proved. Since A and $A^\#$ have the same fraction field, every element α of $A^\#$ can be written as a fraction $\alpha = a/s$ with a, s in A, and then $s\alpha$ is in A. Since $A^\#$ is a finite A-module, one can find a nonzero element s in A such that $s\alpha$ is in A for all α in $A^\#$. Then $sA^\# \subset A$. $\square$

nodecurve **4.3.3. Example.** *(normalization of a nodal cubic curve)* The algebra $A = \mathbb{C}[u,v]/(v^2 - u^3 - u^2)$ can be embedded into the one-variable polynomial algebra $B = \mathbb{C}[x]$, by $u = x^2 - 1$ and $v = x^3 - x$. The fraction fields of A and B are equal because $x = v/u$, and the equation $x^2 = u{+}1$ shows that x is integral over A. The algebra B is normal, so it is the normalization of A.

The plane curve $C = \operatorname{Spec} A$ has a node at the origin $p = (0,0)$, and $\operatorname{Spec} B$ is the affine line $\mathbb{A}^1$. The inclusion $A \subset B$ defines an integral morphism $\mathbb{A}^1 \to C$ whose fibre over p is the point pair $x = \pm 1$, and that is bijective at all other points. I think of C as the variety obtained by gluing the points $x = \pm 1$ of the affine line together.

In this example, the effect of normalization can be visualized geometrically, but. This this is fairly unusual. Normalization is an algebraic process., and its effect on geometry may be subtle. $\square$

ufdnormal **4.3.4. Lemma.** **(i)** *A unique factorization domain is normal. In particular, a polynomial algebra over a field is normal.*
(ii) *A localization A_s of a normal domain A is normal.*
(iii) *Let $s_1, ..., s_k$ be nonzero elements of a domain A that generate the unit ideal. If the localizations A_{s_i} are normal for all i, then A is normal.*

proof. **(i)** Let A be a unique factorization domain, and let β be an element of its fraction field that is integral over A. Say that

eqntwo (4.3.5)
$$\beta^n + a_1\beta^{n-1} + \cdots + a_{n-1}\beta + a_n = 0$$

with a_i in A. We write $\beta = r/s$, where r and s are relatively prime elements of A. Multiplying by s^n gives us the equation
$$r^n = -s\,(a_1 r^{n-1} + \cdots + a_n s^{n-1})$$
This equation shows that if a prime element of A divides s, it also divides r. Since r and s are relatively prime, there is no such prime element. So s is a unit, and β is in A.

(ii) Let β be an element of the fraction field of A that is integral over A_s. There will be a polynomial relation of the form (4.3.5), where the coefficients a_i are elements of A_s. The element $\gamma = s^k\beta$ is a root of the polynomial

$$\gamma^n + (s^k a_1)\gamma^{n-1} + (s^{2k}a_2)\gamma^{n-2} + \cdots + +(s^{nk}a_n) = 0$$

Since a_i are in A_s, all coefficients in this polynomial will be in A when k is sufficiently large, and then γ will be integral over A. Since A is normal, γ will be in A, and $\beta = s^{-k}\gamma$ will be in A_s.

(iii) This proof follows a common pattern. Suppose that A_{s_i} is normal for every i. If an element β of K is integral over A, it will be in A_{s_i} for all i, and $s_i^n\beta$ will be an element of A when n is large. We can use the same exponent n for all i. Since $s_1, ..., s_k$ generate the unit ideal, so do their powers $s_i^n, ..., s_k^n$. Say that $\sum r_i s_i^n = 1$, with r_i in A. Then $\beta = \sum r_i s_i^n \beta$ is in A. $\square$

We prove Theorem 4.3.1 in a slightly more general form. Let A be a finite type domain with fraction field K, and let L be a finite field extension of K. The *integral closure* of A in L is the set of all elements of L that are integral over A.

intclo **4.3.6. Theorem.** *Let A be a finite type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. The integral closure of A in L is a finite A-module.*

The proof is at the end of the section. It makes use of the characteristic zero hypothesis, though the conclusion of the theorem is true for a finite-type algebra over any field.

4.3.7. Lemma. *Let A be a normal noetherian domain with fraction field K of characteristic zero, and let L be an algebraic field extension of K. An element β of L is integral over A if and only if the monic irreducible polynomial f for β over K has coefficients in A.* about-tracetwo

proof. If the monic polynomial f has coefficients in A, then β is integral over A. Suppose that β is integral over A. We may replace L by any field extension that contains β. So we may replace L by $K[\beta]$. Then L becomes a finite extension of K, which embeds into a Galois extension. So we may replace L by a Galois extension. Let G be its Galois group, and let $\{\beta_1, \ldots, \beta_r\}$ be the G-orbit of β, with $\beta = \beta_1$. The irreducible polynomial for β over K is

(4.3.8) $$f(x) = (x - \beta_1) \cdots (x - \beta_r)$$ orbitpoly

If β is integral over A, all elements of the orbit are integral over A. Therefore the coefficients of f, which are elemetary symmetric functions in the orbit, are integral over A, and since A is normal, they are in A. So f has coefficients in A. $\square$

4.3.9. Example. A (nonconstant) polynomial $f(x, y)$ in the polynomial algebra $A = \mathbb{C}[x, y]$ is *square-free* if it has no nonconstant square factors. dplanenormal

Let f be an irreducible, square-free polynomial, and let B denote the integral extension $\mathbb{C}[x, y, w]/(w^2 - f)$ of A. Let K and L be the fraction fields of A and B, respectively. Then L is a Galois extension of K. Its Galois group is generated by the automorphism σ of order 2 that is defined by $\sigma(w) = -w$. The elements of L have the form $\beta = a + bw$ with a, b in K, and $\sigma(\beta) = \beta' = a - bw$.

We show that B is the integral closure of A in L. Suppose that $\beta = a + bw$ is integral over A, with a, b in K. If $b = 0$, then $\beta = a$. This is an element of A and therefore it is in B. If $b \neq 0$, the irreducible polynomial for β will be

$$(x - \beta)(x - \beta') = x^2 - 2ax + (a^2 - b^2 f)$$

Because β is integral over A, $2a$ and $a^2 - b^2 f$ will be in A, and because the characteristic isn't 2, this is true if and only if a and $b^2 f$ are in A. We write $b = u/v$, with u, v relatively prime elements of A, so $b^2 f = u^2 f / v^2$. If v weren't a constant, then since u and v are relatively prime and f is square-free, v^2 couldn't be canceled from $u^2 f$. So $b^2 f$ wouldn't be in A. From $b^2 f$ in A we can conclude that v is a constant and that b is in A. Summing up, β is integral if and only if a and b are in A, which means that β is in B. $\square$

4.3.10. trace deftrace

Let L be a finite field extension of a field K and let β be an element of K. When L is viewed as a K-vector space, multiplication by β becomes a K-linear operator $L \to L$. The *trace* of that operator will be denoted by $\operatorname{trace}(\beta)$. The trace is a K-linear map $L \to K$.

4.3.11. Lemma. *Let L/K be a field extension of degree n, let $K' = K[\beta]$ be the extension of K generated by an element β of L, and let $f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$ be the irreducible polynomial of β over K. Say that $[L : K'] = d$, so that $n = rd$. Then $\operatorname{trace}(\beta) = -da_1$. If β is an element of K, then $\operatorname{trace}(\beta) = n\beta$.* about-traceone

proof. The set $(1, \beta, \ldots, \beta^{r-1})$ is a K-basis for K'. On this basis, the matrix of multiplication by β has the form illustrated below for the case $r = 3$. Its trace is $-a_1$.

$$M = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix}.$$

Next, if $(u_1, \ldots, u_d)$ is a basis for L over K', the set $\{\beta^i u_j\}$, with $i = 0, \ldots, r-1$ and $j = 1, \ldots, d$, will be a basis for L over K. When this basis is listed in the order

$$(u_1, u_1\beta, ..., u_1\beta^{r-1}; u_2, u_2\beta, \ldots u_2\beta^{r-1}; \ \ldots \ ; u_d, u_d\beta, ..., u_d\beta^{r-1}),$$

the matrix of multiplication by β will be made up of d blocks of the matrix M. $\square$

4.3.12. Corollary. *Let A be a normal domain with fraction field K and let L be a finite field extension of K. If an element β of L is integral over A, its trace is in A.* traceinA

This follows from Lemmas 4.3.7 and 4.3.11. □

formnon-deg **4.3.13. Lemma.** *Let A be a normal noetherian domain with fraction field K* of characteristic zero*, and let L be a finite field extension of K. The form $L \times L \to K$ defined by $\langle \alpha, \beta \rangle = \text{trace}(\alpha\beta)$ is K-bilinear, symmetric, and nondegenerate. If α and β are integral over A, then $\langle \alpha, \beta \rangle$ is an element of A.*

proof. The form is obviously symmetric, and it is K-bilinear because multiplication is K-bilinear and trace is K-linear. A form is nondegenerate if its nullspace is zero, which means that, when $\alpha \neq 0$, there is an element β such that $\langle \alpha, \beta \rangle \neq 0$. Let $\beta = \alpha^{-1}$. Then $\langle \alpha, \beta \rangle = \text{trace}(1)$, which, according to (4.3.11), is the degree $[L : K]$ of the field extension. It is here that the hypothesis on the characteristic of K enters: The degree is a nonzero element of K.

If α and β are integral over A, so is their product $\alpha\beta$ (4.2.2) **(ii)**. Corollary 4.3.12 shows that $\langle \alpha, \beta \rangle$ is an element of A. □

clearde-nom **4.3.14. Lemma.** *Let A be a domain with fraction field K, let L be a field extension of K, and let β be an element of L that is algebraic over K. If β is a root of a polynomial $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with a_i in A, then $\gamma = a_n\beta$ is integral over A.*

proof. One finds a monic polynomial with root γ by substituting $x = y/a_n$ into f and multiplying by a_n^{n-1}. □

normfin-proof **4.3.15. proof of Theorem 4.3.1**

Let A be a finite-type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. We are to show that the integral closure of A in L is a finite A-module.

We use the Noether Normalization Theorem to write A as a finite module over a polynomial subalgebra $R = \mathbb{C}[y_1, \dots, y_d]$. Let F be the fraction field of R. Then K and L will be finite extensions of F. An element of L will be integral over A if and only if it is integral over R (4.2.2) **(iv)**. So the integral closure of A in L is the same as the integral closure of R in L. We may replace A by the algebra R and K by F. Thus we may assume that A is normal.

Let $(v_1, \dots, v_n)$ be a K-basis for L whose elements are integral over the normal algebra A. Such a basis exists because we can multiply any element of L by a nonzero element of K to make it integral (Lemma 4.3.14). Let

mapvector (4.3.16)
$$T : L \to K^n$$

be the map defined by $T(\beta) = \big(\langle v_1, \beta \rangle, \dots, \langle v_n, \beta \rangle\big)$, where $\langle \alpha, \beta \rangle = \text{trace}(\alpha\beta)$, as in Lemma 4.3.13. This map is K-linear. If $\langle v_i, \beta \rangle = 0$ for all i, then because $(v_1, \dots, v_n)$ is a basis for L, $\langle \gamma, \beta \rangle = 0$ for every γ in L, and since the form is nondegenerate, $\beta = 0$. Therefore T is injective.

Let B be the integral closure of A in L. We are to show that B is a finite A-module. The basis elements v_i are in B, and if β is in B, then $v_i\beta$ will be in B too. Then $\langle v_i, \beta \rangle = \text{trace}(v_i b)$ will be in A, and $T(\beta)$ will be in A^n (4.3.13). When we restrict T to B, we obtain an injective map $B \to A^n$ that we denote by T_0. Since T is K-linear, T_0 is A-linear. It is an injective homomorphism of A-modules that maps B isomorphically to its image, a submodule of A^n. Since A is noetherian, every submodule of the finite A-module A^n is a finite module. Therefore the image of T_0 is a finite A-module, and so is the isomorphic module B. □

4.4 Geometry of Integral Morphisms

prmint The main geometric properties of an integral morphism of affine varieties are summarized by the theorems in this section, which show that the geometry is as nice as could be expected.

Let $Y \to X$ be an integral morphism of affine varieties. We say that a closed subvariety D of Y *lies over* a closed subvariety C of X if C is the image of D. Similarly, if $A \to B$ is an integral extension of finite-type domains, we say that a prime ideal Q of B *lies over* a prime ideal P of A if P is the contraction $Q \cap A$. For example, if $Y \to X$ is the morphism of affine varieties that corresponds to a homomorphism $A \to B$, and if a point y of Y has image x in X, then y lies over x, and the maximal ideal $\mathfrak{m}_y$ lies over the maximal ideal $\mathfrak{m}_x$.

4.4.1. Lemma. *Let $A \subset B$ be an integral extension of finite-type domains, and let J be an ideal of B. If J isn't the zero ideal of B, then its contraction $J \cap A$ isn't the zero ideal of A.* contrzero

proof. A nonzero element β of J will be a root of a monic polynomial, say $\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_0 = 0$, with coefficients a_i in A. If $a_0 = 0$, then since B is a domain, we can cancel β from this polynomial. So we may assume that $a_0 \neq 0$. The equation shows that a_0 is in J as well as in A. $\square$

4.4.2. Proposition. *Let $A \to B$ be an integral extension of finite-type domains, and let $Y \to X$ be the corresponding morphism, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$.* QoverP

(i) *Let P and Q be prime ideals of A and B, respectively, let C be the locus of zeros of P in X, and let D be the locus of zeros of Q in Y. Then Q lies over P if and only if D lies over C.*

(ii) *Let Q and Q' be prime ideals of B that lie over the same prime ideal P of A. If $Q \subset Q'$, then $Q = Q'$. If D' and D are closed subvarieties of Y that lie over the same subvariety C of X and if $D' \subset D$, then $D' = D$.*

proof. **(i)** Let $\overline{A} = A/P$ and $\overline{B} = B/Q$. Then $D = \operatorname{Spec} \overline{B}$ and $C = \operatorname{Spec} \overline{A}$. Suppose that Q lies over P. So $P = Q \cap A$. Then the canonical map $\overline{A} \to \overline{B}$ will be injective, and $\overline{B}$ will be generated as $\overline{A}$-module by the residues of a set of generators of the finite A-module B. So $\overline{B}$ is an integral extension of $\overline{A}$, and the map from D to C is surjective (Proposition 4.2.5). Therefore D lies over C. Conversely, if D lies over C, the morphism $D \to C$ is surjective. Then the canonical map $\overline{A} \to \overline{B}$ is injective, and this implies that $Q \cap A = P$.

(ii) Suppose that $Q \subset Q'$, and that Q and Q' lie over P. So $Q \cap A = Q' \cap A = P$. Let $\overline{A} = A/P$ and $\overline{B} = B/Q$ as before, and let $\overline{Q}' = Q'/Q$. Because B is an integral extension of A, $\overline{B}$ is an integral extension of $\overline{A}$, and $\overline{Q}'$ is an ideal of $\overline{B}$. To show that $Q = Q'$, we show that $\overline{Q}' = 0$. According to Lemma 4.4.1, it suffices to show that $\overline{Q}' \cap \overline{A} = 0$. Let $\underline{z}$ be an element of $\overline{Q}' \cap \overline{A}$, and let $z \in Q'$ and $a \in A$ be elements whose residues in $\overline{Q}'$ are equal to $\underline{z}$. Then the residue of $z - a$ is zero, so $z - a$ is in Q and in Q'. Since z and $z - a$ are in Q', so is a. Then a is in $Q' \cap A = P$, and this implies that $\underline{z} = 0$. $\square$

4.4.3. Theorem. *Let $Y \xrightarrow{u} X$ be an integral morphism of affine varieties.* closedimage

(i) *The fibres of u have bounded cardinality.*

(ii) *The image of a closed subset of Y is a closed subset of X, and the image of a closed subvariety of Y is a closed subvariety of X.*

(iii) *The set of closed subvarieties of Y that lie over a closed subvariety C of X is finite and nonempty.*

proof. Say that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. Let $A \subset B$ be the extension that corresponds to the integral morphism u.

(i) *(bounding the fibres)* Let $y_1, ..., y_r$ be points of Y in the fibre over a point x of X, and let $\mathfrak{m}_{y_i}$ and $\mathfrak{m}_x$ be the maximal ideals of B and A at y_i and x, respectively. Then $\mathfrak{m}_x$ is the contraction $\mathfrak{m}_{y_i} \cap A$. To bound the number r, we use the Chinese Remainder Theorem to show that B cannot be spanned as A-module by fewer than r elements. Let k_i and k denote the residue fields $B/\mathfrak{m}_{y_i}$, and $A/\mathfrak{m}_x$, respectively, all of these fields being isomorphic to $\mathbb{C}$. Let $\overline{B} = k_1 \times \cdots \times k_r$. We form a diagram of algebra homomorphisms

$$\begin{tikzcd} B \arrow[r, "\varphi"] & \overline{B} \approx \mathbb{C}^r \\ A \arrow[u] \arrow[r] & k \approx \mathbb{C} \arrow[u] \end{tikzcd}$$

which we interpret as a diagram of A-modules. The minimal number of generators of the A-module $\overline{B}$ is equal to its dimension as k-module, which is r. The Chinese Remainder Theorem asserts that the map φ is surjective, so B cannot be spanned by fewer than r elements.

(ii) (*the image of a closed set is closed*) The image of an irreducible set via a continuous map is irreducible (2.1.15)**(iii)**, so it suffices to show that the image of a closed subvariety is closed. Let D be the closed subvariety of Y that corresponds to a prime ideal Q of B, and let $P = Q \cap A$ be its contraction, which is a prime ideal of A. Let C be the variety of zeros of P in X. The coordinate algebras of the affine varieties D and C are $\overline{B} = B/Q$ and $\overline{A} = A/P$, respectively, and $\overline{B}$ is an integral extension of $\overline{A}$ because B is an integral extension of A. The map $D \to C$ is surjective (4.2.5). Therefore C is the image of D.

(iii) (*subvarieties that lie over a closed subvariety*) Let C be a closed subvariety of X. Its inverse image $Z = u^{-1}C$ is closed in Y. It is the union of finitely many irreducible closed sets, say $Z = D'_1 \cup \cdots \cup D'_k$. Part **(i)** tells us that the image C'_i of D'_i is a closed subvariety of X. Since u is surjective, $C = \bigcup C'_i$, and since C is irreducible, $C'_i = C$ for at least one i. For such an i, D'_i lies over C. Next, any subvariety D that lies over C will be contained in the inverse image Z, and therefore contained in D'_i for some i. Proposition 4.4.2 **(ii)** shows that $D = D'_i$. Therefore the varieties that lie over C are among the varieties D'_i. □

4.5 Dimension

dim Every variety has a dimension, and as is true for the dimension of a vector space, the dimension is important, though it is a very coarse measure. We give two definitions of the dimension of a variety X here, but the proof that they are equivalent requires work.

One definition is that the dimension of a variety X is the transcendence degree of its function field. For now, we'll refer to this as the *t-dimension* of X.

intdim **4.5.1. Corollary.** *Let $Y \to X$ be an integral morphism of affine varieties. The t-dimensions of X and Y are equal.* □

The second definition of dimension is the combinatorial dimension. It is defined as follows: A *chain* of closed subvarieties of a variety X is a strictly decreasing sequence

chntwo (4.5.2) $$C_0 > C_1 > C_2 > \cdots > C_k$$

of closed subvarieties (of irreducible closed sets). The *length* of this chain is defined to be k. The chain is *maximal* if it cannot be lengthened by inserting another closed subvariety, which means that $C_0 = X$, that for $i < k$ there is no closed subvariety D with $C_i > D > C_{i+1}$, and that C_k is a point.

For example, $\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$, where $\mathbb{P}^i$ is the linear subspace of points $(x_0, ..., x_i, 0, ..., 0)$, is a maximal chain in projective space $\mathbb{P}^n$, and its length is n.

Theorem 4.5.6 below shows that all maximal chains of closed subvarieties of a variety X have the same length. The *combinatorial dimension* of X is the length of a maximal chain. We'll refer to it as the *c-dimenson*. Theorem 4.5.6 also shows that the t-dimension and the c-dimension of a variety are equal. When we have proved that theorem, we will refer to the t-dimension and to the c-dimension simply as the *dimension*, and we will use the two definitions interchangeably.

In an affine variety $\operatorname{Spec} A$, a strictly decreasing chain (4.5.2) of closed subvarieties corresponds to a strictly increasing chain

chn **4.5.3.** $P_0 < P_1 < P_2 < \cdots < P_k$

of prime ideals of A of length k, a *prime chain*. Such a prime chain is *maximal* if it cannot be lengthened by inserting another prime ideal, which means that P_0 is the zero ideal, that there is no prime ideal Q with $P_i < Q < P_{i+1}$ for $i < k$, and that P_k is a maximal ideal. The *c-dimension* of a finite-type domain A is the length k of a maximal chain of prime ideals. If $X = \operatorname{Spec} A$, the c-dimensions of X and of A are equal.

The next theorem is the basic tool for studying dimension. The statement is intuitively plausible, but its proof isn't easy. It is a subtle theorem. We have put the proof at the end of this section.

krullthm **4.5.4. Krull's Principal Ideal Theorem.** Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let V be the zero locus in X of a nonzero element α of A. Every irreducible component of V has t-dimension $d-1$.

compmax **4.5.5. Corollary.** *Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let C be a component of the zero locus of a nonzero element α of A. Then among proper closed subvarieties of X, C maximal. There is no closed subvariety D such that $C < D < X$.*

proof. Let $C < D < X$ be closed subvarieties of $X = \operatorname{Spec} A$. Some nonzero element β of A will vanish on D. Then D will be a subvariety of the zero locus of β, so by Krull's Theorem, its t-dimension will be at most $d-1$. Similarly, if $D = \operatorname{Spec} B$, some nonzero element of B will vanish on C, so the t-dimension of C will be at most $d-2$. Krull's Theorem tells us that C can't be the zero locus of a nonzero element of A. □

4.5.6. Theorem. *Let X be a variety of t-dimension d. All chains of closed subvarieties of X have length at most d, and all maximal chains have length d. Therefore the c-dimension and the t-dimension of X are equal.* *dimtheorem*

proof. We do the case that X is affine. Induction allows us to assume that the theorem is true for an affine variety whose t-dimension is less than d. Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let a chain (4.5.2), $C_0 > \cdots > C_k$ of closed subvarieties of X be given. We must show that $k \leq d$ and that $k = d$ if the chain is maximal. We may insert closed subvarieties into the chain where possible. So we may assume that $C_0 = X$. Some nonzero element α of A will vanish on the proper subvariety C_1, and C_1 will be contained in an irreducible component Z of the zero locus of α. If $C_1 < Z$, we insert Z into the chain. So we may assume that $C_1 = Z$. Then by Krull's Theorem, C_1 will have t-dimension $d-1$. Induction applies to the chain $C_1 > \cdots > C_k$ of closed subvarieties of C_1. Its length $k-1$ is most $d-1$, and is equal to $d-1$ if it is a maximal chain. The assertion to be proved for the given chain follows. $\square$

Theorem 4.5.6 for an arbitrary variety follows from the next lemma.

4.5.7. Lemma. *Let X' be an open subvariety of a variety X. There is a bijective correspondence between chains $C_0 > \cdots > C_k$ of closed subvarieties of X such that $C_k \cap X' \neq \emptyset$ and chains $C'_0 > \cdots > C'_k$ of closed subvarieties of X'. Given the chain $\{C_i\}$ in X, the chain $\{C'_i\}$ in X' is defined by $C'_i = C_i \cap X'$. Given a chain C'_i in X', the corresponding chain in X consists of the closures C_i in X of the varieties C'_i.* *restrictchain*

proof. Suppose given a chain C_i, and that $C_k \cap X' \neq \emptyset$. Then for every i, the intersection $C'_i = C_i \cap X'$ is a dense open subset of the irreducible closed set C_i (2.1.13). It is an irreducible closed subset of X', and its closure in X is C_i. Since $C_i > C_{i+1}$, it is also true that $C'_i > C'_{i+1}$. Therefore $C'_0 > \cdots > C'_k$ is a chain of closed subsets of X'. Conversely, if $C'_0 > \cdots > C'_k$ is a chain in X', the closures in X form a chain in X. (See (2.1.14).) $\square$

From now on, we use the word *dimension* to denote either the t-dimension or the c-dimension, and we denote the dimension of a variety by $\dim X$. A *curve* is a variety of dimension one and a *surface* is a variety of dimension two.

4.5.8. Examples. (i) The polynomial algebra $\mathbb{C}[x_0, \ldots, x_n]$ in $n+1$ variables has dimension $n+1$. The chain of prime ideals dimpolyalg

4.5.9. $0 < (x_0) < (x_0, x_1) < \cdots < (x_0, \ldots, x_n)$ primechain

is a maximal prime chain. When the irrelevant ideal $(x_0, ..., x_n)$ is removed from this chain, it corresponds to the maximal chain

$$\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$$

of closed subvarieties of projective space $\mathbb{P}^n$, which has dimension n.

(ii) The maximal chains of closed subvarieties of $\mathbb{P}^2$ have the form $\mathbb{P}^2 > C > p$, where C is a plane curve and p is a point. $\square$

If (4.5.2) is a maximal chain in X, then

4.5.10. $C_1 > C_2 > \cdots > C_k$ chaini

will be a maximal chain in the variety C_1. So when X has dimension k, the dimension of C_1 will be $k-1$. Similarly, let $P_0 < P_1 < \cdots < P_k$ be a maximal chain of prime ideals in a finite-type domain A, let $\overline{A} = A/P_1$ and let $\overline{P}_j$ denote the image P_j/P_1 of P_j in $\overline{A}$, for $j \geq 1$. Then

$$\overline{0} = \overline{P}_1 < \overline{P}_2 < \cdots < \overline{P}_k$$

will be a maximal prime chain in $\overline{A}$, and therefore the dimension of the domain $\overline{A}$ will be $k-1$.

4.5.11. Corollary. *Let X be a variety.* *integraldimequal*

(i) *If X' is an open subvariety of X, then $\dim X' = \dim X$.*

(ii) *If $Y \to X$ is an integral morphism of varieties, then $\dim Y = \dim X$.*

(iii) *If Y is a proper closed subvariety of X, then $\dim Y < \dim X$.* $\square$

One more term: A closed subvariety C of a variety X has *codimension* 1 if $\dim C = \dim X - 1$, or if $C < X$ and there is no closed set D with $C < D < X$. A prime ideal P of a noetherian domain has *codimension* 1 if it isn't the zero ideal, and if there is no prime ideal Q with $(0) < Q < P$.

In the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, the prime ideals of codimension 1 are the principal ideals generated by irreducible polynomials.

Krullproof **4.5.12. proof of Krull's Theorem**

We are given an affine variety $X = \operatorname{Spec} A$ of t-dimension d, a nonzero element α of A, and an irreducible component C of the zero locus of α. We are to show that the t-dimension of C is $d-1$.

lemmaone **4.5.13. Lemma.** *Krull's Theorem is true when X is an affine space.*

proof. Here A is the polynomial ring $\mathbb{C}[x_1, ..., x_d]$. Let $\alpha_1 \cdots \alpha_k$ be the factorization of the polynomial α into irreducible polynomials, and let V_i be the zero locus of α_i. The irreducible factors generate prime ideals of A, so V_i are irreducible, and V is their union. We replace α by an irreducible factor, say α_1, and we relabel it as f. Then C becomes the zero locus $V_1 = V(f)$, and its coordinate algebra is $\overline{A} = A/(f)$. We may assume that f is monic in x_d (1.3.20). Then $\overline{A}$ is an integral extension of $\mathbb{C}[x_1, ..., x_{d-1}]$. The t-dimension of C is $d-1$. $\square$

lemmatwo **4.5.14. Lemma.** *To prove Krull's Theorem, it suffices to prove it when the coordinate ring A is normal and the zero locus of α is irreducible.*

proof. We use notation as above. Let $A^\#$ be the normalization of A and let $X^\# = \operatorname{Spec} A^\#$. We recall (4.3.1) that $A^\#$ is an integral extension of A. The t-dimensions of $X^\#$ and X are equal. Let V' and V be the zero loci of α in $X^\#$ and in X, respectively. Then V' is the inverse image of V in $X^\#$. The map $V' \to V$ is surjective because the integral morphism $X^\# \to X$ is surjective.

Let $D_1, ..., D_k$ be the irreducible components of V', and let C_i be the image of D_i in X. The closed sets C_i are irreducible (4.4.3) **(ii)**, and their union is V. So at least one C_i is equal to the chosen component C. Let D be a component of V' whose image is C. The map $D \to C$ is also an integral morphism, so the t-dimensions of D and C are equal. We may therefore replace X by $X^\#$ and C by D. Hence we may assume that A is normal.

Next, we write the zero locus of α as $C \cup \Delta$, where C is the chosen component, and Δ is the union of the other components. We choose an element s of A that is identically zero on Δ but not identically zero on C. Inverting s eliminates all points of Δ, but $C_s = X_s \cap C$ won't be empty. If X is normal, so is X_s (4.3.4). Since localization doesn't change t-dimension, we may replace X and C by X_s and C_s, respectively. $\square$

We go to the proof of Krull's Theorem. We assume that $X = \operatorname{Spec} A$ is a normal affine variety of dimension d, and that the zero locus of α is an irreducible closed set C. We are to prove that the t-dimension of C is $d-1$.

We apply the Noether Normalization Theorem. Let $X \xrightarrow{v} S$ be an integral morphism to an affine space $S = \operatorname{Spec} R$ of dimension d, where R is a polynomial ring $\mathbb{C}[u_1, ..., u_d]$.

Let K and F be the function fields of X and S, respectively, and let $f(t)$ be the monic irreducible polynomial for α over F. The coefficients of f are in R (4.3.7). Let L be a splitting field of f over K, and let $\alpha_1, ..., \alpha_r$ be the roots of f in L, with $\alpha_1 = \alpha$. So $f(t) = (t-\alpha_1) \cdots (t-\alpha_r)$.

Let B be the integral closure of A in L, and let $Y = \operatorname{Spec} B$. Then $R \subset A \subset B$ and $Y \xrightarrow{u} X \xrightarrow{v} S$. The morphisms u, v, and $w = vu$ are integral morphisms. The Galois group G of L/F operates on Y, and S is the space of G-orbits. Let β denote the product $\alpha_1 \cdots \alpha_r$, the constant term of f.

lemmathree **4.5.15. Lemma.** *Let Z be the zero locus of β in S. The morphism $X \xrightarrow{v} S$ maps the zero locus C of α surjectively to Z.*

$$\begin{array}{ccccc} & & C & \longrightarrow & Z \\ & & \downarrow & & \downarrow \\ Y & \xrightarrow{u} & X & \xrightarrow{v} & S \end{array}$$

We assume the lemma for the moment. The'd the zero locus Z of β will be irreducible because C is irreducible (2.1.15). Lemma 4.5.13 shows that the t-dimension of Z is $d-1$. Therefore the t-dimension of C is at least

$d-1$, and since it is a proper closed subset of X, it is less than d. So the t-dimension of C is equal to $d-1$, as Krull's Theorem asserts. $\square$

proof of Lemma 4.5.15 The element β of R defines functions on S, X, and Y, the functions on X and Y being obtained from the function on S by composition with the maps v and w, respectively. We denote all of those functions by β. If y is a point of Y, $x = uy$, and $s = wy$, then $\beta(y) = \beta(x) = \beta(s)$. Similarly, α defines functions on X and on Y that we denote by α. If $x = uy$, then $\alpha(y) = \alpha(x)$.

Let x be a point of C, and let $s = vx$. So $\alpha(x) = 0$. Since α divides β, $\beta(x) = \beta(s) = 0$. This shows that s is a point of Z. So Z contains the image of C.

For the other inclusion, let z be a point of Z, let y be a point of Y such that $wy = z$, and let $x = uy$, so that $y \to x \to z$. Then $\beta(z) = \beta(x) = \beta(y) = 0$. The fibre of Y over z is the G-orbit of y. Since the function β on Y is obtained from a function on S, it vanishes at every point of that orbit. On Y, $\beta = \alpha_1 \cdots \alpha_k$. Therefore $\alpha_i(y) = 0$ for at least one i. Let σ be an element of G such that $\alpha_i = \sigma\alpha$. We recall that $[\sigma\alpha](y) = \alpha(y\sigma)$ (2.7.8). So $\alpha(y\sigma) = 0$. We replace y by $y\sigma$ and we replace x accordingly. We will still have $y \to x \to z$, and now $\alpha(y) = \alpha(x) = 0$. So x is a point of C with image z. Since z can be any point of Z, the map $C \to Z$ is surjective. $\square$

4.6 Chevalley's Finiteness Theorem

4.6.1. finite morphisms

finmorph prodagain

A morphism $Y \xrightarrow{u} X$ of affine varieties $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ is a *finite morphism* if the corresponding homomorphism $A \xrightarrow{\varphi} B$ makes B into a finite A-module. The difference between a finite morphism and an integral morphism of affine varieties, as defined in Section 4.2, is that for a finite morphism, the homomorphism φ needn't be injective. If u is a finite morphism and φ is injective, B will be an integral extension of A, and u will be an integral morphism. We extend these definitions to varieties that aren't necessarily affine here.

The *restriction* of a morphism $Y \xrightarrow{u} X$ to an open subset X' of X is the induced morphism $Y' \to X'$, where Y' is the inverse image of X'.

4.6.2. Definition. A morphism of varieties $Y \xrightarrow{u} X$ is a *finite morphism* if X can be covered by affine open subsets X^i such that the restriction of u to each X^i is a finite morphism of affine varieties. Similarly, a morphism u is an *integral morphism* if X can be covered by affine open sets X^i to which the restriction of u is an integral morphism of affine varieties. deffin-morph

An integral morphism is a finite morphism. The composition of finite morphisms is a finite morphism. The inclusion of a closed subvariety into a variety is a finite morphism.

When X is affine, Definitions 4.6.1 and 4.6.2 both apply. Proposition 4.6.4, which is below, shows that the two definitions are equivalent.

4.6.3. Lemma. **(i)** *Let $A \xrightarrow{\varphi} B$ be a homomorphism of finite-type domains that makes B into a finite A-module, and let s be a nonzero element of A. Then B_s is a finite A_s-module.* *restfin-morph*
(ii) *Using Definition 4.6.2, the restriction of a finite (or an integral) morphism $Y \xrightarrow{u} X$ to an open subset of a variety X is a finite (or an integral) morphism.*

proof. **(i)** Here, since s is in A, B_s is to be interpreted as the localization of the A-module B. This localization can also be obtained by localizing the algebra B with respect to the image $\varphi(s)$, provided that it isn't zero. If $\varphi(s) = 0$, then s annihilates B, so $B_s = 0$. In either case, a set of elements that spans B as A-module will span B_s as A_s-module, so B_s is a finite A_s-module.

(ii) Say that X is covered by affine open sets to which the restriction of u is a finite morphism. The localizations of these open sets form a basis for the Zariski topology on X, so X' can be covered by such localizations. Part **(i)** shows that the restriction of u to X' is a finite morphism. $\square$

4.6.4. Proposition. *Let $Y \xrightarrow{u} X$ be a finite (or an integral) morphism, as defined in (4.6.2), and let X_1 be an affine open subset of X. The restriction of u to X_1 is a finite (or an integral) morphism of affine varieties, as defined in (4.2.1).* *onecov-erfinite*

Since the proof is fairly long, we've put it at the end of the section.

Let X be a variety. In the next theorem, we abbreviate the notation for a product of a variety V with X, writing

$$\widetilde{V} = V \times X$$

Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$ with coordinates $y_0, ..., y_n$

chevfin **4.6.5. Chevalley's Finiteness Theorem.** *Let X be a variety, let Y be a closed subvariety of the product $\widetilde{\mathbb{P}} = \mathbb{P} \times X$, and let π denote the projection $Y \to X$, respectively. If every fibre of π is finite, then π is a finite morphism.* □

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\pi"'] & \widetilde{\mathbb{P}} \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

pro-jchevfin **4.6.6. Corollary.** **(i)** *Let Y be a projective variety, and let $Y \xrightarrow{\pi} X$ be a morphism. If every fibre of π is finite, then π is a finite morphism.*
(ii) *Let Y be a projective curve. Every nonconstant morphism $Y \xrightarrow{\pi} X$ is a finite morphism.*

proof. **(i)** This follows from the theorem when one replaces Y by the graph of π in $\widetilde{Y} = Y \times X$. The graph is isomorphic to Y. If Y is a closed subvariety of $\mathbb{P}$, the graph will be a closed subvariety of $\widetilde{\mathbb{P}}$ (Proposition 3.5.24). □

We use the next two lemmas in the proof of Chevalley's Theorem.

Let $y_0, ..., y_n$ be coordinates in $\mathbb{P} = \mathbb{P}^n$, and let $A[y_0, ..., y_n]$ be the algebra of polynomials in y with coefficients in A. In analogy for terminology used with complex polynomials, we say that a polynomial with coefficients in A is *homogeneous* if it is homogeneous as a polynomial in y, and that an ideal of $A[y]$ that can be generated by such homogeneous polynomials is a *homogeneous ideal*.

hompoly-coeffA **4.6.7. Lemma. (i)** *Let $X = \operatorname{Spec} A$ be an affine variety, and let Y be a nonempty subset of $\widetilde{\mathbb{P}} = \mathbb{P} \times X$. The ideal $\mathcal{I}$ of elements of $A[y]$ that vanish at every point of Y is a homogeneous radical ideal. If Y is a closed subvariety of $\widetilde{\mathbb{P}}$, then $\mathcal{I}$ is a prime ideal.*
(ii) *If the zero locus of a homogeneous ideal $\mathcal{I}$ of $A[y]$ is empty, then $\mathcal{I}$ contains a power of the irrelevant ideal $(y_0, ..., y_n)$ of $A[y]$.*

proof. This is similar to Propositions 3.2.6 and 2.4.12. Let $y_0, ..., y_n$ be coordinates in $\mathbb{P}$, let $\mathbb{A}$ be the affine space of dimension $n+1$ with those coordinates, and let o be the origin in $\mathbb{A}$. Let Z be the inverse image of Y in $\widetilde{\mathbb{A}} = \mathbb{A} \times X$, and let Z' be the complement of $\widetilde{o} = o \times X$ in Z. Because Y isn't empty, $\mathcal{I}$ is the ideal of all polynomials that vanish on Z', and also the ideal of all polynomials that vanish on Z. Proposition 2.4.12 shows that $\mathcal{I}$ is a prime ideal.

If the zero locus of $\mathcal{I}$ in $\mathbb{P} \times X$ is empty, the zero locus in $\widetilde{\mathbb{A}}$ will be contained in $\widetilde{o}$. The radical of $\mathcal{I}$ will contain the ideal of $\widetilde{o}$ in $A[y]$, which is the irrelevant ideal. □

powerso-fyenuf **4.6.8. Lemma.** *Let A be a finite type domain, let I be an ideal of the polynomial algebra $A[u_1, ..., u_n]$, and let k be a positive integer. Suppose that, for each $i = 1, ..., n$, there is a polynomial $g_i(u_1, ..., u_n)$ of degree at most $k-1$, and with coefficients in A, such that $u_i^k - g_i(u)$ is in I. Then $B = A[u]/I$ is a finite A-module.*

proof. Let's denote the residue of u_i in B by the same symbol u_i. In B, we will have $u_i^k = g_i(u)$. Any monomial m of degree at least nk in $u_1, ..., u_n$ will be divisible by u_i^k for at least one i. Then in B, m is equal to a polynomial in $u_1, ..., u_n$ of degree less than nk, with coefficients in A. It follows by induction that the monomials in $u_1, ..., u_n$ of degree at most $nk-1$ span B as an A-module. □

proof of Chevelley's Finiteness Theorem. This is Schelter's proof.
We are given a closed subvariety Y of $\widetilde{\mathbb{P}} = \mathbb{P} \times X$, with $\mathbb{P} = \mathbb{P}^n$, and the fibres of Y over X are finite sets. We are to prove that the projection $Y \to X$ is a finite morphism. By induction, we may assume that the theorem is true when $\mathbb{P}$ is a projective space of dimension $n-1$.

We may suppose that X is affine, say $X = \operatorname{Spec} A$ (see Definition 4.6.2).

Case 1. There is a hyperplane H in $\mathbb{P}$ such that, in $\widetilde{\mathbb{P}}$, Y is disjoint from $\widetilde{H} = H \times X$. This is the main case.

We adjust coordinates $y_0, ..., y_n$ in $\mathbb{P}$ so that H is the hyperplane at infinity $\{y_0 = 0\}$. Because Y is disjoint from $\widetilde{H}$, it is a subset of the affine variety $\widetilde{\mathbb{U}}^0 = \mathbb{U}^0 \times X$, $\mathbb{U}^0$ being the standard open set in $\mathbb{P}$. The coordinate algebra of $\widetilde{\mathbb{U}}^0$ is $A[u_1, ..., u_n]$, here $u_i = y_i/y_0$. Since Y is irreducible and closed in $\widetilde{\mathbb{P}}$, it is also a closed subvariety of $\widetilde{\mathbb{U}}^0$. So Y is affine, and its coordinate algebra B is a quotient of $A[u]$, say $B = A[u]/P$.

Since Y is a subset of $\widetilde{\mathbb{U}}^0$, a homogeneous polynomial $f(y_0, ..., y_n)$ in $A[y]$ vanishes on Y if and only if its dehomogenization $F(u_1, ..., u_n) = f(1, u_1, ..., u_n)$ vanishes there. If $\mathcal{P}$ is the homogeneous prime ideal of $A[y]$ whose zero set in $\widetilde{\mathbb{P}}$ is Y, then the prime ideal P of Y in $\widetilde{\mathbb{U}}^0$ is its dehomogenization. The prime ideal $\mathcal{Q}$ of $\widetilde{H}$ in $\widetilde{\mathbb{P}}$ is the principal ideal generated by y_0.

By hypothesis, $Y \cap \widetilde{H}$ is empty. Therefore the sum $\mathcal{I} = \mathcal{P} + \mathcal{Q}$ contains a power of the irrelevant ideal $\mathcal{M} = (y_0, ..., y_n)$ of $A[y]$. Say that $\mathcal{M}^k \subset \mathcal{I}$. Then y_i^k is in $\mathcal{I}$, for $i = 0, ..., n$. So there are polynomial equations

(4.6.9) $$y_i^k = f_i(y) + y_0 g_i(y)$$ fplusg

with f_i in $\mathcal{P}$ of degree k and g_i in $A[y]$ of degree $k-1$, both f_i and g_i being homogeneous. We dehomogenize those equations. Let $F_i = f_i(1, u_1, ..., u_n)$ and $G_i = g_i(1, u_1, ..., u_n)$. Then $u_i^k = F_i + G_i$. The important points are that F_i vanishes on Y, and that the degree of G_i is at most $k-1$. Therefore $u_i^k - G_i = 0$ is true in the coordinate algebra B of Y. Lemma 4.6.8 shows that $Y \to X$ is a finite morphism. This completes the proof of Case 1.

Case 2. the general case.

We have taken care of the case in which there exists a hyperplane H such that Y is disjoint from $\widetilde{H}$. The next lemma shows that we can cover the given variety X by open subsets to which this special case applies. Then Lemma 4.6.3 and Proposition 4.6.4 apply, to complete the proof. $\square$

4.6.10. Lemma. *Let Y be a closed subvariety of $\widetilde{\mathbb{P}}$, with $\mathbb{P} = \mathbb{P}^n$, and suppose that the projection $Y \xrightarrow{\pi} X$ has finite fibres. Suppose also that Chevalley's Theorem has been proved for closed subvarieties of $\mathbb{P}^{n-1} \times X$. For every point p of X, there is an open neighborhood X' of p in X, and there is a hyperplane H in $\mathbb{P}$ such that $Y' = \pi^{-1}X'$ is disjoint from $\widetilde{H}$.* avoidh-plane

proof. Let p be a point of X, and let $\widetilde{q} = (\widetilde{q}_1, ..., \widetilde{q}_r)$ be the finite set of points of Y making up the fibre over p. We project $\widetilde{q}$ from $\widetilde{\mathbb{P}}$ to $\mathbb{P}$, obtaining a finite set $q = (q_1, ..., q_r)$ of points of $\mathbb{P}$, and we choose a hyperplane H in $\mathbb{P}$ that avoids this finite set. Then $\widetilde{H}$ avoids the fibre $\widetilde{q}$. Let Z denote the closed set $Y \cap \widetilde{H}$. Because the fibres of Y over X are finite, so are the fibres of Z over X. By hypothesis, Chevalley's Theorem is true for subvarieties of $\mathbb{P}^{n-1} \times X$, and $\widetilde{H}$ is isomorphic to $\mathbb{P}^{n-1} \times X$. It follows that, for every component Z' of Z, the morphism $Z' \to X$ is a finite morphism, and therefore its image is closed in X (Theorem 4.4.3). Thus the image of Z is a closed subset of X that doesn't contain p. Its complement is the required neighborhood of p. $\square$

4.6.11. -8.5cm **proof of Proposition 4.6.4** proof-covfin

We'll do the case of an integral morphism. The case of a finite morphism is similar.

Step 1. Preliminaries.

We are given a morphism $Y \xrightarrow{u} X$, and we are given an affine covering $\{X^i\}$ of X, such that, for every i, the restriction u^i of u to X^i is an integral morphism of affine varieties. We are to show that the restriction of u to any affine open subset X_1 of X is an integral morphism of affine varieties.

The affine open set X_1 is covered by the affine open sets $X_1 \cap X^i$, and the restriction of u to $X_1 \cap X^i$ can also be obtained by restricting u^i. So the restriction is an integral morphism (4.6.3). We may replace X by X_1 and X^i by $X_1 \cap X^i$. Since the localizations of an affine variety form a basis for its Zariski topology, we may assume that X^i are localizations of X. Then what is to be proved is this:

A morphism $Y \xrightarrow{u} X$ is given in which $X = \operatorname{Spec} A$ is affine. There are elements $s_1, ..., s_k$ that generate the unit ideal of A such that, for every i, the inverse image Y^i of the localization $X^i = X_{s_i}$ is affine, and its coordinate algebra B_i is an integral extension of the localized algebra $A_i = A_{s_i}$. We must show that Y is affine, and that its coordinate algebra B is an integral A-module.

Step 2. The algebra of regular functions on Y.

We assume that X is affine, $X = \operatorname{Spec} A$. Let B be the algebra of regular functions on Y. If Y is affine, B will be its coordinate algebra, and Y will be its spectrum. However, we don't know that Y is affine.

By hypothesis, the inverse image Y^i of X^i is the spectrum of an integral A_i-algebra B_i. So B and B_i are subalgebras of the function field of Y. Since the localizations X^i cover X, the affine varieties Y^i cover Y. A function is regular on Y if and only if it is regular on each Y^i, and therefore

$$B = \bigcap B_i$$

Step 3. The coordinate algebra B_j of Y^j is a localization of B.

We denote the images in B of the elements s_i by the same symbols, and we show that B_j is the localization $B[s_j^{-1}]$. The localization X^i is the set of points of X at which $s_i \neq 0$. Its inverse image Y^i is the set of points of Y at which $s_i \neq 0$, and $Y^j \cap Y^i$ is the set of points of Y^j at which $s_i \neq 0$. So the coordinate algebra of the affine variety $Y^j \cap Y^i$ is the localization $B_j[s_i^{-1}]$. Then

$$B[s_j^{-1}] \overset{(1)}{=} \bigcap_i B_i[s_j^{-1}] \overset{(2)}{=} \bigcap_i B_j[s_i^{-1}] \overset{(3)}{=} B_j[s_j^{-1}] \overset{(4)}{=} B_j$$

The explanation of the numbered equalities is as follows:

(1) A rational function β is in $B_i[s_j^{-1}]$ if $s_j^n\beta$ is in B_i for large n, and we can use the same exponent n for all $i = 1, ..., r$. Then β is in $\bigcap_i B_i[s_j^{-1}]$ if and only if, for some n, $s_j^n\beta$ is in $\bigcap_i B_i = B$. So β is in $\bigcap_i B_i[s_j^{-1}]$ if and only if it is in $B[s_j^{-1}]$.

(2) $B_i[s_j^{-1}] = B_j[s_i^{-1}]$ because $Y^j \cap Y^i = Y^i \cap Y^j$.

(3),(4) Since s_j is one of the elements s_i, $\bigcap_i B_j[s_i^{-1}] \subset B_j[s_j^{-1}]$. For all i, $B_j \subset B_j[s_i^{-1}]$. Moreover, s_j doesn't vanish on Y^j. It is a unit in B_j, and therefore $B_j[s_j^{-1}] = B_j$. Then $B_j \subset \bigcap_i B_j[s_i^{-1}] \subset B_j[s_j^{-1}] = B_j$.

Step 4. B is an integral extension of A.

With $A_i = A_{s_i}$ as before, we choose a finite set $(b_1, ..., b_n)$ of elements of B that generates the A_i-module B_i for every i. We can do this because we can span the finite A_i-module $B_i = B[s_i^{-1}]$ by finitely many elements of B, and there are finitely many algebras B_i. We show that the set $(b_1, ..., b_n)$ generates the A-module B.

Let x be an element of B. Then x is in B_i, so it is a combination of $(b_1, ..., b_n)$ with coefficients in A_i. For large k, $s_i^k x$ will be a combination of those elements with coefficients in A, say

$$s_i^k x = \sum_\nu a_{i,\nu} b_\nu$$

with $a_{i,\nu}$ in A. We can use the same exponent k for all i. The powers s_i^k generate the unit ideal. With $\sum r_i s_i^k = 1$,

$$x = \sum_i r_i s_i^k x = \sum_i r_i \sum_\nu a_{i,\nu} b_\nu$$

The right side is a combination of $b_1, ..., b_n$ with coefficients in A.

Step 5. Y is affine.

The algebra B of regular functions on Y is a finite-type domain because it is a finite module over the finite-type domain A. Let $\widetilde{Y} = \operatorname{Spec} B$. The fact that B is the algebra of regular functions on Y gives us a morphism $Y \xrightarrow{\epsilon} \widetilde{Y}$ (Corollary 3.6.4). Restricting to the open subset X^j of X gives us a morphism $Y^j \xrightarrow{\epsilon^j} \widetilde{Y}^j$ in which, since $B_j = B[s_j^{-1}]$, Y^j and $\widetilde{Y}^j$ are both equal to $\operatorname{Spec} B_j$. Therefore ϵ^j is an isomorphism. Corollary 3.5.13 **(ii)** shows that ϵ is an isomorphism. So Y is affine, and by ***Step 4***, its coordinate algebra B is an integral A-module. $\square$

4.7 Double Planes

4.7.1. affine double planes dplane affplanes

Let A be the polynomial algebra $\mathbb{C}[x,y]$ and let X be the affine plane $\operatorname{Spec} A$. An affine double plane is a locus of the form $w^2 = f(x,y)$ in the affine 3-space with coordinates w,x,y, where f is a square-free polynomial in x,y (see Example 4.3.9). The affine double plane is $Y = \operatorname{Spec} B$, where $B = \mathbb{C}[w,x,y]/(w^2-f)$, and the inclusion $A \subset B$ gives us an integral morphism $Y \to X$.

We use w,x,y to denote both the variables and their residues in B. As in (4.3.9), B is a normal domain of dimension two, and a free A-module with basis $(1,w)$. It has an automorphism σ of order 2, defined by $\sigma(a+bw) = a-bw$.

The fibres of Y over X are the σ-orbits in Y. If $f(x_0,y_0) \neq 0$, the fibre over the point x_0 of X consists of two points, and if $f(x_0,y_0) = 0$, it consists of one point. The reason that Y is called a *double plane* is that most points of the plane X are covered by two points of Y. The *branch locus* of the covering, which will be denoted by Δ, is the (possibly reducible) curve $\{f=0\}$ in X. The fibres over the *branch points*, the points of Δ, are single points.

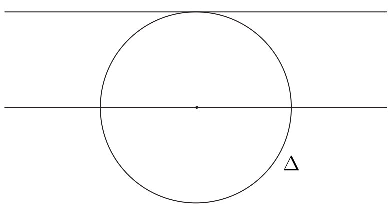
If a closed subvariety D of Y lies over a curve C in X, then $D' = D\sigma$ also lies over C. The curves D and D' may be equal or not. They will have dimension one, and we call them curves too. Let g be the irreducible polynomial whose zero locus in X is C. Krull's Theorem tells us that the components of the zero locus of g in Y have dimension one. If a point q of Y lies over a point p of C, then q and $q\sigma$ are the only points of Y lying over p. One of them will be in D, the other in D'. So the inverse image of C is $D \cup D'$. There are no isolated points in the inverse image, and there is no room for another curve.

If $D = D'$, then D is the only curve lying over C. Otherwise, there will be two curves D and D' that lie over C. In that case, we say that C *splits* in Y.

A curve C in the plane X will be the zero set of a principal prime ideal P of the polynomial algebra A, and if D lies over C, it will be the zero set of a prime ideal Q of B that lies over P (4.4.2) **(i)**.

4.7.2. Example. Let $f(x,y) = x^2+y^2-1$. The double plane $Y : \{w^2 = x^2+y^2-1\}$ is an *affine quadric* in $\mathbb{A}^3$. Its branch locus Δ in the affine plane X is the curve $\{x^2+y^2=1\}$. circleexample

The line $C_1 : \{y=0\}$ in X meets the branch locus Δ transversally at the points $(x,y) = (\pm 1, 0)$, and when we set $y=0$ in the equation for Y, we obtain $w^2 = x^2-1$. The polynomial w^2-x^2+1 is irreducible, so y generates a prime ideal of B. On the other hand, the line $C_2 : \{y=1\}$ is tangent to Δ at the point $(0,1)$, and it splits. When we set $y=1$ in the equation for Y, we obtain $w^2 = x^2$. The locus $\{w^2=x^2\}$ is the union of the two lines $\{w=x\}$ and $\{w=-x\}$ that lie over C_1. The prime ideals of B that correspond to these lines aren't principal ideals.



$\square$

This example illustrates a general fact: If a curve intersects the branch locus transversally it doesn't split. We explain this now.

4.7.3. local analysis localanal

Suppose that a plane curve $C : \{g=0\}$ and the branch locus $\Delta : \{f=0\}$ of a double plane $w^2 = f$ meet at a point p. We adjust coordinates so that p becomes the origin $(0,0)$, and we write

$$f(x,y) = \sum a_{ij}x^iy^j = a_{10}x + a_{01}y + a_{20}x^2 + \cdots$$

Since p is a point of Δ, the constant coefficient of f is zero. If the two linear coefficients aren't both zero, p will be a smooth point of Δ, and the tangent line to Δ at p will be the line $\{a_{10}x + a_{01}y = 0\}$. Similarly, writing $g(x,y) = \sum b_{ij}x^iy^j$, the tangent line to C, if p is a smooth point of C, is the line $\{b_{10}x + b_{01}y = 0\}$.

Let's suppose that the two tangent lines are defined and distinct — that Δ and C intersect transversally at p. We change coordinates once more, to make the tangent lines the coordinate axes. After adjusting by scalar factors, the polynomials f and g will have the form

$$f(x,y) = x + u(x,y) \quad \text{and} \quad g(x,y) = y + v(x,y),$$

where u and v are polynomials all of whose terms have degree at least 2.

Let $X_1 = \operatorname{Spec} \mathbb{C}[x_1, y_1]$ be another affine plane. The map $X_1 \to X$ defined by the substitution $x_1 = x + u(x,y)$, $y_1 = y + v(x,y)$ is invertible analytically near the origin, because the Jacobian matrix

jacob (4.7.4)
$$\left(\frac{\partial(x_1,y_1)}{\partial(x,y)} \right)_{(0,0)}$$

at the origin is the identity matrix. When we make that substitution, Δ becomes the locus $\{x_1 = 0\}$ and C becomes the locus $\{y_1 = 0\}$. In this local analytic coordinate system, the equation $w^2 = f$ that defines the double plane becomes $w^2 = x_1$. When we restrict it to C by setting $y_1 = 0$, x_1 becomes a local coordinate function on C. The restriction of the equation remains $w^2 = x_1$. The inverse image Z of C can't be split analytically. So it doesn't split algebraically either.

splitnot-transversal **4.7.5. Corollary.** *A curve that intersects the branch locus transversally at some point doesn't split.* □

This isn't a complete analysis. When a curve C and the branch locus Δ are tangent at every point of intersection, C may split or not, and which possibility occurs cannot be decided locally in most cases. However, one case in which a local analysis suffices to decide splitting is that C is a line. Let t be a coordinate in a line L, so that $L \approx \operatorname{Spec} \mathbb{C}[t]$. The restriction of the polynomial f to L will give us a polynomial $\overline{f}(t)$ in t. A root of $\overline{f}$ corresponds to an intersection of L with Δ, and a multiple root corresponds to an intersection at which L and Δ are tangent, or at which Δ is singular. The line L will split if and only if the polynomial $w^2 - \overline{f}$ factors, i.e., if and only if $\overline{f}$ is a square in $\mathbb{C}[t]$. This will be true if and only if every root of $\overline{f}$ has even multiplicity — if and only if the intersection multiplicity of L and Δ at every intersection point is even.

projdplane 4.7.6. projective double planes

Let X be the projective plane $\mathbb{P}^2$, with coordinates x_0, x_1, x_2. A projective double plane is a locus of the form

wtdplane (4.7.7)
$$y^2 = f(x_0, x_1, x_2)$$

where f is a square-free, homogeneous polynomial of *even degree* $2d$. To regard (4.7.7) as a homogeneous equation, we must assign weight d to the variable y (see 1.6.9). Then, since we have weighted variables, we must work in a *weighted projective space* $\mathbb{WP}$ with coordinates x_0, x_1, x_2, y, where x_i have weight 1 and y has weight d. A point of this weighted space is represented by a nonzero vector (x_0, x_1, x_2, y), with the equivalence relation that, for all nonzero λ, $(x_0, x_1, x_2, y) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda^d y)$. The points of the projective double plane Y are the points of $\mathbb{WP}$ that solve the equation (4.7.7).

The projection $\mathbb{WP} \to X$ that sends (x_0, x_1, x_2, y) to (x_0, x_1, x_2) is defined at all points except at $(0,0,0,1)$. If (x,y) solves (4.7.7) and if $x = 0$, then $y = 0$ too. So $(0,0,0,1)$ isn't a point of Y. The projection is defined at all points of Y. The fibre of the morphism $Y \to X$ over a point x consists of points (x,y) and $(x,-y)$, which will be equal if and only if x lies on the *branch locus* of the double plane, the (possibly reducible) plane curve $\Delta : \{f = 0\}$ in X. The map $\sigma : (x,y) \rightsquigarrow (x,-y)$ is an automorphism of Y, and points of X correspond bijectively to σ-orbits in Y.

Since the double plane Y is embedded into a weighted projective space, it isn't presented to us as a projective variety in the usual sense. However, it can be embedded into a projective space in the following way: The projective plane X can be embedded by a Veronese embedding of higher order, using as coordinates the monomials $m = (m_1, m_2, \ldots)$ of degree d in the variables x. This embeds X into a projective space $\mathbb{P}^N$ where $N = \binom{d+2}{2} - 1$. When we add a coordinate y of weight d, we obtain an embedding of the weighted projective space $\mathbb{WP}$ into $\mathbb{P}^{N+1}$, that sends the point (x,y) to (m,y). The double plane can be realized as a projective variety by this embedding.

When $Y \to X$ is a projective double plane, then, as with affine double planes, a curve C in X may split in Y or not. If C has a transversal intersection with the branch locus Δ, it will not split. On the other hand, a line, all of whose intersections with the branch locus Δ have even multiplicity, will split.

4.7.8. Corollary. *Let Y be a generic quartic double plane — a double plane whose branch locus Δ is a generic quartic curve. The lines in X that split in Y are the bitangent lines to Δ.* $\square$ *bitansplit*

4.7.9. homogenizing an affine double plane homogdplane

To construct a projective double plane from an affine double plane, we write the affine double plane as

$$w^2 = F(u_1, u_2) \tag{4.7.10}$$ relabelaffdplane

for some nonhomogeneous polynomial F. We suppose that F has even degree $2d$, and we homogenize F, setting $u_i = x_i/x_0$. We multiply both sides of this equation by x_0^{2d} and set $y = x_0^d w$. This produces an equation of the form (4.7.7), where y has weight d and f is the homogenization of F.

If F has odd degree $2d-1$, one needs to multiply F by x_0 in order to make the substitution $y = x_0^d w$ permissible. When one does this, the line at infinity becomes a part of the branch locus.

4.7.11. cubic surfaces and quartic double planes cubicisdplane

Let $\mathbb{P}^3$ be the ordinary projective 3-space with coordinates x_0, x_1, x_2, z of weight one, and let X be be the projective plane $\mathbb{P}^2$ with coordinates x_0, x_1, x_2. We consider the projection $\mathbb{P}^3 \xrightarrow{\pi} X$ that sends (x, z) to x. It is defined at all points except at the center of projection $q = (0, 0, 0, 1)$, and its fibres are the lines through q, with q deleted.

Let S be a cubic surface in $\mathbb{P}^3$, the locus of zeros of an irreducible homogeneous cubic polynomial $g(x, z)$, and supppose that q **is** a point of S. Then the coefficient of z^3 in g will be zero, so g will be quadratic in z: $g(x, z) = az^2 + bz + c$, where a, b, c are homogeneous polynomials in x, of degrees $1, 2, 3$, respectively. The defining equation $g = 0$ for S becomes

$$az^2 + bz + c = 0 \tag{4.7.12}$$ projequation

The discriminant $f(x) = b^2 - 4ac$ of g with respect to z is a homogeneous polynomial of degree 4 in x. Let Y be the projective double plane

$$y^2 = b^2 - 4ac \tag{4.7.13}$$ quarticdplane

in which y has weight 2.

The quadratic formula solves for z in terms of the chosen square root y of the disriminant, wherever $a \neq 0$:

$$z = \frac{-b + y}{2a} \quad \text{or} \quad y = 2az + b \tag{4.7.14}$$ quadrformula

The formula $y = 2az + b$ remains correct when $a = 0$. It defines a map $S \to Y$. The inverse map $Y \to Z$ given by the quadratic formula (4.7.14) is defined wherever $a \neq 0$. So the cubic surface and the quartic double plane are isomorphic except above the line $\{a = 0\}$ in X.

4.7.15. Lemma. *The discriminants of the cubic polynomials $az^2 + bz + c$ include every homogeneous quartic polynomial $f(x)$ whose divisor of zeros $\Delta : \{f = 0\}$ has at least one bitangent line. So the discriminants of those polynomials form a dense subset of the space of quartic polynomials.* *quarticisgeneric*

proof. Let f be a quartic polynomial whose zero locus has a bitangent line ℓ_0. Then ℓ_0 splits in the double plane $y^2 = f$. If ℓ_0 is the zero set of the homogeneous linear polynomial $a(x)$, then f is congruent to a square, modulo a. There is a homogeneous quadratic polynomial $b(x)$ such that $f \equiv b^2$, modulo a. Then $f = b^2 - 4ac$ for some homogeneous cubic polynomial $c(x)$. The polynomial $g(x, z) = az^2 + bz + c$ has discriminant f.

Conversely, let $g(x, z) = az^2 + bz + c$ be given. The intersections of the line $\ell_0 : \{a = 0\}$ with the discriminant divisor $\Delta : \{b^2 - 4ac = 0\}$ are the solutions of the equations $a = 0$ and $b = 0$. Since the quadratic polynomial b appears as a square in the discriminant, the intersections of ℓ_0 and Δ have even multiplicity. So ℓ_0 will be a bitangent, provided that the locus $b = 0$ meets ℓ_0 two distinct points, and this will be true when g is generic. $\square$

From now on, we suppose that S is a generic cubic surface. With a suitable change of coordinates any point of a generic surface can become the point q, so we may suppose that S and q are both generic. Then S contains only finitely many lines (3.7.18), and those lines won't pass through q.

Let ℓ be a line in the plane X, say the locus of zeros of the linear equation $r_0x_0 + r_1x_1 + r_2x_2 = 0$. The same equation defines a plane H in $\mathbb{P}^3_{x,y}$ that contains q, and the intersection $S \cap H$ is a cubic curve C, possibly reducible, in the plane H. The curve C is the inverse image of ℓ in S.

cubicsplits **4.7.16. Lemma.** *Let S be a generic cubic surface and let q be a generic point of S. The lines L in S correspond bijectively to lines ℓ in X whose inverse images C are reducible cubic curves. If C is reducible, it is the union $L \cup Q$ of a line and a conic.*

proof. A line L in S won't contain q. Its image will be a line ℓ in X, and L will be a component of the inverse image of ℓ. Therefore C will be reducible.

Let ℓ be a line in X. At least one irreducible component of its inverse image C will contain q, and that component won't be a line. So if the cubic C is reducible, it will be the union of a conic Q and a line L, and q will be a point of the conic. Then L will be one of the lines in S. $\square$

Let ℓ_0 be the line $\{a = 0\}$. Its inverse image C_0 onsists of the points (x, y) such that $a = 0$ and $y = \pm b$. The points of C_0 are the solutions in $\mathbb{P}^3$ of the equations $a = 0$ and $bz + c = 0$.

ae-qualzero **4.7.17. Lemma.** *The curve C_0 is irreducible.*

proof. We may adjust coordinates so that a becomes the linear polynomial x_0. When we set $x_0 = 0$ in the polynomial $bz + c$, we obtain a polynomial $\overline{b}z + \overline{c}$, where $\overline{b}$ and $\overline{c}$ are generic homogeneous polynomials in x_1, x_2 of degrees 2 and 3, respectively. Such a polynomial is irreducible, and C_0 is the locus $\overline{b}z + \overline{c} = 0$ in the plane $\{x_0 = 0\}$ of $\mathbb{P}^3$. $\square$

twenty-seven **4.7.18. Theorem.** *A generic cubic surface S in $\mathbb{P}^3$ contains precisely 27 lines.*

This theorem follows from next lemma, which relates the 27 lines in the generic cubic surface S to the 28 bitangents of its generic quartic discriminant curve Δ.

linesplits **4.7.19. Lemma.** *Let S be a generic cubic surface $az^2 + bz + c = 0$, and suppose that coordinates are chosen so that $q = (0, 0, 0, 1)$ is a generic point of S. Let $\Delta : \{b^2 - 4ac = 0\}$ be the quartic discriminant curve in $\mathbb{P}^2_x$, and let Y be the double plane $y^2 = b^2 - 4ac$.*

(i) *The image in X of a line L that is contained in S is a bitangent to the curve Δ. Distinct lines in S have distinct images in X.*

(ii) *The line $\ell_0 : \{a = 0\}$ in X is a bitangent, but it isn't the image of a line in S.*

(iii) *Every bitangent except ℓ_0 is the image of a line in S.*

proof. Let L be a line in S, let ℓ be its image in X, and let C be the inverse image of ℓ in S. Lemma 4.7.16 tells us that C is the union of the line L and a conic. So L is the only line in S that has ℓ as its image. The quadratic formula (4.7.14) shows that, because its inverse image in S is reducible, ℓ splits in the double plane Y too, and therefore it is a bitangent. This proves **(i)**. Moreover, Lemma 4.7.17 shows that ℓ cannot be the line ℓ_0. This proves **(ii)**. If a bitangent ℓ is distinct from ℓ_0, the map $Y \to Z$ given by the quadratic formula is defined except at the finite set $\ell \cap \ell_0$. Since ℓ splits in Y, its inverse image C in S will be reducible, and one component of C is a line in S. This proves **(iii)**. $\square$

4.8 Exercises

4.8.1. Let $A \subset B$ be noetherian domains and suppose that B is a finite A-module. Prove that A is a field if and only if B is a field. chapfourex xfinmod-field

4.8.2. Prove this alternate form of the Nullstellensatz: Let k be an infinite field, and let B be a domain that is a finitely generated k-algebra. If B is a field, then $[B : k] < \infty$. xaltnull

4.8.3. Let α be an element of a domain A, and let $\beta = \alpha^{-1}$. Prove that if β is integral over A, then it is an element of A. xinverseintegral

4.8.4. Let $X = \operatorname{Spec} A$, where $A = \mathbb{C}[x, y, z]/(y^2 - xz^2)$. Identify the normalization of X. xnormaliz

4.8.5. Let G be a finite group of automorphisms of a normal, finite-type domain B, let A be the algebra of invariant elements of B, and let $Y \xrightarrow{u} X$ be the integral morphism of varieties corresponding to the inclusion $A \subset B$. Prove that there is a bijective correspondence between G-orbits of closed subvarieties of Y and closed subvarieties of X. xorbits

4.8.6. Let $A \subset B$ be an extension of finite-type domains such that B is a finite A-module, and let P be a prime ideal of A. Prove that the number of prime ideals of B that lie over P is at most equal to the degree $[L : K]$ of the field extension. xatmostdeg

4.8.7. Let $Y \xrightarrow{u} X$ be a surjective morphism of affine varieties, and let K and L be the function fields of X and Y, respectively. Show that if $\dim Y = \dim X$, there is a nonempty open subset X' of X such that all fibres over points of X' have the same order n, and that $n = [L : K]$. xdimdim

4.8.8. Let A be a finite type domain, $R = \mathbb{C}[t]$, $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} R$. Let $\varphi : A \to R$ be a homomorphism whose image is not $\mathbb{C}$, and let $\pi : Y \to X$ be the corresponding morphism. ximclosed

(i) Show that R is a finite A-module.

(ii) Show that the image of π is a closed subset of X.

4.8.9. Prove that the image of an injective morphism from an affine variety Y to $\mathbb{A}^2$ cannot be the complement of a single point. xmisspoint

4.8.10. Let M be a module over a finite-type domain A, and let α be an element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s = \alpha - c$ is an injective map $M \xrightarrow{s} M$. xmultbyfplusc

4.8.11. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism. xptwoptwo

4.8.12. Let X and Y be varieties with the same function field K. Show that there are nonempty open subsets X' of X and and Y' of Y that are isomorphic. xopenisom

4.8.13. Let k be a field of characteristic zero, let $A \subset B$ be finite typed are k-algebras that are domains, and let $Y \to X$ be the corresponding morphism of affine varieties. Suppose that the fraction field L of B is a finite extension of the fraction field K of A. Prove: xAsintclosed

(i) There is a nonzero element s in A such that A_s is integrally closed.

(ii) There is a nonzero element s in A such that B_s is a finite module over a polynomial ring $A_s[y_1, ..., y_d]$.

(iii) There is a nonzero element $s \in A$ such that all fibres of the morphism $Y_s \to X_s$ consist of d points.

4.8.14. **(i)** Prove directly that the prime chain 4.5.9 is maximal. chainmaximalx

(ii) Prove directly that $\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$ is a maximal chain of closed subsets of $\mathbb{P}^n$.

4.8.15. Work out the proof of Chevalley's Theorem in the case that Y is a closed subset of $\mathbb{P}^1 \times X$ that doesn't meet the locus at infinity $\widetilde{H} = H \times X$. (In $\mathbb{P}^1$, H will be the point at infinity.) Do this in the following way: Say that $X = \operatorname{Spec} A$. Let $B_0 = A[u]$, $B_1 = A[v]$, and $B_{01} = A[u, v]$, where $u = y_1/y_0$ and $v = u^{-1} = y_0/y_1$. Then $\widetilde{U}^0 = U^0 \times X = \operatorname{Spec} B_0$, $\widetilde{U}^1 = \operatorname{Spec} B_1$, and $\widetilde{U}^{01} = \operatorname{Spec} B_{01}$. Let P_1 be the ideal of B_1 that defines $Y \cap \widetilde{U}^1$, and let P_0 be the analogous ideal of B_0. In B_1, the ideal of $\widetilde{H}$ is the principal ideal vB_1. Go over to the open set $\widetilde{U}^0$, and show that the residue of u in the coordinate algebra B_0/P_0 of Y is the root of a monic polynomial. xchevthmdimone

4.8.16. Prove that a nonconstant morphism from a curve Y to $\mathbb{P}^1$ is a finite morphism without appealing to Chevalley's Theorem.

xmapcurvefin

xnotprinc **4.8.17.** With reference to Example 4.7.2, show that the prime ideal that corresponds to the line $w = x$ is not a principal ideal.

xfmorph **4.8.18.** Let Y be a closed subvariety of projective space $\mathbb{P}^n$ with coordinates $y = (y_0, ..., y_n)$, let d be a positive integer, and let $w = (w_0, ..., w_k)$ be homogeneous polynomials in y of degree d with no common zeros on Y. Prove that sending a point q of Y to $(w_0(q), ..., w_k(q))$ defines a finite morphism $Y \xrightarrow{u} \mathbb{P}^k$. Consider the case that w_i are linear polynomials first.

xquadrd-plane **4.8.19.** Identify the projective double planes $y^2 = f(x)$ that are defined by quadratic homogeneous polynomials f.

xdblcurve **4.8.20.** A *double line* is a locus $y^2 = f(x_0, x_1)$ analogous to a double plane (4.7.7), where f is a homogeneous polynomial of even degree $2d$ with distinct roots. What is the genus of a double line?

xstrange **4.8.21.** Let $Y \to X$ be an affine double plane. Most curves C in X intersect the branch locus transversally. Therefore they don't split. On the other hand, most curves D in Y will not be symmetric with respect to the automorphism σ of Y over X. Then there will be two curves D and $D\sigma$ with the same image C in X, and C will split. Explain.

xdcc **4.8.22.** A ring A is said to have the *descending chain condition* (dcc) if every strictly decreasing chain of ideals $I_1 > I_2 > \cdots$ is finite. Let A be a finite type $\mathbb{C}$-algebra. Prove

(i) A has dcc if and only if it is a finite dimensional complex vector space.

(ii) If A has dcc, then it has finitely many maximal ideals, and every prime ideal is maximal.

(iii) If a finite-type algebra A has finitely many maximal ideals, then it has dcc.

(iv) Suppose that A has dcc, let M be an arbitrary A-module, and let I denote the intersection of the maximal ideals of A. If $IM = M$, then $M = 0$. (This might be called a 'Stong Nakayama Lemma'. The usual Nakayama Lemma requires that M be finitely generated.)

xclosured-ual **4.8.23.** Let C be a singular plane curve. Prove that the points of the dual curve C^* are the points of the form L^*, where L is a special line at a point of C (see (1.7.6)).

Chapter 5 STRUCTURE OF VARIETIES IN THE ZARISKI TOPOLOGY

In this chapter, we will see how algebraic curves control the geometry of higher dimensional varieties.

5.1 Local Rings

localrings A *local ring* is a noetherian ring that contains just one maximal ideal. We make a few comments about local rings here though we will be interested mainly in some special ones, the discrete valuation rings that are discussed below.

Let R be a local ring with maximal ideal M. An element of R, not in M, isn't in any maximal ideal, so it is a unit. The quotient R/M is a field called the *residue field* of R. For us, the residue field will, most often, be the field of complex numbers.

The Nakayama Lemma 4.1.3 has a useful version for local rings:

local-nakayama **5.1.1. Local Nakayama Lemma.** *Let R be a local ring with maximal ideal M and residue field $k = R/M$, let V be a finite R-module, and let $\overline{V}$ denote the k-module V/MV.*
(i) *If $\overline{V} = 0$, then $V = 0$.*
(ii) *A set of elements $v_1, ..., v_r$ generates V if the set of its residues $\overline{v}_1, ..., \overline{v}_r$ generates $\overline{V}$.*

proof. **(i)** If $\overline{V} = 0$, then $V = MV$. The usual Nakayama Lemma tells us that M contains an element z such that $1-z$ annihilates V. Since z is in M, $1-z$ is not in M, so $1-z$ is a unit. A unit annihilates V, and therefore $V = 0$.

(ii) We use the fact that V/MV is isomorphic to the tensor product $V \otimes_R k$ (9.1.33). Let U be the submodule of V generated by the elements v_i, and let $W = V/U$. Also, let $\overline{U} = U/MU \approx U \otimes_R k$, and $\overline{W} = W/MW \approx W \otimes_R k$. The quotient modules form an exact sequence $\overline{U} \to \overline{V} \to \overline{W} \to 0$ (9.1.32), and the image of $\overline{U}$ in $\overline{V}$ is generated by the elements $\overline{v}_i$. If the set $\{\overline{v}_i\}$ generates $\overline{V}$, then $\overline{W} = 0$. Then by **(i)**, $W = 0$, and therefore $U = V$. $\square$

gen-modmsq **5.1.2. Corollary.** *Let R be a local ring. A set $z_1, ..., z_k$ of elements generates the maximal ideal M if the set of its residues generates M/M^2.* $\square$

A local domain R with maximal ideal M has *dimension one* if it contains exactly two prime ideals, (0) and M. We describe the *normal* local domains of dimension one in this section. They are the discrete valuation rings that are defined below.

dvr **5.1.3. valuations**

Let K be a field. A *discrete valuation* v on K is a surjective homomorphism $K^\times \xrightarrow{\mathrm{v}} \mathbb{Z}^+$ from the multiplicative group $K^\times$ of nonzero elements of K to the additive group $\mathbb{Z}^+$ of integers:

(5.1.4) $$\mathrm{v}(ab) = \mathrm{v}(a) + \mathrm{v}(b)$$ dval

such that, if a, b are elements of K, and if a, b and $a{+}b$ aren't zero, then

(5.1.5) $$\mathrm{v}(a{+}b) \geq \min\{\mathrm{v}(a), \mathrm{v}(b)\}$$ valcond

The word "discrete" refers to the fact that $\mathbb{Z}^+$ is given the discrete topology. There are other valuations. They are interesting, but less important, and we won't use them. To simplify terminology, we refer to a discrete valuation simply as a *valuation*.

Let r be a positive integer. If v is a valuation and if $\mathrm{v}(a) = r$, then r is the *order of zero of* a, and if $\mathrm{v}(a) = -r$, then r is the *order of pole of* a, with respect to the valuation.

The *valuation ring* R associated to a valuation v on a field K is the subset of K of elements with non-negative value, together with zero:

(5.1.6) $$R = \{a \in K^\times \,|\, \mathrm{v}(a) \geq 0\} \;\cup\; \{0\}$$ valnring

Properties (5.1.4) and (5.1.5) show that R is a ring.

Valuation rings are usually called "discrete valuation rings", but we drop the word discrete.

5.1.7. Proposition. *Valuations of the field $\mathbb{C}(t)$ of rational functions in one variable correspond bijectively to points of the projective line. The valuation ring that corresponds to a point p is the ring of rational functions that are regular at p.* *valsinCt*

beginning of the proof. Let a be a complex number. To define the valuation that corresponds to a point $p:\ \{t = a\}$ of $\mathbb{P}^1$, we write a nonzero polynomial f as $(t-a)^k h$, where $t-a$ doesn't divide h and k is an integer, and we define, $\mathrm{v}(f) = k$. Then we define $\mathrm{v}(f/g) = \mathrm{v}(f) - \mathrm{v}(g)$. You will be able to check that, with this definition, v becomes a valuation whose valuation ring is the algebra of functions that are regular at p (see (2.5.1)). This valuation ring is the *local ring* of $\mathbb{P}^1$ at p. Its elements are rational functions in t whose denominators aren't divisible by $t-a$. The valuation that corresponds to the point of $\mathbb{P}^1$ at infinity is obtained by working with t^{-1} in place of t.

The proof that these are all of the valuations of $\mathbb{C}(t)$ will be given at the end of the section.

5.1.8. Proposition. *Let v be a valuation on a field K, and let x be a nonzero element of K with value $\mathrm{v}(x) = 1$.* *idealsin-valring*

(i) *The valuation ring R of v is a normal local domain of dimension one. Its maximal ideal M is the principal ideal xR. The elements of M are the elements of $K^\times$ with positive value, together with zero:*

$$M = \{a \in K^\times \mid \mathrm{v}(a) > 0\} \cup \{0\}$$

(ii) *The units of R are the elements of $K^\times$ with value zero. Every nonzero element z of K has the form $z = x^k u$, where u is a unit and k is an integer, not necessarily positive.*

(iii) *The proper R-submodules of K are the sets $x^k R$, where k is an integer. The set $x^k R$ consists of zero and the elements of $K^\times$ with value $\geq k$. The sets $x^k R$ with $k \geq 0$ are the nonzero ideals of R. They are the powers of the maximal ideal.*

(iv) *There is no ring properly between R and K: If R' is a ring and if $R \subset R' \subset K$, then either $R = R'$ or $R' = K$.*

proof. We prove **(i)** last.

(ii) Since v is a homomorphism, $\mathrm{v}(u^{-1}) = -\,\mathrm{v}(u)$ for any nonzero u in K. Then u and u^{-1} are both in R, i.e., u is a unit of R, if and only if $\mathrm{v}(u)$ is zero. If z is a nonzero element of K with $\mathrm{v}(z) = k$, then $u = x^{-k}z$ has value zero, so u is a unit, and $z = x^k u$.

(iii) It follows from **(ii)** that x^kR consists of 0 and the elements of $K^\times$ of value at least k. Suppose that a nonzero R-submodule J of K contains an element z with value k. Then $z = ux^k$ and u is a unit, so J contains x^k, and therefore it contains x^kR. If $x^kR < J$, then J contains an element with value $< k$. So if k is the smallest integer such J contains an element with value k, then $J = x^kR$. If there is no minimum value among the elements of J, then J contains x^kR for every k, and $J = K$.

(iv) This follows from **(iii)**. The ring R' will be a nonzero R-submodule of K. If $R' < K$, then $R' = x^kR$ for some k, and if R' contains R, then $k \le 0$. But x^kR isn't closed under multiplication when $k < 0$. So the only possibility is that $k = 0$ and $R = R'$.

(i) Part **(iii)** shows that R is a principal ideal domain, so it is noetherian. Its maximal ideal is $M = xR$. It also follows from **(iii)** that M and $\{0\}$ are the only prime ideals of R. So R is a local ring of dimension 1. If the normalization of R were larger than R, then according to **(iv)**, it would be equal to K, and x^{-1} would be integral over R. There would be a polynomial relation $x^{-r} + a_1x^{-(r-1)} + \cdots + a_r = 0$ with a_i in R. When one multiplies this relation by x^r, one sees that 1 would be a multiple of x. Then x would be a unit. It isn't a unit, because its value is 1. □

character-izedvr **5.1.9. Theorem.**
(i) *A local domain whose maximal ideal is a nonzero principal ideal is a valuation ring.*
(ii) *A normal local domain of dimension* 1 *is a valuation ring.*

proof. **(i)** Let R be a local domain whose maximal ideal M is a nonzero principal ideal, say $M = xR$, with $x \neq 0$, and let y be a nonzero element of R. The integers k such that x^k divides y are bounded (4.1.5). Let x^k be the largest power that divides y. Then $y = ux^k$, where u is in R but not in M. So u is a unit. Every nonzero element z of the fraction field K of R will have the form $z = ux^r$ where u is a unit and r is an integer, possibly negative. This is shown by writing the numerator and denominator of a fraction in such a form.

The valuation whose valuation ring is R is defined by $\mathrm{v}(z) = r$ when $z = ux^r$ with u a unit, as above. Suppose that $z_i = u_ix^{r_i}$ for $i = 1, 2$, where u_i are units and $0 \le r_1 \le r_2$. Then $z_1 + z_2 = \alpha x^{r_1}$ and $\alpha = u_1 + u_2x^{r_2 - r_1}$ is an element of R. Therefore $\mathrm{v}(z_1 + z_2) \ge r_1 = \min\{\mathrm{v}(z_1), \mathrm{v}(z_2)\}$. We also have $\mathrm{v}(z_1z_2) = \mathrm{v}(z_1) + \mathrm{v}(z_2)$. Thus v is a surjective homomorphism. The requirements for a valuation are satisfied.

(ii) The fact that a valuation ring is a normal, one-dimensional local ring is Proposition 5.1.8 **(i)**. We show that a normal local domain R of dimension 1 is a valuation ring by showing that its maximal ideal M is a principal ideal. The proof is rather tricky.

Let z be a nonzero element of M. Because R is a local ring of dimension 1, M is the only prime ideal that contains z, so M is the radical of the principal ideal zR, and $M^r \subset zR$ if r is large (Proposition 2.4.10). Let r be the smallest integer such that $M^r \subset zR$. Then there is an element y in M^{r-1} and not in zR, such that $yM \subset zR$. We restate this by saying that $w = y/z$ isn't in R, but $wM \subset R$. Since M is an ideal, multiplication by an element of R carries wM to wM. So wM is an ideal. Since M is the maximal ideal of the local ring R, either $wM \subset M$, or $wM = R$. If $wM \subset M$, Lemma 4.2.5 shows that w is integral over R. This can't happen because R is normal and w isn't in R. Therefore $wM = R$ and $M = w^{-1}R$. This implies that w^{-1} is in R and that M is a principal ideal. □

local-ringatp **5.1.10. the local ring at a point**

Let $\mathfrak{m}$ be the maximal ideal at a point p of an affine variety $X = \operatorname{Spec} A$. The complement S of $\mathfrak{m}$ in A is a multiplicative system (2.6.7). The prime ideals P of the ring of fractions AS^{-1} are the extensions of the prime ideals Q of A that are contained in $\mathfrak{m}$: $P = QS^{-1}$ (2.6.9). Since $\mathfrak{m}$ is a maximal ideal of A, $\mathfrak{m}S^{-1}$ is the unique maximal ideal of AS^{-1}, and AS^{-1} is a local ring. This ring is called the *local ring of A at p* . It is often denoted by A_p. Lemma 4.3.4 shows that, if A is a normal domain, then A_p is normal.

For example, let $X = \operatorname{Spec} A$ be the affine line, $A = \mathbb{C}[t]$, and let p be the point $t = 0$. The local ring A_p is the ring whose elements are fractions of polynomials $f(t)/g(t)$ with $g(0) \neq 0$.

The local ring at a point p of any variety, not necessarily affine, is the the local ring at p of an affine open neighborhood of p.

intloc **5.1.11. Corollary.** *Let $X = \operatorname{Spec} A$ be an affine variety.*

(i) *The coordinate algebra A is the intersection of the local rings A_p at the points of X.*

$$A = \bigcap_{p \in X} A_p$$ *intersect-codi-monea-gain*

(ii) *The coordinate algebra A is normal if and only if all of its local rings A_p are normal.* □

5.1.12. A note about the overused word local local

A property is true *locally* on a topological space X if every point p of X has an open neighborhood U such that the property is true on U.

In these notes, the words *localize* and *localization* refer to the process of adjoining inverses. The localizations of an affine variety $X = \operatorname{Spec} A$ form a basis for the topology on X. If a property is true locally on X, one can cover X by localizations on which the property is true. There will be elements $s_1, ..., s_k$ of A that generate the unit ideal, such that the property is true on each of the localizations X_{s_i}.

Let A be a noetherian domain. An A-module M is *locally free* if there are elements $s_1, ..., s_k$ that generate the unit ideal of A, such that M_{s_i} is a free A_{s_i}-module for each i. The free modules M_{s_i} will have equal rank (9.1.31). That rank is the *rank* of the locally free A-module M.

An ideal I of a domain A is *locally principal* if A contains elements s_i that generate the unit ideal, such that I_{s_i} is a principal ideal of A_{s_i} for every i. A locally principal ideal is a locally free module of rank one. □

5.1.13. Proposition. *Let M be a finite module over a finite-type domain A, and let p be a point of* $\operatorname{Spec} A$. *If the localized module M_p (2.6.11) is a free A_p-module, then there is an element s, not in $\mathfrak{m}_p$, such that M_s is a free A_s-module.* *generateMinloc*

This is an example of the general principle (2.6.13). □

5.1.14. Note. The notations A_s and A_p are traditional, but rather inconsistent. In the localization A_s, the element s is the one that is inverted, while in the local ring A_p, the elements of the maximal ideal $\mathfrak{m}_p$ are the ones that are *not* inverted. □ inconsist

completion of the proof of Proposition 5.1.7. We show that every valuation v of the function field $\mathbb{C}(t)$ of $\mathbb{P}^1$ corresponds to a point of $\mathbb{P}^1$.

Let R be the valuation ring of v. If $\mathrm{v}(t) < 0$, we replace t by t^{-1}, so that $\mathrm{v}(t) \geq 0$. Then t is an element of R, and therefore $\mathbb{C}[t] \subset R$. The maximal ideal M of R isn't zero. It contains a nonzero fraction g/h of polynomials in t. The denominator h is in R, so M also contains the numerator g. Since M is a prime ideal, it contains a monic irreducible factor of g, of the form $t - a$ for some complex number a. The local ring R_0 of $\mathbb{C}[t]$ at the point $t = a$ is the valuation ring obtained by inverting $t - c$ for all $c \neq a$. When $c \neq a$, the scalar $c-a$ isn't in M, so $t-c$ won't be in M. Since R is a local ring, $t-c$ will be a unit of R for all $c \neq a$. So R_0 is contained in R (5.1.7). There is no ring properly containing R_0 except K (5.1.8), so $R_0 = R$. □

5.2 Smooth Curves

A *curve* is a variety of dimension 1. Its proper closed subsets are finite sets. If $X = \operatorname{Spec} A$ is an affine curve, the prime ideals of A different from the zero ideal are maximal ideals. smaffcurve

Smooth points of a curve are defined in terms of the Jacobian matrix:

5.2.1. Definition. Let $X = \operatorname{Spec} A$, be an affine curve, embedded as a closed subvariety of $\mathbb{A}^n$, so that A is a quotient $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$ of the polynomial ring $\mathbb{C}[x]$, and let p be a point of X. The curve X is *smooth* at p if the Jacobian matrix evaluated at p, $J_p = \left(\frac{\partial f_i}{\partial x_j}\right)_p$, has rank $n-1$. Otherwise, p is a *singular point* of X. defsmooth

A curve X, affine or not, is smooth at a point p if p is a smooth point of an affine open neighborhood. It isn't hard to show that The smoothness of a curve X at a point p is independent of an affine open neighborhood.

Let's suppose that p is the origin. The partial derivative $\left(\frac{\partial f_i}{\partial x_j}\right)_p$ of f_i at p is the coefficient of the monomial x_j in f_i. Writing x an f as the column vectors $(x_1, ..., x_n)^t$ and $(f_1, ..., f_k)^t$,

$$f = J_p x + O(2)$$

where, as usual, $O(2)$ denotes a polynomial all of whose terms have degree at least two.

Jrankn **5.2.2. Corollary.** *Let $f_1, ..., f_k$ be polynomials in $\mathbb{C}[x_1, ..., x_n]$, and let R be the local ring of the polynomial ring at the origin p. The Jacobian matrix J_p at the origin p has rank n if and only if $f_1, ..., f_k$ generate the maximal ideal M of R.*

proof. According to the Local Nakayama Lemma 5.1.1**(ii)**, it is enough to show that the residues of f generates M/M^2, and this is true because $f \equiv J_p x$ modulo M^2. $\square$

smoothif-flp **5.2.3. Proposition.** **(i)** *A curve $X = \operatorname{Spec} A$ is smooth at a point p if and only if the maximal ideal $\mathfrak{m}_p$ of A at p is locally principal.*

(ii) *If X is smooth at p, then X is locally analytically isomorphic to the affine line.*

proof. **(i)** Let $\mathfrak{m} = \mathfrak{m}_p$. If $\mathfrak{m}/\mathfrak{m}^2$ has dimension one, the Local Nakayama Lemma tells us that, in the local ring at p, the maximal ideal is a principal ideal, and therefore that $\mathfrak{m}$ is locally principal (5.1.13).

Say that $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$ and that p is the origin in $\mathbb{A}^n$, as above. Let $M = (x_1, ..., x_n)$ be the maximal ideal of the polynomial ring. So the residues of $\{x_i\}$ form a basis of M/M^2, and a basis of $\mathfrak{m}/\mathfrak{m}^2$ is obtained by setting the residues of f to zero. Since $f = J_p x + O(2)$, we may equally well set the residues of $J_p x$ to zero. Then $\mathfrak{m}/\mathfrak{m}^2$ has dimension one if and only if J_p has rank $n-1$, and if so, then $\mathfrak{m}$ is locally principal, and X is smooth at p.

(ii) This follows from the Implicit Function Theorem.

$\square$

smval **5.2.4. Corollary.** **(i)** *A curve X is smooth at a point p if and only if the local ring at p is a valuation ring, and this is true if and only if the local ring is normal.*

(ii) *An affine curve X is smooth if and only if its coordinate algebra is a normal domain.*

(iii) *A curve has finitely many singular points.*

(iv) *The normalization $X^\#$ of a curve X is a smooth curve, and the finite morphism $X^\# \to X$ becomes an isomorphism when singular points of X and their inverse images are deleted.*

proof. **(i)** This follows from Theorem 5.1.9. $\square$

5.2.5. Note. The definition of smoothness extends in a natural way to varieties of arbitrary dimension. If
smoothdimd $X = \operatorname{Spec} A$ is a variety of dimension d and $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$, a point p of X is a *smooth point* if the Jacobian matrix at p has rank $n - d$. However, to apply this definition, one needs to know that dimension of X is equal to d, not lower, and the dimension may not be apparent (see Exercise 5.7.8).

The characterizations of smoothness of Corollary 5.2.4 have no analogs in higher dimensions. $\square$

Let v_p be the valuation associated to a smooth point p of a curve X and let r be a positive integer. A rational function α on X has a zero of order $r > 0$ at p if $\mathrm{v}_p(\alpha) = r$, and it has a pole of order r at p if $\mathrm{v}_p(\alpha) = -r$.

5.2.6. Proposition. *Let X be a smooth curve with function field K. Every point of $\mathbb{P}^n$ with values in K*
pointsofcurve *defines a morphism $X \to \mathbb{P}^n$.*

proof. A point $(\alpha_0, ..., \alpha_n)$ of $\mathbb{P}^n$ with values in K determines a morphism $X \to \mathbb{P}^n$ if it is a good point, which means that, for every (ordinary) point p of X, there is an index j such that the functions α_i/α_j are regular at p for $i = 0, ..., n$ (3.5.6). This will be true when j is chosen so that the order of zero of α_j at p is the minimal integer among the orders of zero of the elements α_i that are nonzero. $\square$

The next example shows that this proposition doesn't extend to varieties X of dimension greater than one.

nomaptop **5.2.7. Example.** Let X' be the complement of the origin in the affine plane $X = \operatorname{Spec} \mathbb{C}[x, y]$, and let $K = \mathbb{C}(x, y)$ be the function field of X. The vector (x, y) defines a good point of X' with values in K, and therefore a morphism $X' \to \mathbb{P}^1$. If (x, y) were a good point of X then, according to Proposition 3.5.4, at least one of the two rational functions x/y or y/x would be regular at the origin $q = (0, 0)$. This isn't the case, so (x, y) isn't a good point of X. The morphism $X' \to \mathbb{P}^1$ doesn't extend to X. $\square$

ptsvals **5.2.8. Proposition.** *Let $X = \operatorname{Spec} A$ be a smooth affine curve with function field K.*

(i) *The local rings of X are the valuation rings of K that contain A.*

(ii) *The maximal ideals of A are locally principal.*

In fact, it follows from Proposition 5.2.11 below that every ideal of A is locally principal.

proof of Proposition 5.2.8. Since A is a normal domain of dimension one, its local rings are valuation rings that contain A (see Theorem 5.1.9 and Corollary 5.1.11). Let R be a valuation ring of K that contains A, let v be the associated valuation, and let M be the maximal ideal of R. The intersection $M \cap A$ is a prime ideal of A. Since A has dimension 1, the zero ideal is the only prime ideal of A that isn't a maximal ideal. We can multiply by an element of R to clear the denominator of an element of M, obtaining an element of A, while staying in M. So $M \cap A$ isn't the zero ideal. It is the maximal ideal $\mathfrak{m}_p$ of A at a point p of X. The elements of A that aren't in $\mathfrak{m}_p$ aren't in M either. They are invertible in R. So the local ring A_p at p, which is a valuation ring, is contained in R, and is therefore equal to R (5.1.8) **(iii)**. Since M is a principal ideal, the maximal ideal of A_p is principal, and $\mathfrak{m}_p$ is locally principal. □

5.2.9. Proposition. *Let X' and X be smooth curves with the same function field K.*

(i) *A morphism $X' \xrightarrow{f} X$ compatible with the identity map on the function field K maps X' isomorphically to an open subvariety of X.* pointsvalns

(ii) *If X is projective, X' is isomorphic to an open subvariety of X.*

(iii) *If X' and X are both projective, they are isomorphic.*

(iv) *If X is projective, every valuation ring of K is the local ring at a point of X.*

proof. **(i)** Let p be the image in X of a point p' of X', let U be an affine open neighborhood of p in X, and let V be an affine open neighborhood of p' in X' that is contained in the inverse image of U. Say $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$. The morphism f gives us an injective homomorphism $A \to B$, and since p' maps to p, this homomorphism extends to an inclusion of local rings $A_p \subset B_{p'}$. These local rings are valuation rings with the same field of fractions, so they are equal. Since B is a finite-type algebra, there is an element s in A, with $s(p') \neq 0$, such that $A_s = B_s$. Then the open subsets $\operatorname{Spec} A_s$ of X and $\operatorname{Spec} B_s$ of X' are equal. Since p' is arbitrary, X' is a union of open subvarieties of X. So X' is an open subvariety of X.

(ii) The projective embedding $X \subset \mathbb{P}^n$ is defined by a point $(\alpha_0, ..., \alpha_n)$ with values in K. That point also defines a morphism $X' \to \mathbb{P}^n$. If $f(x_0, ..., x_n) = 0$ is a set of defining equations of X in $\mathbb{P}^n$, then $f(\alpha) = 0$ in K. Therefore f vanishes on X' too. So the image of X' is contained in the zero locus of f, which is X. Then **(i)** shows that X' is an open subvariety of X.

(iii) This follows from **(ii)**.

(iv) The local rings of X are normal and they have dimension one. They are valuation rings of K. Let R be any valuation ring of K, let v be the corresponding valuation, and let $\beta = (\beta_0, ..., \beta_n)$ be the point with values in K that defines the projective embedding of X. When we order the coordinates so that $\mathrm{v}(\beta_0)$ is minimal, the ratios $\gamma_j = \beta_j/\beta_0$ will be in R. The coordinate algebra A_0 of the affine variety $X^0 = X \cap \mathbb{U}^0$ is generated by the coordinate functions $\gamma_1, ..., \gamma_n$, so $A_0 \subset R$. Proposition 5.2.8**(i)** tells us that R is the local ring of X^0 at some point. □

5.2.10. Proposition. *Let p be a smooth point of an affine curve $X = \operatorname{Spec} A$, and let $\mathfrak{m}$ and v be the maximal ideal and valuation, respectively, at p. The valuation ring of* v *is the local ring of A at p.* truncate-curve

(i) *The power $\mathfrak{m}^k$ consists of the elements of A whose values are at least k. If I is an ideal of A whose radical is $\mathfrak{m}$, then $I = \mathfrak{m}^k$ for some $k > 0$.*

(ii) *For every $n \geq 0$, the algebras $A/\mathfrak{m}^n$ and R/M^n are isomorphic to the truncated polynomial ring $\mathbb{C}[t]/(t^n)$.*

proof. **(i)** Proposition 5.1.8 tells us that the nonzero ideals of the valuation ring R are powers of its maximal ideal M, and that M^k is the set of elements of R with value $\geq k$. Let I be an ideal of A whose radical is $\mathfrak{m}$, and let k be the minimal value $\mathrm{v}(x)$ of the nonzero elements x of I. We will show that I is the set of all elements of A with value $\geq k$, i.e., that $I = M^k \cap A$. Since we can apply the same reasoning to $\mathfrak{m}^k$, it will follow that $I = \mathfrak{m}^k$.

We must show that if an element y of A has value $\mathrm{v}(y) \geq k$, then it is in I. We choose an element x of I with value k. Then x divides y in R, say $y/x = w$, with w in R. The element w will be a fraction a/s with s and a in A and s not in $\mathfrak{m}$: $sy = ax$. The element s will vanish at a finite set of points $q_1, ..., q_r$, but not at p. We choose an element z of A that vanishes at p but not at any of the points $q_1, ..., q_r$. Then z is in $\mathfrak{m}$, and since the radical of I is $\mathfrak{m}$, some power of z is in I. We replace z by such a power, so that z is in I. By our choice, z and s have no common zeros in X. They generate the unit ideal of A, say $1 = cs + dz$ with c and d in A. Then $y = csy + dzy = cax + dzy$. Since x and z are in I, so is y.

(ii) Since p is a smooth point, the local ring of A at p is the valuation ring R. Let s be an element of A that isn't in $\mathfrak{m}$. Then $A/\mathfrak{m}^k$ will be isomorphic to $A_s/\mathfrak{m}_s^k$. So we may localize A. Doing so suitably, we may suppose that $\mathfrak{m}$ is a principal ideal, say tA. Then $\mathfrak{m}^k = t^kA$. Denoting the maximal ideal of R by M, we also have $M^k = t^kM$. Let B be the subring $\mathbb{C}[t]$ of A, and let $\overline{B}_k = B/t^kB$, $\overline{A}_k = A/\mathfrak{m}^k$, and $\overline{R}_k = R/M^k$. The quotients $t^{k-1}B/t^kB$, $\mathfrak{m}^{k-1}/\mathfrak{m}^k$, and M^{k-1}/M^k are one-dimensional vector spaces. So the map labelled g_{k-1} in the diagram below is bijective.

$$\begin{tikzcd}
0 \arrow[r] & t^{k-1}B/t^kB \arrow[r] \arrow[d, "g_{k-1}"'] & \overline{B}_k \arrow[r] \arrow[d, "f_k"'] & \overline{B}_{k-1} \arrow[r] \arrow[d, "f_{k-1}"'] & 0 \\
0 \arrow[r] & \mathfrak{m}^{k-1}/\mathfrak{m}^k \arrow[r] & \overline{A}_k \arrow[r] & \overline{A}_{k-1} \arrow[r] & 0
\end{tikzcd}$$

By induction on k, we may assume that the map f_{k-1} is bijective, and then f_k is bijective too. So $\overline{B}_k$ and $\overline{A}_k$ are isomorphic. Analogous reasoning shows that $\overline{B}_k$ and $\overline{R}_k$ are isomorphic.

idealsin-curve **5.2.11. Proposition.** *Let $X = \operatorname{Spec} A$ be a smooth affine curve. Every nonzero ideal I of A is a product of powers of maximal ideals: $I = \mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.*

proof. Let I be a nonzero ideal of A. Because X has dimension one, the locus of zeros of I is a finite set $\{p_1, ..., p_k\}$, and $\operatorname{rad} I$ is the intersection $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$ of the maximal ideals $\mathfrak{m}_j$ at p_j. By the Chinese Remainder Theorem, this intersection is the product $\mathfrak{m}_1 \cdots \mathfrak{m}_k$. Moreover, I contains a power of that product, say $I \supset \mathfrak{m}_1^N \cdots \mathfrak{m}_k^N$. Let $J = \mathfrak{m}_1^N \cdots \mathfrak{m}_k^N$. The quotient algebra A/J is the product $B_1 \times \cdots \times B_k$, with $B_j = A/\mathfrak{m}_j^N$, and A/I is a quotient of A/J. Proposition 9.1.8 tells us that A/I is a product $\overline{A}_1 \times \cdots \times \overline{A}_k$, where $\overline{A}_j$ is a quotient of B_j. By Proposition 5.2.10**(ii)**, each B_j is a truncated polynomial ring, so the quotient $\overline{A}_j$ is also a truncated polynomial ring. The kernel of the map $A \to A_j$ is a power of $\mathfrak{m}_j$. The kernel I of the map $A \to \overline{A}_1 \times \cdots \times \overline{A}_k$ is a product of powers of the maximal ideals $\mathfrak{m}_j$. □

jacobtwo **5.2.12. isolated points, again**

Recall that a point q of a topological space Y is an isolated point if the one-point set $\{q\}$ is both open and closed in Y.

noisolat-edpt **5.2.13. Proposition.** *In the classical topology, a curve contains no isolated point.*

This was proved for plane curves in Chapter 1 (Proposition 1.3.19).

isolptofy **5.2.14. Lemma.**

(i) *Let Y' be an open subvariety of a variety Y. A point q of Y' is an isolated point of Y if and only if it is an isolated point of Y'.*

(ii) *Let $Y' \xrightarrow{u'} Y$ be a nonconstant morphism of curves, let q' be a point of Y', and let q be its image in Y. If q is an isolated point of Y, then q' is an isolated point of Y'.*

proof. **(ii)** Because Y' has dimension one, the fibre over q is a finite set. Say that the fibre is $\{q'\} \cup S$, where S is a finite set of points distinct form q. Let Y'' denote the open complement $Y' - S$ of S in Y', and let u'' be the restriction of u' to Y''. The fibre of Y'' over q is the point q'. If $\{q\}$ is open in Y, then because u'' is continuous, $\{q'\}$ will be open in Y''. By **(i)**, $\{q'\}$ is open in Y'. □

proof of Proposition 5.2.13. Let q be a point of a curve Y. Part **(i)** of Lemma 5.2.14 allows us to replace Y by an affine neighborhood of q. Let $Y^\#$ be the normalization of Y. Part **(ii)** of the lemma allows us to replace Y by $Y^\#$. So we may assume that Y is a smooth affine curve. Proposition 5.2.3**(ii)** shows that p is not an isolated point. □

5.3 Constructible Sets

construct

In this section, X will denote a noetherian topological space. Every closed subset of X is a finite union of irreducible closed sets (2.1.16).

The intersection $L = Z \cap U$ of a closed set Z and an open set U is a *locally closed set*. Open sets and closed sets are locally closed.

The following conditions on a subset L of X are equivalent.

- L is locally closed.
- L is a closed subset of an open subset U of X.
- L is an open subset of a closed subset Z of X.

A *constructible set* is a subset that is the union of finitely many locally closed sets.

5.3.1. Examples. constrin-curve

(i) A subset S of a curve X is constructible if and only if it is either a finite set or the complement of a finite set — if and only if it is either closed or open.

(ii) In the affine plane $X = \operatorname{Spec} \mathbb{C}[x, y]$, let U be the complement of the line $\{y = 0\}$, and let p be the origin. The union $U \cup \{p\}$ is constructible, but not locally closed. □

In what follows, Z will denote a closed set, U will denote an open set. and L will denote a locally closed set, such as $Z \cap U$.

5.3.2. Theorem. *The set $\mathbb{S}$ of constructible subsets of a noetherian topological space X is the smallest family of subsets that contains the open sets and is closed under the three operations of finite union, finite intersection, and complementation.* *defloc-closed*

By closure under complementation, we mean that if S is in $\mathbb{S}$, then its complement $S^c = X - S$ is in $\mathbb{S}$.

proof. Let $\mathbb{S}_1$ denote the family of subsets obtained from the open sets by the three operations mentioned in the statement. Open sets are constructible, and using those operations, one can make any constructible set from the open sets. So $\mathbb{S} \subset \mathbb{S}_1$. To show that $\mathbb{S} = \mathbb{S}_1$, we show that the family of constructible sets is closed under those operations.

It is obvious that a finite union of constructible sets is constructible. The intersection of two locally closed sets $L_1 = Z_1 \cap U_1$ and $L_2 = Z_2 \cap U_2$ is locally closed because $L_1 \cap L_2 = (Z_1 \cap Z_2) \cap (U_1 \cap U_2)$. If $S = L_1 \cup \cdots \cup L_k$ and $S' = L'_1 \cup \cdots \cup L'_r$ are constructible sets, the intersection $S \cap S'$ is the union of the locally closed intersections $L_i \cap L'_j$, so it is constructible.

Let S be the constructible set $L_1 \cup \cdots \cup L_k$. Its complement S^c is the intersection of the complements L_i^c of L_i: $S^c = L_1^c \cap \cdots \cap L_k^c$. We have shown that intersections of constructible sets are constructible. So to show that S^c is constructible, it suffices to show that the complement of a locally closed set is constructible. Let L be the locally closed set $Z \cap U$, and let Z^c and U^c be the complements of Z and U, respectively. Then Z^c is open and U^c is closed. The complement L^c of L is the union $Z^c \cup U^c$ of constructible sets, so it is constructible. □

5.3.3. Proposition. *In a noetherian topological space X, every constructible set is a finite union of locally closed sets $L_i = Z_i \cap U_i$, in which the closed sets Z_i are irreducible and distinct.* *contain-sopen*

proof. Let $L = Z \cap U$ be a locally closed set, and let $Z = Z_1 \cup \cdots \cup Z_r$ be the decomposition of Z into irreducible components. Then $L = (Z_1 \cap U) \cup \cdots \cup (Z_r \cap U)$, which is constructible. So every constructible set S is a union of locally closed sets $L_i = Z_i \cap U_i$ in which Z_i are irreducible. Next, suppose that two of the irreducible closed sets are equal, say $Z_1 = Z_2$. Then $L_1 \cup L_2 = (Z_1 \cap U_1) \cup (Z_1 \cap U_2) = Z_1 \cap (U_1 \cup U_2)$ is locally closed. So we may assume that the irreducible closed sets are distinct. □

5.3.4. Lemma. *sinx*

(i) *Let X_1 be a closed subset of a variety X, and let X_2 be its open complement. A subset S of X is constructible if and only if $S \cap X_1$ and $S \cap X_2$ are constructible.*

(ii) *Let X' be an open or a closed subvariety of a variety X.*

a) *If S is a constructible subset of X, then $S \cap X'$ is a constructible subset of X'.*

b) *A subset S' of X' is a constructible subset of X' if and only if it is a constructible subset of X.*

proof. **(iia)** It suffices to prove that, if L is a locally closed subset of X, the intersection $L' = L \cap X'$ is a locally closed subset of X'. If $L = Z \cap U$, then $Z' = Z \cap X'$ is closed in X', and $U' = U \cap X'$ is open in X'. So L', which is equal to $Z' \cap U'$, is locally closed.

(iib) It follows from **a)** that if a subset S' of X' is constructible in X, then it is constructible in X'. To show that a constructible subset of X' is contructible in X, it suffices to show that a locally closed subset $L' = Z' \cap U'$ of X' is locally closed in X. If X' is a closed subset of X, then Z' is a closed subset of X, and $U' = X \cap U$ for some open subset U of X. Since $Z' \subset X'$, $L' = Z' \cap U' = Z' \cap X' \cap U = Z' \cap U$. So L' is locally closed in X. If X' is open in X, then U' is open in X. In that case, let Z be the closure of Z' in X. Then $L' = Z \cap U' = Z \cap X' \cap U' = Z' \cap U'$ (2.1.14). Again, L' is locally closed in X. □

The next theorem illustrates a general fact, that many sets that arise in algebraic geometry are constructible.

imagecon-str

5.3.5. Theorem. *Let $Y \xrightarrow{f} X$ be a morphism of varieties. The inverse image of a constructible subset of X is a constructible subset of Y. The image of a constructible subset of Y is a constructible subset of X.*

proof. The fact that a morphism is continuous implies that the inverse image of a constructible set is constructible. It is less obvious that the image of a constructible set is constructible. To prove that, we keep pecking away, using Lemma 5.3.4, until there is nothing left to do. There may be a shorter proof.

Let S be a constructible subset of Y. Lemma 5.3.4 and Noetherian induction allow us to assume that the theorem is true when S is contained in a proper closed subset of Y, and also when its image $f(S)$ is contained in a proper closed subvariety of X.

Suppose that Y is the union of a proper closed subset Y_1 and its open complement Y_2. The sets $S_i = S \cap Y_i$ are constructible subsets of Y_i. It suffices to show that their images $f(S_i)$ are constructible, and Noetherian induction applies to Y_1. So we may replace Y by the arbitrary nonempty open subvariety $Y' = Y_2$ and S by $S' = S \cap Y'$.

Next, suppose that X is the union of a proper closed subset X_1 and its open complement X_2. Let Y_i, $i = 1, 2$, denote the inverse images of X_i, and let $S_i = S \cap Y_i$. It suffices to show that the images $f(S_i)$ are constructible. Here $f(S_i)$ is contained in X_i, and induction applies to X_1. So we may replace X by the arbitrary open subvariety $X' = X_2$, replacing S accordingly.

Summing up, we may replace X by an arbitrary nonempty open subset X', Y by a nonempty open subset Y' of $f^{-1}(X')$, and S by $S' = S \cap Y'$. We can do this finitely often.

Since a constructible set S is a finite union of locally closed sets, it suffices to show that the image of a locally closed subset S of Y is constructible. So we may suppose that $S = Z \cap U$, where U is open and Z is closed. We replace Y by U and S by $Z \cap U$. Then S becomes a closed subset. If S is a proper closed subset, induction applies. So we may assume that $S = Y$.

We may replace X and Y by nonempty open subsets again, so we may assume that they are affine, say $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, so that the morphism $Y \to X$ corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. If the kernel P of φ were nonzero, the image of Y would be contained in the proper closed subset $V(P)$ in X, to which induction would apply. So we may assume that φ is injective.

Corollary 4.2.11 tells us that, for suitable nonzero element s in A, the localization B_s will be a finite module over a polynomial subring $A_s[y_1, ..., y_k]$. We may replace Y and X by the open subsets $Y_s = \operatorname{Spec} B_s$ and $X_s = \operatorname{Spec} A_s$. Then the maps $Y \to \operatorname{Spec} A[y]$ and $\operatorname{Spec} A[y] \to X$ are both surjective, so $Y = S$ maps surjectively to X. □

5.4 Closed Sets

us-ingcurves

Limits of sequences are often used to analyze subsets of a topological space. In the classical topology, a subset Y of $\mathbb{C}^n$ is closed if, whenever a sequence of points in Y has a limit in $\mathbb{C}^n$, the limit is in Y. In algebraic geometry, curves can be used as substitutes for sequences.

We use this notation:

Cwith-point

5.4.1. *C is a smooth affine curve, q is a point of C, and C' is the complement of q in C.*

The closure of C' will be C, and we think of q as a limit point. In fact, the closure will be C in the classical topology as well as in the Zariski topology, because C has no isolated point. Theorem 5.4.3, which is below, characterizes constructible subset of a variety in terms of such limit points.

The next theorem tells us that there are enough curves to do the job.

5.4.2. Theorem. *Let Y be a constructible subset of a variety X, and let p be a point of its closure $\overline{Y}$. There exists a morphism $C \xrightarrow{f} X$ from a smooth affine curve to X and a point q of C, such that $f(q) = p$. and that the image of $C' = C - \{q\}$ is contained in Y.* enoughcurves

proof. If $X = p$, then $Y = p$. In that case, we may take for f the constant morphism from any curve C to p. So we may assume that X has dimension at least 1. Next, we may replace X by an affine open subset X' that contains p, and Y by $Y' = Y \cap X'$. If $\overline{Y}$ denotes the closure of Y in X, the closure of Y' in X' will be $\overline{Y} \cap X'$, and it will contain p. So we may assume that X is affine, $X = \operatorname{Spec} A$.

Since Y is constructible, it is a union $L_1 \cup \cdots \cup L_k$ of locally closed sets, say $L_i = Z_i \cap U_i$ where Z_i are irreducible closed sets and U_i are open. The closure of Y is the union $Z_1 \cup \cdots \cup Z_k$, and p will be in at least one of the closed sets, say $p \in Z_1$. We replace X by Z_1 and Y by U_1. This reduces us to the case that Y is a nonempty open subset of X.

We use Krull's Theorem to slice X down to dimension 1. Let n denote the dimension of X, and suppose that $n > 1$. Let $D = X - Y$ be the closed complement of the open set Y. The components of D have dimension at most $n - 1$. We choose an element α of the coordinate algebra of X that is zero at p and isn't identically zero on any component of D, except at p itself, if p happens to be a component. Krull's Theorem tells us that every component of the zero locus of α has dimension $n-1$, and at least one of those components, call it V, contains p. If V were contained in D, it would be a component of D because $\dim V = n - 1$ and $\dim D \le n-1$. By our choice of α, this isn't the case. So $V \not\subset D$, and therefore $V \cap Y \neq \emptyset$. Let $W = V \cap Y$. Because V is irreducible and Y is open, W is a dense open subset of V, its closure is V, and p is a point of V. We replace X by V and Y by W. The dimension of X is thereby reduced to $n - 1$.

Thus it suffices to treat the case that X has dimension one. Then X will be a curve that contains p, and Y will be a nonempty open subset of X. The normalization of X will be a smooth curve $X^{\#}$ that comes with an integral, and therefore surjective, morphism to X. Finitely many points of $X^{\#}$ will map to p. We choose for C an affine open subvariety of $X^{\#}$ that contains just one of those points, and we call that point q. $\square$

5.4.3. Theorem *(curve criterion for a closed set) Let Y be a constructible subset of a variety X. The following conditions are equivalent:* closed-crittwo

a) *Y is closed.*

b) *For any morphism $C \xrightarrow{f} X$ from a smooth affine curve to X, the inverse image $f^{-1}Y$ is closed in C.*

c) *Let q be a point of a smooth affine curve C, let $C' = C - \{q\}$, and let $C \xrightarrow{f} X$ be a morphism. If $f(C') \subset Y$, then $f(C) \subset Y$.*

The hypothesis that Y be constructible is necessary here. For example, in the affine line X, the set W of points with integer coordinates isn't constructible, but it satisfies condition **b)**. Any morphism $C' \to X$ whose image is in W will map C' to a point, and therefore it will extend to C.

proof of Theorem 5.4.3. The implications **a)** $\Rightarrow$ **b)** $\Rightarrow$ **c)** are obvious. We prove the contrapositive of the implication **c)** $\Rightarrow$ **a)**. Suppose that Y isn't closed. We choose a point p of the closure $\overline{Y}$ that isn't in Y, and we apply Theorem 5.4.2. There exists a morphism $C \xrightarrow{f} X$ from a smooth curve to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. Since $q \notin Y$, this morphism shows that **(c)** doesn't hold either. $\square$

5.4.4. Theorem. *A constructible subset Y of a variety X is closed in the Zariski topology if and only if it is closed in the classical topology.* class-closed

proof. A Zariski closed set is closed in the classical topology because the classical topology is finer than the Zariski topology. Suppose that a constructible subset Y of X is closed in the classical topology. To show that it is closed in the Zariski topology, we choose a point p of its Zariski closure $\overline{Y}$, and we show that p is a point of Y.

Theorem 5.4.2 tells us that there is a map $C \xrightarrow{f} X$ from a smooth curve C to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. Let C_1 denote the inverse image $f^{-1}(Y)$ of Y. Because C_1 contains C', either $C_1 = C'$ or $C_1 = C$. In the classical topology, a morphism is continuous. Since Y is closed, its inverse image C_1 is closed in C. If C_1 were C', then C' would closed as well as open. Its complement $\{q\}$ would be an isolated point of C. Because a smooth curve contains no isolated point, the inverse image of Y is C, which means that $f(C) \subset Y$ and that $p \in Y$. $\square$

5.5 Projective Varieties are Proper

proper

As has been noted, an important property of projective space in the classical topology, is that it is compact. A variety isn't compact in the Zariski topology unless it is a single point. However, in the Zariski topology, projective varieties have a property closely related to compactness: They are *proper*.

Before defining the concept of a proper variety, we explain an analogous property of compact spaces.

proper-compact **5.5.1. Proposition.** *Let X be a compact space, let Z be a metric space, and let V be a closed subset of $Z{\times}X$. The image of V via the projection $Z{\times}X \to Z$ is closed in Z.*

proof. A metric space is compact if and only if every sequence of points has a convergent subsequence. Let W be the image of V in Z. We show that if a sequence of points z_i of W has a limit $\underline{z}$ in Z, then that limit is in W. For each i, we choose a point p_i of V that lies over z_i. So p_i is a pair (z_i, x_i), x_i being a point of X. Since X is compact, there is a subsequence of the sequence x_i that has a limit $\underline{x}$ in X. Passing to a subsequence of the sequence p_i, we may suppose that x_i has limit $\underline{x}$. Then p_i has limit $\underline{p} = (\underline{z}, \underline{x})$. Since V is closed, $\underline{p}$ is in V, and $\underline{z}$ is in its image W. □

defproper **5.5.2. Definition.** A variety X is *proper* if it has the following property: Let $Z{\times}X$ be the product of X with a variety Z, let π_Z denote the projection $Z{\times}X \longrightarrow Z$, and let V be a closed subvariety of $Z{\times}X$. The image W of V is a closed subvariety of Z.

imclos (5.5.3)
$$\begin{array}{ccc} V & \xrightarrow{\subset} & Z{\times}X \\ \downarrow & & \downarrow{\scriptstyle \pi_Z} \\ W & \xrightarrow{\subset} & Z \end{array}$$

When X is proper, it is also true that the image in Z of any closed subset V of $Z{\times}X$ is closed, because every closed set is a finite union of closed subvarieties.

propim-closed **5.5.4. Corollary.** *Let X be a proper variety, let V be a closed subvariety of X, and let $X \xrightarrow{f} Y$ be a morphism. The image $f(V)$ of V is a closed subvariety of Y.*

proof. In $X{\times}Y$, the graph Γ_f of f is a closed set isomorphic to X, and V corresponds to a subset V' of Γ_f that is closed in Γ_f and in $X{\times}Y$. The points of V' are pairs (x, y) such that $x \in V$ and $y = f(x)$. The image of V' via the projection to $X{\times}Y \to Y$ is the same as the image of V. Since X is proper, the image of V' is closed. □

The next theorem is the most important application of the use of curves to characterize closed sets.

pnproper **5.5.5. Theorem.** *Projective varieties are proper.*

proof. Let X be a projective variety. Suppose we are given a closed subvariety V of the product $Z{\times}X$. We must show that its image W in Z is a closed subvariety of Z (see Diagram 5.5.3). Since V is irreducible, its image is irreducible, so it suffices to show that W is closed. Theorem 5.3.5 tells us that W is a constructible set, and since X is closed in projective space, it is compact in the classical topology. Proposition 5.5.1 tells us that W is closed in the classical topology, and 5.4.4 tells us that W is closed in the Zariski topology too. □

needcover **5.5.6. Note.** Since Theorem 5.5.5 is about algebra, an algebraic proof would be preferable. To make an algebraic proof, one could attempt to replace the limit argument used in the proof of Proposition 5.5.1 by the curve criterion, proceeding as follows: Given a closed subset V of $Z{\times}X$ with image W and a point p in the closure of W, one chooses a map $C \xrightarrow{f} Z$ from a smooth affine curve C to Z and a point q of C such that $f(q) = p$ and $f(C') \subset W$, C' being the complement of q in C. Then one tries to lift this map to a morphism $C \xrightarrow{g} Z{\times}X$ such that $g(C') \subset V$ and $f = \pi \circ g$. Since V is closed, it would contain $g(q)$, and therefore $f(q) = \pi g(q)$ would be in $\pi(V) = W$. However, to find the lifting g, it may be necessary to replace C by a suitable covering. It isn't difficult to make this method work, but it takes longer. This is why we used to the classical topology. □

5.5.7. Corollary. *Let X be a projective variety and let $X \to Y$ be a morphism. The image in Y of a closed subvariety of X is a closed subvariety of Y.* *im-closedtwo*

This follows from the theorem and from and Corollary 5.5.4. □

The next examples show how Theorem 5.5.5 can be used.

5.5.8. Example. (*singular curves*) We parametrize the plane projective curves of a given degree d. The number of monomials $x_0^i x_1^j x_2^k$ of degree $d \ = i+j+k$ is the binomial coefficient $\binom{d+2}{2}$. We label those monomials as $m_0, ..., m_n$, ordered arbitrarily, with $n = \binom{d+2}{2} - 1$. A homogeneous polynomial of degree d will be a combination $\sum z_i m_i$ with complex coefficients z_i, so the homogeneous polynomials f of degree d in x, taken up to scalar factors, are parametrized by the projective space of dimension n with coordinates z. Let's denote that projective space by Z. Points of Z correspond bijectively to divisors of degree d in the projective plane, as defined in (1.3.13). properex

The product variety $Z \times \mathbb{P}^2$ represents pairs (D, p), where D is a divisor of degree d and p is a point of $\mathbb{P}^2$. A variable homogeneous polynomial of degree d in x can be written as a bihomogeneous polynomial $f(z, x) = \sum z_i m_i$ of degree 1 in z and degree d in x. For example, when $d = 2$, then with monomials ordered suitably,

$$f = z_0 x_0^2 + z_1 x_1^2 + z_2 x_2^2 + z_3 x_0 x_1 + z_4 x_0 x_2 + z_5 x_1 x_2$$

The locus Γ: $\{f(z, x) = 0\}$ in $Z \times \mathbb{P}^2$ is closed. A point $z = c$, $x = a$ of Γ is a pair (D, p) such that p is the point (c, a) of the divisor D: $f(c, x) = 0$ and

The set Σ of pairs (D, p) such that p is a singular point of D is also a closed set, because it is defined by the system of equations $f_0(z, x) = f_1(z, x) = f_2(z, x) = 0$, where f_i are the partial derivatives $\frac{\partial f}{\partial x_i}$. (Euler's Formula shows that $f(x, z) = 0$ too.) The partial derivatives f_i are bihomogeneous, of degree 1 in z and degree $d-1$ in x.

The proof of the next proposition becomes easy when one uses the fact that projective space is proper:

5.5.9. Proposition *The singular divisors of degree d form a closed subset S of the projective space Z of all divisors of degree d.* *singclosed*

proof. The subset S is the projection of the closed subset Σ of $Z \times \mathbb{P}^2$. Since $\mathbb{P}^2$ is proper, the image of the closed set Σ is closed. □

5.6 Fibre Dimension

semicont

A function $Y \xrightarrow{\delta} \mathbb{Z}$ from a variety to the integers is a *constructible* function if, for every integer n, the set of points p of Y such that $\delta(p) = n$ is constructible, and δ is an *upper semicontinuous* function if for every n, the set of points p such that $\delta(p) \geq n$ is closed. For brevity, we refer to an upper semicontinuous function as *semicontinuous*, though the term is ambiguous. A function might be lower semicontinuous.

A function δ on a curve Y is semicontinuous if and only if there exists an integer n and a nonempty open subset Y' of Y such that $\delta(p) = n$ for all points p of Y' and $\delta(p) \geq n$ for all points of Y not in Y'.

The next curve criterion for semicontinuous functions follows from the criterion for closed sets.

5.6.1. Proposition. *(curve criterion for semicontinuity) Let Y be a variety. A function $Y \xrightarrow{\delta} \mathbb{Z}$ is semicontinuous if and only if it is a constructible function, and for every morphism $C \xrightarrow{f} Y$ from a smooth curve C to Y, the composition $\delta \circ f$ is a semicontinuous function on C.* □ *uppercrit*

Let $Y \xrightarrow{f} X$ be a morphism of varieties, let q be a point of Y, and let Y_p be the fibre of f over $p = f(q)$. The *fibre dimension* $\delta(q)$ of f at q is the maximum among the dimensions of the components of the fibre that contain q.

5.6.2. Theorem. *(semicontinuity of fibre dimension) Let $Y \xrightarrow{u} X$ be a morphism of varieties, and let $\delta(q)$ denote the fibre dimension at a point q of Y.* *uppersemi*

(i) *Suppose that X is a smooth curve, that Y has dimension n, and that u does not map Y to a single point. Then δ is constant — the nonempty fibres have constant dimension: $\delta(q) = n - 1$ for all $q \in Y$.*

(ii) *Suppose that the image of Y is dense in X. Then it contains a nonempty open subset of X. Let the dimensions of X and Y be m and n, respectively. There is a nonempty open subset X' of X such that $\delta(q) = n - m$ for every point q in the inverse image of X'.*
(iii) *δ is a semicontinuous function on Y.*

We will not make use of this theorem, so we omit the proof. Parts of the proof are given as exercises.

5.7 Exercises

5.7.1. **(i)** Let $X = \operatorname{Spec}\mathbb{C}[x_1,...,x_n]/(f_1,...,f_k)$ be a smooth affine curve and let p be a point of X. What is the tangent line to X at p? chapfiveex

(ii) The two-dimensional linear space that has the closest contact with X at p is called the *osculating plane*. Determine that plane. What is the smallest possible order of contact with X?

5.7.2. Prove that a variety contains no isolated point. xnoisolpt

5.7.3. Let v be a (discrete) valuation on a field K that contains the complex numbers. Prove that every nonzero complex number c has value zero. xcvalzero

5.7.4. Prove that the ring $k[[x,y]]$ of formal power series (2.8.6) with coefficients in a field k is a local ring and a unique factorization domain. xfseries

5.7.5. Let A be a normal finite-type domain. Prove that the ring of fractions A_P of A at a prime ideal P of codimension 1 is a valuation ring. xlocval-ring

5.7.6. Let A be the polynomial ring $\mathbb{C}[x_1,...,x_n]$, and let P be the principal ideal generated by an irreducible polynomial $f(x_1,...,x_n)$. The local ring A_P consists of fractions g/h of polynomials in which g is arbitrary, and h can be any polynomial not divisible by f. Describe the valuation v associated to this local ring.

5.7.7. Let $X = \operatorname{Spec} A$ be an affine curve, with $A = \mathbb{C}[x_0,...,x_n]/P$, and let x_i also denote the residues of the variables in A. Let p be a point of X. We adjust coordinates so that p is the origin $(0,...,0)$ and that are otherwise generic. Let $z_i = x_i/x_0, \quad i = 1,...,n$, let $B = \mathbb{C}[x_0,z_1,z_2,...,z_n]$, and let $Y = \operatorname{Spec} B$. The inclusion $A \subset B$ defines a morphism $Y \to X$ the *blowup* of p in X. There will be finitely many points of Y in the fibre over p, and there will be at least one such point. We choose a point p_1 of the fibre, we replace X by Y and p by p_1 and repeat. Prove that this blowing up process yields a curve that is smooth above p in finitely many steps. xblowups-ing

5.7.8. In the space $\mathbb{A}^{n\times n}$ of $n\times n$ matrices, let X be the locus of idempotent matrices: $P^2 = P$. The general linear group GL_n operates on X by conjugation. xidemp-matr

(i) Decompose X into orbits for the operation of GL_n, and prove that the orbits are closed subsets of $\mathbb{A}^{n\times n}$.

(ii) Determine the dimensions of the orbits.

5.7.9. Let $f(x,y)$ and $g(x,y)$ be polynomials. Show that if g divides the partial derivatives f_x and f_y, then f is constant on the locus $g = 0$. xfconst

5.7.10. Prove that, when a variety X is covered by countably many constructible sets, a finite number of those sets cover X. xcount-cover

5.7.11. Let S be a multiplicative system in a finite-type domain R, and let A and B be finite-type domains that contain R as subring. Let R', A',and B' be the rings of S-fractions of R, A, and B, respectively. Prove: xsimple-loc

(i) If a set of elements $\alpha_1,...,\alpha_k$ generates A as R-algebra, it also generates A' as R'-algebra.

(ii) Let $A' \xrightarrow{\varphi'} B'$ be a homomorphism. For suitable s in S, there is a homomorphism $A_s \xrightarrow{\varphi_s} B_s$ whose localization is φ'. If φ' is injective, so is φ_s. If φ' is surjective or bijective, there will be an s such that φ_s is surjective or bijective, respectively.

(iii) If $A' \subset B'$ and if B' is a finite A'-module, then for suitable s in S, $A_s \subset B_s$, and B_s is a finite A_s-module.

5.7.12. What is the dimension of the Grassmanian $\mathbf{G}(m,n)$? xdimgrass

5.7.13. Let G denote the Grassmanian $G(2,4)$ of lines in $\mathbb{P}^3$, and let $[\ell]$ denote the point of G that corresponds to the line ℓ. In the product variety $G\times G$ of pairs of lines, let Z denote the set of pairs $[\ell_1],[\ell_2]$ whose intersection isn't empty. Prove that Z is a closed subset of $G\times G$. xclosed-inGG

5.7.14. Prove Theorem 5.5.9 directly, without appealing to Theorem 5.5.5. singular-closed

5.7.15. With reference to (5.5.6), let $X = \mathbb{P}^1$ and $Z = \mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$. Find a closed subset V of $Z\times X$ whose image is Z, such that the identity map $Z \to Z$ can't be lifted to a map $Z \to V$. cantlift

5.7.16. Is the constructibility hypothesis in 5.6.1 necessary? xupper xnumberofpoints **5.7.17.** Let $Y \xrightarrow{f} X$ be a morphism with finite fibres, and for p in X, let $N(p)$ be the number of points in the fibre $f^{-1}(p)$. Prove that N is a constructible function on X.

xpropproj **5.7.18.** Prove that a (quasiprojective) variety X that is proper is projective.

xredclosed **5.7.19.** There are 10 monomials of degree 3 in x_0, x_1, x_2, so the homogeneous polynomials of degree 3 form a vector space of dimension 10. Let Z be the corresponding projective space of dimension 9. Its points are classes of nonzero homogeneous cubic polynomials, up to scalar factor. Prove that the subset of classes of reducible polynomials is closed.

xtanlineprs **5.7.20.** With coordinates x_0, x_1, x_2 in the plane $\mathbb{P}$ and s_0, s_1, s_2 in the dual plane $\mathbb{P}^*$, let C be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where f is an irreducible homogeneous polynomial in x. Let Γ be the locus of pairs (x, s) of $\mathbb{P} \times \mathbb{P}^*$ such that $x \in C$ and the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$ is the tangent line to C at x. Prove that Γ is a closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

xsurfwithline **5.7.21.** Prove that the surfaces of degree d in $\mathbb{P}^3$ that contain a line form a closed subset of the space that parametrizes all surfaces of degree d.

xcovercurve **5.7.22.** Let U be an affine open subset of a curve X, and let Z be a finite subset of U. Prove that there is an affine open set V that contains no point of Z, such that $X = U \cup V$.

xsemicont **5.7.23.** Let X be a smooth curve, and let $f : Y \to X$ be a morphism. Prove that the fibre dimension is a semicontinuous function.

xfibdim **5.7.24.** Let $f : Y \to X$ be a morphism of varieties. Suppose we know that the fibre dimension is a constructible function. Use the curve criterion to show that the fibre dimension is semicontinuous.

Chapter 6 MODULES

6.1 The Structure Sheaf.

strsh

We use the words *category* and *functor* in this chapter. If you aren't familiar with those concepts, please read about them. You won't need to know much. Learn the definitions and look at a few examples.

We associate a category *(opens)* to a variety X. Its objects are the open subsets of X, and its morphisms are inclusions. If U and V are open sets and if $V \subset U$, there is a unique morphism $V \to U$ in *(opens)*, and if $V \not\subset U$ there is no morphism $V \to U$.

The *structure sheaf* $\mathcal{O}_X$ on a variety X is the functor

strshdef (6.1.1)
$$\textit{(opens)}^\circ \xrightarrow{\mathcal{O}_X} \textit{(algebras)}$$

from open sets to algebras, that sends a nonempty open set U to the ring of regular functions on U. The ring of sections on the empty subset is the zero ring. When one is speaking of the structure sheaf, the ring of regular functions on U is denoted by $\mathcal{O}_X(U)$. Thus $\mathcal{O}_X(U)$ will be a subring of the function field of X. When it is clear which variety is being studied, we may write $\mathcal{O}$ for $\mathcal{O}_X$.

The elements of $\mathcal{O}_X(U)$, the regular functions on U, are *sections* of the structure sheaf $\mathcal{O}_X$ on U, and the elements of $\mathcal{O}_X(X)$, the rational functions that are regular everywhere, are *global sections*.

When $V \to U$ is a morphism in *(opens)*, $\mathcal{O}_X(U)$ will be contained in $\mathcal{O}_X(V)$. The inclusion

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

is the homomorphism that makes $\mathcal{O}_X$ into a functor. Note that $\mathcal{O}_X$ reverses arrows. If $V \to U$, then $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A functor that reverses arrows is a *contravariant* functor. The superscript $^\circ$ in (6.1.1) is a customary notation to indicate that the functor is contravariant.

If $V \subset U$ are open subsets of X, then $\mathcal{O}_X(V) = \mathcal{O}_U(V)$.

6.2 $\mathcal{O}$-Modules

module

On an affine variety $\operatorname{Spec} A$, one works with A-modules. There is no need to do anything different. One can't do this when a variety X isn't affine. The best one can do is work with modules on the affine open subsets. An $\mathcal{O}_X$-*module* associates a module to every affine open subset.

We introduce a subcategory *(affines)* of the category *(opens)* associated to a variety X. The objects of *(affines)* are the affine open subsets of X, and the morphisms are localizations. A morphism $V \to U$ in *(opens)* is a morphism in *(affines)* if U is affine and V is a localization of U, a subset of the form U_s.

If $V \subset U$ are affine open subsets of a variety X, say $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, then $A \subset B$, and if V is the localization U_s, then $B = A_s$. But if B isn't a localization of A, it won't be clear how to construct B from A. The relationship between A and B will remain obscure. This is why we restrict attention to localizations.

a brief review of localization

- Let $X = \operatorname{Spec} A$ be an affine variety. The intersection of two localizations $X_s = \operatorname{Spec} A_s$ and $X_t = \operatorname{Spec} A_t$ is the localization $X_{st} = \operatorname{Spec} A_{st}$.
- Let $W \subset V \subset U$ be affine open subsets of a variety X. If V is a localization of U and W is a localization of V, then W is a localization of U.
- The affine open subsets form a basis for the topology on a variety X, and the localizations of an affine variety form a basis for its topology.
- If U and V are affine open subsets of X, the open sets W that are localizations of U as well as localizations of V, form a basis for the topology on $U \cap V$.

See Section 2.6 for these properties.

We will need a notation that allows us to work when both the module and its ring of scalars vary. Let R and R' be rings. A *homomorphism* from an R module M to an R'-module M' consists of a ring homomorphism $R \xrightarrow{f} R'$ and a homomorphism of abelian groups $M \xrightarrow{\varphi} M'$ that is *compatible* with f, in the sense that, if $m' = \varphi(m)$ and $r' = f(r)$, then $\varphi(rm) = r'm' \quad (= f(r)\varphi(m))$. We use the symbol *(modules)* for the category whose objects are modules over rings, and whose morphisms are such homomorphisms.

The ring homomorphism $R \xrightarrow{f} R'$ involved in a homomomorphism of modules is usually clear from context, and when it is clear, we may suppress notation for it, denoting a homomorphism from an R-module M to an R'-module M' by the symbol used for the map $M \to M'$ (which is φ here).

6.2.1. Definition. An *$\mathcal{O}$-module* $\mathcal{M}$ on a variety X is a (contravariant) functor defO-modtwo

$$\textit{(affines)}^\circ \xrightarrow{\mathcal{M}} \textit{(modules)}$$

such that, for every affine open subset U of X,
- $\mathcal{M}(U)$ is an $\mathcal{O}(U)$-module,
- when s is a nonzero element of $\mathcal{O}(U)$, the module $\mathcal{M}(U_s)$ is the localization $\mathcal{M}(U)_s$ of $\mathcal{M}(U)$, and
- the homomorphism $\mathcal{M}(U) \to \mathcal{M}(U_s)$ that makes $\mathcal{M}$ into a functor is the canonical map from a module to its localization.

This definition may seem complicated, perhaps too complicated for comfort, but don't worry. When a module has a natural definition, the data involved are taken care of automatically. This will become clear as we see some examples.

6.2.2. Note. To say that $\mathcal{M}(U_s)$ **is** the localization of $\mathcal{M}(U)$, isn't completely correct. One should say that $\mathcal{M}(U_s)$ and $\mathcal{M}(U)_s$ are *canonically isomorphic*. The map $\mathcal{M}(U) \to \mathcal{M}(U_s)$ induces a map from the localization $\mathcal{M}(U)_s$ to $\mathcal{M}(U_s)$, and that map should be an isomorphism. But it is almost always permissible to identify canonically isomorphic objects, and we do so here. $\square$ isloc

6.2.3. Terminology. defmod-hom

A *section* of an $\mathcal{O}$-module $\mathcal{M}$ on an affine open set U is an element of $\mathcal{M}(U)$, and the *module of sections* on U is $\mathcal{M}(U)$.

When U_s is a localization of U, the image of a section m on U via the map $\mathcal{M}(U) \to \mathcal{M}(U_s)$ is the *restriction* of m to U_s.

An $\mathcal{O}$-module $\mathcal{M}$ is a *finite $\mathcal{O}$-module* if $\mathcal{M}(U)$ is a finite $\mathcal{O}(U)$-module for every affine open set U.

A *homomorphism* $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of $\mathcal{O}$-modules consists of homomorphisms of $\mathcal{O}(U)$-modules

$$\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)$$

for each affine open subset U such that, when s is a nonzero element of $\mathcal{O}(U)$, the homomorphism $\varphi(U_s)$ is the localization of the homomorphism $\varphi(U)$.

A sequence of homomorphisms

exseqsheaves (6.2.4)
$$\mathcal{M} \to \mathcal{N} \to \mathcal{P}$$

of $\mathcal{O}$-modules on a variety X is *exact* if the sequence $\mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U)$ is exact for every affine open subset U of X. □

moduleexamples **6.2.5. Examples. (i)** The *free module* $\mathcal{O}^k$ is the $\mathcal{O}$-module whose sections on an affine open set U are the elements of the free $\mathcal{O}(U)$-module $\mathcal{O}(U)^k$. In particular, $\mathcal{O}$ is an $\mathcal{O}$-module.

(ii) The *kernel, image*, and *cokernel* of a homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ are among the operations that can be made on $\mathcal{O}$-modules. The kernel $\mathcal{K}$ of φ is the $\mathcal{O}$-module defined by $\mathcal{K}(U) = \ker\big(\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)\big)$ for every affine open set U, and the image and cokernel are defined analogously. Many operations such as these are compatible with localization.

(iii) *modules on a point.* Let's denote the affine variety $\operatorname{Spec}\mathbb{C}$, a point, by p. The point has just one nonempty open set, the whole space p. It is affine, and $\mathcal{O}_p(p) = \mathbb{C}$. So $\mathcal{M}(p)$ is a (complex) vector space. To define an $\mathcal{O}_p$-module $\mathcal{M}$, that vector space can be assigned arbitrarily. One may say that a module on the point **is** a complex vector space.

(iv) *submodules and ideals.* A *submodule* $\mathcal{N}$ of an $\mathcal{O}$-module $\mathcal{M}$ is an $\mathcal{O}$-module such that $\mathcal{N}(U)$ is a submodule of $\mathcal{M}(U)$ for every affine open set U. An *ideal* $\mathcal{I}$ of the structure sheaf $\mathcal{O}$ is submodule of $\mathcal{O}$.

If Y is a closed subvariety of a variety X, the *ideal of* Y is the submodule of $\mathcal{O}$ whose sections on an affine open subset U of X are the rational functions on X that are regular on U and that vanish on $Y \cap U$. We use the notation $V(\mathcal{I})$ for the zero set in X of an ideal $\mathcal{I}$ in the structure sheaf $\mathcal{O}_X$. A point p of X is in $V(\mathcal{I})$ if, whenever an affine open subset U contains p, all elements of $\mathcal{I}(U)$ vanish at p. When $\mathcal{I}$ is the ideal of functions that vanish on a closed subvariety Y, $V(\mathcal{I}) = Y$.

The *maximal ideal* at a point p of X, which we denote by $\mathfrak{m}_p$, is an ideal. If an affine open subset U contains p, its coordinate algebra $\mathcal{O}(U)$ will have a maximal ideal, whose elements are the regular functions that vanish at p. That maximal ideal is the module of sections $\mathfrak{m}_p(U)$ of $\mathfrak{m}_p$ on U. If U doesn't contain p, then $\mathfrak{m}_p(U) = \mathcal{O}(U)$.

(v) *the residue field module.* Let p be a point of a variety X. The residue field module κ_p is defined as follows: If an affine open subset U of X contains p, then $\mathcal{O}(U)$ has a residue field $k(p)$, and $\kappa_p(U) = k(p)$. If U doesn't contain p, then $\kappa_p(U) = 0$. There is a homomorphism of $\mathcal{O}$-modules $\mathcal{O} \to \kappa_p$ whose kernel is the maximal ideal $\mathfrak{m}_p$. □

6.3 The Sheaf Property

nonaffinesections We show here that an $\mathcal{O}$-module $\mathcal{M}$ on a variety X can be extended to a functor

$$\textit{(opens)}^\circ \xrightarrow{\widetilde{\mathcal{M}}} \textit{(modules)}$$

on all open subsets of X with these properties:

extensprop **6.3.1.**

- $\widetilde{\mathcal{M}}(Y)$ is an $\mathcal{O}(Y)$-module for every open subset Y.
- When U is an affine open set, $\widetilde{\mathcal{M}}(U) = \mathcal{M}(U)$.
- $\widetilde{\mathcal{M}}$ has the *sheaf property* that is described below.

The tilde $\sim$ is used for clarity. When we have finished the discussion, we will use the same notation for the functor on *(affines)* and for its extension to *(opens)*.

We use terminology that was introduced in (6.2.3). If U is an open subset of X, an element of $\widetilde{\mathcal{M}}(U)$ is a *section* of $\widetilde{\mathcal{M}}$ on U, and if $V \xrightarrow{j} U$ is an inclusion of open subsets, the associated homomorphism $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is the *restriction* from U to V.

When $V \xrightarrow{j} U$ is an inclusion of open sets, the restriction to V of a section m on U may be denoted by $j^\circ m$. However, the restriction operation occurs very often. Because of this, we usually use the same symbol for a section and for its restriction. Also, if an open set V is contained in two open sets U and U', and if m and m' are sections of $\widetilde{\mathcal{M}}$ on U and U', respectively, we say that m and m' are *equal on* V if their restrictions to V are equal. For example, we say $m = 0$ on V if the restriction of m to V is zero. $\square$

6.3.2. Theorem. *An $\mathcal{O}$-module $\mathcal{M}$ extends uniquely to a functor* *extendO-mod*

$$(opens)^\circ \xrightarrow{\widetilde{\mathcal{M}}} (modules)$$

that has the sheaf property (6.3.4). Moreover, for every nonempty open set U, $\widetilde{\mathcal{M}}(U)$ is an $\mathcal{O}(U)$-module, and for every inclusion $V \to U$ of nonempty open sets, the map $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is compatible with the map $\mathcal{O}(U) \to \mathcal{O}(V)$.

The proof of this theorem isn't especially difficult, but there are quite a few things to check. In order not to break up the discussion, we have put the proof into Section 6.8 at the end of the chapter.

An important point: Though the theorem describes the sections of an $\mathcal{O}$-module on every open set, one always works with the affine open sets. We may sometimes want the sections of an $\mathcal{O}$-module on a non-affine open set, but most of the time, the non-affine open sets are just along for the ride.

We drop the tilde now. Let $\{U^i\}$ be an affine covering of an open set Y, and let's assume that our covering families are finite. The *sheaf property* for this covering and for $\mathcal{M}$ asserts that an element m of $\mathcal{M}(Y)$ corresponds to a set of elements m_i in $\mathcal{M}(U^i)$ such that the restrictions of m_j and m_i to U^{ij} are equal. If the affine open subsets U^i are indexed by $i = 1, ..., n$, the sheaf property asserts that an element of $\mathcal{M}(Y)$ is determined by a vector $(m_1, ..., m_n)$ with m_i in $\mathcal{M}(U^i)$, such that $m_i = m_j$ on U^{ij}.

6.3.3. Let β be the map $\prod_i \mathcal{M}(U^i) \to \prod_{i,j} \mathcal{M}(U^{ij})$ that sends a vector $(m_1, ..., m_n)$ to the $n \times n$ matrix (z_{ij}), where z_{ij} is the difference $m_j - m_i$ of restrictions of the sections m_j and m_i to U^{ij}. The sheaf property asserts that $\mathcal{M}(Y)$ is the kernel of β. sheafker

6.3.4. *The Sheaf Property* for a functor $\mathcal{M}$ and for the affine covering $\{U^i\}$ of a nonempty open set Y states that the sequence sheaf-proptwo

(6.3.5)
$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha} \prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^{ij})$$

is exact, where α is the product of the restriction maps, and β is the difference map described in (6.3.3). So $\mathcal{M}(Y)$ is mapped isomorphically to the kernel of β.

In short, the sheaf property asserts that sections are determined locally: A section on a nonempty open set Y is determined by its restrictions to the open subsets U^i of an affine covering of Y.

Note. With notation as above, the open sets U^{ij} are affine, and there is a morphism $U^{ij} \to U^i$ in *(opens)* because U^{ij} is contained in U^i. However, this morphism needn't be a localization. If it isn't a localization, it won't be a morphism in *(affines)*, and the restriction maps $\mathcal{M}(U^i) \to \mathcal{M}(U^{ij})$ won't be a part of the structure of an $\mathcal{O}$-module. To make sense of the sheaf property, we need a definition of the restriction map for an arbitrary inclusion of affine open subsets. This point will be addressed in **Step 2** of the proof of Theorem 6.3.2, so we won't worry about it here. notloc

6.3.6. Note. Let $\{U^i\}$ be an affine covering of Y. Then, with $U^{ij} = U^i \cap U^j$, we will have $U^{ii} = U^i$ and $U^{ij} = U^{ji}$. These coincidences lead to redundancy in the statement (6.3.13) of the sheaf property. If the indices are $i = 1, ..., k$, we only need to look at intersections U^{ij} with $i < j$. The product $\mathcal{M}(\mathbf{U}_1) = \prod_{i,j} \mathcal{M}(U^{ij})$ that appears in the sheaf property can be replaced by the product with increasing pairs of indices $\prod_{i<j} \mathcal{M}(U^{ij})$. For instance, if an open set Y is covered by two affine open sets U and V, the sheaf property for this covering is an exact sequence twoopensets

$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha} \big[\mathcal{M}(U) \times \mathcal{M}(V)\big] \xrightarrow{\beta} \big[\mathcal{M}(U \cap U) \times \mathcal{M}(U \cap V) \times \mathcal{M}(V \cap U) \times \mathcal{M}(V \cap V)\big]$$

The exact sequence

twoopens (6.3.7)
$$0 \to \mathcal{M}(Y) \longrightarrow \big[\mathcal{M}(U)\times\mathcal{M}(V)\big] \xrightarrow{-,+} \mathcal{M}(U\cap V)$$

is equivalent. $\square$

strsheafPn **6.3.8. Example.**
Let A denote the polynomial ring $\mathbb{C}[x,y]$, and let V be the complement of a point p in the affine space $X = \operatorname{Spec} A$. This is an open set, but it isn't affine. We cover V by two localizations of X: $X_x = \operatorname{Spec} A[x^{-1}]$ and $X_y = \operatorname{Spec} A[y^{-1}]$. The sheaf property (6.3.7) for $\mathcal{O}_X$ and for this covering is equivalent to an exact sequence

$$0 \to \mathcal{O}_X(V) \to A[x^{-1}]\times A[y^{-1}] \to A[(xy)^{-1}]$$

It shows that a regular function on V is in the intersection $A[x^{-1}] \cap A[y^{-1}]$, which is equal to A. Therefore the sections of the structure sheaf $\mathcal{O}_X$ on V are the elements of A. They are the same as the sections on X. $\square$

We have been working with nonempty open sets. This is almost always permissible, but when a module $\mathcal{M}$ on *(affines)* is extended to a module on all open sets, the empty set should be included. The next lemma takes care of the empty set.

emptyset **6.3.9. Lemma.** *The only section of an $\mathcal{O}$-module $\mathcal{M}$ on the empty set is the zero section: $\mathcal{M}(\emptyset) = \{0\}$. In particular, $\mathcal{O}(\emptyset)$ is the zero ring.*

proof. This follows from the sheaf property. The empty set $\emptyset$ is covered by the empty covering, the covering indexed by the empty set. Therefore $\mathcal{M}(\emptyset)$ is contained in an empty product. If we want the empty product to be a module, we have no choice but to define it to be zero. Then $\mathcal{M}(\emptyset)$ is zero too.

If you find this reasoning pedantic, you can take $\mathcal{M}(\emptyset) = \{0\}$ as an axiom. $\square$

omod **6.3.10. families of open sets**

It is convenient to have a more compact notation for the sheaf property, and for this, we introduce symbols to represent families of open sets. Say that $\mathbf{U}$ and $\mathbf{V}$ represent families of open sets $\{U^i\}$ and $\{V^\nu\}$, respectively. A *morphism* of families $\mathbf{V} \to \mathbf{U}$ consists of a morphism from each V^ν to one of the subsets U^i. Such a morphism will be given by a map of index sets $\nu \rightsquigarrow i_\nu$, such that $V^\nu \subset U^{i_\nu}$.

There may be more than one morphism $\mathbf{V} \to \mathbf{U}$, because a subset V^ν may be contained in more than one of the subsets U^i. To define a morphism, one must make a choice among those subsets. For example, let $\mathbf{U} = \{U^i\}$, and let V be an open set. For each i such that $V \subset U^i$, there is a morphism $V \to \mathbf{U}$ that sends V to U^i. There will be a unique morphism $\mathbf{U} \to V$ in the other direction if $U^i \subset V$ for all i.

We extend a functor *(opens)*$^\circ \xrightarrow{\mathcal{M}}$ *(modules)* to families: If $\mathbf{U} = \{U^i\}$, we define

section-sonfamily (6.3.11)
$$\mathcal{M}(\mathbf{U}) = \prod \mathcal{M}(U^i).$$

Then a morphism of families $\mathbf{V} \xrightarrow{f} \mathbf{U}$ defines a map $\mathcal{M}(\mathbf{U}) \xleftarrow{f^\circ} \mathcal{M}(\mathbf{V})$ in a way that is fairly obvious, though notation for it is clumsy. If f maps V^ν to U^i, and if $(u_1, ..., u_r)$ is section of $\mathcal{M}(\mathbf{U})$, the map f° restricts u_i to V^ν.

To write the sheaf property in terms of families of open sets, let $\mathbf{U}_0 = \{U^i\}$ be an affine covering of an open set Y, and let $\mathbf{U}_1$ denote the family $\{U^{ij}\}$ of intersections: $U^{ij} = U^i \cap U^j$. The intersections are also affine, and there are two sets of inclusions

$$U^{ij} \subset U^i \quad \text{and} \quad U^{ij} \subset U^j$$

These inclusions give us two morphisms $\mathbf{U}_1 \xrightarrow{d_0,d_1} \mathbf{U}_0$ of families: $U^{ij} \xrightarrow{d_0} U^j$ and $U^{ij} \xrightarrow{d_1} U^i$. We also have a morphism $\mathbf{U}_0 \to Y$, and the two composed morphisms $\mathbf{U}_1 \xrightarrow{d_i} \mathbf{U}_0 \to Y$ are equal. These morphisms form what we all a *covering diagram*

(6.3.12) $$Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$$ covdiagr

When we apply a functor *(opens)* $\xrightarrow{\mathcal{M}}$ *(modules)* to this covering diagram, we obtain a sequence

(6.3.13) $$0 \to \mathcal{M}(Y) \xrightarrow{\alpha_{\mathbf{U}}} \mathcal{M}(\mathbf{U}_0) \xrightarrow{\beta_{\mathbf{U}}} \mathcal{M}(\mathbf{U}_1)$$ defbeta

where $\alpha_{\mathbf{U}}$ is the restriction map and $\beta_{\mathbf{U}}$ is the difference of the maps induced by the two morphisms $d_0, d_1 : \mathbf{U}_1 \rightrightarrows \mathbf{U}_0$. The sheaf property for the covering $\mathbf{U}_0$ of Y is the assertion that this sequence is exact, which means that $\alpha_{\mathbf{U}}$ is injective, and that its image is the kernel of $\beta_{\mathbf{U}}$.

6.4 More Modules

From here on, we work with the extension of an $\mathcal{O}$-module $\mathcal{M}$ to all open sets, and we denote the extension by the same symbol $\mathcal{M}$. moremodules

6.4.1. some homomorphisms defker

- Homomorphisms of free $\mathcal{O}_X$-modules, $\mathcal{O}^n \to \mathcal{O}^m$ correspond to $m \times n$-matrices of global sections of $\mathcal{O}$.
- Homomorphisms $\mathcal{O} \xrightarrow{\varphi} \mathcal{M}$ correspond bijectively to global sections of $\mathcal{M}$.

This is analogous to the fact that, if M is an A-module, multiplication by an element of A defines a homomorphism $M \to M$. Similarly,

- Multiplication by a global section f of $\mathcal{O}$ defines a homomorphism $\mathcal{M} \xrightarrow{f} \mathcal{M}$.

6.4.2. Proposition. *Let $\{U^i\}$ be an affine covering of a variety X.* *injoncover*

(i) *An $\mathcal{O}$-module $\mathcal{M}$ is the zero module if and only if $\mathcal{M}(U^i) = 0$ for every i.*

(ii) *A homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of $\mathcal{O}$-modules is injective, surjective, or bijective if and only if the maps $\mathcal{M}(U^i) \xrightarrow{\varphi(U^i)} \mathcal{N}(U^i)$ are injective, surjective, or bijective, respectively, for every i.*

proof. **(i)** Let V be an open subset of X. We cover each intersection $V \cap U^i$ by affine open sets $V^{i\nu}$ that are localizations of U^i. These sets, taken together, cover V. If $\mathcal{M}(U^i) = 0$, then the localizations $\mathcal{M}(V^{i\nu})$ are zero too. The sheaf property shows that the map $\mathcal{M}(V) \to \prod \mathcal{M}(V^{i\nu})$ is injective, and therefore that $\mathcal{M}(V) = 0$.

(ii) This follows from **(i)**, because a homomorphism φ is injective or surjective if and only if its kernel or its cokernel is zero. □

6.4.3. kernel kerthm

As we have noted, many operations that one makes on modules over a ring are compatible with localization, and can therefore be made on $\mathcal{O}$-modules. However, the sections over a non-affine open set are almost *never* determined by an operation. One must use the sheaf property. The kernel of a homomorphism is among the few exceptions to this general rule.

6.4.4. Proposition. *Let X be a variety, and let $\mathcal{K}$ be the kernel of a homomorphism of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{N}$, so that the there is an exact sequence $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{N}$. For every open subset Y of X, the sequence of sections* *leftexact-section*

(6.4.5) $$0 \to \mathcal{K}(Y) \to \mathcal{M}(Y) \to \mathcal{N}(Y)$$ *lexsect*

is exact.

proof. We choose covering $Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$, and inspect the corresponding diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{K}(\mathbf{U}_0) & \longrightarrow & \mathcal{M}(\mathbf{U}_0) & \longrightarrow & \mathcal{N}(\mathbf{U}_0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{K}(\mathbf{U}_1) & \longrightarrow & \mathcal{M}(\mathbf{U}_1) & \longrightarrow & \mathcal{N}(\mathbf{U}_1)
\end{array}$$

where the vertical maps are the difference maps β described in (6.3.3). The rows are exact because U^i and U^{ij} are affine. This is the definition of exactness. The sheaf property asserts that the kernels of the vertical maps form the sequence (6.4.5). The sequence of kernels is exact because taking kernels is a left exact operation (see (9.1.22)). $\square$

The section functor isn't right exact. When $\mathcal{M} \to \mathcal{N}$ is a surjective homomorphism of $\mathcal{O}$-modules, the map $\mathcal{M}(Y) \to \mathcal{N}(Y)$ often fails to be surjective, unless Y is affine. There is an example below (6.4.10) **(iii)**).

coverpone

6.4.6. modules on the projective line

The projective line $X = \mathbb{P}^1$ is covered by the standard open sets $\mathbb{U}^0$ and $\mathbb{U}^1$. Their intersection $\mathbb{U}^{01}$ is a localization of $\mathbb{U}^0$ and of $\mathbb{U}^1$. Say that $\mathbb{U}^i = \operatorname{Spec} A_i$, for $i = 0, 1, 01$. Let $\mathcal{M}$ be an $\mathcal{O}$-module, and let $M_i = \mathcal{M}(\mathbb{U}^i)$. The sheaf property, in the form (6.3.7), tells us that, to give a global section of an $\mathcal{O}$-module $\mathcal{M}$, it suffices to give elements of M_0 and M_1 whose restrictions to M_{01} are equal.

With $u = x_1/x_0$, the coordinate algebras are $\mathcal{O}(\mathbb{U}^0) = A_0 = \mathbb{C}[u]$, $\mathcal{O}(\mathbb{U}^1) = A_1 = \mathbb{C}[u^{-1}]$, and $\mathcal{O}(\mathbb{U}^{01}) = A_{01} = \mathbb{C}[u, u^{-1}]$. A global section of $\mathcal{O}$ is determined by polynomials f and g such that $f(u) = g(u^{-1})$ in A_{01}. The only such polynomials are the constants. So the constants are the only rational functions that are regular everywhere on $\mathbb{P}^1$. I think we knew this.

When $\mathcal{M}$ is an $\mathcal{O}$-module, $M_0 = \mathcal{M}(\mathbb{U}^0)$ and $M_1 = \mathcal{M}(\mathbb{U}^1)$ will be modules over A_0 and A_1, respectively. The A_{01}-module $M_{01} = \mathcal{M}(\mathbb{U}^{01})$ can be obtained by localizing M_0 and also by localizing M_1. If $v = u^{-1}$. then

$$M_0[u^{-1}] \approx M_{01} \approx M_1[v^{-1}]$$

A global section of $\mathcal{M}$ is determined by elements m_1 in M_1 and m_2 in M_2, that become equal in the common localization M_{01}. The next proposition shows that this data determines the module $\mathcal{M}$.

modonpone

6.4.7. Proposition. *With notation as above, let M_0, M_1, and M_{01} be modules over the algebras A_0, A_1, and A_{01}, respectively, and let $M_0[u^{-1}] \xrightarrow{\varphi_0} M_{01}$ and $M_1[v^{-1}] \xrightarrow{\varphi_1} M_{01}$ be isomorphisms of A_{01}-modules. There is a unique $\mathcal{O}_X$-module $\mathcal{M}$ such that $\mathcal{M}(\mathbb{U}^0) = M_0$ and $\mathcal{M}(\mathbb{U}^1) = M_1$.*

The proof is at the end of this chapter.

Suppose that M_0 and M_1 are free modules over A_0 and A_1. Their common localization M_{01} will be a free A_{01}-module. A basis $\mathbf{B}_0$ of the A_0-module M_0 will also be a basis of the A_{01}-module M_{01}, and a basis $\mathbf{B}_1$ of M_1 will be a basis of M_{01}. When regarded as bases of M_{01}, $\mathbf{B_0}$ and $\mathbf{B_1}$ will be related by an invertible A_{01}-matrix P, and as Proposition 6.4.7 tells us, that matrix determines $\mathcal{M}$ up to isomorphism. When M_i have rank one, P will be an invertible 1×1 matrix in the Laurent polynomial ring $A_{01} = \mathbb{C}[u, u^{-1}]$, a unit of that ring. The units in A_{01} are scalar multiples of powers of u. Since the scalar can be absorbed into one of the bases, an $\mathcal{O}$-module of rank 1 is determined, up to isomorphism, by a power of u. It is one of the twisting modules that will be described below, in Section 6.7. $\square$

tensprodmodule

6.4.8. tensor products

Tensor products are compatible with localization. If M and N are modules over a domain A and s is a nonzero element of A, the canonical map $(M \otimes_A N)_s \to M_s \otimes_{A_s} N_s$ is an isomorphism (9.1.34). Therefore the tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ of $\mathcal{O}$-modules is defined. On an affine open set U, $[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](U)$ is the tensor product $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$.

For any open subset V of X, there is a canonical map

tensprodmap

$$\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \to [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V) \tag{6.4.9}$$

This map is an equality when V is affine. To describe the map for arbitrary V, we cover V by a family $\mathbf{V}_0$ of affine open sets. The family $\mathbf{V}_1$ of intersections also consists of affine open sets. We form a diagram

$$\begin{array}{ccccccc}
 & & \mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) & \xrightarrow{f} & \mathcal{M}(\mathbf{V}_0) \otimes_{\mathcal{O}(\mathbf{V}_0)} \mathcal{N}(\mathbf{V}_0) & \xrightarrow{g} & \mathcal{M}(\mathbf{V}_1) \otimes_{\mathcal{O}(\mathbf{V}_1)} \mathcal{N}(\mathbf{V}_1) \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
0 & \longrightarrow & [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V) & \longrightarrow & [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](\mathbf{V}_0) & \xrightarrow{g} & [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](\mathbf{V}_1)
\end{array}$$

The vertical maps b and c are equalities. The bottom row is exact, and the composition gf is zero. So f maps $\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V)$ to the kernel of g, which is equal to $[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)$. The map a is (6.4.9). When V isn't affine, this map needn't be either injective or surjective.

6.4.10. Examples. These examples illustrate the failure of bijectivity of the map (6.4.9). residue-fieldmodule

(i) Let p and q be distinct points of the projective line X, and let κ_p and κ_q be the residue field modules. Then $\kappa_p(X) = \kappa_q(X) = \mathbb{C}$, so $\kappa_p(X) \otimes_{\mathcal{O}(X)} \kappa_q(X) = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}$. But $\kappa_p \otimes_{\mathcal{O}} \kappa_q = 0$. The map (6.4.9),

$$\kappa_p(X) \otimes_{\mathcal{O}(X)} \kappa_q(X) \to [\kappa_p \otimes_{\mathcal{O}} \kappa_q](X)$$

is the zero map. It isn't injective.

(ii) Let p a point of a variety X, and let $\mathfrak{m}_p$ and κ_p be the maximal ideal and residue field modules at p. There is an exact sequence of $\mathcal{O}$-modules

$$0 \to \mathfrak{m}_p \to \mathcal{O} \xrightarrow{\pi_p} \kappa_p \to 0 \tag{6.4.11}$$ ideal-sheafatp

In this case, the sequence of global sections is exact.

(iii) Let p and q be the points $(1, 0)$ and $(0, 1)$ of the projective line $\mathbb{P}^1$. We form a homomorphism

$$\mathfrak{m}_p \times \mathfrak{m}_q \xrightarrow{\varphi} \mathcal{O}$$

φ being the map $(a, b) \to b - a$. On the open set $\mathbb{U}^0$, $\mathfrak{m}_q$ is isomorphic to $\mathcal{O}$. So φ is surjective on $\mathbb{U}^0$. Similarly, φ is surjective on $\mathbb{U}^1$. Since $\mathbb{U}^0$ and $\mathbb{U}^1$ cover $\mathbb{P}^1$, φ is surjective (6.4.2). The only global section of $\mathfrak{m}_p \times \mathfrak{m}_q$ is zero, while $\mathcal{O}$ has the nonzero global section 1. The map of global sections determined by φ isn't surjective. $\square$

6.4.12. the function field module ffldmod

Let F be the function field of a variety X. The sections of the *function field module* $\mathcal{F}$ on a nonempty open set U are the elements of F. It is called a *constant* $\mathcal{O}$-module because $\mathcal{F}(U)$ is the same for every nonempty U. It isn't finite module unless X is a point.

Tensoring with the function field module: Let $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X, and let $\mathcal{F}$ be the function field module. Let $U = \operatorname{Spec} A$ be an affine open set and let $M = \mathcal{M}(U)$. The module of sections of the tensor product module $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$ on U is the F-vector space $M \otimes_A F$. On a localization U_s, the module of sections will be $M_s \otimes_{A_s} F$, and because s is invertible in F, this is the same as $M \otimes_A F$. The sections of $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$ are independent of the affine open set U. So $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$ is a constant $\mathcal{O}$-module.

When $\mathcal{M}$ is a torsion module, i.e., when the modules of sections of $\mathcal{M}$ of any affine open set U is a torsion module, the tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$ will be zero.

6.4.13. $\mathcal{O}$-modules on affine varieties modaff

6.4.14. Proposition. *Let $X = \operatorname{Spec} A$ be an affine variety. Sending an $\mathcal{O}$-module $\mathcal{M}$ to the A-module $\mathcal{M}(X)$ of its global sections defines an equivalence of categories between $\mathcal{O}$-modules and A-modules.* *moduleon-affines*

proof. We must invert the functor *$\mathcal{O}$–(modules) $\to$ A–(modules)* that sends $\mathcal{M}$ to $\mathcal{M}(X)$. Given an A-module M, the corresponding $\mathcal{O}$-module $\mathcal{M}$ is defined as follows: Let $U = \operatorname{Spec} B$ be an affine open subset of X. The inclusion $U \subset X$ corresponds to an algebra homomorphism $A \to B$. We define $\mathcal{M}(U)$ to be the B-module $B \otimes_A M$. If s is a nonzero element of B, then $B_s \otimes_A M$ is canonically isomorphic to the localization $(B \otimes_A M)_s$ of $B \otimes_A M$. Therefore $\mathcal{M}$ is an $\mathcal{O}$-module, and $\mathcal{M}(X) = M$.

If $\mathcal{M}$ is an $\mathcal{O}$-module and $\mathcal{M}(X) = M$, $\mathcal{M}$ is the $\mathcal{O}$-module defined from M as above. $\square$

6.4.15. limits of $\mathcal{O}$-modules omodlim

A *directed set of $\mathcal{O}$-modules* on a variety X is a sequence

$$\mathcal{M}_\bullet = \{\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \cdots\}$$

of homomorphisms of $\mathcal{O}$-modules. For every affine open set U, the $\mathcal{O}(U)$-modules $\mathcal{M}_n(U)$ form a directed set, as defined in (9.1.35). The *direct limit* $\varinjlim \mathcal{M}_\bullet$ is defined by simply taking the limit for each affine open set: $[\varinjlim \mathcal{M}_\bullet](U) = \varinjlim [\mathcal{M}_\bullet(U)]$. The limit operation is compatible with localization because localization is a tensor product (see 9.1.38), so $\varinjlim \mathcal{M}_\bullet$ is an $\mathcal{O}$-module. In fact, the equality $[\varinjlim \mathcal{M}_\bullet](U) = \varinjlim [\mathcal{M}_\bullet(U)]$ will be true for every open set, not only for affine open sets.

module-Hom

6.4.16. the module $\underline{\mathrm{Hom}}$

Let M and N be modules over a ring A. The set of homomorphisms $M \to N$, which is usually denoted by $\mathrm{Hom}_A(M,N)$, becomes an A-module with some fairly obvious laws of composition: If φ and ψ are homomorphisms and a is an element of A, then $\varphi{+}\psi$ and $a\varphi$ are defined by

homis-module

$$[\varphi{+}\psi](m) = \varphi(m) + \psi(m) \quad \text{and} \quad [a\varphi](m) = a\varphi(m) \qquad (6.4.17)$$

Because φ is a module homomorphism, it is also true that $\varphi(m_1){+}\varphi(m_2) = \varphi(m_1{+}m_2)$, and $a\varphi(m) = \varphi(am)$.

Nequal-sHom

6.4.18. Lemma. (i) *An A-module N is canonically isomorphic to the module $\mathrm{Hom}_A(A,N)$. The homomorphism $A \xrightarrow{\varphi} N$ that corresponds to an element v of N is multiplication by v: $\varphi(a) = av$. The element of N that corresponds to a homomorphism $A \xrightarrow{\varphi} N$ is $v = \varphi(1)$.*
(ii) *$\mathrm{Hom}_A(A^k,N)$ is isomorphic to N^k, and $\mathrm{Hom}_A(A^k,A^\ell)$ is isomorphic to the module $A^{\ell\times k}$ of $\ell \times k$ A-matrices.* □

homfinite

6.4.19. Lemma. *As a functor in two variables, Hom_A is left exact and* ***contravariant*** *in the first variable: For any A-module N, an exact sequence $M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_3 \to 0$ of A-modules induces an exact sequence*

$$0 \to \mathrm{Hom}_A(M_3,N) \xrightarrow{\circ b} \mathrm{Hom}_A(M_2,N) \xrightarrow{\circ a} \mathrm{Hom}_A(M_1,N)$$

The functor Hom_A is left exact and **covariant** *in the second variable.* □

6.4.20. Corollary. *Let A be a noetherian ring. If M and N are finite A-modules, then $\mathrm{Hom}_A(M,N)$ is a*
homfinitetwo *finite A-module.*

proof. Because M is finitely generated, there is a surjective map $A^k \to M$. It induces an injective map $\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(A^k,N) \approx N^k$. So $\mathrm{Hom}_A(M,N)$ is isomorphic to a submodule of the finite module N^k. Therefore it is a finite module. □

localize-Hom

6.4.21. Lemma. *Let M and N be modules over a noetherian domain A, and suppose that M is a finite module. Let s be a nonzero element of A. The localization $(\mathrm{Hom}_A(M,N))_s$ is canonically isomorphic to $\mathrm{Hom}_{A_s}(M_s,N_s)$. The analogous statement is true for fractions formed using a multiplicative system S.*

proof. Since $\mathrm{Hom}_A(A,M) \approx M$, it is true that $(\mathrm{Hom}_A(A,M))_s \approx M_s \approx \mathrm{Hom}_{A_s}(A_s,M_s)$ and that $(\mathrm{Hom}_A(A^k,M))_s \approx M_s^k \approx \mathrm{Hom}_{A_s}(A_s^k,M_s)$.

Since M is a finite module, it has a presentation $A^\ell \to A^k \to M \to 0$, whose localization $A_s^\ell \to A_s^k \to M_s \to 0$, is a presentation of the A_s-module M_s. The sequence

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(A^k,N) \to \mathrm{Hom}_A(A^\ell,N)$$

is exact, and so is its localization. The lemma follows from the case that $M = A^k$. □

The lemma shows that, when $\mathcal{M}$ and $\mathcal{N}$ are finite $\mathcal{O}$-modules on a variety X, an $\mathcal{O}$-module of homomorphisms $\mathcal{M} \to \mathcal{N}$ is defined. This $\mathcal{O}$-module may be denoted by $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. When $U = \mathrm{Spec}\, A$ is an affine open set, $M = \mathcal{M}(U)$, and $N = \mathcal{N}(U)$, the module of sections of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ on U is the A-module $\mathrm{Hom}_A(M,N)$. The analogues of Lemma 6.4.18 and Lemma 6.4.19 are true for $\underline{\mathrm{Hom}}$:

leftexco

6.4.22. Corollary. (i) *An $\mathcal{O}$-module $\mathcal{M}$ on a variety Y is isomorphic to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O},\mathcal{M})$.*
(ii) *The functor $\underline{\mathrm{Hom}}$ is left exact and* ***contravariant*** *in the first variable, and it is left exact and* ***covariant*** *in the second variable.* □

dropHom

6.4.23. Notation. The notations Hom and $\underline{\mathrm{Hom}}$ are cumbersome. It seems permissible to drop the symbol Hom, and to write $_A(M,N)$ for $\mathrm{Hom}_A(M,N)$. Similarly, if $\mathcal{M}$ and $\mathcal{M}$ are $\mathcal{O}$-modules on a variety X, we may write $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ or $_X(\mathcal{M},\mathcal{N})$ for $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$.

dualmod

6.4.24. the dual module

Let $\mathcal{M}$ be a locally free $\mathcal{O}$-modules on a variety X. The *dual module* $\mathcal{M}^*$ is the $\mathcal{O}$-module of homomorphisms $\mathcal{M} \to \mathcal{O}$:

$$\mathcal{M}^* = {}_{\mathcal{O}}(\mathcal{M},\mathcal{O})$$

A section of $\mathcal{M}^*$ on an affine open set U is an $\mathcal{O}(U)$-module homomorphism $\mathcal{M}(U) \to \mathcal{O}(U)$.

The dualizing operation is contravariant. A homomorphism $\mathcal{M} \to \mathcal{N}$ of locally free $\mathcal{O}$-modules induces a homomorphism $\mathcal{M}^* \leftarrow \mathcal{N}^*$. Composition of maps $\mathcal{N} \to \mathcal{O}$ and $\mathcal{M} \to \mathcal{N}$ produces a map $\mathcal{M} \to \mathcal{O}$.

If $\mathcal{M}$ is a free module with basis $v_1, ..., v_k$, then $\mathcal{M}^*$ is also free, with the *dual basis* $v_1^*, ..., v_k^*$, defined by

$$v_i^*(v_i) = 1 \quad \text{and} \quad v_i^*(v_j) = 0 \quad \text{if } i \neq j$$

When $\mathcal{M}$ is locally free, $\mathcal{M}^*$ is locally free. The dual $\mathcal{O}^*$ of the structure sheaf $\mathcal{O}$ is isomorphic to $\mathcal{O}$.

6.4.25. Corollary. *Let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules.* bidualm
(i) *The module of homomorphisms $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ is canonically isomorphic to the tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}$.*
(ii) *$\mathcal{M}$ is canonically isomorphic to its bidual: $(\mathcal{M}^*)^* \approx \mathcal{M}$.*
(iii) *The tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}'^*$ is isomorphic to $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})^*$.*

proof. **(i)** We identify $\mathcal{N}$ with the module $_{\mathcal{O}}(\mathcal{O},\mathcal{N})$. Given sections φ of $\mathcal{M}^* = {}_{\mathcal{O}}(\mathcal{M},\mathcal{O})$ and γ of $\mathcal{N} = {}_{\mathcal{O}}(\mathcal{O},\mathcal{N})$, the composition $\gamma\varphi$ is a section of $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. This composition is bilinear, so it defines a map $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N} \to {}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. To show that this map is an isomorphism is a local problem, so we may assume that $Y = \operatorname{Spec} A$ is affine and that $\mathcal{M}$ and $\mathcal{N}$ are free modules of ranks k an ℓ, respectively. Then both $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}$ and $_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ are the modules of $\ell \times k$ matrices with entries in A. □

6.4.26. Proposition. *Let X be a variety.* dualseq

(i) *Let $\mathcal{P} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$ be homomorphisms of $\mathcal{O}$-modules whose composition gf is the identity map $\mathcal{P} \to \mathcal{P}$. So f is injective and g is surjective. Then $\mathcal{N}$ is the direct sum of the image of f, which is isomorphic to $\mathcal{P}$, and the kernel $\mathcal{K}$ of g: $\mathcal{N} \approx \mathcal{P} \oplus \mathcal{K}$.*

(ii) *Let $0 \to \mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P} \to 0$ be an exact sequence of $\mathcal{O}$-modules. If $\mathcal{P}$ is locally free, the dual modules form an exact sequence $0 \to \mathcal{P}^* \to \mathcal{N}^* \to \mathcal{M}^* \to 0$.*

proof. **(i)** This follows from the analogous statement about modules over a ring.

(ii) The sequence $0 \to \mathcal{P}^* \to \mathcal{N}^* \to \mathcal{M}^*$ is exact because $\underline{\mathrm{Hom}}$ is left exact. The zero on the right comes from the fact that, when $\mathcal{P}$ is locally free, there is an affine covering on which it is free. When $\mathcal{P}$ is free, the given sequence splits. To define a splitting, one lifts a basis $(p_1, ..., p_r)$ of $\mathcal{P}$, choosing, for each i, elements n_i that map to p_i. Then sending p_i to n_i maps $\mathcal{P}$ to $\mathcal{N}$, and the composition with g is the identity. So $\mathcal{N}$ is isomorphic to $\mathcal{M} \oplus \mathcal{P}$ (9.1.25). Therefore the map $\mathcal{N}^* \to \mathcal{M}^*$ is surjective. □

6.5 Direct Image

directimage

6.5.1. affine morphisms

affinemorph

6.5.2. Definition. An *affine morphism* is a morphism $Y \xrightarrow{f} X$ of varieties with the property that the inverse image $f^{-1}(U)$ of every affine open subset U of X is an affine open subset of Y. □ defaffmorph

The following are examples of affine morphisms:

- the inclusion of an affine open subset Y into X,
- the inclusion of a closed subvariety Y into X,
- a finite morphism, an integral morphism.

But if Y is the complement of a point of the plane X, the inclusion of Y into X is not an affine morphism.

As one sees from the examples, affine morphisms form a rather miscellaneous collection. Hwoever, the concept is convenient.

6.5.3. Definition. Let $Y \xrightarrow{f} X$ be an affine morphism and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The *direct image* $f_*\mathcal{N}$ of $\mathcal{N}$ is the $\mathcal{O}_X$-module defined by $[f_*\mathcal{N}](U) = \mathcal{N}(f^{-1}U)$ for every an affine open subset U of X. Scalar multiplication by a section of $\mathcal{O}_X(U)$ is multiplication by its image in $\mathcal{O}_Y(f^{-1}U)$, which is equal to $[f_*\mathcal{O}_Y](U)$. defdirim

The direct image generalizes restriction of scalars in modules over rings. Recall that, if $A \xrightarrow{\varphi} B$ is an algebra homomorphism and ${}_BN$ is a B-module, scalar multiplication by an element a of A on the restricted module ${}_AN$ is defined to be scalar multiplication by its image $\varphi(a)$.

dirim-restsc **6.5.4. Lemma.** **(i)** *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $Y \xrightarrow{f} X$ be the morphism defined by an algebra homomorphism $A \xrightarrow{\varphi} B$. If $\mathcal{N}$ is the $\mathcal{O}_Y$-module determined by a B-module ${}_BN$, its direct image $f_*\mathcal{N}$ is the $\mathcal{O}_X$-module determined by the A-module ${}_AN$.*

(ii) *Let $Y \xrightarrow{f} X$ be an affine morphism of varieties, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The direct image $f_*\mathcal{N}$ is an $\mathcal{O}_X$-module.*

proof. **(ii)** Let $U' \to U$ be an inclusion of affine open subsets of X, and let $V = f^{-1}U$ and $V' = f^{-1}U'$. These are affine open subsets of Y. The inclusion $V' \to V$ gives us a homomorphism $\mathcal{N}(V) \to \mathcal{N}(V')$, and therefore a homomorphism $[f_*\mathcal{N}](U) \to [f_*\mathcal{N}](U')$. To verify that $f_*\mathcal{N}$ is an $\mathcal{O}_X$-module, one must show that if U is an affine open subset of X and s is a nonzero element of $\mathcal{O}_X(U)$, then $[f_*\mathcal{N}](U_s)$ is obtained by localizing $[f_*\mathcal{N}](U)$. Let V be the inverse image of U and let s' be the image of s in $\mathcal{O}_Y(V)$. Then $[f_*\mathcal{N}](U_s) = \mathcal{N}(V_{s'}) = \mathcal{N}(V)_{s'}$, provided that $s' \neq 0$. If $s' = 0$, then both $[f_*\mathcal{N}](U_s)$ and $\mathcal{N}(V_{s'})$ will be zero. $\square$

fs-taraffmorph **6.5.5. Lemma.** **(i)** *Let $Y \xrightarrow{f} X$ be an affine morphism and let $\mathcal{N} \to \mathcal{N}' \to \mathcal{N}''$ be an exact sequence of $\mathcal{O}_Y$-modules. The sequence $f_*\mathcal{N} \to f_*\mathcal{N}' \to f_*\mathcal{N}''$ of direct images is exact.*

(ii) *Direct images are compatible with limits: If $\mathcal{M}_\bullet$ is a directed set of $\mathcal{O}$-modules, then $\varinjlim (f_*\mathcal{M}_\bullet) \approx f_*(\varinjlim \mathcal{M}_\bullet)$.* $\square$

Two important cases of direct image are that f is the inclusion of a closed subvariety, or of an affine open subvariety. We discuss those special cases now.

extbyzero **6.5.6. extension by zero – the inclusion of a closed subvariety**

Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety into a variety X. The direct image $i_*\mathcal{N}$ of an $\mathcal{O}_Y$-module $\mathcal{N}$ is also called the *extension by zero* of $\mathcal{N}$. If U is an affine open subset of X then, because i is an inclusion map, $i^{-1}U = U \cap Y$. Therefore

$$[i_*\mathcal{N}](U) = \mathcal{N}(U \cap Y)$$

The term “extension by zero” refers to the fact that, when an affine open set U of X doesn't meet Y, the intersection $U \cap Y$ is empty, and the module of sections of $[i_*\mathcal{N}](U)$ is zero. So $i_*\mathcal{N}$ is zero on the complement of Y.

extfrom-point **6.5.7. Examples.**

(i) Let $p \xrightarrow{i} X$ be the inclusion of a point into a variety. When we view the residue field $k(p)$ as a module on p, its extension by zero is the residue field module κ_p.

(ii) Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety, and let $\mathcal{I}$ be the ideal of Y in $\mathcal{O}_Y$. The extension by zero of the structure sheaf on Y fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

So $i_*\mathcal{O}_Y$ is isomorphic to the quotient module $\mathcal{O}_X/\mathcal{I}$. $\square$

izero **6.5.8. Proposition.** *Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X, and let $\mathcal{I}$ be the ideal of Y. Let $\mathbb{M}$ denote the subcategory of the category of $\mathcal{O}_X$-modules that are annihilated by $\mathcal{I}$. Extension by zero defines an equivalence of categories*

$$\mathcal{O}_Y - modules \xrightarrow{i_*} \mathbb{M}$$

proof. Let U be an affine open subset of X. The intersection $U \cap Y = V$ is a closed subvariety of U. Let α be an element of $i_*\mathcal{N}(U)\ \ \big(=\mathcal{N}(V)\big)$. If f is a section of $\mathcal{O}_X$ on U and $\overline{f}$ is its restriction to V, then $f\alpha = \overline{f}\alpha$. If f is in $\mathcal{I}(U)$, then $\overline{f} = 0$ and therefore $f\alpha = 0$. So the extension by zero of an $\mathcal{O}_Y$-module is annihilated by $\mathcal{I}$. The direct image $i_*\mathcal{N}$ is an object of $\mathbb{M}$.

To complete the proof, we start with an $\mathcal{O}_X$-module $\mathcal{M}$ that is annihilated by $\mathcal{I}$, and we construct an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $i_*\mathcal{N}$ is isomorphic to $\mathcal{M}$.

Let Y' be an open subset of Y. The topology on Y is induced from the topology on X, so $Y' = X_1 \cap Y$ for some open subset X_1 of X. We try to set $\mathcal{N}(Y') = \mathcal{M}(X_1)$. To show that this is well-defined, we show that if X_2 is another open subset of X and if $Y' = X_2 \cap Y$, then $\mathcal{M}(X_2)$ is isomorphic to $\mathcal{M}(X_1)$. Let $X_3 = X_1 \cap X_2$. Then it is also true that $Y' = X_3 \cap Y$. Since $X_3 \subset X_1$, we have a map $\mathcal{M}(X_1) \to \mathcal{M}(X_3)$. It suffices to show that this map is an isomorphism, because the same reasoning will give us an isomorphism $\mathcal{M}(X_2) \to \mathcal{M}(X_3)$.

The complement $U = X_1 - Y'$ of Y' in X_1 is an open subset of X_1 and of X, and $U \cap Y = \emptyset$. We cover U by a set $\{U^i\}$ of affine open sets. The restriction of $\mathcal{I}$ to each of the sets U^i is the unit ideal, and since $\mathcal{I}$ annihilates $\mathcal{M}$, $\mathcal{M}(U^i) = 0$. Then X_1 is covered by the open sets $\{U^i\}$ together with X_3. The sheaf property shows that $\mathcal{M}(X_1)$ is isomorphic to $\mathcal{M}(X_3)$.

The rest of the proof, checking compatibility with localization and verifying that $\mathcal{N}$ is determined up to isomorphism, is boring. $\square$

6.5.9. inclusion of an affine open subset includeopen

Let $Y \xrightarrow{j} X$ be the inclusion of an affine open subvariety Y into a variety X.

Before going to the direct image, we mention a rather trivial operation, the *restriction* of an $\mathcal{O}_X$- module from X to the open set Y. By definition, the sections of the restricted module on a subset U of Y are simply the elements of $\mathcal{M}(U)$. This makes sense because open subsets of Y are also open subsets of X. We use subscript notation for restriction, writing $\mathcal{M}_Y$ for the restriction of an $\mathcal{O}_X$-module $\mathcal{M}$ to Y, and denoting the given module $\mathcal{M}$ by $\mathcal{M}_X$ when that seems advisable for clarity. If U is an open subset of Y, then

(6.5.10) $$\mathcal{M}_Y(U) = \mathcal{M}_X(U)$$ restrtoopen

The restriction of the structure sheaf $\mathcal{O}_X$ to the open set Y is the structure sheaf $\mathcal{O}_Y$ on Y. So the subscript notation is permissible.

Now the direct image: Let $Y \xrightarrow{j} X$ be the inclusion of an affine open subvariety Y, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The inverse image of an open subset U of X is the intersection $Y \cap U$, which is affine. So the direct image $j_*\mathcal{N}$ is defined by

$$[j_*\mathcal{N}](U) = \mathcal{N}(Y \cap U)$$

For example, $[j_*\mathcal{O}_Y](U)$ is the algebra of rational functions on X that are regular on $Y \cap U$.

6.5.11. Example. Let $X_s \xrightarrow{j} X$ be the inclusion of a localization X_s into an affine variety $X = \operatorname{Spec} A$. Modules on X correspond to their global sections, which are A-modules. Similarly, modules on X_s correspond to A_s-modules. Let $\mathcal{M}_X$ be the $\mathcal{O}_X$-module that corresponds to an A-module M. When we restrict $\mathcal{M}_X$ to the open set X_s, we obtain the $\mathcal{O}_{X_s}$-module $\mathcal{M}_{X_s}$ that corresponds to the A_s-module M_s. The module M_s is also the module of global sections of $j_*\mathcal{M}_{X_s}$ on X: examplledirectimage

$$[j_*\mathcal{M}_{X_s}](X) \overset{def}{=} \mathcal{M}_{X_s}(X_s) = M_s$$

The localization M_s is made into an A-module by restriction of scalars. $\square$

6.5.12. Proposition. *Let $Y \xrightarrow{j} X$ be the inclusion of an affine open subvariety Y into a variety X.* *fstarexact*

(i) *The restriction from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules is an exact operation.*

(ii) *The direct image j_* is an exact functor.*

(iii) *Let $\mathcal{M}_X$ be an $\mathcal{O}_X$-module, and let $\mathcal{M}_Y$ be its restriction to Y. There is a canonical homomorphism $\mathcal{M}_X \to j_*[\mathcal{M}_Y]$ from $\mathcal{M}_X$ to the direct image of $\mathcal{M}_Y$.*

(iv) *Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Then $[j_*\mathcal{N}]_Y = \mathcal{N}$. The restriction to Y of its direct image $j_*\mathcal{N}$ is equal to $\mathcal{N}$.*

proof. **(ii)** Let U be an affine open subset of X, and let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be an exact sequence of $\mathcal{O}_Y$-modules. The sequence $j_*\mathcal{M}(U) \to j_*\mathcal{N}(U) \to j_*\mathcal{P}(U)$ is the same as the sequence $\mathcal{M}(U \cap Y) \to \mathcal{N}(U \cap Y) \to \mathcal{P}(U \cap Y)$, though the scalars have changed. Since U and Y are affine, $U \cap Y$ is affine. By definition of exactness, this sequence is exact.

(iii) Let U be open in X. Then $j_*\mathcal{M}_Y(U) = \mathcal{M}(U \cap Y)$. Since $U \cap Y \subset U$, $\mathcal{M}(U)$ maps to $\mathcal{M}(U \cap Y)$.

(iv) An open subset V of Y is also open in X, and $[j_*\mathcal{N}]_Y(V) = [j_*\mathcal{N}](V) = \mathcal{N}(V \cap Y) = \mathcal{N}(V)$. $\square$

dirim-standaffine **6.5.13. Example.** Let $X = \mathbb{P}^n$, and let j denote the inclusion of the standard affine open subset $\mathbb{U}^0$ into X. The direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ is the algebra of rational functions that are allowed to have poles on the hyperplane at infinity. The sections of the direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ on an open subset W of X are the regular functions on $W \cap \mathbb{U}^0$:

$$[j_*\mathcal{O}_{\mathbb{U}^0}](W) \;=\; \mathcal{O}_{\mathbb{U}^0}(W \cap \mathbb{U}^0) \;=\; \mathcal{O}_X(W \cap \mathbb{U}^0)$$

Say that we write a rational function α on X as a fraction g/h of relatively prime polynomials. Then α is a section of $\mathcal{O}_X$ on W if h doesn't vanish at any point of W, and α is a section of $[j_*\mathcal{O}_{\mathbb{U}^0}]$ on W if h doesn't vanish on $W \cap \mathbb{U}^0$. Arbitrary powers of x_0 can appear in the denominator of a section of $j_*\mathcal{O}_{\mathbb{U}^0}$. $\square$

6.6 Support

annand-supp

annihilators. Let M be a module over a ring A. The annihilator I of an element m of M is the set of elements α of A such that $\alpha m = 0$. It is an ideal of A that is often denoted by $\operatorname{ann}(m)$. The *annihilator* of an A-module M is the set of elements of A such that $aM = 0$. It is an ideal too.

support. We define support only for finite modules. Let A be a finite-type domain and let $X = \operatorname{Spec} A$. The support C of a finite A-module M is the locus of zeros of its annihilator I, the set of points p of X such that $I \subset \mathfrak{m}_p$. The support of a finite module is a closed subset of X.

The next lemma allows us to extend the concepts of annihilator and support to finite $\mathcal{O}$-modules on a variety X.

localize-support **6.6.1. Lemma.** *Let $X = \operatorname{Spec} A$ be an affine variety, let I be the annihilator of an element m of an A-module M, and let s be a nonzero element of A. The annihilator of m in the localized module M_s is the localized ideal I_s. If M is a finite module with support C, the support of M_s is the intersection $C \cap X_s$.* $\square$

If $\mathcal{I}$ is the annihilator of a finite $\mathcal{O}$-module $\mathcal{M}$ on a variety X, the *support* of $\mathcal{M}$ is defined to be the closed subset $V(\mathcal{I})$. For example, the support of the residue field module κ_p is the point p. The support of the maximal ideal $\mathfrak{m}_p$ at p is the whole variety X. If C is the support of $\mathcal{M}$, and if an open subset U of X doesn't meet C, then $\mathcal{M}(U) = 0$.

support-dimzero **6.6.2. $\mathcal{O}$-modules with support of dimension zero**

suppfinite **6.6.3. Proposition.** *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a variety X.*

(i) *Suppose that the support of $\mathcal{M}$ is a single point p, let $M = \mathcal{M}(X)$, and let U be an affine open subset of X. If U contains p, then $\mathcal{M}(U) = M$, and if U doesn't contain p, then $\mathcal{M}(U) = 0$.*

(ii) *(Chinese Remainder Theorem) If the support of $\mathcal{M}$ is a finite set $\{p_1, ..., p_k\}$, then $\mathcal{M}$ is the direct sum $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ (or the direct product) of $\mathcal{O}$-modules supported at the points p_i.*

proof. **(i)** Let $\mathcal{I}$ be the annihilator of an $\mathcal{O}$-module $\mathcal{M}$ whose support is p. So the locus $V(\mathcal{I})$ is p. If p isn't contained in U then, when we restrict $\mathcal{M}$ to U, we obtain an $\mathcal{O}_U$-module whose support is empty. So $\mathcal{I}(U)$ is the unit ideal, and the restriction of $\mathcal{M}$ to U is the zero module.

Next, suppose that p is contained in U, and let V denote the complement of p in X. We cover X by a finite set $\{U^i\}$ of affine open sets such that $U = U^1$, and such that $U^i \subset V$ if $i > 1$. By what has been shown, $\mathcal{M}(U^i) = 0$ if $i > 1$. The sheaf property for this covering shows that $\mathcal{M}(X) \approx \mathcal{M}(U)$. $\square$

Note. If i denotes the inclusion of a point p into a variety X, one might suppose that an $\mathcal{O}$-module $\mathcal{M}$ supported at p will be the extension by zero of a module on the point p (a vector space). However, this won't be true unless $\mathcal{M}$ is annihilated by the maximal ideal $\mathfrak{m}_p$. $\square$

6.7 Twisting

The twisting modules that we define here are among the most important modules on projective space $\mathbb{P}^d$. twisting-modules

As before, a homogeneous fraction of degree n in $x_0, ..., x_d$ is a fraction g/h of homogeneous polynomials, such that $\deg\, g - \deg\, h = n$. If g and h are relatively prime, the fraction g/h is regular on an open set V if and only if h isn't zero at any point of V.

The definition of the *twisting module* $\mathcal{O}(n)$ is this: The sections of $\mathcal{O}(n)$ on an open subset V of $\mathbb{P}^d$ are the homogeneous fractions of degree n that are regular on V. So in particular, $\mathcal{O}(0) = \mathcal{O}$.

6.7.1. Proposition. *Odismodule*

(i) *Let V be an affine open subset of $\mathbb{P}^d$ that is contained in the standard affine open set $\mathbb{U}^0$. The sections of the twisting module $\mathcal{O}(n)$ on V form a free module of rank 1 with basis x_0^n over the coordinate algebra $\mathcal{O}(V)$.*
(ii) *The twisting module $\mathcal{O}(n)$ is an $\mathcal{O}$-module.*

proof. **(i)** Let V be an open set contained in $\mathbb{U}^0$, and let $\alpha = g/h$ be a section of $\mathcal{O}(n)$ on V, with g, h relatively prime. Then $f = \alpha x_0^{-n}$ has degree zero. It is a rational function. Since $V \subset \mathbb{U}^0$, x_0 doesn't vanish at any point of V. Since α is regular on V, f is a regular function on V, and $\alpha = f x_0^n$.

(ii) It is clear that $\mathcal{O}(n)$ is a contravariant functor. We verify compatibility with localization. Let $V = \operatorname{Spec} A$ be an affine open subset of X and let s be a nonzero element of A. We must show that $[\mathcal{O}(n)](V_s)$ is the localization of $[\mathcal{O}(n)](V)$, or that, if β is a section of $\mathcal{O}(n)$ on V_s, then $s^k\beta$ is a section on V when k is sufficiently large.

We cover V by the affine open sets $V^i = V \cap \mathbb{U}^i$. It suffices to show that $s^k\beta$ is a section on V^i for every i. Since V_s^0 is contained in $\mathbb{U}^0$, part **(i)** tells us that β can be written uniquely in the form $f x_0^n$, where f is a regular function on V_s^0 and n is an integer. Then $s^k f$ is a regular function on V^0 when k is large, and $s^k\alpha = s^k f x_0^n$ is a section of $\mathcal{O}(n)$ on V^0. The analogous statement is true for every index $i = 0, ..., d$. $\square$

As part **(i)** of the proposition shows, $\mathcal{O}(n)$ is quite similar to the structure sheaf. However, though it is a free module on each of the sets $\mathbb{U}^i$, $\mathcal{O}(n)$ is only *locally* free.

6.7.2. Proposition. *When $n \geq 0$, the global sections of the twisting module $\mathcal{O}(n)$ on $\mathbb{P}^d$ are the homogeneous polynomials of degree n. When $n < 0$, the only global section of $\mathcal{O}(n)$ is the zero section.* *sectionsofOn*

proof. A nonzero global section u of $\mathcal{O}(n)$ will restrict to a section on the standard affine open set $\mathbb{U}^0$. Since elements of $\mathcal{O}(\mathbb{U}^0)$ are homogeneous fractions of degree zero whose denominators are powers of x_0, and since $[\mathcal{O}(n)](\mathbb{U}^0)$ is a free module with basis x_0^n, we will have $u = f x_0^n$ for some function f that is regular on $\mathbb{U}^0$, and f can be written as a homogeneous fraction of degree zero whose denominator is a power of x_0, say $f = g/x_0^r$. So u will have the form g/x_0^s, where $s = r - n$. Similarly, restriction to $\mathbb{U}^1$ shows that u has the form g_1/x_1^ℓ. It follows that $k = \ell = 0$ and that u is a polynomial of degree n. $\square$

6.7.3. Examples. prodhomfr

(i) The product uv of homogeneous fractions of degrees r and s has degree $r+s$, and if u and v are regular on an open set V, so is their product uv. So multiplication defines a homomorphism of $\mathcal{O}$-modules

(6.7.4) $$\mathcal{O}(r) \oplus \mathcal{O}(s) \to \mathcal{O}(r+s)$$ oros

(ii) Multiplication by a homogeneous polynomial f of degree n defines an injective homomorphism

(6.7.5) $$\mathcal{O}(k) \xrightarrow{f} \mathcal{O}(k+n)$$ $\square$ multbyfone

The twisting modules $\mathcal{O}(n)$ have a second interpretation. They are isomorphic to the modules that we denote by $\mathcal{O}(nH)$, of rational functions with poles of order at most n on the hyperplane $H : \{x_0 = 0\}$ at infinity.

The definition of $\mathcal{O}(nH)$ is this: Its sections on an open set V are the rational functions f such that $x_0^n f$ is a section of $\mathcal{O}(n)$ on V. Multiplication by x_0^n defines an isomorphism

(6.7.6) $$\mathcal{O}(nH) \xrightarrow{x_0^n} \mathcal{O}(n)$$ OnOnH

If a rational function f is a section of $\mathcal{O}(nH)$ on an open set V, and if we write f as a homogeneous fraction g/h of degree zero, with g, h relatively prime, the denominator h may have x_0^k, with $k \le n$, as factor. The other factors of h cannot vanish anywhere on V. If f is a *global* section of $\mathcal{O}(nH)$, the denominator h has the form cx_0^k, with $c \in \mathbb{C}$ and $k \le n$, so f can be written as a homogeneous fraction g/x_0^k of degree zero.

Since x_0 doesn't vanish at any point of the standard open set $\mathbb{U}^0$, the sections of $\mathcal{O}(nH)$ on an open subset V of $\mathbb{U}^0$ are simply the regular functions on V. Using the subscript notation (6.5.9) for restriction to an open set,

jstarOH (6.7.7)
$$\mathcal{O}(nH)_{\mathbb{U}^0} = \mathcal{O}_{\mathbb{U}^0}$$

Let V be an open subset of one of the other standard affine open sets, say of $\mathbb{U}^1$. The ideal of $H \cap \mathbb{U}^1$ in $\mathbb{U}^1$ is the principal ideal generated by $v_0 = x_0/x_1$, and the element v_0 generates the ideal of $H \cap V$ in V too. If f is a rational function, then because x_1 doesn't vanish on $\mathbb{U}^1$, the function fv_0^n will be regular on V if and only if the homogeneous fraction fx_0^n is regular there, i.e., if an only if f is a section of $\mathcal{O}(nH)$ on V. We say that such a function f has a pole of order at most n on H because v_0 generates the ideal of H in V.

The isomorphic $\mathcal{O}$-modules $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ are interchangeable. The twisting module $\mathcal{O}(n)$ is often better because its definition is independent of coordinates. On the other hand, $\mathcal{O}(nH)$ can be convenient because its restriction to $\mathbb{U}^0$ is the structure sheaf.

curveideal **6.7.8. Proposition.** *Let Y be a hypersurface of degree n in $\mathbb{P}^d$, the zero locus of an irreducible homogeneous polynomial f of degree n. Let $\mathcal{I}$ be the ideal of Y, and let $\mathcal{O}(-n)$ be the twisting module. Multiplication by f defines an isomorphism $\mathcal{O}(-n) \xrightarrow{f} \mathcal{I}$.*

proof. We choose coordinates so that f isn't isn't divisible by any of the coordinate variables x_i.

If α is a section of $\mathcal{O}(-n)$ on an open set V, then $f\alpha$ will be a regular function on V that vanishes on $Y \cap V$. Therefore the image of the multiplication map $\mathcal{O}(-n) \xrightarrow{f} \mathcal{O}$ is contained in $\mathcal{I}$. The multiplication map is injective because $\mathbb{C}[x_0, ..., x_d]$ is a domain. To show that it is an isomorphism, it suffices to show that its restrictions to the standard affine open sets $\mathbb{U}^i$ are surjective (6.4.2). We work with $\mathbb{U}^0$, as usual.

Because x_0 desn't divide f, $Y \cap \mathbb{U}^0$ will be a nonempty, and therefore dense, open subset of Y. The sections of $\mathcal{O}$ on $\mathbb{U}^0$ are the homogeneous fractions g/x_0^k of degree zero. Such a fraction is a section of $\mathcal{I}$ on $\mathbb{U}^0$ if and only if g vanishes on $Y \cap \mathbb{U}^0$. If so, then since $Y \cap \mathbb{U}^0$ is dense in Y and since the zero set of g is closed, g will vanish on Y, and therefore it will be divisible by f, say $g = fq$. The sections of $\mathcal{I}$ on $\mathbb{U}^0$ have the form fq/x_0^k. They are in the image of the map $\mathcal{O}(-n) \xrightarrow{f} \mathcal{I}$, so f is an isomorphism. □

The proposition has an interesting corollary:

idealsisom **6.7.9. Corollary.** *When regarded as $\mathcal{O}$-modules, the ideals of all hypersurfaces of degree n in $\mathbb{P}^d$ are isomorphic.* □

twistmodule **6.7.10. twisting a module**

deftwistm **6.7.11. Definition** Let $\mathcal{M}$ be an $\mathcal{O}$-module on projective space $\mathbb{P}^d$, and let $\mathcal{O}(n)$ be the twisting module. The (nth) *twist* of $\mathcal{M}$ is the tensor product $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(n)$. Similarly, $\mathcal{M}(nH) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(nH)$. If X is a closed subvariety of $\mathbb{P}^d$ and $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $\mathcal{M}(n)$ and $\mathcal{M}(nH)$ are obtained by twisting the extension of $\mathcal{M}$ by zero. (See the equivalence of categories (6.5.8).)

twistexact **6.7.12. Lemma.** *The operation of twisting is an exact functor on $\mathcal{O}$-modules. If S is an exact sequence of $\mathcal{O}$-modules, the sequences $S(n)$ and $S(nH)$ are exact.*

proof. This is true because $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ are locally free of rank one (6.7.1). □

Since x_0^n is a basis of $\mathcal{O}(n)$ on $\mathbb{U}^0$, a section of $\mathcal{M}(n)$ on an open subset V of $\mathbb{U}^0$ can be written as a sum of elements of the form $m \otimes gx_0^n$, where g is a regular function on V and m is a section of $\mathcal{M}$ on V (6.7.1). The function g can be moved over to m, so a section of $\mathcal{M}(n)$ can also be written as a sum of elements of the form $m \otimes x_0^n$.

The modules $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ form directed sets that are related by a diagram

(6.7.13)
$$\begin{array}{ccccccc}
\mathcal{O} & \xrightarrow{\subset} & \mathcal{O}(H) & \xrightarrow{\subset} & \mathcal{O}(2H) & \xrightarrow{\subset} & \cdots \\
\| & & \Big\downarrow{\scriptstyle x_0} & & \Big\downarrow{\scriptstyle x_0^2} & & \\
\mathcal{O} & \xrightarrow{x_0} & \mathcal{O}(1) & \xrightarrow{x_0} & \mathcal{O}(2) & \longrightarrow & \cdots
\end{array}$$
Oninclusions

In this diagram, the vertical arrows are bijections and the horizontal arrows are injections. The limit of the upper directed set is the module whose sections on an open set V are rational functions that can have arbitrary poles on $H \cap V$, and are otherwise regular. This is also the module $j_*\mathcal{O}_{\mathbb{U}^0}$, where j denotes the inclusion of the standard affine open set $\mathbb{U}^0$ into X (see (6.5.12) **(iii)**):

(6.7.14)
$$\varinjlim_n \mathcal{O}(nH) = j_*\mathcal{O}_{\mathbb{U}^0}$$
limequalsjstar

Tensoring (6.7.13) with $\mathcal{M}$ gives us the diagram

(6.7.15)
$$\begin{array}{ccccccc}
\mathcal{M} & \longrightarrow & \mathcal{M}(H) & \longrightarrow & \mathcal{M}(2H) & \longrightarrow & \cdots \\
\| & & \Big\downarrow{\scriptstyle x_0} & & \Big\downarrow{\scriptstyle x_0^2} & & \\
\mathcal{M} & \xrightarrow{1\otimes x_0} & \mathcal{M}(1) & \xrightarrow{1\otimes x_0} & \mathcal{M}(2) & \longrightarrow & \cdots
\end{array}$$
Mnmaps

Here the vertical maps are bijective, but $\mathcal{M}$ may have torsion. The horizontal maps may not be injective.

Let $\mathbb{U} = \mathbb{U}^0$. Since tensor products are compatible with limits,

(6.7.16)
$$\varinjlim_n \mathcal{M}(nH) \overset{(1)}{\approx} \mathcal{M} \otimes_{\mathcal{O}} \big(\varinjlim_n \mathcal{O}(nH)\big) \approx \mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}} \overset{(2)}{\approx} j_*\mathcal{M}_{\mathbb{U}}$$
limjstar

The isomorphism (1) comes from the fact that tensor produts are compatible with limits, and (2) is part **(ii)** of the next lemma.

6.7.17. Lemma. *Let $\mathcal{M}$ be an $\mathcal{O}$-module on $\mathbb{P}^d$, and let j be the inclusion of $\mathbb{U}^0$ into $\mathbb{P}^d$.* *Mjstarequaljstar M*
(i) *For every k, the restriction of $\mathcal{M}(kH)$ to $\mathbb{U}^0$ is the same as the restriction $\mathcal{M}^0_{\mathbb{U}}$ of $\mathcal{M}$, and the restriction of $j_*\mathcal{M}_{\mathbb{U}^0}$ to $\mathbb{U}^0$ is also $\mathcal{M}_{\mathbb{U}^0}$. The restriction of the map $\mathcal{M}(kH) \to j_*(\mathcal{M}_{\mathbb{U}^0})$ to $\mathbb{U}^0$ is the identity map.*
(ii) *The direct image $j_*\mathcal{M}_{\mathbb{U}^0}$ is isomorphic to $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0}$.*

proof. **(i)** Because $H \cap \mathbb{U}^0$ is empty, the restrictions of $\mathcal{M}(kH)$ and $\mathcal{M}$ to $\mathbb{U}^0$ are equal. The fact that the restriction of $j_*\mathcal{M}_{\mathbb{U}^0}$ is also equal to $\mathcal{M}_{\mathbb{U}^0}$ is Proposition 6.5.12 **(iv)**.

(ii) Suppose given a section of $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0}$ of the form $\alpha \otimes f$ on an open set V, where α is a section of $\mathcal{M}$ on V and f is a section of $j_*\mathcal{O}_{\mathbb{U}^0}$ on V, a regular function on $V \cap \mathbb{U}^0$. We denote the restriction of α to $V \cap \mathbb{U}^0$ by the same symbol α. Then αf will be a section of $\mathcal{M}$ on $V \cap \mathbb{U}^0$ and therefore a section of $j_*\mathcal{M}_{\mathbb{U}^0}$ on V. The map $(\alpha, f) \to \alpha f$ is $\mathcal{O}$-bilinear, so it corresponds to a homomorphism $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0} \to j_*\mathcal{M}_{\mathbb{U}^0}$. To show that this homomorphism is an isomorphism, it suffices to verify that it restricts to an isomorphism on each of the standard affine open sets $\mathbb{U}^i$. The restrictions of $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0}$ and $j_*\mathcal{M}_{\mathbb{U}^0}$ to $\mathbb{U}^0$ are both equal to $\mathcal{M}_{\mathbb{U}^0}$. So that case is trivial. We look at $\mathbb{U}^1$. On that open set, $[j_*\mathcal{M}_{\mathbb{U}^0}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^{01})$, and with $v_0 = x_0/x_1$, $[j_*\mathcal{O}_{\mathbb{U}^0}](\mathbb{U}^1) = \mathcal{O}(\mathbb{U}^{01}) = \mathcal{O}(\mathbb{U}^1)[v_0^{-1}]$. By definition of the tensor product,

$$[\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^1) \otimes_{\mathcal{O}(\mathbb{U}^1)} \mathcal{O}(\mathbb{U}^{01}) = \mathcal{M}(\mathbb{U}^1)[v_0^{-1}] = \mathcal{M}(\mathbb{U}^{01}) = [j_*\mathcal{M}_{\mathbb{U}^0}](\mathbb{U}^1) \qquad \square$$

6.7.18. generating an $\mathcal{O}$-module generators

A set $m = (m_1, ..., m_k)$ of global sections of an $\mathcal{O}$-module on a variety X defines a map

(6.7.19)
$$\mathcal{O}^k \xrightarrow{m} \mathcal{M}$$
gen

that sends a section $(\alpha_1, ..., \alpha_k)$ of $\mathcal{O}^k$ on an open set to the combination $\sum \alpha_i m_i$. The global sections $m_1, ..., m_k$ are said to *generate* $\mathcal{M}$ if this map is surjective. If the sections generate $\mathcal{M}$, their restrictions generate the $\mathcal{O}(U)$-module $\mathcal{M}(U)$ for every affine open set U. They may fail to generate $\mathcal{M}(U)$ when U isn't affine.

generate-example

6.7.20. Example. Let $X = \mathbb{P}^1$. For $n \geq 0$, the global sections of the twisting module $\mathcal{O}(n)$ are the polynomials of degree n in the coordinate variables x_0, x_1 (6.7.2). Consider the map $\mathcal{O} \oplus \mathcal{O} \xrightarrow{(x_0^n, x_1^n)} \mathcal{O}(n)$. On $\mathbb{U}^0$, $\mathcal{O}(n)$ has basis x_0^n. So the map is surjective on $\mathbb{U}^0$. Similarly, it is surjective on $\mathbb{U}^1$. It is a surjective map on all of X (6.4.2). The global sections x_0^n, x_1^n generate $\mathcal{O}(n)$. However, the global sections of $\mathcal{O}(n)$ are the homogeneous polynomials of degree n. When $n > 1$, the two sections x_0^n, x_1^n don't span the space of global sections. □

The next theorem explains the importance of the twisting operation.

gentwist

6.7.21. Theorem. *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. For sufficiently large k, the twist $\mathcal{M}(k)$ is generated by global sections.*

proof. We may assume that X is the projective space $\mathbb{P}^d$. To show that $\mathcal{M}(k)$ is generated by its global sections, it suffices to show that for each $i = 0, ..., n$, the restrictions of the global sections generate the $\mathcal{O}(\mathbb{U}^i)$-module $[\mathcal{M}(k)](\mathbb{U}^i)$ (6.4.2). We work with the index $i = 0$, and we replace $\mathcal{M}(k)$ by the isomorphic module $\mathcal{M}(kH)$. Recall that $\varinjlim_k \mathcal{M}(kH) = j_*\mathcal{M}_{\mathbb{U}^0}$ (6.7.16).

We have maps $\mathcal{M} \xrightarrow{1\otimes x_0^k} \mathcal{M}(kH) \to j_*\mathcal{M}_{\mathbb{U}^0}$, and they are isomorphisms when restricted to $\mathbb{U}^0$. Let $A_0 = \mathcal{O}(\mathbb{U}^0)$ and $M_0 = \mathcal{M}(\mathbb{U}^0)$, which is a finite A_0-module because $\mathcal{M}$ is a finite $\mathcal{O}$-module. We choose a finite set of generators $m_1, ..., m_r$ for the A_0-module M_0. The elements of M_0, and in particular, the chosen generators, are global sections of $j_*\mathcal{M}_{\mathbb{U}^0}$. Since $\varinjlim_k \mathcal{M}(kH) = j_*\mathcal{M}_{\mathbb{U}^0}$, they are represented by global sections $m_1', ..., m_r'$ of $\mathcal{M}(kH)$, when k is large. The restrictions of $\mathcal{M}(kH)$ and $\mathcal{M}$ to $\mathbb{U}_0$ are equal, and the restrictions of m_i' to $\mathbb{U}^0$ is equal to the restriction of m_i. So the restrictions of $m_1', ..., m_r'$ generate M_0 too. Therefore M_0 is generated by global sections of $\mathcal{M}(kH)$, as was to be shown. □

6.8 Extending a Module: proof

globalsections

Recall that, when $\{U^i\}$ is an affine covering of an open subset Y, the sheaf property for $\mathcal{M}$ asserts that the sequence

sheafpropthee

$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha} \prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^{ij}) \tag{6.8.1}$$

is exact, where α is the product of the restriction maps, and β is the difference map described in (6.3.3). So $\mathcal{M}(Y)$ is mapped isomorphically to the kernel of β. But, as mentioned in (6.3), the morphism $U^i \to U^{ij}$ needn't be a localization. This point is addressed below, in **Step 2**.

You will want to read about sheaves sometime, but we make do with the minimum here. The proof we give is a bootstrap operation.

sheafcoveraffine

6.8.2. Step 1. *The sheaf property for a covering $\{U^i\}$ of an affine variety Y by localizations.*

We suppose that an affine open subset $Y = \operatorname{Spec} A$ of X is covered by a family of localizations $\mathbf{U}_0 = \{U_{s_i}\}$. Let $\mathcal{M}$ be an $\mathcal{O}$-module, and let M, M_i, and M_{ij} denote the modules of sections of $\mathcal{M}$ on Y, U_{s_i}, and $U_{s_i s_j}$, respectively. The sheaf property for the covering diagram $Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$ is the exact sequence

localize-module

$$0 \to M \xrightarrow{\alpha} \prod M_i \xrightarrow{\beta} \prod M_{ij} \tag{6.8.3}$$

where α sends an element m of M to the vector $(m, ..., m)$ of its images in $\prod_i M_i$, and the difference map β sends a vector $(m_1, ..., m_k)$ in $\prod_i M_i$ to the matrix (z_{ij}), with $z_{ij} = m_j - m_i$ in M_{ij} (6.3.4). We must show that this sequence is exact.

exactness at M: Since the open sets U^i cover Y, the elements $s_1, ..., s_k$ generate the unit ideal. Let m be an element of M that maps to zero in every M_i. Then there exists an n such that $s_i^n m = 0$, and we can use the same exponent n for all i. The elements also s_i^n generate the unit ideal. Writing $\sum a_i s_i^n = 1$, we have $m = \sum a_i s_i^n m = \sum a_i 0 = 0$.

exactness at $\prod M_i$: Let m_i be elements of M_i such that $m_j = m_i$ in M_{ij} for all i, j. We must find an element w in M that maps to m_j in M_j for every j.

We write m_i as a fraction: $m_i = s_i^{-n}x_i$, or $x_i = s_i^n m_i$, with x_i in M. Since we can replace s_i by a power, we may assume that $n = 1$. The equation $m_j = m_i$ in M_{ij} tells us that $s_i x_j = s_j x_i$ in M_{ij}. Since M_{ij} is the localization $M_{s_i s_j}$, $(s_i s_j)^r s_i x_j = (s_i s_j)^r s_j x_i$, or $s_i^{r+1} s_j^r x_j = s_j^{r+1} s_i^r x_i$, will be true in M, if r is large.

We adjust the notation: Let $\widetilde{x}_i = s_i^r x_i$, and $\widetilde{s}_i = s_i^{r+1}$. Then in M, $\widetilde{x}_i = \widetilde{s}_i m_i$ and $\widetilde{s}_i \widetilde{x}_j = \widetilde{s}_j \widetilde{x}_i$. The elements $\widetilde{s}_i$ generate the unit ideal. So there is an equation in A, of the form $\sum a_i \widetilde{s}_i = 1$.

Let $w = \sum a_i \widetilde{x}_i$. This is an element of M, and in M,

$$\widetilde{x}_j = \big(\sum_i a_i \widetilde{s}_i\big)\widetilde{x}_j = \sum_i a_i \widetilde{s}_j \widetilde{x}_i = \widetilde{s}_j w$$

So $\widetilde{x}_j = \widetilde{s}_j w$ and also $\widetilde{x}_j = \widetilde{s}_j m_j$. Since $\widetilde{s}_j$ is invertible on $\mathbb{U}^j$, $w = m_j$, in M_j. Since j is arbitrary, w is the required element of M. □

6.8.4. Step 2. *extension of an $\mathcal{O}$-module to all maps between affine open sets.* extendallmaps

Let X be a variety, and let *(Affines)* denote the category whose objects are the affine open subsets of X, and whose morphisms are all inclusions. The two categories *(affines)* and *(Affines)* have the same objects, but there are more morphisms in *(Affines)*. We want to s extend an $\mathcal{O}_X$-module $\mathcal{M}$ to a functor *(Affines)*$^\circ \to$ *(modules)* that has the sheaf property.

Let $V \to U$ be a morphism in *(Affines)*, and let V' be an affine subset of V that is a localization of U, and therefore also a localization of V. The structure of $\mathcal{O}$-module on $\mathcal{M}$ gives us localization maps $\mathcal{M}(U) \xrightarrow{a} \mathcal{M}(V')$ and $\mathcal{M}(V) \xrightarrow{b} \mathcal{M}(V')$.

6.8.5. Lemma. *With notation as above, there is a unique homomorphism of modules $\mathcal{M}(U) \xrightarrow{\delta} \mathcal{M}(V)$, compatible with the ring homomorphism $\mathcal{O}(U) \to \mathcal{O}(V)$, and such that, for every localization V' of U that is contained in V, $b\delta = a$.* *findmap*

$$\begin{tikzcd}
\mathcal{M}(U) \arrow[r, "a"] \arrow[d, "\delta"'] & \mathcal{M}(V') \arrow[d, equal] \\
\mathcal{M}(V) \arrow[r, "b"] & \mathcal{M}(V')
\end{tikzcd}$$

proof. We cover V by a family $\mathbf{V}_0 = \{V^1, ..., V^r\}$ of open sets that are localizations of U and of V. Then V^{ij} are localizations of V^i and of V^j. The covering diagram $\mathbf{V}_1 \rightrightarrows \mathbf{V}_0 \to V$, combined with the map $V \to U$, gives us maps $\mathcal{M}(U) \xrightarrow{\gamma} \mathcal{M}(\mathbf{V}_0) \xrightarrow{\beta} \mathcal{M}(\mathbf{V}_1)$, and because the two maps $\mathbb{V}_1 \rightrightarrows U$ are equal, the image of γ is contained in the kernel of β. As Proposition 6.8.2 tells us, that kernel is $\mathcal{M}(V)$. This gives us a map $\mathcal{M}(U) \xrightarrow{\delta} \mathcal{M}(V)$. Since the map $\mathcal{M}(U) \to \mathcal{M}(\mathbf{V}_0)$ is a module homomorphism, so is δ.

We show that δ is independent of the choice of the covering $\mathbf{V}_0$. One can go from one affine covering to another in a finite number of steps, each of which adds or deletes a single affine open set. So to prove independence of the map $\mathcal{M}(U) \to \mathcal{M}(V)$ defined above, it suffices to relate a covering $\mathbf{V}_0$ to the covering $\mathbf{W}_0 = \{V^1, ..., V^r, Z\}$ obtained by adding one localization Z of U that is contained in V to $\mathbf{V}_0$. Let $\mathbf{W}_1$ be the family of intersections of pairs of elements of $\mathbf{W}_0$. The inclusion $\mathbf{V}_0 \subset \mathbf{W}_0$ defines a map $\mathcal{M}(\mathbf{W}_0) \to \mathcal{M}(\mathbf{V}_0)$, and a map $\mathcal{M}(\mathbf{W}_1) \to \mathcal{M}(\mathbf{V}_1)$. This gives us a diagram

$$\begin{tikzcd}
\mathcal{M}(U) \arrow[r] \arrow[d, equal] & \mathcal{M}(\mathbf{V}_0) \arrow[r, "\beta_{\mathbf{V}}"] & \mathcal{M}(\mathbf{V}_1) \\
\mathcal{M}(U) \arrow[r] & \mathcal{M}(\mathbf{W}_0) \arrow[r, "\beta_{\mathbf{W}}"] \arrow[u, "a"] & \mathcal{M}(\mathbf{W}_1) \arrow[u, "b"]
\end{tikzcd} \tag{6.8.6}$$

comparecovtwo

in which $\mathcal{M}(U)$ is mapped to the kernels of $\beta_{\mathbf{W}}$ and $\beta_{\mathbf{V}}$, both of which are equal to $\mathcal{M}(V)$. Looking at the diagram, one sees that the maps $\mathcal{M}(U) \to \mathcal{M}(V)$ defined using the two coverings $\mathbf{V}_0$ and $\mathbf{W}_0$ are the same.

Thus we have defined a unique module homomorphism $\mathcal{M}(U) \to \mathcal{M}(V)$. Since we can include an arbitrary localization V' of U in a covering, this is the required homomorphism. □

We have thus defined a homomorphism $\mathcal{M}(U) \to \mathcal{M}(V)$ for every inclusion of affine open sets. We verify compatibility with composition of maps. Let $W \subset V \subset U$ be affine open sets, and let $\mathcal{M}(U) \xrightarrow{f} \mathcal{M}(V)$,

$\mathcal{M}(V) \xrightarrow{g} \mathcal{M}(W)$ and $\mathcal{M}(U) \xrightarrow{h} \mathcal{M}(W)$ be the maps we have defined. We want to show that $h = gf$. We choose a covering $\mathbf{W}_0 = \{W^i\}$ of W by open sets that arelocalizations of U. The $\mathcal{O}$-module structure gives us maps $\mathcal{M}(W) \xrightarrow{a} \mathcal{M}(\mathbf{W}_0)$, $\mathcal{M}(V) \xrightarrow{b} \mathcal{M}(\mathbf{W}_0)$, and $\mathcal{M}(U) \xrightarrow{c} \mathcal{M}(\mathbf{W}_0)$, and Lemma 6.8.5 tells us that $ag = b$, $bf = c$, and $ah = c$. Then $agf = ah$. Since $\mathbf{W}_0$ is an arbitrary family of localizations of U, h is uniquely determined by the equation $ah = c$. So $gf = h$.

To show that this extended functor has the sheaf property for an arbitrary affine covering $\mathbf{V}_0 = \{V^i\}$ of an affine variety U, we let $\mathbf{W}_0$ be the covering obtained by covering each V^i by localizations of U. We form a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{V}_0) & \xrightarrow{\beta_{\mathbf{V}}} & \mathcal{M}(\mathbf{V}_1) \\ & & \| & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow \\ 0 & \longrightarrow & \mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \xrightarrow{\beta_{\mathbf{W}}} & \mathcal{M}(\mathbf{W}_1) \end{array}$$

The sheaf property to be verified is that the top row of this diagram is exact. Since $\mathbf{W}_0$ is an affine covering of the affine variety U, the bottom row is exact. Because $\mathbf{W}_0$ covers $\mathbf{V}_0$, $\mathbf{W}_1$ covers $\mathbf{V}_1$ as well. So the maps a and b are injective. It follows that the top row is exact. $\square$

glsecaffine **6.8.7. Corollary.** *If U is an affine open set, the module of sections on U, as defined above, is $\mathcal{M}(U)$.*

proof. When U is affine the identity map is a covering family. $\square$

extmodule **6.8.8. Step 3.** *extension of an $\mathcal{O}$-module to the category (opens)*

We prove Theorem 6.3.2 now. The theorem asserts that an $\mathcal{O}$-module $\mathcal{M}$ extends uniquely to a functor

$$\textit{(opens)}^\circ \xrightarrow{\widetilde{\mathcal{M}}} \textit{(modules)}$$

that has the sheaf property (6.3.4). Moreover, for every open set U, $\widetilde{\mathcal{M}}(U)$ is an $\mathcal{O}(U)$-module, and for every inclusion $V \to U$ of nonempty open sets, the map $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is compatible with the map $\mathcal{O}(U) \to \mathcal{O}(V)$.

Some notation for the proof: Let Y be an open subset of X, and let $\mathbf{V}_0 = \{V^1, ...V^r\}$ be an affine covering of Y. Suppose that a covering diagram $Y \leftarrow \mathbf{V}_0 \leftleftarrows \mathbf{V}_1$ and an $\mathcal{O}$-module $\mathcal{M}$ are given, and let $\mathcal{M}(\mathbf{V}_0) \xrightarrow{\beta_{\mathbf{V}}} \mathcal{M}(\mathbf{V}_1)$ be the difference map. We denote the kernel of $\beta_{\mathbf{V}}$ by $\mathbf{K}_{\mathbf{V}}$. We want to define $\widetilde{\mathcal{M}}(Y) = \mathbf{K}_{\mathbf{V}}$. When we show that $\mathbf{K}_{\mathbf{V}}$ doesn't depend on the covering $\mathbf{V}_0$, it will follow that $\widetilde{\mathcal{M}}$ is well-defined, and that it has the sheaf property.

As in **Step 2**, it suffices to relate a covering $\mathbf{V}_0$ to the covering $\mathbf{W}_0 = \{V^1, ..., V^r, Z\}$ obtained by adding one affine open subset Z of Y to $\mathbf{V}_0$. The inclusion $\mathbf{V}_0 \subset \mathbf{W}_0$ defines maps $\mathcal{M}(\mathbf{W}_0) \to \mathcal{M}(\mathbf{V}_0)$ and $\mathcal{M}(\mathbf{W}_1) \to \mathcal{M}(\mathbf{V}_1)$. This gives us a map $\mathbf{K}_{\mathbf{W}} \to \mathbf{K}_{\mathbf{V}}$. We will show that, for any element $(v_1, ..., v_r)$ in the kernel $\mathbf{K}_{\mathbf{V}}$, there is a unique element z in $\mathcal{M}(Z)$ such that $(v_1, ..., v_r, z)$ is in the kernel $\mathbf{K}_{\mathbf{W}}$. This will show that $\mathbf{K}_{\mathbf{W}}$ and $\mathbf{K}_{\mathbf{V}}$ are isomorphic.

Let $U^i = V^i \cap Z$, $i = 1, ..., r$. Since $\mathbf{V}_0$ is an affine covering of Y, $\mathbf{U}_0 = \{U^i\}$ is an affine covering of Z. Let u_i denote the restriction of the section v_i to W^i. Since $(v_1, ..., v_r)$ is in the kernel $\mathbf{K}_{\mathbf{V}}$, i.e., $v_i = v_j$ on V^{ij}, it is also true that $u_i = u_j$ on the smaller open set W^{ij}. So $(u_1, ..., u_r)$ is in the kernel $\mathbf{K}_{\mathbf{W}}$, and since $\mathbf{W}_0$ is an affine covering of the affine variety Z, **Step 2** tells us that $\mathbf{K}_{\mathbf{U}} = \mathcal{M}(Z)$. So there is a unique element z in $\mathcal{M}(Z)$ that restricts to $w = u_i$ on W^i for each i. We show that, with this element z, $(v_1, ..., v_r, z)$ is in the kernel $\mathbf{K}_{\mathbf{W}}$.

When the subsets in the family $\mathbf{W}_1$ are listed in the order

$$\mathbf{W}_1 = \{V^i \cap V^j\}_{ij}, \{Z \cap V^j\}_j, \{V^i \cap Z\}_i, \{Z \cap Z\}$$

the difference map $\beta_{\mathbf{W}}$ sends $(v_1, ..., v_r, z)$ to $\big[(v_j - v_i), (v_j - z), (z - v_i), 0\big]$, the sections being restricted appropriately. Here $v_i = v_j$ on $V^i \cap V^j$ because $(v_1, ..., v_r)$ is in the kernel $\mathbf{K}_{\mathbf{V}}$. By definition, $v_j = u_j = z$ on $V^j \cap Z = W^j$. So $(v_1, ..., v_r, z)$ is in $\mathbf{K}_{\mathbf{W}}$.

It remains to prove that this process defines a functor. However, this proof has no interesting features, and we won't use the functorality, so we omit it. $\square$

6.8.9. proof of Proposition 6.4.7.

The proposition states that an $\mathcal{O}$-module on $\mathbb{P}^1$ is determined by modules M_0 and M_1 over the algebras $A_0 = \mathbb{C}[u]$ and $A_1 = \mathbb{C}[u^{-1}]$ and an isomorphism $M_0[u^{-1}] \to M_1[v^{-1}]$. Proposition 6.4.14 shows that M_i defines $\mathcal{O}$-modules $\mathcal{M}_i$ on $\mathbb{U}^i$ for $i = 0, 1$, and the restrictions of $\mathcal{M}_0$ and $\mathcal{M}_1$ to $\mathbb{U}^{01}$ are isomorphic. Let's denote all of these modules by $\mathcal{M}$. Then $\mathcal{M}$ is defined on any open set that is contained, either in $\mathbb{U}^0$ or in $\mathbb{U}^1$.

Let V be an arbitrary open set V, and let $V^i = V \cap \mathbb{U}^i$, for $i = 0, 1, 01$. We define $\mathcal{M}(V)$ to be the kernel of the map $[\mathcal{M}(V^0) \times \mathcal{M}(V^1)] \to \mathcal{M}(V^{01})$. With this definition, $\mathcal{M}$ becomes a functor. We must verify the sheaf property, and the notation gets confusing. We suppose given an affine covering $\{W^\nu\}$ of V. We denote that covering by $\mathbf{W}_0$. The corresponding covering diagram is $V \leftarrow \mathbf{W}_0 \leftleftarrows \mathbf{W}_1$.

For $i = 0, 1, 01$, let $\mathbf{W}_j^i = \mathbf{W}_j \cap \mathbb{U}^i$. We form a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \longrightarrow & \mathcal{M}(\mathbf{W}_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V^0) \times \mathcal{M}(V^1) & \longrightarrow & \mathcal{M}(\mathbf{W}_0^0) \times \mathcal{M}(\mathbf{W}_0^1) & \longrightarrow & \mathcal{M}(\mathbf{W}_1^0) \times \mathcal{M}(\mathbf{W}_1^1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V^{01}) & \longrightarrow & \mathcal{M}(\mathbf{W}_0^{01}) & \longrightarrow & \mathcal{M}(\mathbf{W}_1^{01})
\end{array}$$

The columns are exact by our definition of $\mathcal{M}$, and the second and third rows are exact because the open sets making up $\mathbf{W}_j^i$ are contained in $\mathbb{U}^0$ or $\mathbb{U}^1$. Since kernel is a left exact operation, the top row is exact too. This is the sheaf property. $\square$

6.9 Exercises

chapsixex xsect-snotfin **6.9.1.** Let $A = \mathbb{C}[x, y]$ and let U be the complement of the origin in the affine plane $X = \operatorname{Spec} A$.

(i) Let $\mathcal{M}$ be the $\mathcal{O}_X$-module that correponds to the A-module $M = A/yA$. Show that, though $\mathcal{M}$ is a finite $\mathcal{O}$-module, $\mathcal{M}(U)$ isn't a finite $\mathcal{O}(U)$-module.

(ii) Show that, for any $k \geq 1$, the homomorphism $\mathcal{O} \times \mathcal{O} \xrightarrow{(x,y)^t} \mathcal{O}$ is surjective on U, but that the induced map $\mathcal{O}(U) \times \mathcal{O}(U) \to \mathcal{O}(U)$ isn't surjective.

xnottensor **6.9.2.** Let V be the complement of the origin in the affine plane $X = \operatorname{Spec} A$, $\quad A = \mathbb{C}[x, y]$, and let M be the A-module A/yA. Identify $\mathcal{M}(V)$, and show that it is not the module $A \otimes_A M = M$.

xsimple-mod **6.9.3.** An R-module is *simple* if it is nonzero and if it has no proper submodules. Prove that a simple module over a finite type $\mathbb{C}$-algebra is a complex vector space of dimension 1.

xcomplvin **6.9.4.** Determine $\mathcal{O}_{\mathbb{P}}(V)$ when V is the complement of a finite set in $\mathbb{P}^d$.

xMisten-sor **6.9.5.** Let $U' \subset U$ be affine open subsets of a variety X, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Say that $\mathcal{O}(U) = A$, $\mathcal{O}(U') = A'$, $\mathcal{M}(U) = M$, and $\mathcal{M}(U') = M'$. Prove that $M' = M \otimes_A A'$.

xmultf **6.9.6.** Let $\mathcal{M}$ be an $\mathcal{O}$-module on projective space. Describe the kernel and cokernel of the multiplication map $\mathcal{M}(k) \xrightarrow{f} \mathcal{M}(k + d)$ by a homogeneous polynomial f of degree d.

xsectwist **6.9.7.** Let $X = \mathbb{P}^2$. What are the sections of the twisting module $\mathcal{O}_X(n)$ on the open complement of the line $\{x_1 + x_2 = 0\}$?

xlimM **6.9.8.** Let s be an element of a domain A, and let M be an A-module. Identify the limit of the directed set $M \xrightarrow{s} M \xrightarrow{s} M \cdots$.

xgenrel **6.9.9.** Let $R = \mathbb{C}[x_0, x_1, x_2]$, and let $f = x_0^2 - x_1 x_2$.

(i) Determine generators and defining relations for the ring $R_{\{f\}}$ of homogeneous fractions of degree zero whose denominators are powers of f.

(ii) Prove that the twisting module $\mathcal{O}(1)$ isn't a free module on the open subset $\mathbb{U}_{\{f\}}$ of $\mathbb{P}^2$ at which $f \neq 0$.

notfree **6.9.10.** Let $R = \mathbb{C}[x, y, z]$, let $X = \mathbb{P}^2$, and let $s = z^2 - xy$. Determine the degree one part of R_s, and prove that $\mathcal{O}(1)$ is not free on X_s.

xtwistsi-som **6.9.11.** In the description (6.4.6) of modules over the projective line, we considered the standard affine open sets $\mathbb{U}^0$ and $\mathbb{U}^1$. Interchanging these open sets changes the variable t to t^{-1}, and it changes the matrix P accordingly. Does it follow that the $\mathcal{O}$-modules of rank 1 defined by (t^k) and by (t^{-k}) are isomorphic?

xmultin-ject **6.9.12.** Let M be a finite module over a finite-type domain A, and let α be a nonzero element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s = \alpha - c$ defines an injective map $M \xrightarrow{s} M$.

xcoher-prop **6.9.13.** *the coherence property.* Let Y be an open subset of a variety X, let s be a nonzero regular function on Y. Prove that, if $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $\mathcal{M}(Y_s)$ is the localization $\mathcal{M}(Y)_s$ of $\mathcal{M}(Y)$. (This is a requirement for an $\mathcal{O}$-module, when Y is affine.)

xgendirim **6.9.14.** Using Exercise 6.9.13, extend the definition of direct image to an arbitrary morphism of varieties.

Chapter 7 COHOMOLOGY

cohomology

7.1 Cohomology

cohqcoh

This chapter is taken from Serre's classic 1956 paper "Faisceaux Algébriques Cohérents", in which Serre showed how the Zariski topology could be used to define cohomology of $\mathcal{O}$-modules.

To save time, we define cohomology only for $\mathcal{O}$-modules. Anyway, the Zariski topology has limited use for cohomology with other coefficients. In the Zariski topology, the constant coefficient cohomology $H^q(X,\mathbb{Z})$ on a variety X is zero for all $q>0$.

Let $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X. The *zero-dimensional cohomology* of $\mathcal{M}$ is the space $\mathcal{M}(X)$ of its global sections. When speaking of cohomology, one denotes that space by $H^0(X,\mathcal{M})$. The functor

$$\textit{(}\mathcal{O}\textit{-modules)} \xrightarrow{H^0} \textit{(vector spaces)}$$

that carries an $\mathcal{O}$-module $\mathcal{M}$ to $H^0(X,\mathcal{M})$ is left exact: If

SS (7.1.1) $$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$$

is a short exact sequence of $\mathcal{O}$-modules, the associated sequence of global sections

globalsect (7.1.2) $$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{N}) \to H^0(X,\mathcal{P})$$

is exact, but unless X is affine, the map $H^0(X,\mathcal{N}) \to H^0(X,\mathcal{P})$ needn't be surjective. The *cohomology* is a sequence of functors *(*$\mathcal{O}$*-modules)* $\xrightarrow{H^q}$ *(vector spaces)*,

$$H^0,\ H^1,\ H^2,\ \dots$$

beginning with H^0, one for each dimension, that compensates for the lack of exactness in the way that is explained in **(a)** and **(b)** below:

(a) To every short exact sequence (7.1.1) of $\mathcal{O}$-modules, there is an associated long exact *cohomology sequence*

HSS (7.1.3) $$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{N}) \to H^0(X,\mathcal{P}) \xrightarrow{\delta^0}$$

$$\xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to H^1(X,\mathcal{N}) \to H^1(X,\mathcal{P}) \xrightarrow{\delta^1} \dots$$

$$\dots \xrightarrow{\delta^{q-1}} H^q(X,\mathcal{M}) \to H^q(X,\mathcal{N}) \to H^q(X,\mathcal{P}) \xrightarrow{\delta^q} \cdots$$

The maps δ^q in this sequence are the *coboundary maps*.

(b) A diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{M}' \arrow[r] & \mathcal{N}' \arrow[r] & \mathcal{P}' \arrow[r] & 0 \end{tikzcd}$$

whose rows are short exact sequences of $\mathcal{O}$-modules, induces a map of cohomology sequences

(7.1.4)

$$\begin{tikzcd} \cdots \arrow[r] & H^q(X,\mathcal{N}) \arrow[r] \arrow[d] & H^q(X,\mathcal{P}) \arrow[r, "\delta^q"] \arrow[d] & H^{q+1}(X,\mathcal{M}) \arrow[r] \arrow[d] & H^{q+1}(X,\mathcal{N}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^q(X,\mathcal{N}') \arrow[r] & H^q(X,\mathcal{P}') \arrow[r, "\delta^q"] & H^{q+1}(X,\mathcal{M}') \arrow[r] & H^{q+1}(X,\mathcal{N}) \arrow[r] & \cdots \end{tikzcd}$$

deltadiagram

A sequence of functors H^q, $q = 0, 1, 2, ...$ that has these properties is called a *cohomological functor*. Cohomology is a cohomological functor.

Most of the cohomology sequence (7.1.3 and the diagram 7.1.4 are consequences of the fact that the H^q are functors. The only additional data are the coboundary maps δ^q and their properties.

Unfortunately, there is no canonical construction of cohomology. There is a construction in Section 7.4, but it isn't canonical. One needs to look at an explicit construction occasionally, but most of the time, it is best to work with the characteristic properties of cohomology that are described in Section 7.3 below.

The one-dimensional cohomology H^1 has an interesting interpretation that you can read about if you like. We won't use it. The cohomology in dimension greater than one has no useful direct interpretation.

7.2 Complexes

complexes

Complexes are used in the construction of cohomology, so we discuss them here.

A *complex* $V^\bullet$ of vector spaces is a sequence of homomorphisms of vector spaces

(7.2.1) $$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$ complex-one

indexed by the integers, such that the composition $d^n d^{n-1}$ of adjacent maps is zero, which means that for every n, the image of d^{n-1} is contained in the kernel of d^n. The q-dimensional *cohomology* of the complex $V^\bullet$ is the quotient

(7.2.2) $$\mathbf{C}^q(V^\bullet) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1}).$$ cohcoplx-one

The cohomology can be regarded as a measure of non-exactness of the complex. A complex whose cohomology is zero is an exact sequence.

A finite sequence of homomorphisms $V^k \xrightarrow{d^k} V^{k+1} \to \cdots \xrightarrow{d^{r-1}} V^r$ such that the compositions $d^i d^{i-1}$ are zero for $i = k, ..., r-1$, can be made into a complex by defining $V^n = 0$ for all other integers n. For example, a homomorphism of vector spaces $V^0 \xrightarrow{d^0} V^1$ can be made into the complex

$$\cdots \to 0 \to V^0 \xrightarrow{d^0} V^1 \to 0 \to \cdots$$

For this complex, the cohomology $\mathbf{C}^0$ is the kernel of d^0, $\mathbf{C}^1$ is its cokernel, and $\mathbf{C}^q$ is zero for all other q.

In the complexes that arise here, V^q will always be zero when $q < 0$.

A *map* $V^\bullet \xrightarrow{\varphi} V'^\bullet$ of complexes is a collection of homomorphisms $V^n \xrightarrow{\varphi^n} V'^n$ making a diagram

$$\begin{tikzcd}
\arrow[r] & V^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & V^n \arrow[r, "d^n"] \arrow[d, "\varphi^n"'] & V^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"'] & \cdots \\
\arrow[r] & V'^{n-1} \arrow[r, "d'^{n-1}"] & V'^n \arrow[r, "d'^n"] & V'^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

A map of complexes induces maps on the cohomology

$$\mathbf{C}^q(V^\bullet) \to \mathbf{C}^q(V'^\bullet)$$

because $\ker d^q$ maps to $\ker d'^q$ and $\operatorname{im} d^q$ maps to $\operatorname{im} d'^q$.

An *exact sequence* of complexes

exseqcplx (7.2.3) $$\cdots \to V^\bullet \xrightarrow{\varphi} V'^\bullet \xrightarrow{\psi} V''^\bullet \to \cdots$$

is a sequence of maps in which the sequences

(7.2.4) $$\cdots \to V^q \xrightarrow{\varphi^q} V'^q \xrightarrow{\psi^q} V''^q \to \cdots$$

are exact for every q.

cohcplx **7.2.5. Proposition.**
Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ *be a short exact sequence of complexes. For every q, there are maps* $\mathbf{C}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{C}^{q+1}(V^\bullet)$ *such that the sequence*

$$0 \to \mathbf{C}^0(V^\bullet) \to \mathbf{C}^0(V'^\bullet) \to \mathbf{C}^0(V''^\bullet) \xrightarrow{\delta^0} \mathbf{C}^1(V^\bullet) \to \mathbf{C}^1(V'^\bullet) \to \mathbf{C}^1(V''^\bullet) \xrightarrow{\delta^1} \mathbf{C}^2(V^\bullet) \to \quad \cdots$$

is exact.

The proof is below. This long exact sequence is the *cohomology sequence* associated to the short exact sequence of complexes. If

$$\begin{tikzcd}
0 \arrow[r] & U^\bullet \arrow[r] \arrow[d] & U'^\bullet \arrow[r] \arrow[d] & U''^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V^\bullet \arrow[r] & V'^\bullet \arrow[r] & V''^\bullet \arrow[r] & 0
\end{tikzcd}$$

is a diagram of complexes whose rows are short exact sequences, the diagrams

$$\begin{tikzcd}
\mathbf{C}^q(U''^\bullet) \arrow[r, "\delta^q"] \arrow[d] & \mathbf{C}^{q+1}(U^\bullet) \arrow[d] \\
\mathbf{C}^q(V''^\bullet) \arrow[r, "\delta^q"] & \mathbf{C}^{q+1}(V^\bullet)
\end{tikzcd}$$

commute. Thus a map of short exact sequences induces a map of cohomology sequences, and the set of functors $\{\mathbf{C}^q\}$ is a cohomological functor on the category of complexes.

snakeco-homology **7.2.6. Example.** We make the Snake Lemma 9.1.22 into a cohomology sequence. Suppose given a diagram

$$\begin{tikzcd}
& V^0 \arrow[r, "u"] \arrow[d, "f"'] & V'^0 \arrow[r] \arrow[d, "f'"'] & V''^0 \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & V^1 \arrow[r, "u"] & V'^1 \arrow[r] & V''^1 &
\end{tikzcd}$$

with exact rows. We form the complex $V^\bullet :\ 0 \to V^0 \xrightarrow{f} V^1 \to 0$ so that $\mathbf{C}^0(V^\bullet) = \ker f$ and $\mathbf{C}^1(V^\bullet) = \operatorname{coker} f$, and we do the analogous thing for the maps f' and f''. Having done that, the Snake Lemma becomes an exact sequence

$$\mathbf{C}^0(V^\bullet) \to \mathbf{C}^0(V'^\bullet) \to \mathbf{C}^0(V''^\bullet) \to \mathbf{C}^1(V^\bullet) \to \mathbf{C}^1(V'^\bullet) \to \mathbf{C}^1(V''^\bullet) \qquad \square$$

proof of Proposition 7.2.5. Let $V^\bullet$ be a complex:

$$\cdots \to V^{q-1} \xrightarrow{d^{q-1}} V^q \xrightarrow{d^q} V^{q+1} \xrightarrow{d^{q+1}} \cdots$$

The image B^q of d^{q-1} and the kernel Z^q of d^q are subspaces of V^q. The elements of B^q are called *coboundaries* and the elements of Z^q are the *cocycles*. The cohomology is

$$\mathbf{C}^q(V^\bullet) = Z^q/B^q \ \big(= (cocycles)/(coboundaries)\big)$$

7.2.7. Lemma. *Let D^q be the cokernel of d^{q-1}, the quotient V^q/B^q,* factordq
(i) *There is a unique map $D^q \xrightarrow{f^q} Z^{q+1}$ such that the map $V^q \xrightarrow{d^q} V^{q+1}$ becomes a composition of three maps*

$$V^q \xrightarrow{\pi^q} D^q \xrightarrow{f^q} Z^{q+1} \xrightarrow{i^{q+1}} V^{q+1}$$

where π^q is the projection from V^q to its quotient D^q, and i^{q+1} is the inclusion of Z^{q+1} into V^{q+1}.
(ii) *With f^q as in* **(i)***,*

$$\mathbf{C}^q(V^\bullet) = \ker f^q \quad and \quad \mathbf{C}^{q+1}(V^\bullet) = \operatorname{coker} f^q$$

proof. **(i)** The kernel of d^q contains B^q and the image of d^q is contained in Z^{q+1}. So d^q factors as indicated.

(ii) Since $D^q = V^q/B^q$, the kernel of f^q is $Z^q/B^q = \mathbf{C}^q$. The image of d^q is B^{q+1}, and this is also the image of f^q. So the cokernel of f^q is $Z^{q+1}/B^{q+1} = \mathbf{C}^{q+1}$. □

Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ be a short exact sequence of complexes, and let f^q be the map defined in 7.2.7. In the diagram below, the top row is exact because D^q, D'^q, D''^q are cokernels, and cokernel is a right exact operation. The bottom row is exact because Z^q, Z'^q, Z''^q are kernels, and kernel is left exact:

$$\begin{tikzcd}
 & D^q \arrow[r] \arrow[d, "f^q"'] & D'^q \arrow[r] \arrow[d, "f'^q"'] & D''^q \arrow[r] \arrow[d, "f''^q"'] & 0 \\
0 \arrow[r] & Z^{q+1} \arrow[r] & Z'^{q+1} \arrow[r] & Z''^{q+1} &
\end{tikzcd}$$

The Snake Lemma, together with (7.2.7)**(ii)**, gives us an exact sequence

$$\mathbf{C}^q(V^\bullet) \to \mathbf{C}^q(V'^\bullet) \to \mathbf{C}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{C}^{q+1}(V^\bullet) \to \mathbf{C}^{q+1}(V'^\bullet) \to \mathbf{C}^{q+1}(V''^\bullet)$$

The cohomology sequence (7.2.5) associated to the short exact sequence of complexes is obtained by splicing these sequences together.

7.3 Characteristic Properties

charprop

The cohomology of $\mathcal{O}$-modules, the sequence of functors H^0, H^1, H^2, $\cdots$

$$(\mathcal{O}\text{-}modules) \xrightarrow{H^q} (vector\ spaces)$$

is characterized by the three properties below. The first two have already been mentioned.

7.3.1. Characteristic Properties of Cohomology charpropone

1. $H^0(X, \mathcal{M})$ is the space $\mathcal{M}(X)$ of global sections of $\mathcal{M}$.
2. The sequence $H^0, H^1, H^2, \cdots$ is a cohomological functor on $\mathcal{O}$-modules: A short exact sequence of $\mathcal{O}$-modules produces a long exact cohomology sequence.
3. Let $Y \xrightarrow{f} X$ be the inclusion of an affine open subset Y into X, let $\mathcal{N}$ be an $\mathcal{O}_Y$-module, and let $f_*\mathcal{N}$ be its direct image on X. The cohomology $H^q(X, f_*\mathcal{N})$ is zero for all $q > 0$.

Note. On an affine variety X, the global section functor is exact: When $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is a short exact sequence of $\mathcal{O}_X$-modules, the sequence

$$0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{N}) \to H^0(X, \mathcal{P}) \to 0$$

is exact (6.2.4). There is no need for higher cohomology $H^1, H^2, \cdots$. One may as well define $H^q(X, \cdot) = 0$ for $q > 0$ when X is affine. This is the third characteristic property for the identity map $X \to X$, and the third property is based on this observation. Intuitively, it tells us that allowing poles on the complement of an affine open set kills cohomology in positive dimension.

existco-hom **7.3.2. Theorem.** *There exists a cohomology theory with the properties (7.3.1), and it is unique up to unique isomorphism.*

The proof is in the next section.

cohze-roaffine **7.3.3. Corollary.** *If X is an affine variety, $H^q(X, \mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.*

As explained above, this follows from the third characteristic property for the identity map $X \to X$. $\square$

thirdex **7.3.4. Example.** This example shows how the third characteristic property can be used. Let j be inclusion of the standard affine $\mathbb{U}^0$ into $X = \mathbb{P}$. Then $\varinjlim_n \mathcal{O}(nH) \approx j_*\mathcal{O}_{\mathbb{U}^0}$, where $\mathcal{M}_{\mathbb{U}^0}$ is the restriction of $\mathcal{M}$ to $\mathbb{U}^0$ (6.7.14). The third property tells us that the cohomology H^q of the direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ is zero when $q > 0$. We will see below (7.4.24) that cohomology commutes with direct limits. Therefore $\varinjlim_n H^q(X, \mathcal{O}_X(nH))$ is zero when $q > 0$, and so is $\varinjlim_n H^q(X, \mathcal{O}_X(n))$. $\square$

cohdirsum **7.3.5. Lemma.** *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules on a variety X. The cohomology of the direct sum $\mathcal{M} \oplus \mathcal{N}$ is canonically isomorphic to the direct sum $H^q(X, \mathcal{M}) \oplus H^q(X, \mathcal{N})$.*

In this statement, one could substitute just about any functor for H^q, and since the direct sum and the direct product are equal, one could substitute $\times$ for $\oplus$

proof. We have homomorphisms of $\mathcal{O}$-modules $\mathcal{M} \xrightarrow{i_1} \mathcal{M} \oplus \mathcal{N} \xrightarrow{\pi_1} \mathcal{M}$ and analogous homomorphisms $\mathcal{N} \xrightarrow{i_2} \mathcal{M} \oplus \mathcal{N} \xrightarrow{\pi_2} \mathcal{N}$. The direct sum is characterized by these maps, together with the relations $\pi_1 i_1 = id_{\mathcal{M}}$, $\pi_2 i_2 = id_{\mathcal{N}}$, $\pi_2 i_1 = 0$, $\pi_1 i_2 = 0$, and $i_1\pi_1 + i_2\pi_2 = id_{\mathcal{M}\oplus\mathcal{N}}$ (see (9.1.25)). Applying the functor H^q gives analogous homomorphisms relating $H^q(\mathcal{M})$, $H^q(\mathcal{N})$, and $H^q(\mathcal{M} \oplus \mathcal{N})$. Therefore $H^q(\mathcal{M} \oplus \mathcal{N}) \approx H^q(\mathcal{M}) \oplus H^q(\mathcal{N})$. $\square$

7.4 Existence of Cohomology

constrcoh

The proof of existence and uniqueness of cohomology are based on the following facts:

• The intersection of two affine open subsets of a variety is affine.

• A sequence $\cdots \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to \cdots$ of $\mathcal{O}$-modules on a variety X is exact if and only if, for every affine open subset U, the sequence of sections $\cdots \to \mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U) \to \cdots$ is exact. (This is the definition of exactness.)

We begin by choosing an arbitrary affine covering $\mathbf{U} = \{U^\nu\}$ of our variety X by finitely many affine open sets U^ν, and we use this covering to describe the cohomology. When we have shown that the cohomology is unique, we will know that it is independent of the choice of covering.

Let $\mathbf{U} \xrightarrow{j} X$ denote the family of inclusions $U^\nu \xrightarrow{j^\nu} X$ of our chosen affine open sets into X. If $\mathcal{M}$ is an $\mathcal{O}$-module and $\mathcal{M}_{U^\nu}$ is its restriction to U^ν (6.5.10), $\mathcal{R}_{\mathcal{M}}$ will denote the $\mathcal{O}$-module $\prod j^\nu_* \mathcal{M}_{U^\nu}$. We could also write $\mathcal{R}_{\mathcal{M}} = j_*\mathcal{M}_{\mathbf{U}}$. As has been noted, there is a canonical map $\mathcal{M} \to j^\nu_*\mathcal{M}_{U^\nu}$, and therefore a canonical map $\mathcal{M} \to \mathcal{R}_{\mathcal{M}}$.

defcalr **7.4.1. Lemma.** **(i)** *Let X' be an open subset of X. The module of sections of $\mathcal{R}_{\mathcal{M}}$ on X' is $\prod_\nu \mathcal{M}(X' \cap U^\nu)$. The space of global sections $\mathcal{R}_{\mathcal{M}}(X)$, which is $H^0(X, \mathcal{R}_{\mathcal{M}})$, is the product $\prod_\nu \mathcal{M}(U^\nu)$.*
(ii) *The canonical map $\mathcal{M} \to \mathcal{R}_{\mathcal{M}}$ is injective. Thus, if $\mathcal{S}_{\mathcal{M}}$ denotes the cokernel of that map, there is a short exact sequence of $\mathcal{O}$-modules*

MRze-roMone (7.4.2)
$$0 \to \mathcal{M} \to \mathcal{R}_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}} \to 0$$

(iii) *For any cohomology theory with the characteristic properties and for any $q > 0$, $H^q(X, \mathcal{R}_{\mathcal{M}}) = 0$.*

proof. **(i)** This is seen by going through the definitions:

$$\mathcal{R}(X') \;=\; \textstyle\prod_\nu [j^\nu_* \mathcal{M}_{U^\nu}](X') = \prod_\nu \mathcal{M}_{U^\nu}(X' \cap U^\nu) = \prod_\nu \mathcal{M}(X' \cap U^\nu).$$

(ii) Let X' be an open subset of X. The map $\mathcal{M}(X') \to \mathcal{R}_\mathcal{M}(X')$ is the product of the restriction maps $\mathcal{M}(X') \to \mathcal{M}(X' \cap U^\nu)$. Because the open sets U^ν cover X, the intersections $X' \cap U^\nu$ cover X'. The sheaf property of $\mathcal{M}$ tells us that the map $\mathcal{M}(X') \to \prod_\nu \mathcal{M}(X' \cap U^\nu)$ is injective.

(iii) This follows from the third characteristic property. $\square$

7.4.3. Lemma. (i) *A short exact sequence* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *of* $\mathcal{O}$*-modules embeds into a diagram* Rcminjective

(7.4.4)
$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\
\mathcal{R}_\mathcal{M} \arrow[r] \arrow[d] & \mathcal{R}_\mathcal{N} \arrow[r] \arrow[d] & \mathcal{R}_\mathcal{P} \arrow[d] \\
\mathcal{S}_\mathcal{M} \arrow[r] & \mathcal{S}_\mathcal{N} \arrow[r] & \mathcal{S}_\mathcal{P}
\end{tikzcd}$$
ttdiagr

whose rows and columns are short exact sequences. (We have suppressed the surrounding zeros.)
(ii) *The sequence of global sections* $0 \to \mathcal{R}_\mathcal{M}(X) \to \mathcal{R}_\mathcal{N}(X) \to \mathcal{R}_\mathcal{P}(X) \to 0$ *is exact.*

proof. **(i)** We are given that the top row of the diagram is a short exact sequence, and we have seen that the columns are short exact sequences. To show that the middle row

(7.4.5) $$0 \to \mathcal{R}_\mathcal{M} \to \mathcal{R}_\mathcal{N} \to \mathcal{R}_\mathcal{P} \to 0$$ Rsequence

is exact, we must show that if X' is an affine open subset of X, the sections on X' form an short exact sequence. The sections are explained in Lemma 7.4.1 **(i)**. Since products of exact sequences are exact, we must show that the sequence

$$0 \to \mathcal{M}(X' \cap U^\nu) \to \mathcal{N}(X' \cap U^\nu) \to \mathcal{P}(X' \cap U^\nu) \to 0$$

is exact. This is true because $X' \cap U^\nu$ is an intersection of affine open sets, and is therefore affine.

Now that we know that the first two rows of the diagram are exact, the Snake Lemma tells us that the bottom row is exact.

(ii) The sequence of of global sections referred to in the statement is the product of the sequences

$$0 \to \mathcal{M}(U^\nu) \to \mathcal{N}(U^\nu) \to \mathcal{P}(U^\nu) \to 0$$

These sequences are exact because the open sets U^ν are affine. $\square$

7.4.6. uniqueness of cohomology uniquecoh

Suppose that a cohomology with the characteristic properties (7.3.1) is given, and let $\mathcal{M}$ be an $\mathcal{O}$-module. The cohomology sequence associated to the sequence $0 \to \mathcal{M} \to \mathcal{R}_\mathcal{M} \to \mathcal{S}_\mathcal{M} \to 0$ is

$$0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{R}_\mathcal{M}) \to H^0(X, \mathcal{S}_\mathcal{M}) \xrightarrow{\delta^0} H^1(X, \mathcal{M}) \to H^1(X, \mathcal{R}_\mathcal{M}) \to \cdots$$

Lemma 7.4.1 **(iii)** tells us that $H^q(X, \mathcal{R}_\mathcal{M}) = 0$ when $q > 0$. So this cohomology sequence breaks up into an exact sequence

(7.4.7) $$0 \to H^0(X, \mathcal{M}) \to H^0(X, \mathcal{R}_\mathcal{M}) \to H^0(X, \mathcal{S}_\mathcal{M}) \xrightarrow{\delta^0} H^1(X, \mathcal{M}) \to 0$$ hone

and isomorphisms

(7.4.8) $$0 \to H^q(X, \mathcal{S}_\mathcal{M}) \xrightarrow{\delta^q} H^{q+1}(X, \mathcal{M}) \to 0$$ hq for every $q > 0$. The first three terms of the sequence (7.4.7), and the arrows connecting them, depend on our choice of covering of X, but the important point is that they don't depend on the cohomology. So that sequence determines $H^1(X, \mathcal{M})$ up to unique isomorphism as the cokernel of a map that is independent of the cohomology. This this is true for every $\mathcal{O}$-module $\mathcal{M}$, including for the module $\mathcal{S}_{\mathcal{M}}$. Therefore it is also true that $H^1(X, \mathcal{S}_{\mathcal{M}})$ is determined uniquely. This being so, $H^2(X, \mathcal{M})$ is determined uniquely for every $\mathcal{M}$, by the isomorphism (7.4.8), with $q = 1$. The isomorphisms (7.4.8) determine the rest of the cohomology up to unique isomorphism by induction on q. □

existcoh **7.4.9. construction of cohomology**

One can use the sequence (7.4.2) and induction to construct cohomology, but it seems clearer to proceed by iterating the construction of $\mathcal{R}_{\mathcal{M}}$.

Let $\mathcal{M}$ be an $\mathcal{O}$-module. We rewrite the exact sequence (7.4.2), labeling $\mathcal{R}_{\mathcal{M}}$ as $\mathcal{R}^0_{\mathcal{M}}$, and $\mathcal{S}_{\mathcal{M}}$ as $\mathcal{M}^1$:

MtoR-toMone (7.4.10)
$$0 \to \mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{M}^1 \to 0$$

and we repeat the construction with $\mathcal{M}^1$. Let $\mathcal{R}^1_{\mathcal{M}} = \mathcal{R}^0_{\mathcal{M}^1}$ $(= j_*\mathcal{M}^1_{\mathbb{U}})$, so that there is an exact sequence

cmonese-quence (7.4.11)
$$0 \to \mathcal{M}^1 \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{M}^2 \to 0$$

analogous to the sequence (7.4.10), with $\mathcal{M}^2 = \mathcal{R}^1_{\mathcal{M}}/\mathcal{M}^1$. We combine the sequences (7.4.10) and (7.4.11) into an exact sequence

RMone (7.4.12)
$$0 \to \mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{M}^2 \to 0$$

and we let $\mathcal{R}^2_{\mathcal{M}} = \mathcal{R}^0_{\mathcal{M}^2}$. Continuing in this way, we construct modules $\mathcal{R}^k_{\mathcal{M}}$ that form an exact sequence

RM (7.4.13)
$$0 \to \mathcal{M} \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{R}^2_{\mathcal{M}} \to \cdots$$

The next lemma follows by induction from (7.4.1)**(iii)** and (7.4.3)**(i,ii)**.

Rcmexact **7.4.14. Lemma.**

(i) *Let $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ be a short exact sequence of $\mathcal{O}$-modules. For every n, the sequences*

$$0 \to \mathcal{R}^n_{\mathcal{M}} \to \mathcal{R}^n_{\mathcal{N}} \to \mathcal{R}^n_{\mathcal{P}} \to 0$$

are exact, and so are the sequences of global sections

$$0 \to \mathcal{R}^n_{\mathcal{M}}(X) \to \mathcal{R}^n_{\mathcal{N}}(X) \to \mathcal{R}^n_{\mathcal{P}}(X) \to 0$$

(ii) *If $H^0,\ H^1,\ \ldots$ is a cohomology theory, then $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ for all n and all $q > 0$.* □

An exact sequence such as (7.4.13) is called a *resolution* of $\mathcal{M}$, and because $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ when $q > 0$, it is an *acyclic resolution*.

Continuing with the proof of existence, we consider the complex of $\mathcal{O}$-modules that is obtained by replacing the term $\mathcal{M}$ in (7.4.13) by 0. Let $\mathcal{R}^\bullet_{\mathcal{M}}$ denote that complex:

Rse-quencetwo (7.4.15)
$$\mathcal{R}^\bullet_{\mathcal{M}} \;=\; 0 \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{R}^2_{\mathcal{M}} \to \cdots$$

The complex $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ of its global sections

RMX (7.4.16)
$$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X) \to \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$

can also be written as

$$0 \to H^0(X, \mathcal{R}^0_{\mathcal{M}}) \to H^0(X, \mathcal{R}^1_{\mathcal{M}}) \to H^0(X, \mathcal{R}^2_{\mathcal{M}}) \to \cdots$$

The complex $\mathcal{R}^\bullet_{\mathcal{M}}$ becomes the resolution (7.4.13) when the module $\mathcal{M}$ is inserted. So it is an exact sequence except at $\mathcal{R}^0_{\mathcal{M}}$. However, the global section functor is only left exact, and the sequence (7.4.16) of global sections $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ needn't be exact anywhere. It is a complex though, because $\mathcal{R}^\bullet_{\mathcal{M}}$ is a complex. The composition of adjacent maps is zero.

Recall that the cohomology of a complex $0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots$ of vector spaces is $\mathbf{C}^q(V^\bullet) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1})$, and that $\{\mathbf{C}^q\}$ is a cohomological functor on complexes.

7.4.17. Definition. The cohomology of an $\mathcal{O}$-module $\mathcal{M}$ is the cohomology of the complex $\mathcal{R}^\bullet_{\mathcal{M}}(X)$: definecoh

$$H^q(X,\mathcal{M}) = \mathbf{C}^q(\mathcal{R}^\bullet_{\mathcal{M}}(X))$$

Thus if we denote the maps in the complex (7.4.16) by d^q:

$$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \xrightarrow{d^0} \mathcal{R}^1_{\mathcal{M}}(X) \xrightarrow{d^1} \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$

then $H^q(X,\mathcal{M}) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1})$.

7.4.18. Lemma. *Let X be an affine variety. With cohomology defined as above, $H^q(X,\mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.* *affineco-hzero*

proof. When X is affine, the sequence of global sections of the exact sequence (7.4.13) is exact. □

To show that our definition gives the unique cohomology, we verify the three characteristic properties. Since the sequence (7.4.13) is exact and since the global section functor is left exact, $\mathcal{M}(X)$ is the kernel of the map $\mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X)$. This kernel is also equal to $\mathbf{C}^0(\mathcal{R}^\bullet_{\mathcal{M}}(X))$, so our cohomology has the first property: $H^0(X,\mathcal{M}) = \mathcal{M}(X)$.

To show that we obtain a cohomological functor, we apply Lemma 7.4.14 to conclude that, for a short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$, the spaces of global sections

(7.4.19)
$$0 \to \mathcal{R}^\bullet_{\mathcal{M}}(X) \to \mathcal{R}^\bullet_{\mathcal{N}}(X) \to \mathcal{R}^\bullet_{\mathcal{P}}(X) \to 0,$$
crtwo

form an exact sequence of complexes. The cohomology $H^q(X,\,\cdot\,)$ is a cohomological functor because cohomology of complexes is a cohomological functor. This is the second characteristic property.

We make a digression before verifying the third characteristic property.

Let $Y \xrightarrow{f} X$ be an affine morphism of varieties, let $U \xrightarrow{j} X$ be the inclusion of an open subvariety into X, and let V be the inverse image $f^{-1}U$, which is an open subvariety of Y. These varieties and maps form a diagram

(7.4.20)
$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$
jstarfstar

Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Its direct image $f_*\mathcal{N}$ is an $\mathcal{O}_X$-module. When we restrict $f_*\mathcal{N}$ to U, we obtain an $\mathcal{O}_U$-module $[f_*\mathcal{N}]_U$. We can obtain an $\mathcal{O}_U$-module in a second way: First restrict the module $\mathcal{N}$ to the open subset V of Y, and then take its direct image. This gives us the $\mathcal{O}_U$-module $g_*[\mathcal{N}_V]$.

7.4.21. Lemma. *The $\mathcal{O}_U$-modules $g_*[\mathcal{N}_V]$ and $[f_*\mathcal{N}]_U$ are equal.* *jstarfs-tartwo*

proof. Let U' be an open subset of U, and let $V' = g^{-1}U'$. Then

$$[f_*\mathcal{N}]_U(U') = [f_*\mathcal{N}](U') = \mathcal{N}(V') = \mathcal{N}_V(V') = [g_*[\mathcal{N}_V]](U')$$
□

7.4.22. Proposition. *Let $Y \xrightarrow{f} X$ be an affine morphism, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Let $\mathbf{V} = \{V^i\}$ be the covering of Y that is the inverse image of our chosen covering $\mathbf{U}$ of X. Let $H^q(X,\,\cdot\,)$ be the cohomology defined as in (7.4.17), and let $H^q(Y,\,\cdot\,)$ be the cohomology that is defined in the analogous way, using the covering $\mathbf{V}$ of Y. Then $H^q(X, f_*\mathcal{N})$ is isomorphic to $H^q(Y,\mathcal{N})$.* *affinedi-rectimage*

proof. The main part of this proof consists of untangling the notation.

To compute the cohomology of $f_*\mathcal{N}$ on X, we substitute $f_*\mathcal{N}$ for $\mathcal{M}$ into (7.4.17):

$$H^q(X, f_*\mathcal{N}) = \mathbf{C}^q(\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)).$$

To compute the cohomology of $\mathcal{N}$ on Y, we let

$$\mathcal{R}'^0_{\mathcal{N}} = i_*[\mathcal{N}_{\mathbf{V}}]$$

where $V \xrightarrow{i} Y$ is as in Diagram 7.4.20, and we continue, to construct a resolution

$$0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{R}'^1_{\mathcal{N}} \to \cdots$$

(The prime is there to remind us that $\mathcal{R}'$ is defined using the covering $\mathbf{V}$ of Y.) Let $\mathcal{R}'^{\bullet}_{\mathcal{N}}$ be the complex that is obtained by replacing the term $\mathcal{N}$ by zero. Then

$$H^q(Y,\mathcal{N}) = \mathbf{C}^q(\mathcal{R}'^{\bullet}_{\mathcal{N}}(Y)).$$

It suffices to show that the complexes of global sections $\mathcal{R}^{\bullet}_{f_*\mathcal{N}}(X)$ and $\mathcal{R}'^{\bullet}_{\mathcal{N}}(Y)$ are isomorphic. If so, we will have

$$H^q(X, f_*\mathcal{N}) = \mathbf{C}^q(\mathcal{R}^{\bullet}_{f_*\mathcal{N}}(X)) \approx \mathbf{C}^q(\mathcal{R}'^{\bullet}_{\mathcal{N}}(Y)) = H^q(Y,\mathcal{N})$$

as required.

By definition of the direct image, $[f_*\mathcal{R}'^q_{\mathcal{N}}](X) = \mathcal{R}'^q_{\mathcal{N}}(Y)$. So we must show that $[\mathcal{R}^q_{f_*\mathcal{N}}](X)$ is isomorphic to $[f_*\mathcal{R}'^q_{\mathcal{N}}](X)$, and it suffices to show that $\mathcal{R}^q_{f_*\mathcal{N}} \approx f_*\mathcal{R}'^q_{\mathcal{N}}$. Let i be the map $\mathbf{V} \to Y$. We look back at the definition of the modules $\mathcal{R}^0$, written in the form (7.4.10). On Y, the analogous sequence for $\mathcal{N}$ on Y is

$$0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{N}^1 \to 0$$

where $\mathcal{R}'^0_{\mathcal{N}} = i_*[\mathcal{N}_{\mathbf{V}}]$. Since f is an affine morphism, the direct image of this sequence

$$0 \to f_*\mathcal{N} \to f_*\mathcal{R}'^0_{\mathcal{N}} \to f_*\mathcal{N}^1 \to 0$$

is exact. We substitute $U = \mathbf{U}$ and $V = \mathbf{V}$ into Diagram 7.4.20, in which $fi = jg$. Then

$$f_*\mathcal{R}'^0_{\mathcal{N}} = f_*i_*[\mathcal{N}_{\mathbf{V}}] = (fi)_*[\mathcal{N}_{\mathbf{V}}] = j_*g_*[\mathcal{N}_{\mathbf{V}}] \xrightarrow{(1)} j_*[f_*\mathcal{N}]_{\mathbb{U}} = \mathcal{R}^0_{f_*\mathcal{N}}$$

the equality (1) being Lemma 7.4.21. So $f_*\mathcal{R}'^0_{\mathcal{N}} = \mathcal{R}^0_{f_{\mathcal{N}}}$. Now induction on q completes the proof. $\square$

We go back to verify the third characteristic property of cohomology, that when $Y \xrightarrow{f} X$ is the inclusion of an affine open subset, $H^q(X, f_*\mathcal{N}) = 0$ for all $\mathcal{O}_Y$-modules $\mathcal{N}$ and all $q > 0$. The inclusion of an affine open set is an affine morphism, so $H^q(Y,\mathcal{N}) = H^q(X, f_*\mathcal{N})$ (7.4.22). Since Y is affine, $H^q(Y,\mathcal{N}) = 0$ for all $q > 0$ (7.4.18). $\square$

cohextbyO **7.4.23. Corollary.** *Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. With cohomology defined as above, $H^q(Y,\mathcal{N})$ and $H^q(X, i_*\mathcal{N})$ are isomorphic for every q.* $\square$

Proposition 7.4.22 is one of the places where a specific construction of cohomology is used. The characteristic properties don't apply directly. The next proposition is another such place.

cohlimit **7.4.24. Lemma.** *Cohomology is compatible with direct limits of $\mathcal{O}$-modules. For all q, $H^q(X, \varinjlim \mathcal{M}_\bullet) \approx \varinjlim H^q(X, \mathcal{M}_\bullet)$.*

proof. The direct and inverse image functors and the global section functor are all compatible with direct limits, and $\varinjlim$ is exact (9.1.36). So the module $\mathcal{R}^q_{\varinjlim \mathcal{M}_\bullet}$ that is used to compute the cohomology of $\varinjlim \mathcal{M}_\bullet$ is isomorphic to $\varinjlim [\mathcal{R}^q_{\mathcal{M}_\bullet}]$, and $\mathcal{R}^q_{\varinjlim \mathcal{M}_\bullet}(X)$ is isomorphic to $\varinjlim [\mathcal{R}^q_{\mathcal{M}_\bullet}](X)$. $\square$

7.5 Cohomology of the Twisting Modules

cohprojsp

We will see here that the cohomology $H^q(\mathbb{P}^n, \mathcal{O}(d))$ of the twisting modules $\mathcal{O}(d)$ on $\mathbb{P}^n$ is zero for most values of q. This fact will help to determine the cohomology of other modules.

Lemma 7.4.18 about vanishing of cohomology on an affine variety, and Lemma 7.4.22 about the direct image via an affine morphism, were stated using a particular affine covering. Since we know that cohomology is unique, that particular covering is irrelevant. Though it isn't strictly necessary, we restate those lemmas here as a corollary:

7.5.1. Corollary. **(i)** *On an affine variety X, $H^q(X, \mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.* *affineco-hzerotwo*

(ii) *Let $Y \xrightarrow{f} X$ be an affine morphism. If $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then $H^q(X, f_*\mathcal{N})$ and $H^q(Y, \mathcal{N})$ are isomorphic. If Y is an affine variety, $H^q(X, f_*\mathcal{N}) = 0$ for all $q > 0$.* □

7.5.2. Corollary. *Let $X \xrightarrow{i} \mathbb{P}^n$ be the embedding of a projective variety into projective space, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. For all q, the cohomology $H^q(X, \mathcal{M})$ of $\mathcal{M}$ on X is isomorphic to the cohomology $H^q(\mathbb{P}^n, i_*\mathcal{M})$ of its extension by zero to $\mathbb{P}^n$.* □ *cohXcohP*

Recall that, on projective space, $\mathcal{M}(d) \approx \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(d)$, The twisting modules $\mathcal{O}(d)$ and the twists $\mathcal{M}(d) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(d)$ are isomorphic to $\mathcal{O}(dH)$ and $\mathcal{M}(dH)$, respectively (6.7.11). If j is the inclusion of the standard open set $\mathbb{U}^0$ into $\mathbb{P}^n$, then $\varinjlim_d \mathcal{O}(dH) \approx j_*\mathcal{O}_{\mathbb{U}^0}$ and $\varinjlim_d \mathcal{M}(dH) \approx j_*\mathcal{M}_{\mathbb{U}^0}$ (6.7.16).

On a projective variety X, the twist $\mathcal{M}(d)$ of an $\mathcal{O}_X$-module $\mathcal{M}$ is obtained by twisting its extension by zero.

7.5.3. Corollary. **(i)** *Let j denote the inclusion $\mathbb{U}^0 \xrightarrow{j} \mathbb{P}^n$. For all $q > 0$, $H^q(\mathbb{P}^n, j_*\mathcal{M}_{\mathbb{U}^0}) = 0$.* *Onse-quencetwo*
(ii) *For all projective varieties X, all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$, $\varinjlim_d H^q(X, \mathcal{M}(d)) = 0$.*
In particular, $H^q(\mathbb{P}^n, j_*\mathcal{O}_{\mathbb{U}^0}) = 0$ and $\varinjlim_d H^q(X, \mathcal{O}(d)) = 0$.

proof. **(i)** follows from the facts that $\mathbb{U}^0$ is affine and that the inclusion j is an affine morphism, and **(ii)** follows from **(i)** and diagram (6.7.15), because $\mathcal{M}(d)$ is isomorphic to $\mathcal{M}(dH)$, and because cohomology is compatible with direct limits (7.4.24). □

7.5.4. Notation. We denote the dimension of $H^q(X, \mathcal{M})$ by $\mathbf{h}^q\mathcal{M}$, or by $\mathbf{h}^q(X, \mathcal{M})$ if there is ambiguity about the variety. In Section 7.7, we will see that, when $\mathcal{M}$ is a finite $\mathcal{O}$-module on a projective variety, $\mathbf{h}^q\mathcal{M}$ will be finite for every q. □ dimnota-tion

7.5.5. Theorem. *cohOd*
(i) *For $d \geq 0$, $\mathbf{h}^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{d+n}{n}$ and $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}(d)) = 0$ if $q \neq 0$.*
(ii) *For $r > 0$, $\mathbf{h}^n(\mathbb{P}^n, \mathcal{O}(-r)) = \binom{r-1}{n}$ and $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}(-r)) = 0$ if $q \neq n$.*

The case $d = 0$ in **(i)** asserts that $\mathbf{h}^0(\mathbb{P}^n, \mathcal{O}) = 1$ and $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}) = 0$ for all $q > 0$, and the case $r = 1$ in **(ii)** asserts that $\mathbf{h}^q(\mathbb{P}^n, \mathcal{O}(-1)) = 0$ for all q.

proof. We have described the global sections of $\mathcal{O}(d)$ before: When $d \geq 0$, $H^0(X, \mathcal{O}(d))$ is the space of homogeneous polynomials of degree d in the coordinate variables. Its dimension is $\binom{d+n}{n}$, and when $d < 0$, $H^0(\mathbb{P}^n, \mathcal{O}(d)) = 0$. (See (6.7.2).)

Let $X = \mathbb{P}^n$, and let Y be the hyperplane at infinity $\{x = 0\}$, which is a projective space of dimension $n-1$, and let $Y \xrightarrow{i} X$ be the inclusion of Y into X. By induction on n, we may assume that the theorem has been proved for Y.

(i) *the case* $d \geq 0$.

We consider the exact sequence

(7.5.6) $$0 \to \mathcal{O}_X(-1) \xrightarrow{x_0} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$ basecase

and its twists

(7.5.7) $$0 \to \mathcal{O}_X(d-1) \xrightarrow{x_0} \mathcal{O}_X(d) \to i_*\mathcal{O}_Y(d) \to 0$$ Od

The twisted sequences are exact because they are obtained by tensoring (7.5.6) with the $\mathcal{O}$-modules $\mathcal{O}(d)$, which is locally free $\mathcal{O}$-modules of rank one. Because the inclusion i is an affine morphism, $H^q(X, i_*\mathcal{O}_Y(d)) \approx H^q(Y, \mathcal{O}_Y(d))$.

The monomials of degree d in $n+1$ variables form a basis of the space of global sections of $\mathcal{O}_X(d)$. Deleting monomials divisible by x_0 produces a basis of $\mathcal{O}_Y(d)$. So every global section of $\mathcal{O}_Y(d)$ is the restriction of a global section of $\mathcal{O}_X(d)$. In the cohomology sequence associated to (7.5.7), the sequence of global sections

$$0 \to H^0(X, \mathcal{O}_X(d-1)) \xrightarrow{x_0} H^0(X, \mathcal{O}_X(d)) \to H^0(Y, \mathcal{O}_Y(d)) \to 0$$

is exact. the map $H^1(X, \mathcal{O}_X(d-1)) \longrightarrow H^1(X, \mathcal{O}_X(d))$ is injective.

By induction on the dimension of X, $H^q(Y, \mathcal{O}_Y(d)) = 0$ for $d \geq 0$ and $q > 0$. When combined with the injectivity noted above, the cohomology sequence of (7.5.7) shows that the maps $H^q(X, \mathcal{O}_X(d-1)) \to H^q(X, \mathcal{O}_X(d))$ are bijective for every $q > 0$. Since the limits are zero (7.5.3), $H^q(X, \mathcal{O}_X(d)) = 0$ for all $d \geq 0$ and all $q > 0$.

(ii) *the case* $d < 0$, *or* $r > 0$.
We use induction on the integers r and n. We suppose the theorem proved for a given r, and we substitute $d = -r$ into the sequence (7.5.7):

Or (7.5.8)
$$0 \to \mathcal{O}_X(-(r+1)) \xrightarrow{x_0} \mathcal{O}_X(-r) \to i_*\mathcal{O}_Y(-r) \to 0$$

For $r = 0$, this sequence is $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$. In the associated cohomology sequence, the terms $H^q(X, \mathcal{O}_X)$ and $H^q(Y, \mathcal{O}_Y)$ are zero when $q > 0$, and $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = \mathbb{C}$. Therefore $H^q(X, \mathcal{O}_X(-1)) = 0$ for every q. This proves **(ii)** for $r = 1$.

Our induction hypothesis is that, $\mathbf{h}^n(X, \mathcal{O}(-r)) = \binom{r-1}{n}$ and $\mathbf{h}^q = 0$ if $q \neq n$. By induction on n, we may suppose that $\mathbf{h}^{n-1}(Y, \mathcal{O}(-r)) = \binom{r-1}{n-1}$ and that $\mathbf{h}^q = 0$ if $q \neq n-1$.

Instead of displaying the cohomology sequence associated to (7.5.8), we assemble the dimensions of the cohomology into a table, in which the asterisks stand for entries that are to be determined:

co-hdimstwo (7.5.9)

	$\mathcal{O}_X(-(r+1))$	$\mathcal{O}_X(-r)$	$i_*\mathcal{O}_Y(-r)$
$\mathbf{h}^0$:	$*$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{h}^{n-2}$:	$*$	0	0
$\mathbf{h}^{n-1}$:	$*$	0	$\binom{r-1}{n-1}$
$\mathbf{h}^n$:	$*$	$\binom{r-1}{n}$	0

The second column is determined by induction on r, and the third column by induction on n. The exact cohomology sequence shows that, in the first column, all entries except the last are zero, and that

$$\mathbf{h}^n(X, \mathcal{O}(-(r+1))) = \binom{r-1}{n-1} + \binom{r-1}{n}$$

The right side of this equation is equal to $\binom{r}{n}$. $\square$

7.6 Cohomology of Hypersurfaces

cohhyper

The cohomology of a plane projective curve: Let $C \xrightarrow{i} X$ be the inclusion of a plane curve of degree k into the plane $X = \mathbb{P}^2$. The ideal of functions that vanish on C is isomorphic to the twisting module $\mathcal{O}_X(-k)$ (6.7.8). So one has an exact sequence

coh-planecurvetwo (7.6.1)
$$0 \to \mathcal{O}_X(-k) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0$$

The table below displays the dimensions of the cohomology. The first two columns are determined by Theorem 7.5.5.

(7.6.2) cohdims

	$\mathcal{O}_X(-k)$	$\mathcal{O}_X$	$i_*\mathcal{O}_C$
$\mathbf{h}^0:$	0	1	$*$
$\mathbf{h}^1:$	0	0	$*$
$\mathbf{h}^2:$	$\binom{k-1}{2}$	0	0

Since the inclusion of C into X is an affine morphism, $\mathbf{h}^q(C, \mathcal{O}_C) = \mathbf{h}^q(X, i_*\mathcal{O}_C)$. Therefore

(7.6.3) $$\mathbf{h}^0(C, \mathcal{O}_C) = 1, \quad \mathbf{h}^1(C, \mathcal{O}_C) = \binom{k-1}{2}, \quad \text{and} \quad \mathbf{h}^q = 0 \quad \text{when} \quad q > 1$$ coh-planecurvethre

The dimension of $H^1(C, \mathcal{O}_C)$, which is $\binom{k-1}{2}$, is called the *arithmetic genus* of C. It is usually denoted by p_a or $p_a(C)$. We will see later (8.8.2) that the arithmetic genus of smooth curve is equal to its topological genus: $p_a = g$. But the arithmetic genus of a plane curve of degree k is equal to $\binom{k-1}{2}$ when the curve C is singular too.

We restate the results as a corollary.

7.6.4. Corollary. *For a plane curve C of degree k, $\mathbf{h}^0\mathcal{O}_C = 1$, $\mathbf{h}^1\mathcal{O}_C = \binom{k-1}{2} = p_a$, and $\mathbf{h}^q = 0$ if $q \neq 0, 1$.* □ *coh-planecurve*

The fact that $\mathbf{h}^0\mathcal{O}_C = 1$ tells us that the only rational functions that are regular everywhere on C are the constants. It follows that a plane curve is connected in the Zariski topology, and it hints at a fact that will be proved later, that a plane curve is connected in the classical topology, though it isn't a proof of that fact.

To determine cohomology of a curve embedded in a higher dimensional projective space, we will need to know that its cohomology is finite-dimensional, which is Theorem 7.7.3 below, and that it is zero in dimension greater than one, which is Theorem 7.7.1, also below. The cohomology of projective curves will be studied in Chapter 8.

One can make a similar computation for the hypersurface Y in $X = \mathbb{P}^n$ defined by an irreducible homogeneous polynomial f of degree k. The ideal of such a hypersurface is isomorphic to $\mathcal{O}_X(-k)$ (6.7.8), so there is an exact sequence

$$0 \to \mathcal{O}_X(-k) \xrightarrow{f} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

Since we know the cohomology of $\mathcal{O}_X(-k)$ and of $\mathcal{O}_X$, and since $H^q(X, i_*\mathcal{O}_Y) \approx H^q(Y, \mathcal{O}_Y)$, we can use this sequence to compute the dimensions of the cohomology of $\mathcal{O}_Y$.

7.6.5. Corollary. *Let Y be a hypersurface of dimension d and degree k in a projective space of dimension $d+1$. Then $\mathbf{h}^0(Y, \mathcal{O}_Y) = 1$, $\mathbf{h}^d(Y, \mathcal{O}_Y) = \binom{k-1}{d+1}$, and $\mathbf{h}^q(Y, \mathcal{O}_Y) = 0$ for all other q.* □ *cohhyper-surface*

In particular, when S is the surface in $\mathbb{P}^3$ defined by an irreducible polynomial of degree k, $\mathbf{h}^0(S, \mathcal{O}_S) = 1$, $\mathbf{h}^1(S, \mathcal{O}_S) = 0$, $\mathbf{h}^2(S, \mathcal{O}_S) = \binom{k-1}{3}$, and $\mathbf{h}^q = 0$ if $q > 2$.

For a surface S in a higher dimensional projective space, it is still true that $\mathbf{h}^0 = 1$ and that $\mathbf{h}^q = 0$ if $q > 2$, but $\mathbf{h}^1(S, \mathcal{O}_S)$ may be nonzero. The dimensions $\mathbf{h}^1(S, \mathcal{O}_S)$ and $\mathbf{h}^2(S, \mathcal{O}_S)$ are invariants of S that are somewhat analogous to the genus of a curve. In classical terminology, $\mathbf{h}^2(S, \mathcal{O}_S)$ is the *geometric genus* p_g and $\mathbf{h}^1(S, \mathcal{O}_S)$ is the *irregularity* q. The *arithmetic genus* p_a of S is defined to be

(7.6.6) $$p_a \ = \ \mathbf{h}^2(S, \mathcal{O}_S) - \mathbf{h}^1(S, \mathcal{O}_S) = p_g - q$$ patwo

Therefore the irregularity q is $p_g - p_a$. When S is a surface in $\mathbb{P}^3$, the irregularity is zero, and $p_g = p_a$.

In modern terminology, it might seem more natural to replace the arithmetic genus by the Euler characteristic of the structure sheaf $\chi(\mathcal{O}_S)$, which is defined to be $\sum_q (-1)^q \mathbf{h}^q \mathcal{O}_S$ (see (7.7.7) below). The Euler characteristic of the structure sheaf on a curve is

$$\chi(\mathcal{O}_C) = \mathbf{h}^0(C, \mathcal{O}_C) - \mathbf{h}^1(C, \mathcal{O}_C) = 1 - p_a$$

and on a surface S it is

$$\chi(\mathcal{O}_S) = \mathbf{h}^0(S, \mathcal{O}_S) - \mathbf{h}^1(S, \mathcal{O}_S) + \mathbf{h}^2(S, \mathcal{O}_S) = 1 + p_a$$

But because of tradition, the arithmetic genus is used quite often.

7.7 Three Theorems about Cohomology

threethms cohsupport **7.7.1. Theorem.** *Let X be a projective variety, and let $\mathcal{M}$ be a finite $\mathcal{O}_X$-module. If the support of $\mathcal{M}$ (6.6) has dimension k, then $H^q(X, \mathcal{M}) = 0$ for all $q > k$. In particular, if X has dimension n, then $H^q(X, \mathcal{M}) = 0$ for all $q > n$.*

largetwist **7.7.2. Theorem.** *Let $\mathcal{M}(d)$ be the twist of a finite $\mathcal{O}_X$-module $\mathcal{M}$ on a projective variety X. For sufficiently large d, $H^q(X, \mathcal{M}(d)) = 0$ for all $q > 0$.*

findim **7.7.3. Theorem.** *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. The cohomology $H^q(X, \mathcal{M})$ is a finite-dimensional vector space for every q.*

descind **7.7.4. Notes.** As is true for everthing in this chapter, these theorems are due to Serre.

(a) The first theorem asserts that the highest dimension in which cohomology of an $\mathcal{O}_X$-module on a projective variety X can be nonzero is the dimension of X. It is also true that, on any projective variety X of dimension n, there will be $\mathcal{O}_X$-modules $\mathcal{M}$ such that $H^n(X, \mathcal{M}) \neq 0$. In contrast, in the classical topology on a projective variety X of dimnsion n, the constant coefficient cohomology $H^{2n}(X_{class}, \mathbb{Z})$ isn't zero. As we have mentioned, in the Zariski topology, the cohomology $H^q(X_{zar}, \mathbb{Z})$ with constant coefficients is zero for every $q > 0$. When X is an affine variety, the cohomology of any $\mathcal{O}_X$-module is zero when $q > 0$.

(b) The third theorem tells us that the space $H^0(X, \mathcal{M})$ of global sections of a finite $\mathcal{O}$-module on a projective variety is finite-dimensional. This is one of the most important consequences of the theorem, and it isn't easy to prove directly. Cohomology needn't be finite-dimensional on a variety that isn't projective. For example, on an affine variety $X = \operatorname{Spec} A$, $H^0(X, \mathcal{O}) = A$ isn't finite-dimensional unless X is a point. When X is the complement of a point in $\mathbb{P}^2$, $H^1(X, \mathcal{O})$ isn't finite-dimensional.

(c) The proofs of the second and third theorems have an interesting structure. The first theorem allows us to use *descending* induction in the proofs, beginning with the fact that $H^k(X, \mathcal{M}) = 0$ when k is greater than the dimension of X. $\square$

In these theorems, we are given that X is a closed subvariety of a projective space $\mathbb{P}^n$. We can replace an $\mathcal{O}_X$-module by its extension by zero to $\mathbb{P}^n$. This doesn't change the cohomology or the dimension of support. The twist $\mathcal{M}(d)$ of an $\mathcal{O}_X$-module that is referred to in the second theorem is defined in terms of the extension by zero. So we may assume that X is a projective space.

The proofs are based on the cohomology of the twisting modules (7.5.5) and on the vanishing of the limit $\varinjlim_d H^q(X, \mathcal{M}(d))$ for $q > 0$ (7.5.3).

proof of Theorem 7.7.1 (*vanishing in large dimension*)

Here $\mathcal{M}$ is a finite $\mathcal{O}$-module whose support S has dimension at most k. We are to show that $H^q(X, \mathcal{M}) = 0$ when $q > k$. We choose coordinates so that the hyperplane $H : x_0 = 0$ doesn't contain any component of S. Then $H \cap S$ has dimension at most $k - 1$. We inspect the multiplication map $\mathcal{M}(-1) \xrightarrow{x_0} \mathcal{M}$. The kernel $\mathcal{K}$ and cokernel $\mathcal{Q}$ are annihilated by x_0, so the supports of $\mathcal{K}$ and $\mathcal{Q}$ are contained in H. Since they are also in S, those supports have dimension at most $k-1$. We can apply induction on k to them. In the base case $k = 0$, the supports of $\mathcal{K}$ and $\mathcal{Q}$ will be empty, and their cohomology will be zero.

We break the exact sequence $0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{M} \to \mathcal{Q} \to 0$ into two short exact sequences by introducing the kernel $\mathcal{N}$ of the map $\mathcal{M} \to \mathcal{Q}$:

kercoker (7.7.5)
$$0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{N} \to 0 \quad \text{and} \quad 0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0$$

The induction hypothesis applies to $\mathcal{K}$ and to $\mathcal{Q}$. It tells us that $H^q(X, \mathcal{K}) = 0$ and $H^q(X, \mathcal{Q}) = 0$, when $q \geq k$. For $q > k$, the cohomology sequences associated to the two exact sequences give us bijections

$$H^q(X, \mathcal{M}(-1)) \to H^q(X, \mathcal{N}) \quad \text{and} \quad H^q(X, \mathcal{N}) \to H^q(X, \mathcal{M})$$

Therefore the composed map $H^q(X, \mathcal{M}(-1)) \to H^q(X, \mathcal{M})$ is bijective, and this is true for every $\mathcal{O}$-module whose support has dimension $\leq k$, including for the $\mathcal{O}$-module $\mathcal{M}(d)$. For every $\mathcal{O}$-module $\mathcal{M}$ whose support has dimension at most k, the canonical map $H^q(X, \mathcal{M}(d-1)) \to H^q(X, \mathcal{M}(d))$ is bijective for all d and all

$q > k$. According to (7.5.3), the limit $\varinjlim_d H^q(X, \mathcal{M}(d))$ is zero. It follows that $H^q(X, \mathcal{M}(d)) = 0$ for all d when $q > k$, and in particular, $H^q(X, \mathcal{M}) = 0$.

proof of Theorem 7.7.2 (*vanishing for a large twist*)

Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. We recall that $\mathcal{M}(r)$ is generated by global sections when r is sufficiently large (6.7.21). Choosing generators gives us a surjective map $\mathcal{O}^n \to \mathcal{M}(r)$. Let $\mathcal{N}$ be the kernel of this map. When we twist the exact sequence $0 \to \mathcal{N} \to \mathcal{O}^n \to \mathcal{M}(r) \to 0$, we obtain short exact sequences

(7.7.6) $$0 \to \mathcal{N}(d) \to \mathcal{O}(d)^n \to \mathcal{M}(d{+}r) \to 0$$ threetwos

for every $d \geq 0$. These sequences are useful because $H^q(X, \mathcal{O}(d)) = 0$ when $d \geq 0$ and $q > 0$ (7.5.5).

To prove Theorem 7.7.2, we must show this for all $q > 0$:

(*) *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module. For sufficiently large d, $H^q(X, \mathcal{M}(d)) = 0$.*

Let n be the dimension of X. By Theorem 7.7.1, $H^q(X, \mathcal{M}) = 0$ for any $\mathcal{O}$-module $\mathcal{M}$, when $q > n$. In particular, $H^q(X, \mathcal{M}(d)) = 0$ when $q > n$. This leaves a finite set of integers $q = 1, ..., n$ to consider, and it suffices to consider them one at a time. If (*) is true for each individual q, there will be a d that is sufficiently large so that (*) is true for each of the integers $q = 1, ..., n$ at the same time, and therefore for all positive integers q, as the theorem asserts. We use descending induction on q, the base case being $q = n+1$, for which (*) is true with $d = 0$. We suppose that (*) is true for every finite $\mathcal{O}$-module $\mathcal{M}$ when $q = p+1$, and that $p > 0$, and we show that (*) is true for every finite $\mathcal{O}$-module $\mathcal{M}$ when $q = p$.

We substitute $q = p$ into the cohomology sequence associated to the sequence (7.7.6). The relevant part of that sequence is

$$\to H^p(X, \mathcal{O}(d)^n) \to H^p(X, \mathcal{M}(d{+}r)) \xrightarrow{\delta^p} H^{p+1}(X, \mathcal{N}(d)) \to$$

Since p is positive, $H^p(X, \mathcal{O}(d)) = 0$ for all $d \geq 0$. The map δ^p is injective. We note that $\mathcal{N}$ is a finite $\mathcal{O}$-module. So our induction hypothesis applies to it. The induction hypothesis tells us that, when d is large, $H^{p+1}(X, \mathcal{N}(d)) = 0$ and therefore $H^p(X, \mathcal{M}(d{+}r)) = 0$. The particular form of the integer $d{+}r$ isn't useful. Our conclusion is that, for every finite $\mathcal{O}$-module $\mathcal{M}$, $H^p(X, \mathcal{M}(d_1)) = 0$ when d_1 is large enough. $\square$

proof of Theorem 7.7.3 (*finiteness of cohomology*)

This proof uses ascending induction on the dimension of support and descending induction on the degree d of a twist. As has been mentioned, it isn't easy to prove directly that the space $H^0(X, \mathcal{M})$ of global sections is finite-dimensional.

Let $\mathcal{M}$ be an $\mathcal{O}$-module whose support has dimension at most k. We go back to the sequences (7.7.5) and their cohomology sequences, in which the supports of $\mathcal{K}$ and $\mathcal{Q}$ have dimension $\leq k{-}1$. Ascending induction on the dimension of support allows us to assume that $H^r(X, \mathcal{K})$ and $H^r(X, \mathcal{Q})$ are finite-dimensional for all r. Denoting finite-dimensional spaces ambiguously by $finite$, the two cohomology sequences become

$$\cdots \to finite \to H^q(X, \mathcal{M}(-1)) \to H^q(X, \mathcal{N}) \to finite \to \cdots$$

and

$$\cdots \to finite \to H^q(X, \mathcal{N}) \to H^q(X, \mathcal{M}) \to finite \to \cdots$$

The first of these sequences shows that $H^q(X, \mathcal{M}(-1))$ has infinite dimension if and only if $H^q(X, \mathcal{N})$ has infinite dimension, and the second sequence shows that $H^q(X, \mathcal{N})$ has infinite dimension if and only if $H^q(X, \mathcal{M})$ has infinite dimenson. Therefore either $H^q(X, \mathcal{M}(-1))$ and $H^q(X, \mathcal{M})$ are both finite-dimensional, or else they are both infinite-dimensional. This applies to the twisted modules $\mathcal{M}(d)$ as well as to $\mathcal{M}$: $H^q(X, \mathcal{M}(d{-}1))$ and $H^q(X, \mathcal{M}(d))$ are both finite-dimensional or both infinite-dimensional.

Suppose that $q > 0$. Then $H^q(X, \mathcal{M}(d)) = 0$ when d is large enough (Theorem 7.7.2). Since the zero space is finite-dimensional, we can use the sequence together with descending induction on d, to conclude that $H^q(X, \mathcal{M}(d))$ is finite-dimensional for every finite module $\mathcal{M}$ and every d. In particular, $H^q(X, \mathcal{M})$ is finite-dimensional.

This leaves the case that $q = 0$. To prove that $H^0(X, \mathcal{M})$ is finite-dimensional, we put $d = -r$ with $r > 0$ into the sequence (7.7.6):

$$0 \to \mathcal{N}(-r) \to \mathcal{O}(-r)^m \to \mathcal{M} \to 0$$

The corresponding cohomology sequence is

$$0 \to H^0(X, \mathcal{N}(-r)) \to H^0(X, \mathcal{O}(-r))^m \to H^0(X, \mathcal{M}) \xrightarrow{\delta^0} H^1(X, \mathcal{N}(-r)) \to \cdots .$$

Here $H^0(X, \mathcal{O}(-r))^m = 0$, and we've shown that $H^1(X, \mathcal{N}(-r))$ is finite-dimensional. It follows that $H^0(X, \mathcal{M})$ is finite-dimensional, and this completes the proof.

Notice that the finiteness of H^0 comes out only at the end. The higher cohomology is essential for this proof. $\square$

eulerchar **7.7.7. Euler characteristic**

Theorems 7.7.1 and 7.7.3 allow us to define the Euler characteristic of a finite module on projective variety.

The *Euler characteristic* of a finite $\mathcal{O}$-module $\mathcal{M}$ on a projective variety X is the alternating sum of the dimensions of the cohomology:

chi (7.7.8)
$$\chi(\mathcal{M}) = \sum (-1)^q \mathbf{h}^q(X, \mathcal{M}).$$

This makes sense because $\mathbf{h}^q(X, \mathcal{M})$ is finite for every q, and is zero when q is large.

Try not to confuse the Euler characterstic of an $\mathcal{O}$-module with the topological Euler characteristic of the variety X.

7.7.9. Proposition. *If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is a short exact sequence of finite $\mathcal{O}$-modules on a projective variety X, then $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P}) = 0$. If $0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \cdots \to \mathcal{M}_n \to 0$ is an exact sequence of finite $\mathcal{O}$-modules, the alternating sum $\sum (-1)^i \chi(\mathcal{M}_i)$ is zero.*

altsum **7.7.10. Lemma.** *Let $0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ be an exact sequence of finite dimensional vector spaces. The alternating sum $\sum (-1)^q \dim V^q$ is zero.* $\square$

proof of Proposition 7.7.9. **(i)** Let n be the dimension of X. The cohomology sequence associated to the sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is

$$0 \to H^0(\mathcal{M}) \to H^0(\mathcal{N}) \to H^0(\mathcal{P}) \to H^1(\mathcal{M}) \to \quad \cdots \quad \to H^n(\mathcal{N}) \to H^n(\mathcal{P}) \to 0$$

The lemma tells us that the alternating sum of its dimensions is zero. That alternating sum is also equal to $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P})$.

For the second assertion, we denote the given sequence by $\mathbb{S}_0$ and the alternating sum $\sum_i (-1)^i \chi(\mathcal{M}_i)$ by $\chi(\mathbb{S}_0)$. Let $\mathcal{N} = \mathcal{M}_1/\mathcal{M}_0$. The sequence $\mathbb{S}_0$ decomposes into the two exact sequences

$$\mathbb{S}_1 : \ 0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{N} \to 0 \quad \text{ and } \quad \mathbb{S}_2 : \ 0 \to \mathcal{N} \to \mathcal{M}_2 \to \cdots \to \mathcal{M}_k \to 0$$

One sees directly that $\chi(\mathbb{S}_0) = \chi(\mathbb{S}_1) - \chi(\mathbb{S}_2)$, so the assertion follows from **(i)** by induction on n. $\square$

7.8 Bézout's Theorem

bezout

As an application of cohomology, we prove Bézout's Theorem. We restate that theorem here:

bezoutrestated **7.8.1. Bézout's Theorem.** *Let Y and Z be distinct curves, of degrees m and n, respectively, in the projective plane X. The number of intersection points $Y \cap Z$, when counted with an appropriate multiplicity, is equal to swhi mn. Moreover, the multiplicity is 1 at a point at which Y and Z intersect transversally.*

The definition of the multiplicity will emerge during the proof.

intersectlines **Note.** Let f and g be relatively prime homogeneous polynomials. When one replaces Y and Z by their divisors of zeros (1.3.13), the theorem remains true whether or not they are irreducible, and the proof isn't significantly different from the one we give here. For example, suppose that f and g are products of linear polynomials, so that Y is the union of m lines and Z is the union of n lines, and suppose that those lines are distinct. Since distinct lines intersect transversally in a single point, there are mn intersection points of multiplicity 1. $\square$

proof of Bézout's Theorem. We suppress notation for the extension by zero from Y or Z to the plane X, denoting the direct images of $\mathcal{O}_Y$ and $\mathcal{O}_Z$ by the same symbols. Let f and g be the irreducible homogeneous polynomials whose zero loci are Y and Z. Multiplication by f defines a short exact sequence

$$0 \to \mathcal{O}_X(-m) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

This exact sequence describes $\mathcal{O}_X(-m)$ as the ideal $\mathcal{I}$ of regular functions that vanish on Y, and there is a similar sequence describing the module $\mathcal{O}_X(-n)$ as the ideal $\mathcal{J}$ of Z. The zero locus of the ideal $\mathcal{I}+\mathcal{J}$ is the intersection $Y \cap Z$. It is a finite set of points $\{p_1, ..., p_k\}$.

Let $\overline{\mathcal{O}}$ denote the quotient $\mathcal{O}_X/(\mathcal{I}+\mathcal{J})$. Its support is the finite set $Y \cap Z$, and therefore $\overline{\mathcal{O}}$ is isomorphic to a direct sum $\bigoplus \overline{\mathcal{O}}_i$, where each $\overline{\mathcal{O}}_i$ is a finite-dimensional algebra whose support is p_i (6.6.2). The *intersection multiplicity* of Y and Z at p_i is defined to be the dimension of $\overline{\mathcal{O}}_i$, and this is also the dimension of the space of its global sections. Let's denote the intersection multiplicity by μ_i. The dimension of $H^0(X, \overline{\mathcal{O}})$ is the sum $\mu_1 + \cdots + \mu_k$, and $H^q(X, \overline{\mathcal{O}}) = 0$ for all $q > 0$ (Theorem 7.7.1). The Euler characteristic $\chi(\overline{\mathcal{O}})$ is equal to $\mathbf{h}^0(X, \overline{\mathcal{O}})$. We'll show that $\chi(\overline{\mathcal{O}}) = mn$, and therefore that $\mu_1 + \cdots + \mu_k = mn$. This will prove Bézout's Theorem.

We form a sequence, in which $\mathcal{O}$ stands for $\mathcal{O}_X$:

(7.8.2) $$0 \to \mathcal{O}(-m-n) \xrightarrow{(g,f)^t} \mathcal{O}(-m) \times \mathcal{O}(-n) \xrightarrow{(-f,g)} \mathcal{O} \xrightarrow{\pi} \overline{\mathcal{O}} \to 0$$ bezoutresolution

In order to interpret the maps in this sequence as matrix multiplication, with homomorphisms acting on the left, a section of $\mathcal{O}(-m) \times \mathcal{O}(-n)$ should be represented as a column vector $(u, v)^t$, u and v being sections of $\mathcal{O}(-m)$ and $\mathcal{O}(-n)$, respectively.

7.8.3. Lemma. *The sequence (7.8.2) is exact.* *bresolexact*

proof. We choose coordinates so that none of the points making up $Y \cap Z$ lie on the coordinate axes. To prove exactness, it suffices to show that the sequence of sections on each of the standard open sets is exact. We look at the index 0 as usual, denoting $\mathbb{U}^0$ by $\mathbb{U}$. Let A be the algebra of regular functions on $\mathbb{U}$, which is the polynomial algebra $\mathbb{C}[u_1, u_2]$, with $u_i = x_i/x_0$. We identify $\mathcal{O}(k)$ with $\mathcal{O}(kH)$, H being the hyperplane at infinity. The restriction of the module $\mathcal{O}(kH)$ to $\mathbb{U}$ is the same as the restriction $\mathcal{O}_{\mathbb{U}}$ of $\mathcal{O}$. Its sections on $\mathbb{U}$ are the elements of A. Let $\overline{A}$ be the algebra of sections of $\overline{\mathcal{O}}$ on $\mathbb{U}$. Since f and g are relatively prime, so are their dehomogenizations $F = f(1, u_1, u_2)$ and $G = g(1, u_1, u_2)$. The sequence of sections of (7.8.2) on $\mathbb{U}$ is

$$0 \to A \xrightarrow{(G,F)^t} A \times A \xrightarrow{(-F,G)} A \to \overline{A} \to 0$$

and the only place at which exactness of this sequence isn't obvious is at $A \times A$. Suppose that $(u, v)^t$ is in the kernel of the map $(-F, G)$, i.e., that $Fu = Gv$. Since F and G are relatively prime, F divides v, G divides u, and $v/F = u/G$. Let $w = v/F = u/G$. Then $(u, v)^t = (G, F)^t w$. So $(u, v)^t$ is the image of w. $\square$

We go back to the proof of Bézout's Theorem. Since cohomology is compatible with products (7.3.5), $\chi(\mathcal{M} \times \mathcal{N}) = \chi(\mathcal{M}) + \chi(\mathcal{N})$. Proposition 7.7.9 **(ii)**, applied to the exact sequence (7.8.2), tells us that the alternating sum

(7.8.4) $$\chi(\mathcal{O}(-m-n)) - \chi(\mathcal{O}(-m)) - \chi(\mathcal{O}(-n)) + \chi(\mathcal{O}) - \chi(\overline{\mathcal{O}})$$ chiforbezout

is zero. Solving for $\chi(\overline{\mathcal{O}})$ and applying Theorem 7.5.5,

$$\chi(\overline{\mathcal{O}}) = \binom{n+m-1}{2} - \binom{m-1}{2} - \binom{n-1}{2} + 1$$

The right side of this equation evaluates to mn. This completes the proof. $\square$

We still need to explain the assertion that the multiplicity at a transversal intersection p is equal to 1. The intersection at p will be transversal if and only if $\mathcal{I}+\mathcal{J}$ generates the maximal ideal $\mathfrak{m}$ of $A = \mathbb{C}[y, z]$ at p

locally. If so, then the component of $\overline{\mathcal{O}}$ supported at p will have dimension 1, and the intersection multiplicity at p will be 1.

When Y and Z are lines, we may choose affine coordinates so that p is the origin in the plane $X = \operatorname{Spec} A$ and the curves are the coordinate axes $\{z = 0\}$ and $\{y = 0\}$. The variables y, z generate the maximal ideal at the origin.

Suppose that Y and Z intersect transverally at p, but that they aren't lines. We choose affine coordinates so that p is the origin and that the tangent directions of Y and Z at p are the coordinate axes. The affine equations of Y and Z will have the form $y_1 = 0$ and $z_1 = 0$, where $y_1 = y + g(y, z)$ and $z_1 = z + h(y, z)$, g and h being polynomials all of whose terms have degree at least 2. Because Y and Z may intersect at points other than p, the elements y_1 and z_1 may fail to generate the maximal ideal $\mathfrak{m}$ at p. However, they do generate the maximal ideal locally. To show this, it suffices to show that they generate the maximal ideal M in the local ring R at p. According to Corollary 5.1.2, it suffices to show that y_1 and z_1 generate M/M^2, and this is true because y_1 and z_1 are congruent to y and z modulo M^2. □

7.9 Uniqueness of the Coboundary Maps

unique-cobound

In Section 7.4, we constructed a cohomology $\{H^q\}$ that has the characteristic properties, and we showed that the functors H^q are unique. We haven't shown that the coboundary maps δ^q that appear in the cohomology sequences are unique. We go back to do this now.

To make it clear that there is something to show, we note that the cohomology sequence (7.1.3) remains exact when a coboundary map δ^q is multiplied by -1. Why can't we define a new collection of coboundary maps by changing some signs? The reason we can't do this is that we used the coboundary maps δ^q in (7.4.7) and (7.4.8), to identify $H^q(X, \mathcal{M})$. Having done that, we aren't allowed to change δ^q for the particular short exact sequences (7.4.2). We show that the coboundary maps for those sequences determine the coboundary maps for every short exact sequence of $\mathcal{O}$-modules

$$(A) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \to 0$$

The sequences (7.4.2) were rewritten as (7.4.10):

$$(B) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{R}^0_{\mathcal{M}} \longrightarrow \mathcal{M}^1 \to 0$$

To show that the coboundaries for the sequence (A) are determined uniquely, we relate it to the sequence (B), for which the coboundary maps are fixed. We map the sequences (A) and (B) to a third exact sequence

$$(C) \qquad 0 \to \mathcal{M} \xrightarrow{\psi} \mathcal{R}^0_{\mathcal{N}} \longrightarrow \mathcal{Q} \to 0$$

where ψ is the composition of the injective maps $\mathcal{M} \to \mathcal{N} \to \mathcal{R}^0_{\mathcal{N}}$ and $\mathcal{Q}$ is the cokernel of ψ.

First, we inspect the diagram

$$\begin{tikzcd}
(A) & \mathcal{M} \arrow[r] \arrow[d, equal] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\
(C) & \mathcal{M} \arrow[r, "\psi"] & \mathcal{R}^0_{\mathcal{N}} \arrow[r] & \mathcal{Q}
\end{tikzcd}$$

and its diagram of coboundary maps

$$\begin{tikzcd}
(A) & H^q(X, \mathcal{P}) \arrow[r, "\delta^q_A"] \arrow[d] & H^{q+1}(X, \mathcal{M}) \arrow[d, equal] \\
(C) & H^q(X, \mathcal{Q}) \arrow[r, "\delta^q_C"] & H^{q+1}(X, \mathcal{M})
\end{tikzcd}$$

This diagram shows that the coboundary map δ^q_A for the sequence (A) is determined by the coboundary map δ^q_C for (C).

Next, we inspect the diagram

$$
\begin{array}{cccccc}
(B) & \mathcal{M} & \longrightarrow & \mathcal{R}^0_{\mathcal{M}} & \longrightarrow & \mathcal{M}^1 \\
 & \| & & \big\downarrow u & & \big\downarrow v \\
(C) & \mathcal{M} & \xrightarrow{\psi} & \mathcal{R}^0_{\mathcal{N}} & \longrightarrow & \mathcal{Q}
\end{array}
\tag{7.9.1}
$$

BtoC

and its diagram of coboundary maps

$$
\begin{array}{cccc}
(B) & H^q(X,\mathcal{M}^1) & \xrightarrow{\delta^q_B} & H^{q+1}(X,\mathcal{M}) \\
 & \big\downarrow & & \| \\
(C) & H^q(X,\mathcal{Q}) & \xrightarrow{\delta^q_C} & H^{q+1}(X,\mathcal{M})
\end{array}
$$

When $q > 0$, δ^q_C and δ^q_B are bijective because the cohomology of $\mathcal{R}^0_{\mathcal{M}}$ and $\mathcal{R}^0_{\mathcal{N}}$ is zero in positive dimension. Then δ^q_C is determined by δ^q_B, and so is δ^q_A.

We have to look more closely to settle the case $q = 0$. The map labeled u in (7.9.1) is injective. The Snake Lemma shows that v is injective, and that the cokernels of u and v are isomorphic. We denote both of those cokernels by $\mathcal{R}^0_{\mathcal{P}}$. When we add the cokernels to the diagram, and pass to cohomology, we obtain a diagram whose relevant part is

$$
\begin{array}{cccccc}
(B) & H^0(X,\mathcal{R}^0_{\mathcal{M}}) & \longrightarrow & H^0(X,\mathcal{M}^1) & \xrightarrow{\delta^0_B} & H^1(X,\mathcal{M}) \\
 & \big\downarrow u & & \big\downarrow v & & \| \\
(C) & H^0(X,\mathcal{R}^0_{\mathcal{N}}) & \xrightarrow{\beta} & H^0(X,\mathcal{Q}) & \xrightarrow{\delta^0_C} & H^1(X,\mathcal{M}) \\
 & \big\downarrow \gamma & & \big\downarrow w & & \\
 & H^0(X,\mathcal{R}^0_{\mathcal{P}}) & = & H^0(X,\mathcal{R}^0_{\mathcal{P}}) & &
\end{array}
$$

Its rows and columns are exact. We want to show that the map δ^0_C is determined uniquely by δ^0_B. It is determined by δ^0_B on the image of v and it is zero on the image of β. To show that δ^0_C is determined by δ^0_B, it suffices to show that the images of v and β together span $H^0(X,\mathcal{Q})$. This follows from the fact that γ is surjective (7.4.3). We could omit the verification of surjectivity, but here it is: Note that $\gamma = w\beta$. Let q be an element of $H^0(\mathcal{Q})$. Since γ is surjective, there is an element r in $H^0(\mathcal{R}^0_{\mathcal{N}})$ such that $wq = \gamma r$ Then $wq = w\beta r$, and $w(q - \beta r) = 0$. So $q - \beta r$ is in the image of v. Thus δ^0_C is determined by δ^0_B, and so is δ^0_A. $\square$

7.10 Exercises

chapsex-cplex xcohcom-plpt **7.10.1.** The complement X of the point $(0,0,1)$ in $\mathbb{P}^2$ is covered by the two open standard open sets $\mathbb{U}^0, \mathbb{U}^1$. Use this covering to compute the cohomology $H^q(X, \mathcal{O}_X)$.

xcech-Hone **7.10.2.** Let U, V be affine open sets that cover a variety X. Using (6.3.7), construct an exact sequence $0 \to H^0(X, \mathcal{M}) \to \mathcal{M}(U) \oplus \mathcal{M}(V) \xrightarrow{(-,+)} \mathcal{M}(U \cap V) \to H^1(X, \mathcal{M}) \to 0$, and prove that $H^q(X, \mathcal{M}) = 0$ if $q > 1$.

xeuler-complex **7.10.3.** Let $0 \to V_0 \to \cdots \to V_n \to 0$ be a complex of finite-dimensional vector spaces. Prove that $\sum_i (-1)^i \dim V_i = \sum (-1)^q \mathbf{C}^q(V^\bullet\}$.

xglobsec-sexact **7.10.4.** Let $0 \to \mathcal{M}_0 \to \cdots \to \mathcal{M}_k \to 0$ be an exact sequence of $\mathcal{O}$-modules on a variety X. Prove that if $H^q(\mathcal{M}_i) = 0$ for all $q > 0$ and all i, the sequence of global sections is exact.

xcousin **7.10.5.** *the Cousin Problem.*

(i) Detemine the cohomology of the function field module $\mathcal{F}$ on a projective variety.

(ii) Let X be a projective space, and let $\{V^i\}$, $i = 1, ..., k$ be an open covering of X. Suppose that rational functions f_i are given, such that $f_i - f_j$ is a regular function on $V^i \cap V^j$ for all i and j. The Cousin Problem asks for a rational function $\widetilde{f}$ such that $\widetilde{f} - f_i$ is a regular function on V^i for every i. Analyze this problem making use of the exact sequence $0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{Q} \to 0$, where $\mathcal{Q}$ is the quotient $\mathcal{F}/\mathcal{O}$.

xcohdi-mone **7.10.6.** Prove that if a variety X is covered by two affine open sets, then $H^q(X, \mathcal{M}) = 0$ for every $\mathcal{O}$-module $\mathcal{M}$ and every $q > 1$.

nodes-cusps **7.10.7.** Let C be a plane curve of degree d with δ nodes and κ cusps, and let $C^\#$ be its normalization. Determine the genus of $C^\#$.

xcohd-blplane **7.10.8.** Let $f(x_0, x_1, x_2)$ be an irreducible homogeneous polynomial of degree $2d$, and let Y be the projective double plane $y^2 = f(x_0, x_1, x_2)$. Compute the cohomology $H^q(Y, \mathcal{O}_Y)$.

xABBA **7.10.9.** Let A, B be 2×2 variable matrices, let P be the polynomial ring $\mathbb{C}[a_{ij}, b_{ij}]$. and let R be the quotient of P by the ideal that expresses the condition $AB = BA$. Show that R has a resolution as P-module of the form $0 \to P^2 \to P^3 \to P \to R \to 0$. (Hint: Write the equations in terms of $a_{11} - a_{22}$ and $b_{11} - b_{22}$.)

xregfn-const **7.10.10.** Prove that a regular function on a projective variety is constant.

xalgbez **7.10.11.** *an algebraic version of Bézout's Theorem.* Let $R = \mathbb{C}[x, y, z]$, and let f and g be homogeneous polynomials in R, of degrees m and n, respectively. The quotient algebra $A = R/(f, g)$ inherits a grading by degree: $A = A_0 \oplus A_1 \oplus \cdots$, where A_n is the image of the space of homogeneous polynomials of degree n, together with 0.

(i) Show that the sequence

$$0 \to R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)^t} R \to A \to 0$$

is exact.

(ii) Prove that $\dim A_k = mn$ for all sufficiently large k.

(iii) Explain in what way this is an algebraic version of Bézout's Theorem.

xpascal **7.10.12.** Let p_1, p_2, p_3 and q_1, q_2, q_3 be distinct points on a conic C, and let L_{ij} be the line through p_i and q_j.

(i) Let g and h be the homogeneous cubic polynomials whose zero loci are $L_{12} \cup L_{13} \cup L_{23}$ and $L_{21} \cup L_{31} \cup L_{32}$, respectively, and let x be another point on C. Show that there is a scalar c such that the cubic $f = g + ch$ vanishes at x as well as at the six given points. What does Bézout's Theorem tell us about this cubic f?

(ii) *Pascal's Theorem* asserts that the three intersection points $r_1 = L_{23} \cap L_{32}$, $r_2 = L_{31} \cap L_{13}$, and $r_3 = L_{12} \cap L_{21}$ lie on a line. Prove Pascal's Theorem.

(iii) Let $Z_1, ..., Z_6$ be six lines that are tangent to C. Let $p_{12} = Z_1 \cap Z_2$, $p_{23} = Z_2 \cap Z_3$, $p_{34} = Z_3 \cap Z_4$, $p_{45} = Z_4 \cap Z_5$, $p_{56} = Z_5 \cap Z_6$, and $p_{61} = Z_6 \cap Z_1$. Think of the six lines as sides of a 'hexagon', with vertices p_{ij}. The 'main diagonals' are the lines D_1 through p_{12} and p_{45}, D_2 through p_{23} and p_{56}, and D_3 through p_{61} and p_{34}. *Brianchon's Theorem* asserts that the main diagonals have a common point. Prove this by studying the dual configuration in $\mathbb{P}^*$.

7.10.13. Let $X = \mathbb{P}^d$ and let $Y \xrightarrow{\pi} X$ be a finite morphism. Prove that Y is a projective variety. Do this by showing that, for large n, the global sections of $\mathcal{O}_Y(nH) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(nH)$ define a map to projective space whose image is isomorphic to Y. xfiniteproj

7.10.14. **(i)** Let R be the polynomial ring $\mathbb{C}[x, y, z]$, let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of degrees m and n, and with no common factor, and let $A = R/(f, g)$. Show that the sequence xmod-fandg

$$0 \to R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)^t} R \to A \to 0$$

is exact.

(ii) Let Y be a normal affine variety coordinate algebra B. Let I be an ideal of B generated by two elements u, v, and let X be the locus $V(I)$ in Y. Suppose that $\dim X \le \dim Y - 2$. Use the fact that $B = \bigcap B_Q$ where Q ranges over prime ideals of codimension 1 to prove that this sequence is exact:

$$0 \to B \xrightarrow{(v,-u)^t} B^2 \xrightarrow{(u,v)} B \to B/I \to 0$$

7.10.15. Let I be the ideal of $\mathbb{C}[x_0, x_1, x_2, x_3]$ generated by two homogeneous polynomials f, g, of dimensions d, e respectively, and assume that the locus $X = V(\mathcal{I})$ in $\mathbb{P}^3$ has dimension 1. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$. Multiplication by f and g defines a map $\mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O}$. Let $\mathcal{A}$ be the cokernel of this map. xmodfg

(i) Construct an exact sequence

$$0 \to \mathcal{O}(-d-e) \to \mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O} \to \mathcal{A} \to 0.$$

(ii) Show that X is a connected subset of $\mathbb{P}^3$ in the Zariski topology, i.e., that it is not the union of two proper disjoint Zariski-closed subsets.

(iii) Determine the genus of X in the case that X is a smooth curve.

7.10.16. A curve in $\mathbb{P}^3$ that is the zero locus of a homogeneous prime ideal generated by two elements is a *complete intersection*. Determine the genus of a smooth complete intersection when the generators have degrees r and s. xcplt-inttwo

7.10.17. *a theorem of Max Noether.* (Max Noether was Emmy Noether's father.) Let f and g be homogeneous polynomials in $x_0, ..., x_k$, of degrees r and s, respectively, with $k \ge 2$. Suppose that the locus $X : \{f = g = 0\}$ in $\mathbb{P}^k$ consists of distinct points if $k = 2$, or is a closed subvariety of codimension 2 if $k > 2$. A theorem that is called the AF+BG Theorem, asserts that, if a homogeneous polynomial p of degree n vanishes on X, there are homogeneous polynomials a and b such that $p = af + bg$. Prove Noether's theorem. xafplusbg

7.10.18. Let xmi-norstwo

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix}$$

be a 3×2 matrix whose entries are homogeneous quadratic polynomials in four variables $x_0, ..., x_3$. Let $M = (m_1, m_2, m_3)$ be the 1×3 matrix of minors

$$m_1 = u_{21}u_{32} - u_{22}u_{31}, \quad m_2 = -u_{11}u_{32} + u_{12}u_{31}, \quad m_3 = u_{11}u_{22} - u_{12}u_{21}$$

The matrices U and M give us a sequence

$$0 \to \mathcal{O}(-6)^2 \xrightarrow{U} \mathcal{O}(-4)^3 \xrightarrow{M} \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0$$

where $\mathcal{I}$ is the ideal generated by the minors.

(i) Suppose that the above sequence is exact, and that the locus of zeros of I in $\mathbb{P}^3$ is a curve. Determine the genus of that curve.

(ii) Prove that, if the locus is a curve, the sequence is exact.

7.10.19. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathcal{O}_X(m, n)$ denote the $\mathcal{O}_X$-module whose sections on an open subset W of X are the bihomogeneous fractions of bidegree m, n that are regular on W. xOmn

(i) Determine the cohomology of $\mathcal{O}_X(m, n)$.

(ii) Describe the smooth projective curves C that admit two distinct maps of degree 2 to $\mathbb{P}^1$.

Chapter 8 THE RIEMANN-ROCH THEOREM FOR CURVES

rrcurves

We study a classical problem of algebraic geometry, to determine the rational functions with given poles on a smooth projective curve. This is often difficult. If $p_1, ..., p_k$ are points of the curve, the rational functions whose poles have orders at most r_i at p_i form a vector space, and one is happy when one can determine the dimension of that space. The most important tool for determining the dimension is the Riemann-Roch Theorem.

8.1 Divisors

divtwo Smooth affine curves were discussed in Chapter 5. It was shown there that an affine curve is smooth if its local rings are valuation rings, or if its coordinate ring is a normal domain. An arbitrary curve is smooth if it has an open covering by smooth affine curves.

We take a brief look at modules on a smooth curve. Recall that a module M over a domain A is torsion-free if its only torsion element is zero: If $a \in A$ and $m \in M$ are nonzero, then $am \neq 0$. This definition is extended to $\mathcal{O}$-modules by applying it to the affine open subsets.

lfree **8.1.1. Lemma.** *Let Y be a smooth curve.*
(i) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is locally free if and only if it is torsion-free.*
(ii) *An $\mathcal{O}_Y$-module $\mathcal{M}$ that isn't torsion-free has a nonzero global section.*

proof of Lemma 8.1.1 **(i)**. We may assume that Y is affine, $Y = \operatorname{Spec} B$, and that $\mathcal{M}$ is the $\mathcal{O}$-module associated to a B-module M. Let $\widetilde{B}$ be the local ring of B at a point q, and let $\widetilde{M}$ be the localization of M at q, which is isomorphic to the tensor product $M \otimes_B \widetilde{B}$. If M is a torsion-free B-module, then $\widetilde{M}$ is a torsion-free module over the valuation ring $\widetilde{B}$. It suffices to show that, for every point q of Y, $\widetilde{M}$ is a free $\widetilde{B}$-module (2.6.13). The next sublemma does this.

vringpid **8.1.2. Sublemma.** *A finite, torsion-free module $\widetilde{M}$ over a valuation ring $\widetilde{B}$ is a free module.*

proof. It is easy to prove this directly. Or, one can use the fact that every finite, torsion-free module over a principal ideal domain is free. A valuation ring is a principal ideal domain because its nonzero ideals are powers of its maximal ideal, and the maximal ideal is a principal ideal. □

proof of Lemma 8.1.1 **(ii)** If $\mathcal{M}$ isn't torsion-free, then on some affine open subset U, there will be nonzero elements m in $\mathcal{M}(U)$ and a in $\mathcal{O}(U)$, such that $am = 0$. Let Z be the finite set of zeros of a in U. We choose an affine open set V that doesn't contain any points of Z, and such that $Y = U \cup V$. The proof of existence of such an open set is Exercise 5.7.22). Then a is invertible on the intersection $W = U \cap V$, and since $am = 0$, the restriction of m to W is zero.

The open sets U and V cover Y, so the sheaf property for this covering can be written as the exact sequence

$$0 \to \mathcal{M}(Y) \to \mathcal{M}(U) \times \mathcal{M}(V) \xrightarrow{-,+} \mathcal{M}(W)$$

(see Lemma 6.3.6). In this sequence, the section $(m, 0)$ of $\mathcal{M}(U) \times \mathcal{M}(V)$ maps to zero in $\mathcal{M}(W)$. Therefore it is the image of a nonzero global section of $\mathcal{M}$. □

8.1.3. Lemma. *Let Y be a smooth curve. Every nonzero ideal $\mathcal{I}$ of $\mathcal{O}_Y$ is a product of powers of maximal ideals: $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.* *idealprodmax*

proof. This follows for any smooth curve from the case that the curve is affine, which is Proposition 5.2.11. □

8.1.4. divisors divsect

A *divisor* on a smooth curve Y is a finite combination

$$D = r_1 q_1 + \cdots + r_k q_k$$

where r_i are integers and q_i are points. It is an element of the abelian group that has the points of Y as a $\mathbb{Z}$-basis.

The *degree* of the divisor D is the sum $r_1 + \cdots + r_k$ of the coefficients. The *support* of D is the set of points q_i such that $r_i \neq 0$.

The *restriction* of a divisor $D = r_1 q_1 + \cdots + r_k q_k$ to an open subset Y' is the divisor obtained from D by deleting points of the support that aren't in Y'. For example, let $D = q$. The restriction to Y' is q if q is in Y', and it is zero if q is not in Y'.

A divisor $D = \sum r_i q_i$ is *effective* if all of its coefficients r_i are non-negative, and D is *effective on an open set* Y' if its restriction to Y' is effective — if $r_i \geq 0$ when q_i is a point of Y'.

Let $D = \sum r_i p_i$ and $E = \sum s_i p_i$ be divisors. We my write $E \geq D$ if $s_i \geq r_i$ for all i, or if $E - D$ is effective. With this notation, $D \geq 0$ means that D is effective.

8.1.5. the divisor of a function divfn

The *divisor of* a nonzero rational function f on a smooth curve Y is

$$\operatorname{div}(f) = \sum_{q \in Y} \mathrm{v}_q(f)\, q$$

where v_q is the valuation of K associated to the point q. The divisor is written here as a sum over all points q, but it becomes a finite sum when we disregard terms with coefficient zero, because f has finitely many zeros and poles. The coefficients will be zero at all other points.

The map

$$K^\times \xrightarrow{\operatorname{div}} (divisors)^+ \tag{8.1.6}$$ Ktodiv

that sends a nonzero rational function to its divisor is a homomorphism from the multiplicative group $K^\times$ of nonzero elements of K to the additive group of divisors:

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$$

The divisor of a rational function is a *principal divisor*. The image of the map (8.1.6) is the set of principal divisors.

As before, if r is a positive integer, a nonzero rational function f has a *zero of order* r at q if $\mathrm{v}_q(f) = r$, and it has a *pole of order* r at q if $\mathrm{v}_q(f) = -r$. The divisor of f is the difference of two effective divisors:

$$\operatorname{div}(f) = zeros(f) - poles(f)$$

A rational function f is regular on Y if and only if $\operatorname{div}(f)$ is effective — if and only if $poles(f) = 0$.

Every divisor is locally principal: There is an affine covering $\{Y^i\}$ of Y such that the restriction of D is a principal divisor on each Y^i. This is true because the maximal ideals of Y are locally principal. If f is a generator of the maximal ideal at a point q, then $\operatorname{div}(f) = q$.

Two divisors D and E are said to be *linearly equivalent* if their difference $D - E$ is a principal divisor. For instance, the divisors $zeros(f)$ and $poles(f)$ of a rational function f are linearly equivalent.

levelsets **8.1.7. Lemma.** *Let f be a rational function on a smooth curve Y. For all complex numbers c, the divisors $zeros(f - c)$, the level sets of f, are linearly equivalent.*

proof. The functions $f - c$ have the same poles as f. □

divterm **8.1.8. review of terminology**

divisor: an integer combination of points: $D = r_1 q_1 + \cdots + r_k q_k$.

effective divisor: The divisor D is effective if $r_i \geq 0$ for all i.

principal divisor: The divisor of a rational function. The divisor of f is the sum $\sum_q \mathrm{v}_q(f) q$.

linearly equivalent divisors: Two divisors D and E are linearly equivalent if $D - E$ is a principal divisor.

restriction of the divisor D to an open set: The restriction to the open set U is the sum of the terms $r_i q_i$ such that q_i is a point of U.

support of the divisor D: the points q_i such that $r_i \neq 0$.

zeros and poles of a divisor: The zeros of the divisor D are the points q_i such that $r_i > 0$. The poles are the points q_i such that $r_i < 0$.

mod-uleOD **8.1.9. the module $\mathcal{O}(D)$**

To analyze functions with given poles on a smooth curve Y, we associate an $\mathcal{O}$-module $\mathcal{O}(D)$ to a divisor D. It is an analog of the module $\mathcal{O}(nH)$ on projective space. The nonzero sections of $\mathcal{O}(D)$ on an open set V are the rational functions f such that the the divisor $\operatorname{div}(f) + D$ is effective on V — such that its restriction to V is effective:

ODV (8.1.10)
$$\begin{aligned}[\mathcal{O}(D)](V) &= \{f \mid \operatorname{div}(f) + D \text{ is effective on } V\} \cup \{0\} \\ &= \{f \mid poles(f) \leq D \text{ on } V\} \cup \{0\}\end{aligned}$$

So $\mathcal{O}(D)$ is a submodule of the function field module $\mathcal{F}$. When the divisor D is effective, the global sections of $\mathcal{O}(D)$ are the rational functions whose poles are bounded by D. They are the solutions of the classical problem mentioned at the beginning of the chapter.

Let D be the divisor $\sum r_i q_i$. If an open set V contains q_i such that $r_i > 0$, a section of $\mathcal{O}(D)$ on V may have a pole of order at most r_i at q_i, and if $r_i < 0$ a section must have a zero of order at least $-r_i$ at q_i. For example, the sections of the module $\mathcal{O}(-q)$ on an open set V that contains q are the regular functions on V that are zero at q. So $\mathcal{O}(-q)$ is the maximal ideal $\mathfrak{m}_q$. Similarly, the sections of $\mathcal{O}(q)$ on an open set V that contains q are the rational functions that have a pole of order at most 1 at q and are regular at every other point of V. The sections of $\mathcal{O}(-q)$ and of $\mathcal{O}(q)$ on an open set V that doesn't contain p are the regular functions on V. Points that aren't in an open set V impose no conditions on sections. A section of $\mathcal{O}(D)$ on V can have arbitrary zeros or poles at points not in V.

The fact that a section of $\mathcal{O}(D)$ is allowed to have a pole at q_i when $r_i > 0$ contrasts with the divisor of a function. If $\operatorname{div}(f) = \sum r_i q_i$, then $r_i > 0$ means that f has a zero at q_i. When $\operatorname{div}(f) = D$, f will be a global section of $\mathcal{O}(-D)$.

ODinvert **8.1.11. Lemma. (i)** *Let D be the principal divisor $\operatorname{div}(g)$. Then $\mathcal{O}(D)$ is the free $\mathcal{O}$-module of rank 1 with basis g^{-1}.*

(ii) *For any divisor D on a smooth curve, $\mathcal{O}(D)$ is a locally free module of rank one.*

proof. **(i)** Let D be the divisor of a rational function g. The sections of $\mathcal{O}(D)$ on an open set U are the rational functions f such that $\operatorname{div}(f) + D = \operatorname{div}(fg) \geq 0$ on U — the functions f such that fg is a section of $\mathcal{O}$ on U, or such that f is a section of $g^{-1}\mathcal{O}$.

(ii) Every divisor is locally principal, $\square$

8.1.12. Proposition. *Let D and E be divisors on a smooth curve Y.* *LisOD*

(i) *The map $\mathcal{O}(D)\otimes_{\mathcal{O}}\mathcal{O}(E) \to \mathcal{O}(D+E)$ that sends $f\otimes g$ to the product fg is an isomorphism.*

(ii) *$\mathcal{O}(D) \subset \mathcal{O}(E)$ if and only if $E \geq D$.*

proof. **(i)** It is enough to verify this locally, so we may assume that Y is affine and that the supports of D and E contain just one point, say $D = rq$ and $E = sq$. We may also assume that the maximal ideal at q is a principal ideal, generated by an element x. Then $\mathcal{O}(D)$, $\mathcal{O}(E)$, and $\mathcal{O}(D+E)$ will be free modules with bases x^r, x^s and x^{r+s}, respectively. $\square$

8.1.13. Proposition. *Let Y be a smooth curve.* *idealOD*

(i) *The nonzero ideals of $\mathcal{O}_Y$ are the modules $\mathcal{O}(-E)$, where E is an effective divisor.*

(ii) *The modules $\mathcal{O}(D)$ are the finite $\mathcal{O}$-submodules of the function field module $\mathcal{F}$ of Y.*

(iii) *The function field module $\mathcal{F}$ is the union of the modules $\mathcal{O}(D)$.*

proof. **(i)** Say that $E = r_1q_1 + \cdots + r_kq_k$, and that $r_i \geq 0$ for all i. A rational function f is a section of $\mathcal{O}(-E)$ if $\operatorname{div}(f) - E$ is effective, which happens when $poles(f) = 0$, and $zeros(f) \geq E$. The same condition describes the elements of the ideal $\mathcal{I} = \mathfrak{m}_1^{r_1} \cdots \mathfrak{m}_k^{r_k}$.

(ii) Let $\mathcal{L}$ be a finite $\mathcal{O}$-submodule of $\mathcal{F}$. Since $\mathcal{L}$ is a finite $\mathcal{O}$-module, then because the local ring is a valuation ring, $\mathcal{L}$ will be generated by one element, a rational function f, in some open neighborhood U of a point q. If D is the divisor of f^{-1} on U, then $\mathcal{L} = \mathcal{O}(D)$ on U. This determines the divisor D uniquely. If D_1 and D_2 are divisors, and $D_1 \neq D_2$, then $\mathcal{O}(D_1) \neq \mathcal{O}(D_2)$. So when $\mathcal{L} = \mathcal{O}(D)$ on U and $\mathcal{L} = \mathcal{O}(D')$ on U', D and D' must agree on $U \cap U'$. Therefore there is a divisor D on the whole curve Y such that $\mathcal{L} = \mathcal{O}(D)$ in a suitable neighbohood U of any point q. This implies that $\mathcal{L} = \mathcal{O}(D)$. $\square$

8.1.14. Proposition. *Let D and E be divisors on a smooth curve Y. Multiplication by a rational function f such that $\operatorname{div}(f)+E-D \geq 0$ defines a homomorphism of $\mathcal{O}$-modules $\mathcal{O}(D) \to \mathcal{O}(E)$, and every homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ is multiplication by such a function.* *mapOD-toOE*

proof. For any $\mathcal{O}$-module $\mathcal{M}$, a homomorphism $\mathcal{O} \to \mathcal{M}$ is multiplication by a global section of $\mathcal{M}$ (6.4.1). So a homomorphism $\mathcal{O} \to \mathcal{O}(E-D)$ will be multiplication by a rational function f such that $\operatorname{div}(f)+E-D \geq 0$. If f is such a function, one obtains a homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ by tensoring with $\mathcal{O}(D)$. $\square$

8.1.15. Corollary. *Let D and E be divisors on a smooth curve Y.* *ODOE*

(i) *The modules $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are isomorphic if and only if D and E are linearly equivalent.*

(ii) *Let f be a rational function on Y, and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$.*

proof. If multiplication by a rational function f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$, the inverse morphism is defined by f^{-1}. Then $\operatorname{div}(f) + E - D \geq 0$ and also $\operatorname{div}(f^{-1}) + D - E = -\operatorname{div}(f) + D - E \geq 0$, so $\operatorname{div}(f) = D - E$. This proves **(i)**, and **(ii)** is the special case that $E = 0$. $\square$

8.1.16. invertible modules invertmod

An *invertible* $\mathcal{O}$-module is a locally free module of rank one — a module that is isomorphic to the module $\mathcal{O}$ in a neighborhood of any point. If D is a divisor on a smooth curve Y, then $\mathcal{O}(D)$ is an invertible $\mathcal{O}$-module. The tensor product $\mathcal{L}\otimes_{\mathcal{O}}\mathcal{M}$ of invertible modules is invertible.

8.1.17. Lemma. *Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module on a smooth curve Y, and let $\mathcal{L}^*$ be the dual module.* *inverse-mod*

(i) *The canonical map $\mathcal{L}^*\otimes_{\mathcal{O}}\mathcal{L} \to \mathcal{O}$ defined by $\gamma\otimes\alpha \mapsto \gamma(\alpha)$ is an isomorphism.*

(ii) *The map $\mathcal{O} \to {}_{\mathcal{O}}(\mathcal{L},\mathcal{L})$ $\big(= \underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{L},\mathcal{L})\big)$ that sends a regular function α to multiplication by α is an isomorphism.*

(iii) *Every nonzero homomorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ to a locally free module $\mathcal{M}$ is injective.*

Because of **(i)**, $\mathcal{L}^*$ may be thought of as an inverse to $\mathcal{L}$. This is the reason for the term 'invertible'. The dual of $\mathcal{O}(D)$ is $\mathcal{O}(-D)$.

proof of Lemma 8.1.17. **(i,ii)** It is enough to verify these assertions in the case that $\mathcal{L}$ is free, isomorphic to $\mathcal{O}$, in which case they are clear.

(iii) The problem is local, so we may assume that the variety is affine, say $Y = \operatorname{Spec} A$, and that $\mathcal{L}$ and $\mathcal{M}$ are free. Then φ becomes a nonzero homomorphism $A \to A^k$, which is injective because A is a domain. $\square$

As Proposition 8.1.13 **(ii)** shows, the only difference between an invertible module $\mathcal{L}$ and a module $\mathcal{O}(D)$ is that $\mathcal{O}(D)$ is a submodule of the function field module $\mathcal{F}$, while $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{F}$ can be any one-dimensional vector space over the function field.

Lequal-sOD **8.1.18. Corollary.** *Every invertible $\mathcal{O}$-module $\mathcal{L}$ on a smooth curve Y is isomorphic to one of the form $\mathcal{O}(D)$.* $\square$

If Y is smooth projective curve, the *degree* of an invertible module $\mathcal{L}$ on Y is defined to be the degree of the divisor D such that $\mathcal{L} \approx \mathcal{O}(D)$.

8.2 The Riemann-Roch Theorem I

rrone

Let Y be a smooth projective curve, and let $\mathcal{M}$ be a finite $\mathcal{O}_Y$-module. We have seen that the cohomology $H^q(Y, \mathcal{M})$ is a finite-dimensional vector space for $q = 0, 1$, and is zero when $q > 1$ (7.7.3),(7.7.1). As before, we denote the dimension of $H^q(Y, \mathcal{M})$ by $\mathbf{h}^q\mathcal{M}$ or $\mathbf{h}^q(Y, \mathcal{M})$.

The Euler characteristic (7.6.6) of a finite $\mathcal{O}$-module $\mathcal{M}$ is

chicurve (8.2.1)
$$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^1\mathcal{M}$$

In particular,
$$\chi(\mathcal{O}_Y) \;=\; \mathbf{h}^0\mathcal{O}_Y - \mathbf{h}^1\mathcal{O}_Y$$

The dimension $\mathbf{h}^1\mathcal{O}_Y$ is the arithmetic genus p_a of Y (7.6). We will see below that $\mathbf{h}^0\mathcal{O}_Y = 1$ (8.2.9). So

chi-curvetwo (8.2.2)
$$\chi(\mathcal{O}) \;=\; 1 - p_a$$

RRcurve **8.2.3. Riemann-Roch Theorem (version 1).** *Let $D = \sum r_i p_i$ be a divisor on a smooth projective curve Y. Then*
$$\chi(\mathcal{O}(D)) \;=\; \chi(\mathcal{O}) + \deg\, D \qquad \big(= \deg\, D + 1 - p_a \big)$$

proof. We analyze the effect on cohomology when a divisor is changed by adding or subtracting a point, by inspecting the inclusion $\mathcal{O}(D{-}p) \subset \mathcal{O}(D)$. The cokernel ϵ of the inclusion map is a one-dimensional vector space supported at p, isomorphic to the residue field module κ_p. We'll write the cokernel as κ_p, though the identification of ϵ with κ_p requires choosing a basis of the one-dimensional module ϵ.

So there is a short exact sequence

addpoint (8.2.4)
$$0 \to \mathcal{O}(D{-}p) \to \mathcal{O}(D) \to \kappa_p \to 0$$

This sequence can be obtained by tensoring the sequence

kappaseq (8.2.5)
$$0 \to \mathfrak{m}_p \to \mathcal{O} \to \kappa_p \to 0$$

with the invertible module $\mathcal{O}(D)$, because $\mathfrak{m}_p = \mathcal{O}(-p)$.

Since the support of κ_p has dimension zero, $H^1(\kappa_p) = 0$, and $H^0(\kappa_p) = \mathbb{C}$. Let's denote the one-dimensional vector space $H^0(Y, \kappa_p)$ by [1]. The cohomology sequence associated to (8.2.4) is

add-pointtwo (8.2.6)
$$0 \to H^0(Y, \mathcal{O}(D{-}p)) \to H^0(Y, \mathcal{O}(D)) \xrightarrow{\gamma} [1] \xrightarrow{\delta} H^1(Y, \mathcal{O}(D{-}p)) \to H^1(Y, \mathcal{O}(D)) \to 0$$

In this exact sequence of vector spaces, one of the maps γ or δ, must be zero. Either

- γ is zero and δ is injective. In this case

$$\mathbf{h}^0\mathcal{O}(D{-}p) = \mathbf{h}^0\mathcal{O}(D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D{-}p) = \mathbf{h}^1\mathcal{O}(D) + 1, \quad \text{or}$$

- δ is zero and γ is surjective. In this case

$$\mathbf{h}^0\mathcal{O}(D){-}p) = \mathbf{h}^0\mathcal{O}(D) - 1 \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D{-}p) = \mathbf{h}^1\mathcal{O}(D)$$

In either case,

(8.2.7) $$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D{-}p)) + 1$$ chichange

We also have $\deg D = \deg(D - p) + 1$. The Riemann-Roch theorem follows. It is true for the module $\mathcal{O}$, and we can get from $\mathcal{O}$ to $\mathcal{O}(D)$ by a finite number of operations, each of which changes the divisor by adding or subtracting a point. Therefore it is true for all D. □

Because $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$, this version of the Riemann-Roch Theorem gives reasonably good control of H^0. It is less useful for controlling H^1. For that, one wants the full Riemann-Roch Theorem, which we call version 2. The full theorem requires some preparation, so we have put it into Section 8.7. However, version 1 has important consequences:

8.2.8. Corollary. *Let p be a point of a smooth projective curve Y. The dimension $\mathbf{h}^0(Y, \mathcal{O}(np))$ tends to infinity with n. Therefore there exist rational functions on Y that have a pole of some order at a single point p, and no other poles.* *onepole*

proof. The sequence (8.2.6) shows that when we go from $\mathcal{O}(np)$ to $\mathcal{O}((n{+}1)p)$, either $\mathbf{h}^0$ increases or else $\mathbf{h}^1$ decreases. Since $\mathbf{h}^1\mathcal{O}(np)$ is finite, the second possibility can occur only finitely often, as n tends to ∞. □

8.2.9. Corollary. *Let Y be a smooth projective curve.* *RRcor*

(i) *The divisor of a rational function on Y has degree zero: The number of zeros is equal to the number of poles.*

(ii) *Linearly equivalent divisors on Y have equal degrees.*

(iii) *A nonconstant rational function on Y takes every value, including infinity, the same number of times, when counted with multiplicity.*

(iv) *A rational function on Y that is regular at every point of Y is a constant: $H^0(Y, \mathcal{O}) = \mathbb{C}$.*

proof. **(i)** Let f be a nonzero rational function and let $D = \mathrm{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$ (8.1.15), so $\chi(\mathcal{O}(D)) = \chi(\mathcal{O})$. On the other hand, by Riemann-Roch, $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D$. Therefore $\deg D = 0$.

(ii) If D and E are linearly equivalent divisors, $\mathcal{O}(D)$ and $co(E)$ are isomorphic.

(iii) The divisor of zeros of the function $f - c$ is linearly equivalent to the divisor of poles of f.

(iv) According to **(iii)**, a nonconstant function must have a pole. □

8.2.10. Corollary. *Let D be a divisor on Y. If $\deg D \geq p_a$, then $\mathbf{h}^0\mathcal{O}(D) > 0$. If $\mathbf{h}^0\mathcal{O}(D) > 0$, then* $\deg D \geq 0$. *RRcortwo*

When the degree d of D is in the range $0 < d < p_a$, $\mathbf{h}^0$ may depend on the particular divisor.

proof of Corollary 8.2.10. If $\deg D \geq p_a$, then $\chi(\mathcal{O}(D)) = \deg D + 1 - p_a \geq 1$, and $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$. If $\mathcal{O}(D)$ has a nonzero global section f, a rational function such that $\mathrm{div}(f){+}D$ is effective, then $\deg(\mathrm{div}(f) + D) \geq 0$, and because the degree of $\mathrm{div}(f)$ is zero, $\deg D \geq 0$. □

8.2.11. Theorem. *With its classical topology, a smooth projective curve Y is a connected, compact, orientable two-dimensional manifold.* *curveconn* *proof.* We prove connectedness here. The other points have been explained before in the case of a plane curve (Theorem 1.7.21), and the proofs for any projective curve are no different.

A topological space is connected if it isn't the union of two disjoint, nonempty, closed subsets. Suppose that, in the classical topology, Y is the union of disjoint, nonempty closed subsets Y_1 and Y_2. Both Y_1 and Y_2 will be compact, two-dimensional manifolds. Let p be a point of of Y_1. There is a nonconstant rational function f whose only pole is at p (Corollary 8.2.8). Then f will be a regular function on the complement of p, and therefore a regular function on the entire compact manifold Y_2.

For review: A point q of the smooth curve Y has a neighborhood V that is analytically equivalent to an open subset U of the affine line X. If a function g on V is analytic, the function on U that corresponds to g is an analytic function of one variable. The maximum principle for analytic functions asserts that, on an open region of the complex plane, the absolute value of a nonconstant analytic function has no maximum. This applies to the open set U and therefore also to the neighborhood V of q. Since q can be any point of Y_2, a nonconstant analytic function cannot have a maximum anywhere on Y_2. On the other hand, since Y_2 is compact, a continuous function does have a maximum. So a function that is analytic on Y_2 must be a constant.

Going back to the rational function f with a single pole p, the restriction of f to Y_2 will be analytic, and therefore constant. When we subtract that constant from f, we obtain a nonconstant rational function on Y with a single pole p that is zero on Y_2. But a rational function on a curve has finitely many zeros. This is a contradiction. □

8.3 The Birkhoff-Grothendieck Theorem

birkgroth This theorem describes the finite, torsion-free modules on the projective line $X = \mathbb{P}^1$.

BGtheorem **8.3.1. Birkhoff-Grothendieck Theorem.** *A finite, locally free (or torsion-free) $\mathcal{O}$-module on the projective line X is isomorphic to a direct sum of twisting modules: $\mathcal{M} \approx \bigoplus \mathcal{O}(n_i)$.*

This theorem was proved by Grothendieck using cohomology. It had been proved earlier by Birkhoff, in the following equivalent form:

Birkhoff Factorization Theorem. Let $A_0 = \mathbb{C}[u]$, $A_1 = \mathbb{C}[u^{-1}]$, and $A_{01} = \mathbb{C}[u, u^{-1}]$. Let P be an invertible A_{01}-matrix. There exist an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1 such that $Q_0^{-1}PQ_1$ is diagonal, and its diagonal entries are integer powers of u.

proof of the Birkhoff-Grothendieck Theorem. This is Grothendieck's proof.

According to Theorem 7.5.5, the cohomology of the twisting modules on the projective line X is $\mathbf{h}^0\mathcal{O} = 1$, $\mathbf{h}^1\mathcal{O} = 0$, and if r is a positive integer,

$$\mathbf{h}^0\mathcal{O}(r) = r+1, \quad \mathbf{h}^1\mathcal{O}(r) = 0, \quad \mathbf{h}^0\mathcal{O}(-r) = 0, \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(-r) = r-1$$

mapstoMbounded **8.3.2. Lemma.** *Let $\mathcal{M}$ be a finite, locally free $\mathcal{O}$-module on X. For sufficiently large r,*
(i) *the only homomorphism $\mathcal{O}(r) \to \mathcal{M}$ is the zero map, and*
(ii) $\mathbf{h}^0(X, \mathcal{M}(-r)) = 0$.

proof. **(i)** A nonzero homomorphism $\mathcal{O}(r) \xrightarrow{\varphi} \mathcal{M}$ from the twisting module $\mathcal{O}(r)$ to the locally free module $\mathcal{M}$ will be injective (8.1.17), and the associated map $H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{M})$ will be injective too, so $\mathbf{h}^0(X, \mathcal{O}(r)) \le \mathbf{h}^0(X, \mathcal{M})$. Since $\mathbf{h}^0(X, \mathcal{O}(r)) = r+1$, r is bounded by the integer $\mathbf{h}^0(X, \mathcal{M}) - 1$.

(ii) A global section of $\mathcal{M}(-r)$ defines a map $\mathcal{O} \to \mathcal{M}(-r)$. Its twist by r will be a map $\mathcal{O}(r) \to \mathcal{M}$. By **(i)**, r is bounded. □

We go to the proof now. We use induction on the rank. We suppose that $\mathcal{M}$ has rank r, that $r > 0$, and that the theorem has been proved for locally free $\mathcal{O}$-modules of rank less than r. The plan is to show that $\mathcal{M}$ has a twisting module as a direct summand, so that $\mathcal{M} = \mathcal{W} \oplus \mathcal{O}(n)$ for some $\mathcal{W}$. Then induction on the rank, applied to $\mathcal{W}$, will prove the theorem.

Twisting is compatible with direct sums, so we may replace $\mathcal{M}$ by a twist $\mathcal{M}(n)$. Instead of showing that $\mathcal{M}$ has a twisting module $\mathcal{O}(n)$ as a direct summand, we show that, after we replace $\mathcal{M}$ by a suitable twist, the structure sheaf $\mathcal{O}$ will be a direct summand.

The twist $\mathcal{M}(n)$ will have a nonzero global section when n is sufficiently large (6.7.21), and it will have no nonzero global section when n is sufficiently negative (8.3.2) **(ii)**. Therefore, when we replace $\mathcal{M}$ by a suitable twist, we will have $H^0(X, \mathcal{M}) \neq 0$ but $H^0(X, \mathcal{M}(-1)) = 0$. We assume that this is true for $\mathcal{M}$.

We choose a nonzero global section m of $\mathcal{M}$ and consider the injective multiplication map $\mathcal{O} \xrightarrow{m} \mathcal{M}$. Let $\mathcal{W}$ be its cokernel, so that we have a short exact sequence

(8.3.3) $$0 \to \mathcal{O} \xrightarrow{m} \mathcal{M} \xrightarrow{\pi} \mathcal{W} \to 0$$ cvcwsequence

8.3.4. Lemma. *Let $\mathcal{W}$ be the $\mathcal{O}$-module that appears in the sequence (8.3.3).* *sectionbasic*
(i) $H^0(X, \mathcal{W}(-1)) = 0$.
(ii) *$\mathcal{W}$ is torsion-free, and therefore locally free.*
(iii) *$\mathcal{W}$ is isomorphic to a direct sum $\bigoplus_{i=1}^{r-1} \mathcal{O}(n_i)$ of twisting modules on $\mathbb{P}^1$, with $n_i \leq 0$.*

proof. **(i)** This follows from the cohomology sequence associated to the twisted sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{M}(-1) \to \mathcal{W}(-1) \to 0$$

because $H^0(X, \mathcal{M}(-1)) = 0$ and $H^1(X, \mathcal{O}(-1)) = 0$.

(ii) If the torsion submodule of $\mathcal{W}$ were nonzero, the torsion submodule of $\mathcal{W}(-1)$ would also be nonzero, and then $\mathcal{W}(-1)$ would have a nonzero global section (8.1.1).

(iii) The fact that $\mathcal{W}$ is a direct sum of twisting modules follows by induction on the rank: $\mathcal{W} \approx \bigoplus \mathcal{O}(n_i)$. Since $H^0(X, \mathcal{W}(-1)) = 0$, we must have $H^0(X, \mathcal{O}(n_i - 1)) = 0$. Therefore $n_i - 1 < 0$, and $n_i \leq 0$. $\square$

We go back to the proof of Theorem 8.3.1, and dualize the sequence (8.3.3). Because $\mathcal{O}^* = \mathcal{O}$, the dual sequence is an exact sequence

$$0 \to \mathcal{W}^* \xrightarrow{\pi^*} \mathcal{M}^* \xrightarrow{m^*} \mathcal{O} \to 0$$

In it, $\mathcal{W}^* \approx \bigoplus \mathcal{O}(n_i)^* \approx \bigoplus \mathcal{O}(-n_i)$, and $-n_i \geq 0$ for all i. Therefore $\mathbf{h}^1 \mathcal{W}^* = 0$. The map $H^0(\mathcal{M}^*) \to H^0(\mathcal{O})$ is surjective. Let α be a global section of $\mathcal{M}^*$ whose image in $\mathcal{O}$ is 1. Multiplication by α defines a map $\mathcal{O} \xrightarrow{\alpha} \mathcal{M}^*$ that splits the sequence, i.e., such that $m^*\alpha$ is the identity map on $\mathcal{O}$ Then $\mathcal{M}^*$ is the direct sum $im(\alpha) \oplus ker(m^*) \approx \mathcal{O} \oplus \mathcal{W}^*$. Therefore $\mathcal{M} \approx \mathcal{W} \oplus \mathcal{O}$. $\square$

8.4 Differentials

diff

We introduce differentials and branched coverings, because they will be used in version 2 of the Riemann-Roch theorem. Why differentials enter into the Riemann-Roch Theorem is something of a mystery, but one important fact is the *Residue Theorem*, which controls the poles of a rational differential. Proofs of Reimann-Roch are often based on the Residue Theorem. We recommend reading about it, though we won't use it. [1]

Try not to get bogged down in the preliminary disussions here. Give the next pages a quick read to learn the terminology. You can look back as needed. Begin to read carefully when you get to Section 8.6.

Let A be an algebra and let M be an A-module. A *derivation* $A \xrightarrow{\delta} M$ is a $\mathbb{C}$-linear map that satisfies the product rule for differentiation, a map that has these properties:

(8.4.1) $$\delta(ab) = a\,\delta b + b\,\delta a\,, \quad \delta(a{+}b) = \delta a + \delta b\,, \quad \text{and} \quad \delta c = 0$$ deriv

for all a, b in A and all c in $\mathbb{C}$. The fact that δ is $\mathbb{C}$-linear, i.e., that it is a homomorphism of vector spaces, follows: Since $\delta c = 0$, $\delta(cb) = c\,\delta b + b\,\delta c = c\,\delta b$.

For example, differentiation $\frac{d}{dt}$ is a derivation $\mathbb{C}[t] \to \mathbb{C}[t]$.

[1] See one of the books by Fulton, Miranda, or Mumford in the bibliography, or for a general treatment, Tate, J., Residues of differentials on curves, Ann Sci ENS 1968.

compde **8.4.2. Lemma.** **(i)** *Let B be an algebra, let $M \xrightarrow{g} N$ be a homomorphism of B-modules, and let $B \xrightarrow{\delta} M$ be a derivation. The composition $B \xrightarrow{g\delta} N$ is a derivation.*
(ii) *Let $A \xrightarrow{\varphi} B$ be an algebra homomorphism, and let $B \xrightarrow{\delta} M$ be a derivation. The composition $A \xrightarrow{\delta\varphi} M$ is a derivation.*
(iii) *Let $A \xrightarrow{\varphi} B$ be a surjective algebra homomorphism, let $B \xrightarrow{h} M$ be a map to a B-module M, and let $d = h\varphi$. If $A \xrightarrow{d} M$ is a derivation, then h is a derivation.* □

The *module of differentials* Ω_A of an algebra A is an A-module generated by elements that are denoted by da, one for each element a of A. The elements of Ω_A are (finite) combinations $\sum b_i\, da_i$, with a_i and b_i in A. The defining relations among the generators da are the ones that make the map $A \xrightarrow{d} \Omega_A$ that sends a to da a derivation: For all a, b in A and all c in $\mathbb{C}$,

defdiff (8.4.3) $$d(ab) = a\,db + b\,da\ , \quad d(a{+}b) = da + db\ , \quad \text{and} \quad dc = 0$$

The elements of Ω_A are the *differentials*.

ho-momderiv **8.4.4. Lemma.**

(i) *When we compose a homomorphism $\Omega_A \xrightarrow{f} M$ of $\mathcal{O}$-modules with the derivation $A \xrightarrow{d} \Omega_A$, we obtain a derivation $A \xrightarrow{fd} M$. This composition with d defines a bijection between module homomorphisms $\Omega_A \to M$ and derivations $A \xrightarrow{\delta} M$.*
(ii) *Ω is a functor: An algebra homomorphism $A \xrightarrow{u} B$ induces a homomorphism $\Omega_A \xrightarrow{v} \Omega_B$ that is compatible with the ring homomorphism u, and that makes a diagram*

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_B \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} \\ A & \xrightarrow{d} & \Omega_A \end{array}$$

(Recall that, when ω is an element of Ω_A and α is an element of A, compatibility of v with u means that $v(\alpha\omega) = u(\alpha)v(\omega)$.)

proof. **(ii)** When Ω_B is made into an A-module by restriction of scalars, the composed map $A \xrightarrow{u} B \xrightarrow{d} \Omega_B$ will be a derivation to which **(i)** applies. □

omegafree **8.4.5. Lemma.** *Let R be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$. The module of differentials Ω_R is a free R-module with basis $dx_1, ..., dx_n$.*

proof. The formula $df = \sum \frac{df}{dx_i} dx_i$ follows from the defining relations. It shows that the elements $dx_1, ..., dx_n$ generate Ω_R. Let V be a free R-module with basis $v_1, ..., v_n$. The product rule for differentiation shows that the map $\delta : R \to V$ defined by $\delta(f) = \frac{\partial f}{\partial x_i} v_i$ is a derivation. It induces a module homomorphism $\Omega_A \xrightarrow{\varphi} V$ that sends dx_i to v_i (8.4.4). Since $dx_1, ..., dx_n$ generate Ω_R and $v_1, ..., v_n$ is a basis of V, φ is an isomorphism. □

omegafreetwo **8.4.6. Proposition.** *Let I be an ideal of an algebra R, let A be the quotient algebra R/I, and let dI denote the set of differentials df with f in I. The subset $N = dI{+}I\Omega_R$ of Ω_R is a submodule, and Ω_A is isomorphic to the quotient Ω_R/N.*

The proposition can be interpreted this way: Suppose that an ideal I is generated by elements $f_1, ..., f_r$ of R, and let $A = R/I$. Then Ω_A is the quotient of Ω_R that is obtained by introducing these two rules:

- $df_i = 0$, and
- multiplication by f_i is zero.

These rules hold in Ω_A because the elements f_i are zero in A.

8.4.7. Example. Let R be the polynomial ring $\mathbb{C}[y]$ in one variable. So Ω_R is a free module with basis dy. Let I be the principal ideal (y^2), and let $A = R/I$. In this case, y^2dy generates the module $I\Omega_A$, and dI is also an R-module. It is generated by the element $2y\,dy$. So N is generated by ydy. If $\overline{y}$ denotes the residue of y in A, $\Omega_A = \Omega_R/N$ is generated by an element $d\overline{y}$, with the relation $\overline{y}\,d\overline{y} = 0$. It isn't the zero module. $\square$ diffmod-square

proof of Proposition 8.4.6. First, $I\Omega_R$ is a submodule of Ω_R, and dI is an additive subgroup of Ω_R. To show that N is a submodule, we must show that scalar multiplication by an element of R maps dI to N, i.e., that if g is in R and f is in I, then $g\,df$ is in N. By the product rule, $g\,df = d(fg) - f\,dg$. Since I is an ideal, fg is in I. Then $d(fg)$ is in dI, and $f\,dg$ is in $I\Omega_R$. So $g\,df$ is in N.

The rules displayed above show that N is contained in the kernel of the surjective map $\Omega_R \xrightarrow{v} \Omega_A$ defined by the homomorphism $R \to A$. Let $\overline{\Omega}$ denote the quotient Ω_R/N. It is an A-module, and v defines a surjective map of A-modules $\overline{\Omega} \xrightarrow{\overline{v}} \Omega_A$, because $N \subset \ker\, v$. We show that $\overline{v}$ is an isomorphism. Let r be an element of R and let $\overline{dr}$ be its image in $\overline{\Omega}$. The composed map $R \xrightarrow{d} \Omega_R \xrightarrow{\pi} \overline{\Omega}$ is a derivation that sends r to $\overline{dr}$, and I is in its kernel. It defines a derivation $R/I = A \xrightarrow{\delta} \overline{\Omega}$, and if a is the residue of r in A, then $\delta(a) = \overline{dr}$. This derivation corresponds to a homomorphism of A-modules $\Omega_A \to \overline{\Omega}$ that sends da to $\overline{dr}$, and that inverts $\overline{v}$. $\square$

8.4.8. Corollary. *If A is a finite-type algebra, then Ω_A is a finite A-module.* *Omegafinite*

This follows from Proposition 8.4.6, because the module of differentials of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is a finite module. $\square$

8.4.9. Lemma. *Let S be a multiplicative system in a domain A. The module $\Omega_{S^{-1}A}$ of differentials of $S^{-1}A$ is canonically isomorphic to the module of fractions $S^{-1}\Omega_A$. In particular, if K is the field of fractions of A, then $K\otimes_A\Omega_A \approx \Omega_K$.* *localizeomega*

We have moved the symbol S^{-1} to the left for clarity. The lemma shows that a finite $\mathcal{O}$-module Ω_Y of differentials on a variety Y is defined such that, when $U = \operatorname{Spec} A$ is an affine open subset of Y, $\Omega_Y(U) = \Omega_A$.

proof of Lemma 8.4.9. The composed map $A \to S^{-1}A \xrightarrow{d} \Omega_{S^{-1}A}$ is a derivation. It defines an A-module homomorphism $\Omega_A \to \Omega_{S^{-1}A}$ which extends to an $S^{-1}A$-homomorphism $S^{-1}\Omega_A \xrightarrow{\varphi} \Omega_{S^{-1}A}$ because scalar multiplication by the elements of S is invertible in $\Omega_{S^{-1}A}$. The relation $ds^{-k} = -ks^{-k-1}ds$ follows from the definition of a differential, and it shows that φ is surjective. The quotient rule

$$\delta(s^{-k}a) = -ks^{-k-1}a\,ds + s^{-k}da$$

defines a derivation $S^{-1}A \xrightarrow{\delta} S^{-1}\Omega_A$, that corresponds to a homomorphism $\Omega_{S^{-1}A} \to S^{-1}\Omega_A$ and that inverts φ. Here, one must show that δ is well-defined, that $\delta(s_1^{-k}a_1) = \delta(s_2^{-\ell}a_2)$ if $s_1^{-\ell}a_1 = s_2^{-k}a_2$, and that δ is a derivation. You will be able to do this. $\square$

8.4.10. Proposition. *Let y be a local generator for the maximal ideal at a point q of a smooth curve Y. In a suitable neighborhood of q, the module Ω_Y of differentials is a free $\mathcal{O}$-module with basis dy. Therefore Ω_Y is an invertible module.* *omegafunct*

proof. We may assume that Y is affine, say $Y = \operatorname{Spec} B$. Let q be a point of Y, and let y be an element of B with $\mathrm{v}_q(y) = 1$. To show that dy generates Ω_B locally, we may localize, so we may suppose that y generates the maximal ideal $\mathfrak{m}$ at q. We must show that after we localize B once more, every differential df with f in B will be a multiple of dy. Let $c = f(q)$, so that $f = c + yg$ for some g in B, and because $dc = 0$, $df = g\,dy + y\,dg$. Here $g\,dy$ is in $B\,dy$ and $y\,dy$ is in $\mathfrak{m}\Omega_B$. This shows that

$$\Omega_B = B\,dy + \mathfrak{m}\Omega_B$$

An element β of Ω_B can be written as $\beta = b\,dy + \gamma$, with b in B and γ in $\mathfrak{m}\Omega_B$. If W denotes the quotient module $\Omega_B/(B\,dy)$, then $W = \mathfrak{m}W$. The Nakayama Lemma tells us that there is an element z in $\mathfrak{m}$ such that $s = 1 - z$ annihilates W. When we replace B by its localization B_s, we will have $W = 0$ and $\Omega_B = B\,dy$, as required.

We must still verify that the generator dy of Ω_B isn't a torsion element. Suppose that $b\,dy = 0$, with $b \neq 0$, then Ω_B will be zero except at the finite set of zeros of b in Y. We choose for q a point at which Ω_B is zero, keeping the rest of the notation unchanged. Let $A = \mathbb{C}[y]/(y^2)$. As was noted in Example 8.4.7, Ω_A isn't the zero module. Proposition 5.2.10 tells us that, at our point q, the algebra $B/\mathfrak{m}_q^2$ is isomorphic to A, and Proposition 8.4.6 tells us that Ω_A is a quotient of Ω_B. Since Ω_A isn't zero, neither is Ω_B. Therefore dy isn't a torsion element. $\square$

8.5 Branched Coverings

cover-curve

By a *branched covering*, we mean an integral morphism $Y \xrightarrow{\pi} X$ of smooth curves. Chevalley's Finiteness Theorem 4.6.6 shows that, if Y is projective, every morphism $Y \to X$ will be a branched covering, unless it maps Y to a point.

Let $Y \to X$ be a branched covering. The function field K of Y will be a finite extension of the function field F of X. The *degree* $[Y:X]$ of the covering is defined to be the degree $[K:F]$ of that field extension. If $X' = \operatorname{Spec} A$ is an affine open subset of X, its inverse image Y' will be an affine open subset $Y' = \operatorname{Spec} B$ of Y, and B will be a locally free A-module, of rank equal to $[Y:X]$.

To describe the fibre of a branched covering $Y \xrightarrow{\pi} X$ over a point p of X, we may localize. We assume that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that the maximal ideal $\mathfrak{m}_p$ of A at a point p is a principal ideal, generated by an element x of A. If a point q of Y lies over p, the *ramification index* at q is defined to be $\mathrm{v}_q(x)$, where v_q is the valuation of the function field K that corresponds to q. We will denote the ramification index by e. Then if y is a local generator for the maximal ideal $\mathfrak{m}_q$ of B at q, we will have

$$x = uy^e$$

where u is a *local unit* — a rational function on Y that is regular and invertible on some open neighborhood of q.

Points of Y whose ramification indices are greater than one are called *branch points*. We also call a point of X a *branch point* of the covering if it is the image of a branch point of Y. A branched covering $Y \to X$ has finitely many branch points.

degoneisom

8.5.1. Corollary. *A branched covering $Y \xrightarrow{\pi} X$ of degree one is an isomorphism.*

proof. When $[Y:X] = 1$, the function fields of Y and X will be equal. Then because $Y \to X$ is an integral morphism and X is normal, $Y = X$. $\square$

The next lemma follows from the Chinese Remainder Theorem.

extendedidealisprod

8.5.2. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Suppose that the maximal ideal $\mathfrak{m}_p$ at a point p of X is a principal ideal, generated by an element x. Let $q_1, ..., q_k$ be the points of Y that lie over p and let $\mathfrak{m}_i$ and e_i be the maximal ideal and ramification index at q_i, respectively.*
(i) *The extended ideal $\mathfrak{m}_pB = xB$ is the product ideal $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.*
(ii) *Let $\overline{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\overline{B} = B/xB$ is isomorphic to the product $\overline{B}_1 \times \cdots \times \overline{B}_k$.*
(iii) *The degree $[Y:X]$ of the covering is the sum $e_1 + \cdots + e_k$ of the ramification indices at the points q_i.*
(iv) *If p isn't a branch point, the fibre over p consists of n points with ramification indices equal to 1, n being the degree $[Y:X]$.* $\square$

nomultzero

8.5.3. Proposition. *Let f be a nonconstant rational function on a smooth projective curve Y, and let d be the degree of the divisor of poles of f. For generic scalars c, the level set $f = c$ has no multiple point.*

proof. We map Y to $X = \mathbb{P}^1$ by the functions $(1, f)$, so that Y becomes a branched covering of X. The level sets of f are the fibres of that morphism. For all but finitely many points of X, the fibres consist of d points of multiplicity one, with $d = [Y:X]$ (see(4.2.7)). $\square$

locanstr

8.5.4. local analytic structure

In the classical topology, the analytic structure of a branched covering $Y \xrightarrow{\pi} X$ is very simple. We describe it here because it is useful and helpful for intuition.

yethroottwo

8.5.5. Proposition. *Locally in the classical topology, Y is analytically isomorphic to the e-th root covering $y^e = x$.*

proof. Let q be a point of Y, let p be its image in X, let x and y be local generators for the maximal ideals $\mathfrak{m}_p$ of $\mathcal{O}_X$ and $\mathfrak{m}_q$ of $\mathcal{O}_Y$, respectively. Let e be the ramification index at q. So $x = uy^e$, where u is a local unit at q. In a neighborhood of q in the classical topology, u will have an analytic e-th root w. The element $y_1 = wy$ also generates $\mathfrak{m}_q$ locally, and $x = y_1^e$. We replace y by y_1. The Implicit Function Theorem tells us that that x and y are local analytic coordinate functions on X and Y (see (9.2)). $\square$

8.5.6. Corollary. *Let $Y \xrightarrow{\pi} X$ be a branched covering, let $\{q_1, ..., q_k\}$ be the fibre over a point p of X, and let e_i be the ramification index at q_i. As a point p' of X approaches p, e_i points that lie over p' approach q_i.* $\square$ *pprimeapproachesp*

8.5.7. suppressing notation for the direct image workonX

When considering a branched covering $Y \xrightarrow{\pi} X$ of smooth curves, we will often pass between an $\mathcal{O}_Y$-module $\mathcal{M}$ and its direct image $\pi_*\mathcal{M}$, and we may want to work primarily on X. Recall that if X' is an open subset X' of X and Y' is its invere image, then

$$[\pi_*\mathcal{M}](X') = \mathcal{M}(Y')$$

One can think of the direct image $\pi_*\mathcal{M}$ as working with $\mathcal{M}$, but looking only at the open subsets Y' of Y that are inverse images of open subsets of X. If we look only at such subsets, the only significant difference between $\mathcal{M}$ and its direct image will be that, when X' is open in X and $Y' = \pi^{-1}X'$, the $\mathcal{O}_Y(Y')$-module $\mathcal{M}(Y')$ is made into an $\mathcal{O}_X(X')$-module by restriction of scalars.

To simplify notation, we will often drop the symbol π_*, and write $\mathcal{M}$ instead of $\pi_*\mathcal{M}$. If X' is an open subset of X, $\mathcal{M}(X')$ will stand for $\mathcal{M}(\pi^{-1}X')$. When denoting the direct image of an $\mathcal{O}_Y$-module $\mathcal{M}$ by the same symbol $\mathcal{M}$, we may refer to it as an $\mathcal{O}_X$-module. In accordance with this convention, we may also write $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$, but we must include the subscript Y.

This abbreviated terminology is analogous to the one used for restriction of scalars in a module. When $A \to B$ is an algebra homomorphism and M is a B-module, the B-module $_BM$ and the A-module $_AM$ obtained from it by restriction of scalars are usually denoted by the same letter M.

8.5.8. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, of degree $n = [Y : X]$. With notation as above,* *BrankArank*

(i) *The direct image of $\mathcal{O}_Y$, also denoted by $\mathcal{O}_Y$, is a locally free $\mathcal{O}_X$-module of rank n.*

(ii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a torsion $\mathcal{O}_Y$-module if and only if its direct image is a torsion $\mathcal{O}_X$-module.*

(iii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module if and only if its direct image is a locally free $\mathcal{O}_X$-module. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module of rank r, then its direct image is a locally free $\mathcal{O}_X$-module of rank nr.* $\square$

8.6 Trace of a Differential

8.6.1. trace of a function tracediff tracefn

Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, and let F and K be the function fields of X and Y, respectively. The trace map $K \xrightarrow{\text{trace}} F$ for a field extension of finite degree has been defined before (4.3.11). If α is an element of K, multiplication by α on the F-vector space K is a linear operator, and $\text{trace}(k)$ is the trace of that operator. The trace is F-linear: If f_i are in F and α_i are in K, then $\text{trace}(\sum f_i\alpha_i) = \sum f_i \,\text{trace}(\alpha_i)$. Moreover, the trace carries regular functions to regular functions: If $X' = \operatorname{Spec} A'$ is an affine open subset of X, with inverse image $Y' = \operatorname{Spec} B'$, then because A' is normal, the trace of an element of B' will be in A' (4.3.7). Using our abbreviated notation $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$, the trace defines a homomorphism of $\mathcal{O}_X$-modules

(8.6.2) $$\mathcal{O}_Y \xrightarrow{\text{trace}} \mathcal{O}_X$$ traceglobal

Analytically, this trace can be described as a sum over the sheets of the covering. Let $n = [Y : X]$. When a point p of X isn't a branch point, there will be n points $q_1, ..., q_n$ of Y lying over p. If U is a small neighborhood of p in X in the classical topology, its inverse image V will consist of disjoint neighborhoods V_i of q_i, each of which maps bijectively to U. The ring of analytic functions on V_i will be isomorphic to the ring $\mathcal{A}$ of analytic functions on U, and the ring of analytic functions on V is isomorphic to the direct sum $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ of n copies of $\mathcal{A}$. If a rational function g on Y is regular on V, its restriction to V can be written as $g = g_1 \oplus \cdots \oplus g_n$, with g_i in $\mathcal{A}_i$. The matrix of left multiplication by g on $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ is the diagonal matrix with entries $g_1, ..., g_n$, and

trsum (8.6.3)
$$\text{trace}(g) = g_1 + \cdots + g_n$$

tracesum **8.6.4. Lemma.** *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, let p be a point of X, let $q_1, ..., q_k$ be the fibre over p, and let e_i be the ramification index at q_i. If a rational function g on Y is regular at the points $q_1, ..., q_k$, its trace is regular at p. Its value at p is $[\text{trace}(g)](p) = e_1 g(q_1) + \cdots + e_k g(q_k)$.*

proof. The regularity was discussed above. If p isn't a branch point, we will have $k = n$ and $e_i = 1$ for all i. In this case, the lemma follows by evaluating (8.6.3). It follows by continuity for any point p. As a point p' approaches p, e_i points q' of Y approach q_i (8.5.6). For each point q' that approaches q_i, the limit of $g(q')$ will be $g(q_i)$. □

traced **8.6.5. trace of a differential**

The structure sheaf is naturally contravariant. A branched covering $Y \xrightarrow{\pi} X$ corresponds to a homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_Y \leftarrow \mathcal{O}_X$. The trace map for functions is a homomorphism in the opposite direction: $\mathcal{O}_Y \xrightarrow{\text{trace}} \mathcal{O}_X$.

Differentials are also naturally contravariant. A morphism $Y \xrightarrow{\pi} X$ induces an $\mathcal{O}_X$-module homomorphism $\Omega_X \to \Omega_Y$ that sends a differential dx on X to a differential on Y that we may denote by dx too (8.4.4). As is true for functions, there is a trace map for differentials in the opposite direction. It is defined below, in (8.6.7), and it will be denoted by τ:

$$\Omega_Y \xrightarrow{\tau} \Omega_X$$

But first, a lemma about the natural contravariant map $\Omega_X \to \Omega_Y$:

dxydy **8.6.6. Lemma.** *Let $Y \to X$ be a branched covering.*

(i) *Let p be the image in X of a point q of Y, let x and y be local generators for the maximal ideals of X and Y at p and q, respectively, and let e be the ramification index at q. As a differential on Y, $dx = vy^{e-1}dy$, where v is a local unit at q.*

(ii) *The canonical homomorphism $\Omega_X \to \Omega_Y$ is injective.*

proof. As we have noted before, $x = uy^e$, for some local unit u. Since dy generates Ω_Y locally, there is a rational function z that is regular at q, such that $du = zdy$. Let $v = yz + eu$. Then

$$dx \;=\; d(uy^e) \;=\; y^e z\, dy + ey^{e-1}u\, dy \;=\; vy^{e-1}dy$$

Since yz is zero at q and eu is a local unit, v is a local unit. See (8.1.17) **(iii)** for part **(ii)**. □

To define the trace for differentials, we begin with differentials of the functions fields F and K of X and Y, respectively. The modules of differentials of those fields are defined in the same way as for any algebra. The $\mathcal{O}_Y$-module Ω_Y is invertible (8.4.10), and when $Y = \operatorname{Spec} B$, the module Ω_K of differentials of K is the localization $S^{-1}\Omega_B$, where S is the multiplicative system of nonzero elements of B. So Ω_K is a free K-module of rank one. Any nonzero differential will form a K-basis. We choose as basis a nonzero F-differential α. Its image in Ω_K, which we denote by α too, will be a K-basis for Ω_K. We could, for instance, take $\alpha = dx$, where x is a local coordinate function on X.

$$\begin{array}{ccc} K & Y & \Omega_Y \\ \uparrow & \downarrow & \downarrow{\scriptstyle \tau} \\ F & X & \Omega_X \end{array}$$

Since α is a basis, elements β of Ω_K can be written uniquely, as

$$\beta = g\alpha$$

where g is an element of K. The *trace* $\Omega_K \xrightarrow{\tau} \Omega_F$ is defined by

(8.6.7) $$\tau(\beta) = \mathrm{trace}(g)\alpha$$ deftrdif

where $\mathrm{trace}(g)$ is the trace of the function g. Since the trace for functions is F-linear, τ is F-linear.

8.6.8. Corollary. *Let x be a local generator for the maximal ideal $\mathfrak{m}_p$ at a point p of X. If the degree $[Y:X]$ of Y over X is n, then when we regard dx as a differential on Y, $\tau(dx) = n\,dx$.* taudx

We need to check that τ is independent of the choice of α. Let α' be another nonzero F-differential. Then $f\alpha' = \alpha$ for some nonzero f in F, and $gf\alpha' = g\alpha$. Since trace is F-linear, $\mathrm{trace}(gf) = f\,\mathrm{trace}(g)$. Then

$$\tau(gf\alpha') = \mathrm{trace}(gf)\alpha' = f\,\mathrm{trace}(g)\alpha' = \mathrm{trace}(g)f\alpha' = \mathrm{trace}(g)\alpha = \tau(g\alpha)$$

Using α' in place of α gives the same value for the trace.

A differenial of the function field K of Y is a *rational differential* on Y. A rational differential β is *regular* at a point q of Y if there is an affine open neighborhood $Y' = \mathrm{Spec}\,B$ of q such that β is an element of Ω_B. If y is a local generator for the maximal ideal $\mathfrak{m}_q$ and if $\beta = g\,dy$, the differential β is regular at q if and only if the rational function g is regular at q.

Let $Y \to X$ be a branched covering of affine varieties, with $X = \mathrm{Spec}\,A$ and $Y = \mathrm{Spec}\,B$, and let p be a point of X. Suppose that the maximal ideal at p is generated by an element x of A, and so that the differential dx generates Ω_A locally. Let $q_1, ..., q_k$ be the points of Y that lie over p, and let e_i be the ramification index at q_i.

8.6.9. Corollary. *With notation as above,* poleorder

(i) *When dx is viewed as a differential on Y, it has a zero of order $e_i - 1$ at q_i.*

(ii) *When a differential β on Y that is regular at q_i is written as $\beta = g\,dx$, where g is a rational function on Y, then g has a pole of order at most $e_i - 1$ at q_i.*

This follows from Lemma 8.6.6 **(i)**. □

8.6.10. Main Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering, let p be a point of X, let $q_1, ..., q_k$ be the points of Y that lie over p, and let β be a rational differential on Y.* tracereg

(i) *If β is regular at the points $q_1, ..., q_k$, its trace $\tau(\beta)$ is regular at p.*

(ii) *If β has a simple pole at q_i and is regular at q_j for all $j \neq i$, then $\tau(\beta)$ is not regular at p.*

proof. **(i)** Let x be a local generator for the maximal ideal at p. We write β as $g\,dx$, where g is a rational function on Y. Suppose that β is regular at the points q_i. Corollary 8.6.9 tells us that g has poles of orders at most $e_i - 1$ at the points q_i. Since x has a zero of order e_i at q_i, the function xg is regular at the points q_i, and its value there is zero. Then $\mathrm{trace}(xg)$ is regular at p, and its value at p is zero (8.6.4). So $x^{-1}\,\mathrm{trace}(xg)$ is a regular function at p. Since trace is F-linear and x is in F, $x^{-1}\,\mathrm{trace}(xg) = \mathrm{trace}(g)$. So $\mathrm{trace}(g)$ and $\tau(\beta) = \mathrm{trace}(g)dx$ are regular at p.

(ii) In this case, g has poles of orders at most e_j–1 at the points q_j when $j \neq i$, and it has a pole of order e_i at q_i. The function xg will be regular at all of the points q_j. Its value at q_j will be zero when $j \neq i$, and not zero when $j = i$. Then $\mathrm{trace}(xg)$ will be regular at p, but not zero there (8.6.4). Therefore $\mathrm{trace}(g) = x^{-1}\,\mathrm{trace}(xg)$ and $\tau(\beta) = \mathrm{trace}(g)dx$ won't be regular at p. □

8.6.11. Corollary. *The trace map (8.6.7) defines a homomorphism of $\mathcal{O}_X$-modules* $\Omega_Y \xrightarrow{\tau} \Omega_X$. □

omegaYandX

Let $Y \xrightarrow{\pi} X$ be a branched covering. As is true for any $\mathcal{O}_Y$-module, Ω_Y is isomorphic to the module of homomorphisms ${}_Y(\mathcal{O}_Y, \Omega_Y)$. The homomorphism $\mathcal{O}_Y \to \Omega_Y$ that corresponds to a section β of Ω_Y on an open set U sends a regular function f on U to $f\beta$. We denote that homomorphism by β too: $\mathcal{O}_Y \xrightarrow{\beta} \Omega_Y$.

8.6.12. Lemma. *Composition with the trace* $\Omega_Y \xrightarrow{\tau} \Omega_X$ *defines a homomorphism of* $\mathcal{O}_X$*-modules* compwtau

(8.6.13) $$\Omega_Y \approx {}_Y(\mathcal{O}_Y, \Omega_Y) \xrightarrow{\tau} {}_X(\mathcal{O}_Y, \Omega_X)$$

proof. An $\mathcal{O}_Y$-linear map becomes an $\mathcal{O}_X$-linear map by restriction of scalars. When we compose an $\mathcal{O}_Y$-linear map β with τ, then because τ is $\mathcal{O}_X$-linear, the result will be $\mathcal{O}_X$-linear. It will be a homomorphism of $\mathcal{O}_X$-modules. □

8.6.14. Theorem. **(i)** *The homomorphism (8.6.12) is an isomorphism.*

traceomega **(ii)** *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. Composition with the trace defines an isomorphism of $\mathcal{O}_X$-modules*

compw-trace (8.6.15)
$$_Y(\mathcal{M},\Omega_{\mathcal{O}_Y}) \xrightarrow{\tau\circ} {}_X(\mathcal{M},\Omega_{\mathcal{O}_X})$$

When one looks carefully, this theorem follows from the Main Lemma.

Note. The domain and range (8.6.15) are to be interpreted as modules on X. When we put the symbols $\underline{\mathrm{Hom}}$ and π_* that we have suppressed into the notation, the map (8.6.15) becomes an isomorphism

$$\pi_*\big(\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{M},\Omega_Y)\big) \xrightarrow{\tau\circ} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\pi_*\mathcal{M},\Omega_X)$$

It suffices to verify the theorem locally, because it concerns modules on X. So we may suppose that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. When the theorem is stated in terms of algebras and modules, it becomes this:

8.6.16. Theorem. *Let $Y \to X$ be a branched covering, with $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.*

traceomegaal-gebra **(i)** *The trace map $\Omega_B = {}_B(B,\Omega_B) \xrightarrow{\tau\circ} {}_A(B,\Omega_A)$ is an isomorphism of A-modules.*

(ii) *For any locally free B-module M, composition with the trace defines an isomorphism of A-module ${}_B(M,\Omega_B) \xrightarrow{\tau\circ} {}_A(M,\Omega_A)$.*

The B-modules Ω_B and ${}_B(M,\Omega_B)$ become A-modules by restriction of scalars, and in ${}_A(M,\Omega_A)$ the B-module M is interpreted as an A-module by restriction of scalars too.

thetaspole **8.6.17. Lemma.** *Let $\mathcal{L} \subset \mathcal{M}$ be an inclusion of invertible modules on a smooth curve Y, let q be a point in the support of the quotient $\mathcal{M}/\mathcal{L}$, and let V be an affine open subset of Y that contains q. Suppose that a rational function f has a simple pole at q and is regular at all other points of V. If α is a section of $\mathcal{L}$ on V, then $f^{-1}\alpha$ is a section of $\mathcal{M}$ on V.*

proof. Working locally, we may assume that $\mathcal{L} = \mathcal{O}$. Then $\mathcal{M} = \mathcal{O}(D)$ for some effective divisor D. Since q is in the support of $\mathcal{M}/\mathcal{L}$, the coefficient of q in D is positive. Therefore $\mathcal{O}(q) \subset \mathcal{O}(D) = \mathcal{M}$. Then α is a section of $\mathcal{O}$, and f^{-1} is a section of $\mathcal{O}(q)$. So $f^{-1}\alpha$ will be a section of $\mathcal{O}(q)$, and therefore a section of $\mathcal{O}(D)$. □

homisB-mod **8.6.18. Lemma.** *Let $A \subset B$ be rings, let M be a B-module, and let N be an A-module. The module ${}_A(M,N)$ of homomorphisms has the structure of a B-module.*

proof. We must define scalar multiplication of a homomorphism $M \xrightarrow{\varphi} N$ of A-modules by an element b of B. The definition is $[b\varphi](m) = \varphi(bm)$. One must show that the map $b\varphi$ is a homomorphism of A-modules $M \to N$, and checkt the axioms for a B-module. You will be able to do this. □

proof of Theorem 8.6.14 Since the theorem is local, we are allowed to localize. We use the algebra version 8.6.16 of the theorem.

(i) Both B and Ω_B are torsion-free, and therefore locally free A-modules. Localizing as needed, we may assume that they are free A-modules, and that Ω_A is a free A-module of rank one with basis of the form dx. Then ${}_A(B,\Omega_A)$ will be a free A-module too. Let's denote ${}_A(B,\Omega_A)$ by Θ. Lemma 8.6.18 tells us that Θ is a B-module. Because B and Ω_A are free A-modules, Θ is a free A-module and a locally free B-module (see (8.5.8)). Since Ω_A has A-rank 1, the A-rank of Θ is the same as the A-rank of B. Therefore the B-rank of Θ is the same as the B-rank of B, which is 1. So Θ is an invertible B-module.

If x is a local coordinate on X, then $\tau(dx) \neq 0$ (8.6.8). The trace map $\Omega_B \xrightarrow{\tau} \Theta$ isn't the zero map. Since domain and range are invertible B-modules, it is an injective homomorphism. Its image, which is isomorphic to Ω_B, is an invertible submodule of the invertible B-module Θ.

To show that $\Omega_B = \Theta$, we apply Lemma 8.6.17 to show that the quotient $\overline{\Theta} = \Theta/\Omega_B$ is the zero module. Suppose not, and let q be a point in the support of $\overline{\Theta}$. Let p be the image of q in X and let $q_1, ..., q_k$ be the fibre over p, with $q = q_1$.

We choose a differential α that is regular at all of the points q_i. If y is a local generator for the maximal ideal at q_1, then $\alpha = g\,dy$, where g is a regular function at q_1. We assume also that α has been chosen so that that $g(q_1) \neq 0$.

Let f be a rational function that is regular on an affine open subset V of Y that contains the points $q_1, ..., q_k$, and such that $f(q_1) = 0$ and $f(q_i) \neq 0$ when $i > 1$. Lemma 8.6.17 tells us that $\beta = f^{-1}\alpha$ is a section of Θ on V, but the Main Lemma 8.6.10 tells us that $\tau(\beta)$ isn't regular at p. This contradiction proves part **(i)** of the theorem. $\square$

proof of Theorem 8.6.14 **(ii)**. We are to show that if $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, composition with the trace defines an isomorphism ${}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \to {}_{\mathcal{O}_X}(\mathcal{M}, \Omega_{\mathcal{O}_X})$. Part **(i)** of the theorem tells us that this is true in when $\mathcal{M} = \mathcal{O}_Y$. Therefore it is also true when $\mathcal{M}$ is a free module $\mathcal{O}_Y^k$. And, since **(ii)** is a statement about $\mathcal{O}_X$-modules, it suffices to prove it locally on X.

8.6.19. Lemma. *Let $q_1, ..., q_k$ be points of a smooth curve Y, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. There is an open subset V of Y that contains the points $q_1, ..., q_k$, such that $\mathcal{M}$ is free on V.* *local-lyfreeonX*

We assume the lemma and complete the proof of the theorem. Let $\{q_1, ..., q_k\}$ be the fibre over a point p of X and let be V as in the lemma. The complement $D = Y - V$ is a finite set whose image Z in X is also finite, and Z doesn't contain p. If U is the complement of Z in X, its inverse image W will be a subset of V that contains the points q_i, and on which $\mathcal{M}$ is free. $\square$

proof of the lemma. With terminology as in Lemma 8.5.2, let $\mathfrak{m}_i$ be the maximal ideal of B at q_i, and let $\overline{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\overline{B} = B/xB$ is isomorphic to the product $\overline{B}_1 \times \cdots \times \overline{B}_k$. Since $\mathcal{M}$ is locally free, $M/\mathfrak{m}_i M = \overline{M}_i$ is a free $\overline{B}_i$-module. Its dimension is the rank r of the B-module M.

If M has rank r, there will be a set of elements $m = (m_1, ..., m_r)$ in M whose residues form a basis of $\overline{M}_i$ for every i. This follows from the Chinese Remainder Theorem. The set m will generate M locally at each of the points. Let M' be the B-submodule of M generated by m. The cokernel of the map $M' \to M$ is zero at the points $q_1, ..., q_k$, and therefore it's support, which is a finite set, is disjoint from those points. When we localize to delete this finite set from X, the set m becomes a basis for M. $\square$

Note. Theorem 8.6.14 is subtle. Unfortunately the proof, though understandable, doesn't give an intuitive explanation of the fact that Ω_Y is isomorphic to ${}_X(\mathcal{O}_Y, \Omega_X)$. To get more insight into that, we would need a better understanding of differentials. My father Emil Artin said: "One doesn't really understand differentials, but one can learn to work with them."

8.7 The Riemann-Roch Theorem II

8.7.1. the Serre dual

roch serredual

Let Y be a smooth projective curve, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. The *Serre dual* of $\mathcal{M}$ is defined to be the module

$$\mathcal{M}^S \;=\; {}_Y(\mathcal{M}, \Omega_Y) \qquad \big(= \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(M, \Omega_Y)\big) \tag{8.7.2}$$

defSerred-ual

Its sections on an open subset U are the homomorphisms of $\mathcal{O}_Y(U)$-modules $\mathcal{M}(U) \to \Omega_Y(U)$, and it can also be written as $\mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y$, where $\mathcal{M}^*$ is the ordinary dual ${}_Y(\mathcal{M}, \mathcal{O}_Y)$. The invertible module Ω_Y is locally isomorphic to $\mathcal{O}_Y$. So the Serre dual $\mathcal{M}^S$ is a locally sree module of the same rank as $\mathcal{M}$. Th Serre bidual $(\mathcal{M}^S)^S$ is isomorphic to $\mathcal{M}$:

$$(\mathcal{M}^S)^S \approx (\mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y)^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \otimes_{\mathcal{O}} \Omega_Y^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \approx \mathcal{M}$$

(See (8.1.17) **(i)**.) For example, $\mathcal{O}_Y^S = \Omega_Y$ and $\Omega_Y^S = \mathcal{O}_Y$.

8.7.3. Riemann-Roch Theorem, version 2. *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module on a smooth projective curve Y, and let $\mathcal{M}^S$ be its Serre dual. Then $\mathbf{h}^0\mathcal{M} = \mathbf{h}^1\mathcal{M}^S$ and $\mathbf{h}^1\mathcal{M} = \mathbf{h}^0\mathcal{M}^S$.* *dualco-hom*

The second assertion follows from the first when one replaces $\mathcal{M}$ by $\mathcal{M}^S$. Thus $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$ and $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y\ (= p_a)$. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then

chitwo (8.7.4)
$$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^0\mathcal{M}^S$$

A more precise statement of the Riemann-Roch Theorem is that $H^1(Y,\mathcal{M})$ and $H^0(Y,\mathcal{M}^S)$ are dual spaces. This becomes important when one wants to apply the theorem to a cohomology sequence, but the fact that the dimensions are equal is enough for many applications. We omit the proof.

Our plan is to prove Theorem 8.7.3 directly for the projective line $\mathbb{P}^1$. This will be easy, because the structure of locally free modules on $\mathbb{P}^1$ is very simple. We derive it for an arbitrary smooth projective curve by projection to $\mathbb{P}^1$. Projection to projective space is a method that was used by Grothendieck in the proof of his general Riemann-Roch Theorem.

Let $X = \mathbb{P}^1$, let Y be a smooth projective curve, and let $Y \xrightarrow{\pi} X$ be a branched covering. Let Serre dual of $\mathcal{M}$, a locally free $\mathcal{O}_Y$-module, be
$$\mathcal{M}_1^S = {}_Y(\mathcal{M},\Omega_Y)$$
The direct image of $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module that we denote by $\mathcal{M}$, and we can form the Serre dual on X. Let
$$\mathcal{M}_2^S = {}_X(\mathcal{M},\Omega_X)$$

abouthom **8.7.5. Corollary.** *The direct image $\pi_*\mathcal{M}_1^S$, also denoted by $\mathcal{M}_1^S$, is isomorphic to $\mathcal{M}_2^S$.*

proof. This is Theorem 8.6.14. □

The corollary allows us to drop the subscripts from $\mathcal{M}^S$. Because a branched covering $Y \xrightarrow{\pi} X$ is an affine morphism, the cohomology of $\mathcal{M}$ and of its Serre dual $\mathcal{M}^S$ can be computed, either on Y or on X. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then $H^q(Y,\mathcal{M}) \approx H^q(X,\mathcal{M})$ and $H^q(Y,\mathcal{M}^S) \approx H^q(X,\mathcal{M}^S)$ (7.4.22).

Thus it is enough to prove Riemann-Roch for the projective line.

dualityfor pone
8.7.6. Riemann-Roch for the projective line

The Riemann-Roch Theorem for the projective line $X = \mathbb{P}^1$ is a consequence of the Birkhoff-Grothendieck Theorem, which tells us that a locally free $\mathcal{O}_X$-module $\mathcal{M}$ on X is a direct sum of twisting modules. To prove Riemann-Roch for the projective line X, it suffices to prove it for the twisting modules $\mathcal{O}_X(k)$.

omegapone **8.7.7. Lemma.** *The module of differentials Ω_X on the projective line X is isomorphic to the twisting module $\mathcal{O}_X(-2)$.*

proof. Let $\mathbb{U}^0 = \operatorname{Spec}\mathbb{C}[x]$ and $\mathbb{U}^1 = \operatorname{Spec}\mathbb{C}[z]$ be the standard open subsets of $\mathbb{P}^1$, with $z = x^{-1}$. On $\mathbb{U}^0$, the module of differentials is free, with basis dx, and $dx = d(z^{-1}) = -z^{-2}dz$ describes the differential dx on $\mathbb{U}^1$. The point at infinity is $p_\infty : \{x_0 = 0\}$, and dx has a pole of order 2 there. It is a global section of $\Omega_X(2p_\infty)$, and as a section of that module, it isn't zero anywhere. Multiplication by dx defines an isomorphism $\mathcal{O} \to \Omega_X(2p_\infty)$ that sends 1 to dx. Tensoring with $\mathcal{O}(-2p_\infty)$ shows that $\mathcal{O}(-2p_\infty)$ is isomorphic to Ω_X. □

twisthom **8.7.8. Lemma.** *Let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules on the projective line X. Then ${}_X(\mathcal{M}(r),\mathcal{N})$ is canonically isomorphic to ${}_X(\mathcal{M},\mathcal{N}(-r))$.*

proof. When we tensor a homomorphism $\mathcal{M}(r) \xrightarrow{\varphi} \mathcal{N}$ with $\mathcal{O}(-r)$, we obtain a homomorphism $\mathcal{M} \to \mathcal{N}(-r)$, and tensoring with $\mathcal{O}(r)$ is the inverse operation. □

The Serre dual $\mathcal{O}(n)^S$ of $\mathcal{O}(n)$ is therefore
$$\mathcal{O}(n)^S \approx {}_X(\mathcal{O}(n),\mathcal{O}(-2)) \approx \mathcal{O}(-2-n)$$
To prove Riemann-Roch for $X = \mathbb{P}^1$, we must show that
$$\mathbf{h}^0\mathcal{O}(n) = \mathbf{h}^1\mathcal{O}(-2-n) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(n) = \mathbf{h}^0\mathcal{O}(-2-n)$$
This follows from Theorem 7.5.5, which computes the cohomology of the twisting modules.

8.8 Using Riemann-Roch

8.8.1. genus appl genussm-curve

There are three closely related numbers associated to a smooth projective curve Y: its *topological genus* g, its *arithmetic genus* $p_a = \mathbf{h}^1\mathcal{O}_Y$, and the *degree* δ of the module of differentials Ω_Y (8.1.18).

8.8.2. Theorem. *The topological genus g and the arithmetic genus p_a of a smooth projective curve Y are equal, and the degree δ of the module Ω_Y is $2g-2$, which is equal to $2p_a-2$.* genus-genus

Thus the Riemann-Roch Theorem 8.2.3 can we written as

(8.8.3) $$\chi(\mathcal{O}(D)) = \deg\, D + 1 - g$$ dplusone-minusg

We'll write it this way when the theorem is proved.

proof. Let $Y \xrightarrow{\pi} X$ be a branched covering with $X = \mathbb{P}^1$, and of degree n. The topological Euler characteristic $e(Y) = 2-2g$, can be computed in terms of the branching data for the covering, as in (1.7.24). Let q_i be the ramification points in Y, and let e_i be the ramification index at q_i. Then e_i sheets of the covering come together at q_i. This decreases the number of points in the fibre of π that contains q_i by $e_i - 1$. So

(8.8.4) $$2-2g = e(Y) \;=\; ne(X) - \sum(e_i - 1) \;=\; 2n - \sum(e_i - 1)$$ degdelta

We compute the degree δ of Ω_Y in two ways. First, the Riemann-Roch Theorem tells us that $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = p_a$ and $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$. So $\chi(\Omega_Y) = -\chi(\mathcal{O}_Y) = p_a - 1$. The Riemann-Roch Theorem also tells us that $\chi(\Omega_Y) = \delta + 1 - p_a$. Therefore

(8.8.5) $$\delta = 2p_a - 2$$ delta

Next, we compute δ by computing the divisor of the differential dx on Y, x being a coordinate on the projective line

Let e_i be the ramification index at a ramification point q_i. Then dx has a zero of order $e_i - 1$ at q_i. At the point of X at infinity, dx has a pole of order 2. Let's choose coordinates so that the point at infinity isn't a branch point. Then there will be n points of Y at which dx has a pole of order 2, n being the degree of Y over X. The degree of Ω_Y is therefore

(8.8.6) $$\delta = \;zeros - \;poles = \sum(e_i - 1) - 2n$$ deltatwo

Combining (8.8.6) with (8.8.4), one sees that $\delta = 2g-2$. Since we also have $\delta = 2p_a - 2$, we conclude that $g = p_a$. □

8.8.7. canonical divisors canondiv

Because the module Ω_Y of differentials on a smooth curve Y is invertible, it is isomorphic to $\mathcal{O}(K)$ for some divisor K (Proposition 8.1.13). Such a divisor is called a *canonical divisor*. The degree of a canonical divisor is $2g-2$, the same as the degree of Ω_Y (8.8.2). It is often convenient to represent Ω_Y as a module $\mathcal{O}(K)$, though the canonical divisor K isn't unique. It is determined only up to linear equivalence.

When written in terms of a canonical divisor K, the Serre dual of an invertible module $\mathcal{O}(D)$ is

(8.8.8) $$\mathcal{O}(D)^S \;=\; {}_Y(\mathcal{O}(D), \mathcal{O}(K)) \;\approx\; \mathcal{O}(D)^* \otimes_{\mathcal{O}} \mathcal{O}(K) \;\approx\; \mathcal{O}(-D) \otimes_{\mathcal{O}} \mathcal{O}(K) \;\approx\; \mathcal{O}(K-D)$$ degK

(8.1.12),(6.4.25). With this notation, the Riemann-Roch Theorem for $\mathcal{O}(D)$ becomes

(8.8.9) $$\mathbf{h}^0\mathcal{O}(D) = \mathbf{h}^1\mathcal{O}(K-D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(K-D)$$ □ RRforOD

8.8.10. curves of genus zero genuszero

Every smooth projective curve of genus zero is isomorphic to the projective line $\mathbb{P}^1$. The proof is an exercise.

A *rational curve* is a curve, smooth or not, whose function field is isomorphic to the field $\mathbb{C}(t)$ of rational functions in one variable. A smooth projective curve of genus zero is a rational curve.

genusone **8.8.11. curves of genus one**

A smooth projective curve of genus $g = 1$ is called an *elliptic curve*.

Let Y be an elliptic curve. The Riemann-Roch Theorem tells us that

$$\chi(\mathcal{O}(D)) = \deg D$$

and that $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = 1$. So Ω_Y has a nonzero global section ω. Also, Ω_Y has degree zero (8.8.2). Therefore the global section doesn't vanish anywhere, and multiplication defines an isomorphism $\mathcal{O} \xrightarrow{\omega} \Omega_Y$.

hrpforel-liptic **8.8.12. Lemma.** *Let D be a divisor of positive degree r on an elliptic curve Y. Then $\mathbf{h}^0\mathcal{O}(D) = r$, and $\mathbf{h}^1\mathcal{O}(D) = 0$. In particular, $\mathbf{h}^0(\mathcal{O}(rp)) = r$ and $\mathbf{h}^1(\mathcal{O}(rp)) = 0$.*

proof. Since Ω_Y is isomorphic to $\mathcal{O}$, $K = 0$ is a canonical divisor, and the Serre dual of $\mathcal{O}(D)$ is $\mathcal{O}(-D)$. Then $\mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(-D)$, which is 0 when the degree of D is positive. $\square$

Now, since $H^0(Y, \mathcal{O}_Y) \subset H^0(Y, \mathcal{O}_Y(p))$, and since both spaces have dimension one, they are equal. So (1) is a basis for $H^0(Y, \mathcal{O}_Y(p))$. We choose a basis $(1, x)$ for the two-dimensional space $H^1(Y, \mathcal{O}_Y(2p))$. Then x isn't a section of $\mathcal{O}(p)$. It has a pole of order precisely 2 at p, and no other pole. Next, we choose a basis $(1, x, y)$ for $H^1(Y, \mathcal{O}_Y(3p))$. So y has a pole of order 3 at p.

The point $(1, x, y)$ of $\mathbb{P}^2$ with values in K determines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^2$. Let u, v, w be coordinates in $\mathbb{P}^2$. The map φ sends a point q distinct from p to $(u, v, w) = (1, x(q), y(q))$. Since Y has dimension one, φ is a finite morphism, and its image Y' is a plane curve (5.5.4).

To determine the image of the point p, we multiply $(1, x, y)$ by $\lambda = y^{-1}$, obtaining the equivalent vector $(y^{-1}, xy^{-1}, 1)$. The rational function y^{-1} has a zero of order 3 at p, and xy^{-1} has a simple zero there. Evaluating at p, we see that the image of p is the point $(0, 0, 1)$.

Let ℓ be a generic line $\{au+bv+cw = 0\}$ in $\mathbb{P}^2$. The rational function $a+bx+cy$ on Y has a pole of order 3 at p and no other pole. It takes every value, including zero, three times, and the three points of Y at which $a + bx + cy$ is zero form the inverse image of ℓ. The only possibilities for the degree of Y' are 1 and 3. Since $1, x, y$ are independent, they don't satisfy a homogeneous linear equation. So Y' isn't a line. It is a cubic curve (1.3.10). The rational functions x, y satisfy a cubic relation f.

We determine the form of the polynomial f. The seven monomials $1, x, y, x^2, xy, y^2, x^3$ have poles of orders $0, 2, 3, 4, 5, 6, 6$ at p, respectively. They are global sections of $\mathcal{O}(6p)$. Riemann-Roch tells us that $\mathbf{h}^0\mathcal{O}_Y(6p) = 6$. So those monomials are linearly dependent. The dependency relation gives us a cubic equation among x and y, of the form

eqnell
$$cy^2 + (a_1x+a_3)y + (a_0x^3+a_2x^2+a_4x+a_6) = 0 \tag{8.8.13}$$

There can be no linear relation mong functions whose orders of pole at p are distinct. So when we delete either x^3 or y^2 from the list of seven monomials, we obtain an independent set — a basis for the six-dimensional space $H^0(Y, \mathcal{O}(6p))$. In the cubic relation, the coefficients c and a_0 aren't zero. We normalize them to 1. Then we eliminate the linear term in y from the relation by substituting $y - \frac{1}{2}(a_1x+a_3)$ for y, and we eliminate the quadratic term in x in the resulting polynomial by substituting $x - \frac{1}{3}a_2$ for x. Bringing the terms in x to the other side of the equation, we are left with a cubic relation of the form

$$y^2 = x^3 + a_4x + a_6$$

The coefficients a_4 and a_6 have been changed, of course.

This derivation seems magical, but one can obtain the result using only the fact that, because the curve is a cubic, there is some relation f among the monomials in x, y of degree ≤ 3. Among those monomials, y^3 has the largest order of pole, namely 9. If y^3 had nonzero coefficient in f, we could solve for it as a combination of the other monomials. But because its order of pole is greater than the other orders of pole, this is impossible. The coefficient of y^3 in f is 0. This being so, the monomial xy^2 becomes the one with largest order of pole, namely 8. So the coefficient of xy^2 in f is zero, and then similarly, the coefficient of x^2y, with a pole of order 7, is also zero. Since the relation f is a cubic polynomial, the only remaining monomial of degree 3, which is x^3, has a nonzero coefficient. Both x^3 and y^2 have poles of order 6, and the remaining monomials $1, x, y, x^2, xy$ have orders of pole less than 6. So y^2 also has a nonzero coefficient, and f has the form (8.8.13) in which c and a_0 are nonzero. The rest of the reduction is the same.

The cubic curve Y' defined by the homogenized equation $y^2z = x^3 + a_4xz^2 + a_6z^3$ is the image of Y. This curve meets a generic line $ax + by + cz = 0$ in three points and, as we saw above, its inverse image in Y consists of three points too. Therefore the morphism $Y \xrightarrow{\varphi} Y'$ is generically injective, and Y is the normalization of Y'. Projection from a singular point shows that a singular curve of degree 3 as a rational curve. Therefore Y' is smooth, and is isomorphic to Y.

8.8.14. the group law on an elliptic curve genusone-group

The points of an elliptic curve Y form an abelian group, once one fixes a point as the identity element.

We choose an identity element and label it o. We write the law of composition as $p \oplus q$, to make a clear distinction between the product in the group, which is a point of Y, and the divisor $p + q$. To define $p \oplus q$, we compute the cohomology of $\mathcal{O}_Y(p{+}q{-}o)$. Lemma 8.8.12 tells us that $\mathbf{h}^0\mathcal{O}_Y(p{+}q{-}o) = 1$. So there is a nonzero function f, unique up to scalar factor, with simple poles at p and q and a simple zero at o. This function has exactly one additional zero. That zero is defined to be the sum $p \oplus q$. In terms of linearly equivalent divisors, $p \oplus q$ is the unique point s such that the divisor $p + q$ is linearly equivalent to $o + s$.

8.8.15. Proposition. *With the law of composition $\oplus$ defined above, an elliptic curve becomes an abelian group.* *grplaw*

The proof makes a good exercise. $\square$

8.8.16. mapping a curve to projective space maptoP

Let Y be a smooth projective curve. We have seen that any set $(f_0, ..., f_n)$ of rational functions on Y, not all zero, defines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ (5.2.6). For review: Let q be a point of Y and let $g_j = f_j/f_i$, where i is an index such that the value $\mathrm{v}_q(f_i)$ is a minimum, for $k = 0, ..., n$. The rational functions g_j are regular at q for all j, and the morphism φ sends the point q to $(g_0(q), ..., g_n(q))$. For example, the inverse image $\varphi^{-1}(\mathbb{U}^0)$ of the standard open set $\mathbb{U}^0$ is the set of points of Y at which the functions $g_j = f_j/f_0$ are regular. If q is such a point, then $\varphi(q) = (1, g_1(q), ..., g_n(q))$.

8.8.17. Lemma. *Let Y be a smooth projective curve, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism to projective space defined by a set $(f_0, ..., f_n)$ of rational functions on Y.* *mapcurve*

(i) *If the space spanned by $\{f_0, ..., f_n\}$ has dimension at least two, then φ isn't a constant morphism to a point.*

(ii) *If $f_0, ..., f_n$ are linearly independent, the image of Y isn't contained in a hyperplane.* $\square$

8.8.18. base points basept

Let D be a divisor on the smooth projective curve Y, and suppose that $\mathbf{h}^0\mathcal{O}(D) = k > 1$. A basis $(f_0, ..., f_k)$ of global sections of $\mathcal{O}(D)$ defines a morphism $Y \to \mathbb{P}^{k-1}$. This is a common way to construct a morphism to projective space, though one could use any set of rational functions that aren't all zero.

If a global section of $\mathcal{O}(D)$ vanishes at a point p of Y, it is a global section of $\mathcal{O}(D{-}p)$. A *base point* of $\mathcal{O}(D)$ is a point of Y at which *every* global section of $\mathcal{O}(D)$ vanishes. A base point can be described in terms of the usual exact sequence

$$0 \to \mathcal{O}(D{-}p) \to \mathcal{O}(D) \to \kappa_p \to 0$$

The point p is a base point if $\mathbf{h}^0\mathcal{O}(D{-}p) = \mathbf{h}^0\mathcal{O}(D)$, or if $\mathbf{h}^1\mathcal{O}(D{-}p) = \mathbf{h}^1\mathcal{O}(D) - 1$.

The *degree* of a nonconstant morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ is defined to be the number of points of the inverse image $\varphi^{-1}H$ of a generic hyperplane H in $\mathbb{P}^n$.

deg-nobasept **8.8.19. Lemma.** *Let D be a divisor on a smooth projective curve Y with $\mathbf{h}^0\mathcal{O}(D) = n+1$, $n > 0$, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism defined by a basis of global sections.*
(i) *The image of φ isn't contained in any hyperplane.*
(ii) *If $\mathcal{O}(D)$ has no base point, the degree of φ is equal to degree of D. If there are base points, the degree is lower.*

proof. **(ii)** We may assume that D is an effective divisor of degree d. If D has no base point, then for every point p in the support of D, $H^0(\mathcal{O}(D-p)) < H^0(\mathcal{O}(D))$. A generic section f of $\mathcal{O}(D)$ will not vanish at any point of D. Then $poles(f)$ will be equal to D, and $zeros(f)$ will have degree d. Proposition 8.5.3 shows that $zeros(f)$ consists of d points. □

exbase-points **8.8.20. Proposition.** *Let K be a canonical divisor on a smooth projective curve Y of genus $g > 0$.*
(i) *$\mathcal{O}(K)$ has no base point.*
(ii) *Every point p of Y is a base point of $\mathcal{O}(K+p)$.*

proof. **(i)** Let p be a point of Y. We apply Riemann-Roch to the exact sequence

$$0 \to \mathcal{O}(K-p) \to \mathcal{O}(K) \to \kappa_p \to 0$$

The Serre duals of $\mathcal{O}(K)$ and $\mathcal{O}(K-p)$ are $\mathcal{O}(K)^S = \mathcal{O}$ and $\mathcal{O}(K-p)^S = \mathcal{O}(p)$, respectively. They form an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p) \to \kappa_p \to 0$$

Because Y has positive genus, there is no rational function on Y with just one simple pole. So $\mathbf{h}^0\mathcal{O} = \mathbf{h}^0\mathcal{O}(p) = 1$. Riemann-Roch tells us that $\mathbf{h}^1\mathcal{O}(K-p) = \mathbf{h}^1\mathcal{O}(K) = 1$. The cohomology sequence

$$0 \to H^0(\mathcal{O}(K-p)) \to H^0(\mathcal{O}(K)) \to [1] \to H^1(\mathcal{O}(K-p)) \to H^1(\mathcal{O}(K)) \to 0$$

shows that $\mathbf{h}^0\mathcal{O}(K-p) = \mathbf{h}^0\mathcal{O}(K) - 1$. So p is not a base point.

(ii) Here, the relevant sequence is

$$0 \to \mathcal{O}(K) \to \mathcal{O}(K+p) \to \kappa_p \to 0$$

The Serre dual of $\mathcal{O}(K+p)$ is $\mathcal{O}(-p)$, which has no global section. Therefore $\mathbf{h}^1\mathcal{O}(K+p) = 0$, while $\mathbf{h}^1\mathcal{O}(K) = \mathbf{h}^0\mathcal{O} = 1$. The cohomology sequence

$$0 \to \mathbf{h}^0\mathcal{O}(K) \to \mathbf{h}^0\mathcal{O}(K+p) \to [1] \to \mathbf{h}^1\mathcal{O}(K) \to \mathbf{h}^1\mathcal{O}(K+p) \to 0$$

shows that $H^0(\mathcal{O}(K+p)) = H^0(\mathcal{O}(K))$. So p is a base point of $\mathcal{O}(K+p)$. □

hyper **8.8.21. hyperelliptic curves**

A *hyperelliptic curve* Y is a smooth projective curve of genus $g \geq 2$ that can be represented as a branched double covering of the projective line — such that there exists a morphism $Y \xrightarrow{\pi} X = \mathbb{P}^1$ of degree two. The term 'hyperelliptic' comes from the fact that every elliptic curve can be represented (though not uniquely) as a double cover of $\mathbb{P}^1$. The global sections of $\mathcal{O}(2p)$, where p is any point of an elliptic curve Y, define a map to $\mathbb{P}^1$ of degree 2.

The topological Euler characteristic of a hyperelliptic curve Y can be computed in terms of the double covering $Y \to X$. The covering will be branched at a finite set of points of X. The branch points are those such that the fibre consists of one point. If there are n branch points, the Euler characteristic is $e(Y) = 2e(X) - n = 4 - n$. Since we know that $e(Y) = 2 - 2g$, the number of branch points is $n = 2g + 2$. When $g = 3$, $n = 8$.

It would take some experimentation to guess that the next remarkable theorem might be true, and some time to find a proof.

8.8.22. Theorem. *Let Y be a hyperelliptic curve, let $Y \xrightarrow{\pi} X = \mathbb{P}^1$ be a branched covering of degree 2. The morphism $Y \xrightarrow{\psi} \mathbb{P}^{g-1}$ defined by the global sections of $\Omega_Y = \mathcal{O}(K)$ factors through π. There is a unique morphism $X \xrightarrow{u} \mathbb{P}^{g-1}$ such that ψ is the composed map $Y \xrightarrow{\pi} X \xrightarrow{u} \mathbb{P}^{g-1}$:* *hyper-canon*

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "\psi"'] & X \arrow[d, "u"] \\ \mathbb{P}^{g-1} \arrow[r, equal] & \mathbb{P}^{g-1} \end{tikzcd}$$

8.8.23. Corollary. *A curve of genus $g \geq 2$ can be presented as a branched covering of $\mathbb{P}^1$ of degree 2 in at most one way.* $\square$ *oned-blcover*

proof of Theorem 8.8.22. Let x be an affine coordinate in X, so that the standard affine open subset $\mathbb{U}^0$ of X is $\operatorname{Spec} \mathbb{C}[x]$. Suppose that the point of X at infinity isn't a branch point of the covering π. The open set $Y^0 = \pi^{-1}\mathbb{U}^0$ will be described by an equation of the form $y^2 = f(x)$, where f is a polynomial of degree $n = 2g+2$ with simple roots, and there will be two points of Y above the point at infinity. They are interchanged by the automorphism $y \to -y$. Let's call those points q_1 and q_2.

We start with the differential dx, which we view as a rational differential on Y. Then $2y\,dy = f'(x)dx$. Since f has simple roots at the branch points, f' doesn't vanish at any of them. Therefore dx has simple zeros on Y at the points at which $y = 0$, the points above the branch points of f on X. We also have a regular function on Y^0 with simple roots at those points, the function y. Therefore the differential $\omega = \frac{dx}{y}$ is regular and nowhere zero on Y^0. Because the degree of a differential on Y is $2g-2$, ω has a total of $2g-2$ zeros at infinity. By symmetry, ω has zeros of order $g-1$ at q_1 and at q_2. Then $K = (g{-}1)q_1 + (g{-}1)q_2$ is a canonical divisor on Y.

Since K has zeros of order $g-1$ at infinity, the rational functions $1, x, x^2, ..., x^{g-1}$, when viewed as functions on Y, are among the global sections of $\mathcal{O}_Y(K)$. They are independent, and there are g of them. Since $\mathbf{h}^0\mathcal{O}_Y(K) = g$, they form a basis of $H^0(\mathcal{O}_Y(K))$. The map $Y \to \mathbb{P}^{g-1}$ defined by the global sections of $\mathcal{O}_Y(K)$ evaluates these powers of x, so it factors through X. $\square$

When Y is a hyperelliptic curve of genus 3, its image via the canonical map will be a plane conic, and if the genus of a hyperelliptic curve is 4, its image will be a twisted cubic in $\mathbb{P}^3$.

8.8.24. canonical embedding canonemb

Let Y be a smooth projective curve of genus $g \geq 2$, and let K be a canonical divisor on Y. Its global sections define a morphism $Y \to \mathbb{P}^{g-1}$. This morphism is called the *canonical map*. We denote the canonical map by ψ, as before. Since $\mathcal{O}(K)$ has no base point, the degree of ψ is the degree $2g-2$ of the canonical divisor. Theorem 8.8.22 shows that, when Y is hyperelliptic, the image of the canonical map is isomorphic to $\mathbb{P}^1$.

8.8.25. Theorem. *Let Y be a smooth projective curve of genus g at least two. If Y is not hyperelliptic, the canonical map embeds Y as a closed subvariety of projective space $\mathbb{P}^{g-1}$.* *canonem-btwo*

proof. We show first that, if the canonical map $Y \xrightarrow{\psi} \mathbb{P}^{g-1}$ isn't injective, then Y is hyperelliptic.

Let p and q be distinct points of Y with the same image: $\psi(p) = \psi(q)$. We choose an effective canonical divisor K whose support doesn't contain p or q, and we inspect the global sections of $\mathcal{O}(K{-}p{-}q)$. Since $\psi(p) = \psi(q)$, any global section of $\mathcal{O}(K)$ that vanishes at p vanishes at q too. Therefore $\mathcal{O}(K-p)$ and $\mathcal{O}(K{-}p{-}q)$ have the same global sections, and q is a base point of $\mathcal{O}(K{-}p)$. We've computed the cohomology of $\mathcal{O}(K-p)$ before: $\mathbf{h}^0\mathcal{O}(K-p) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p) = 1$. Therefore $\mathbf{h}^0\mathcal{O}(K-p-q) = g-1$ and $\mathbf{h}^1\mathcal{O}(K{-}p{-}q) = 2$. The Serre dual of $\mathcal{O}(K{-}p{-}q)$ is $\mathcal{O}(p+q)$, so by Riemann-Roch, $\mathbf{h}^0\mathcal{O}(p+q) = 2$. For any divisor D of degree one on a curve of positive genus, $\mathbf{h}^0(\mathcal{O}(D)) \leq 1$. So $\mathcal{O}(p+q)$ has no base point, and the global sections of $\mathcal{O}(p+q)$ define a morphism $Y \to \mathbb{P}^1$ of degree 2. The curve Y is hyperelliptic.

If Y isn't hyperelliptic, the canonical map is injective, so Y is mapped bijectively to its image Y' in $\mathbb{P}^{g-1}$. This almost proves the theorem, but: Can Y' have a cusp? We must show that the bijective map $Y \xrightarrow{\psi} Y'$ is an isomorphism. We go over the computation made above for a pair of points p, q, this time taking $q = p$. The computation is the same. Since Y isn't hyperelliptic, p isn't a base point of $\mathcal{O}_Y(K-p)$. Therefore

$\mathbf{h}^0\mathcal{O}_Y(K-2p) = \mathbf{h}^0\mathcal{O}_Y(K-p) - 1$. This tells us that there is a global section f of $\mathcal{O}_Y(K)$ that has a zero of order exactly 1 at p. When properly interpreted, this fact shows that ψ doesn't collapse any tangent vector at p, and that ψ is an isomorphism. Since we haven't discussed tangent vectors, we prove this directly.

Since ψ is bijective, the function fields of Y and its image Y' are equal, and Y is the normalization of Y'. Moreover, ψ is an isomorphism except on a finite set. We work locally at a point p' of Y', denoting the unique point of Y that maps to p' by p. When we restrict the global section f of $\mathcal{O}_Y(K)$ found above to the image Y', we obtain an element of the maximal ideal $\mathfrak{m}_{p'}$ of $\mathcal{O}_{Y'}$ at p', that we denote by x. On Y, x has a zero of order 1 at p. Therefore it is a local generator for the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_Y$. We apply the local Nakayama Lemma 5.1.1. Let R' and R denote the local rings at p. We regard R as a finite R'-module. Since x is in $\mathfrak{m}_{p'}$, $R/\mathfrak{m}_{p'}R$ is the residue field of R, which is spanned, as R'-module, by the element 1. The local Nakayama Lemma lemma, with $V = R$ and $M = \mathfrak{m}_{p'}$, tells us that R is spanned by 1, and this shows that $R = R'$. □

lowgenus **8.8.26. some curves of low genus**

curves of genus 2

When Y is a smooth projective curve of genus 2. The canonical map ψ is a map from Y to $\mathbb{P}^1$, of degree $2g-2 = 2$. Every smooth projective curve of genus 2 is hyperelliptic.

curves of genus 3

Let Y be a smooth projective curve of genus 3. The canonical map ψ is a morphism of degree 4 from Y to $\mathbb{P}^2$. If Y isn't hyperelliptic, its image will be a plane curve of degree 4 that is isomorphic to Y. This agrees with the fact that the genus of a smooth projective curve of degree 4 is equal to 3 (1.7.26).

There is another way to arrive at the same result. We go through it because the method can be used for curves of genus 4 or 5.

Let K be a canonical divisor. on a smooth curve of genus $g > 1$. When $d > 1$,

$$\mathbf{h}^1\mathcal{O}(dK) = \mathbf{h}^0\mathcal{O}(K - dK) = 0 \quad \text{and}$$

OdK (8.8.27)
$$\mathbf{h}^0\mathcal{O}(dK) = deg(dK) + 1 - g = (2d-1)(g-1)$$

In our case $g = 3$, and $\mathbf{h}^0\mathcal{O}(dK) = 4d - 2$, when $d > 1$.

The number of monomials of degree d in $n+1$ variables is $\binom{n+d}{n}$. Here $n = 2$, so that number is $\binom{d+2}{2}$.

We assemble this information into a table:

d	0	1	2	3	4	5
monos deg d	1	3	6	10	15	21
$\mathbf{h}^0\mathcal{O}(dK)$	1	3	6	10	14	18

Let $(\alpha_0, \alpha_1, \alpha_2)$ be a basis of $H^0(\mathcal{O}(K))$. The monomials of degree d in $\alpha_0, \alpha_1, \alpha_2$ are global sections of $\mathcal{O}(dK)$. The table shows that there is at least one nonzero homogeneous polynomial $f(x_0, x_1, x_2)$ of degree 4, such that $f(\alpha) = 0$, and Y lies in the zero locus of that polynomial. When Y isn't hyperelliptic, there won't be any linear relation among monomials of degree less than 4. So Y is a quartic curve, and f is, up to scalar factor, the only homogeneous quartic that vanishes on Y. The monomials of degree 4 in α span a space of dimension 14, and therefore they span $H^0(\mathcal{O}(4K))$.

The table also shows that there are (at least) three independent polynomials of degree 5 that vanish on Y. They are x_0f, x_1f, x_2f.

curves of genus 4

When Y is a smooth projective curve of genus 4 that isn't hyperelliptic, the canonical map embeds Y as a curve of degree 6 in $\mathbb{P}^3$. Let's leave the analysis of this case as an exercise.

curves of genus 5

With genus 5, things become more complicated. Let Y be a smooth projective curves of genus 5 that isn't hyperelliptic. The canonical map embeds Y as a curve of degree 8 in $\mathbb{P}^4$. We make a computation analogous to what was done for genus 3. For $d > 1$, the dimension of the space of global sections of $\mathcal{O}(dK)$ is

$$\mathbf{h}^0\mathcal{O}(dK) = 8d - 4$$

and the number of monomials of degree d in 5 variables is $\binom{d+4}{4}$. We form a table:

d	0	1	2	3
monos deg d	1	5	15	35
$\mathbf{h}^0\mathcal{O}(dK)$	1	5	12	20

This table predicts that there are at least three independent homogeneous quadratic polynomials q_1, q_2, q_3 that vanish on the curve Y. Let Q_i be the quadric $\{q_i = 0\}$. Then Y will be contained in the zero locus $Z = Q_1 \cap Q_2 \cap Q_3$.

One can expect the intersection of the three quadrics in $\mathbb{P}^4$ to have dimension 1. If so, a version of Bézout's Theorem asserts that the degree of the intersection will be $2 \cdot 2 \cdot 2 = 8$. Let's skip the proof, which is similar to the proof of the usual Bézout's Theorem. Then the intersection Z will have the same degree as Y, so $Y = Z$. In this case, Y is called a *complete intersection* of the three quadrics. However, it is possible that the intersection Z has dimension 2.

A curve Y that can be represented as a three-sheeted covering of $\mathbb{P}^1$ is called a *trigonal curve*.

8.8.28. Proposition. *A trigonal curve of genus* 5 *isn't isomorphic to an intersection of three quadrics in* $\mathbb{P}^4$. *trigonal*

proof. Given a trigonal curve Y of genus 5, we inspect the morphism of degree 3 to the projective line X. We choose a fibre of that morphism, say the fibre $\{q_1, q_2, q_3\}$ over a point p of X, and we adjust coordinates in X so that p is the point at infinity. With coordinates (x_0, x_1) on X, the rational function $u = x_1/x_0$ on X has poles $D = q_1 + q_2 + q_3$ on Y. So $H^0(Y, \mathcal{O}_Y(D))$ contains 1 and u, and therefore $\mathbf{h}^0\mathcal{O}_Y(D) \geq 2$. By Riemann-Roch, $\chi(\mathcal{O}_Y(D)) = 3+1-g = -1$. So $\mathbf{h}^1\mathcal{O}_Y(D) =\geq 3$, and therefore $\mathbf{h}^0\mathcal{O}_Y(K-D) \geq 3$. There are three independent global sections of $\mathcal{O}_Y(K)$ that vanish on D, say $\alpha_0, \alpha_1, \alpha_2$. We extend this set to a basis $(\alpha_0, ..., \alpha_4)$ of $\mathcal{O}_Y(K)$, and we embed Y into $\mathbb{P}^4$ by that basis. With coordinates $x_0, ..., x_4$ in $\mathbb{P}^4$, the three hyperplanes $H_i : \{x_i = 0\}$, $i = 0, 1, 2$, contain the points q_1, q_2, q_3. The intersection of those hyperplanes is a line L in $\mathbb{P}^4$ that contains the three points.

We go back to the quadrics Q_1, Q_2, Q_3 that contain Y. Since the quadrics contain Y, they contain D. A quadric intersects a line in at most two points unless it contains that line. Therefore each of the quadrics Q_i contains L. So $Q_1 \cap Q_2 \cap Q_3$ contains L. Since the point p over which the fibre of the morphism $Y \to X$ is q_1, q_2, q_3 was arbitrary, $Q_1 \cap Q_2 \cap Q_3$ contains a family of lines L parametrized by X, a ruled surface. It doesn't have dimension one. $\square$

This is the only exceptional case. A curve of genus 5 is hyperelliptic or trigonal, or else it is a complete intersection of three quadrics in $\mathbb{P}^4$, but we omit the proof. We've done enough.

8.9 What is Next

These remarks are meant for someone who has become reasonably comfortable with the material in these notes, and wishes to continue. next

First, spend some time, not too much time, learning about varieties over arbitrary ground fields and schemes. I suggest reading in the books by Hartshorne or Cutkosky that are listed in the references.

If you wish to continue with algebraic geometry, I recommend the book "Complex Algebraic Surfaces" by Beauville. Algebraic surfaces have an interesting intrinsic geometry, formed by curves and their intersections. They were classified by the mathematicians Castelnuovo and Enriques, who spent much of their careers studying them. Enriques wrote, with tongue in cheek: *mentre le curve algebriche sono create da Dio, le superficie invece sono opera del Demonio.*

If you are interested in arithmetic, I recommend reading one of Serre's book, such as "Algebraic Groups and Class Fields". Indeed, one can't go wrong reading any of Serre's books. Even when the book doesn't cover what you think you want, it will be so clear as to be useful.

In either case, a long-range goal should be to understand Mumford's work on geometric invariant theory.

8.10 Exercises

chapeigh.tex xgzero **8.10.1.** Prove that any smooth projective curve of genus zero is isomorphic to the projective line $\mathbb{P}^1$.

xgenusoneOp **8.10.2.** Let Y be a smooth projective curve of genus $p_a = 1$. Use version 1 of the Riemann-Roch Theorem to determine the dimensions of the $\mathcal{O}$-modules $\mathcal{O}(rp)$.

xnotspec **8.10.3.** Let C be a smooth projective curve. Use version 1 of Riemann-Roch to prove

(i) There are positive divisors D such that $\mathbf{h}^1\mathcal{O}(D) = 0$.

(ii) Let D be a positive divisor such that $\mathbf{h}^0\mathcal{O}(D) \geq 2$ and $\mathbf{h}^1\mathcal{O}(D) = 0$. If p is a generic point of C, then $\mathbf{h}^1\mathcal{O}(D-p) = 0$.

(iii) If $d \geq p_a(C)$, there is a positive divisor D of degree $d \geq p_a$, such that $\mathbf{h}^1\mathcal{O}(D) = 0$.

xsmrat **8.10.4.** Let D be a divisor of degree d on a smooth projective curve Y. Show that $\mathbf{h}^0(\mathcal{O}(D)) \leq d+1$, and if $\mathbf{h}^0(\mathcal{O}(D)) = d+1$, then Y is isomorphic to $\mathbb{P}^1$.

8.10.5. Prove that a projective curve Y such that $\mathbf{h}^1(\mathcal{O}_Y) = 0$, smooth or not, is isomorphic to the projective
xHonezero line $\mathbb{P}^1$.

xgentwo **8.10.6.** Let C be a smooth projective curve of genus 2. It follows from Exercise 8.10.3 that there is a positive divisor D of degree 4 such that $\mathbf{h}^0\mathcal{O}(D) = 3$. A basis of global sections of $\mathcal{O}(D)$ defines a morphism $C \to \mathbb{P}^2$.

(i) Prove that the image of C is either a plane curve of degree 4 with a double point, or a conic.

(ii) Prove that C can be represented as a double cover of $\mathbb{P}^1$.

xinvimhyp **8.10.7.** Let D be a divisor on a smooth projective curve Y, and suppose that $\mathbf{h}^0\mathcal{O}(D) > 1$. When Y is mapped to projective space using a basis for $H^0(\mathcal{O}_Y(D))$, what is the inverse image in Y of a hyperplane?

degtwoisconic **8.10.8.** **(i)** Prove that every projective curve of degree 2 is a plane conic.

(ii) Classify projective curves of degree 3.

xlogdif **8.10.9.** For the branched covering $Y \to X$, where $X = \operatorname{Spec}\mathbb{C}[x]$, and Y is the locus $y^e = x$ in $\operatorname{Spec}\mathbb{C}[x,y]$, compute $\tau(dy)$ and $\tau(dy/y)$.

xgenericnotbp **8.10.10.** Let D be a divisor of degree d on a smooth projective curve Y, such that $\mathbf{h}^0\mathcal{O}(D) = k > 0$. Prove that $\mathbf{h}^0\mathcal{O}(D) \leq d+1$, and that if $\mathbf{h}^0\mathcal{O}(D) = d+1$, then X is isomorphic to $\mathbb{P}^1$.

openaffine **8.10.11.** Prove that every nonempty open subset of a smooth affine curve is affine.

xgenustwo **8.10.12.** Let Y be a smooth projective curve of genus 2. Determine the possible dimensions of $H^q(Y, \mathcal{O}(D))$, when D is an effective divisor of degree n.

xgenustwotwo **8.10.13.** Let Y be a curve of genus 2, and let p be a point of Y. Suppose that $\mathbf{h}^1\mathcal{O}(2p) = 0$. Show that there is a basis of global sections of $\mathcal{O}(4p)$ of the form $(1, x, y)$, where x and y have poles of orders 3 and 4 at p. Prove that this basis defines a morphism $Y \to \mathbb{P}^2$ whose image is a singular curve Y' of degree 4.

xproveBG **8.10.14.** The two standard affine open sets $U^0 = \operatorname{Spec} R_0$ and $U^1 = \operatorname{Spec} R_1$, with $R_0 = \mathbb{C}[u]$ with $u = x_1/x_0$ and $R_1 = \mathbb{C}[u^{-1}]$ cover $\mathbb{P}^1$. The intersection U^{01} is the spectrum of the Laurent polynomial ring $R_{01} = \mathbb{C}[u, u^{-1}]$. The units of R_{01} are the monomials cu^k, where k can be any integer.

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible R_{01}-matrix. Prove that there is an invertible R_0-matrix Q, and there is an invertible R_1-matrix P, such that $Q^{-1}AP$ is diagonal.

(ii) Prove the Birkhoff-Grothendieck Theorem for torsion-free $\mathcal{O}_X$-modules of rank 2.

xOmplusOn **8.10.15.** On $\mathbb{P}^1$, when is $\mathcal{O}(m) \oplus \mathcal{O}(n)$ isomorphic to $\mathcal{O}(r) \oplus \mathcal{O}(s)$?

xellgrplaw **8.10.16.** Let Y be an elliptic curve.

(i) Prove that, with the law of composition $\oplus$ defined in (8.8.14), Y is an abelian group.

(ii) Let p be a point of Y. Describe the sum $p \oplus p \cdots \oplus p$ of k copies of p.

(iii) Determine the number of points of order 2 on Y.

(iv) Suppose that Y is a plane curve and that the origin is a flex point. Show that the other the flexes of Y are the points of order 3, and determine the number of points of Y of order 3.

8.10.17. How many real flex points can a real cubic curve have? xrealflex

8.10.18. Prove that a finite $\mathcal{O}$-module on a smooth curve is a direct sum of a torsion module and a locally free module. xmod-dirsum

8.10.19. **(i)** Let $B = A[x]$ be the ring of polynomials in one variable with coefficients in a finite-type domain A. Describe the module Ω_B in terms of Ω_A. xde-scomega
(ii) Let A_s be a localization of A. Describe the module Ω_{A_s} in terms of Ω_A.

8.10.20. Let K be the function field of a variety X of dimension d. Prove that Ω_K is a K-module of dimension d. xomegaK

8.10.21. Let $Y \to X$ be a branched covering, and let p be a point of X whose inverse image in Y consists of one point q. Prove a local analytic version of the main theorem on the trace map for differentials by computation. xprove-trace

8.10.22. Let $Y = \operatorname{Spec} B$ a smooth affine curve, and let y be an element of B. At what points is dy a local generator for Ω_Y? xgenomega

8.10.23. Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth affine curves, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $B \xrightarrow{\delta} {}_A(B, \Omega_A)$ be the composition of the derivation $B \xrightarrow{d} \Omega_B$ with the trace map $\Omega_B \approx {}_B(B, \Omega_B) \xrightarrow{\tau\otimes} {}_A(B, \Omega_A)$. Prove that δ is a derivation from B to the B-module ${}_A(B, \Omega_A)$. xtraced-eriv

8.10.24. **(i)** Let Y be plane curve of degree 5 with a node. Show that the projection of the plane to $\mathbb{P}^1$ with the double point as center of projection represents Y as a trigonal curve of genus 5. xdegfive
(ii) The canonical embedding of a trigonal genus 5 curve Y will have three colinear points $D = p_1 + p_2 + p_3$, Show that $\mathbf{h}^0\mathcal{O}(K-D) = 3$ and that $\mathcal{O}(K-D)$ has no base point. Show that a basis of $H^0\mathcal{O}(K-D)$ maps Y to a curve of degree 5 in $\mathbb{P}^2$ with a double point.

8.10.25. *the basepoint-free trick.* Let D be an effective divisor on a smooth projective curve Y. Suppose that $\mathcal{O}(D)$ has no base point, and that $\mathbf{h}^1\mathcal{O}(D) = 0$. Choose global sections α, β of $\mathcal{O}(D)$ with no common zeros. Prove the following: xbptfre
(i) The sections α, β generate the $\mathcal{O}$-module $\mathcal{O}(D)$, and there is an exact sequence

$$0 \to \mathcal{O}(-D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}(D) \to 0$$

(ii) The tensor product of this sequence with $\mathcal{O}(kD)$ is an exact sequence

$$0 \to \mathcal{O}((k-1)D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}(kD)^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}((k+1)D) \to 0$$

(iii) If $H^1\mathcal{O}((k-1)D) = 0$, every global section of $\mathcal{O}((k+1)D)$ can be obtained as a combination $\alpha u + \beta v$ with $u, v \in H^0\mathcal{O}(kD)$.

8.10.26. Let Y be a smooth projective curve Y of genus g, and let d be an integer. Prove that xgminu-sone
(i) If $d < g - 1$, then for every divisor D of degree d on Y, $\mathbf{h}^1\mathcal{O}(D) > 0$.
(ii) If $d \leq 2g - 2$, there exist divisors D of degree d on Y such that $\mathbf{h}^1\mathcal{O}(D) > 0$.
(iii) If $d \geq g - 1$, there exist divisors D of degree d on Y such that $\mathbf{h}^1\mathcal{O}(D) = 0$.
(iv) If $d > 2g - 2$, then for every divisor D of degree d on Y, $\mathbf{h}^1\mathcal{O}(D) = 0$.

8.10.27. Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves. Prove that its direct image of $\mathcal{O}_Y$ is isomorphic to the direct sum $\mathcal{O}_X \oplus \mathcal{M}$ for some locally free $\mathcal{O}_X$-module $\mathcal{M}$. xOYdi-rectsum

8.10.28. Let Y be a smooth projective curve of genus $g > 1$, and let D be an effective divisor of degree $g+1$ on Y. Suppose that $\mathbf{h}^1\mathcal{O}(D) = 0$ and $\mathbf{h}^0\mathcal{O}(D) = 2$. Let $Y \xrightarrow{\pi} X$ be the morphism to the projective line X defined by a basis $(1, f)$ of $H^0\mathcal{O}(D)$. The $\mathcal{O}_X$-module $\mathcal{O}_Y$ is isomorphic to a direct sum $\mathcal{O}_X \oplus \mathcal{M}$, where $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of rank g (Exercise 8.10.27). decom-poseOY
(i) Let p be the point at infinity of X. Prove that $\mathcal{O}_Y(D)$ is isomorphic to $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(p)$.
(ii) Determine the dimensions of cohomology of $\mathcal{M}$ and of $\mathcal{M}(p)$.
(iii) According to the Birkhoff-Grothendieck Theorem, $\mathcal{M}$ is isomorphic to a sum of twisting modules $\sum_{i=1}^{g} \mathcal{O}_X(r_i)$. Determine the twists r_i.

xsinggenus **8.10.29.** Let C be the plane curve defined by a homogeneous polynomial $f(x,y,z)$ of degree d.

(i) Let C be a plane curve of degree d with a node as its only singularity. Determine the genus of its normalization $C^{\#}$. Do the same for a curve with a cusp.

(ii) One says that C has an *ordinary r-fold point* at $p = (0,0,1)$ if the homogeneous parts f_i of $f(x,y,1)$ are zero for $i < r$, and f_r has distinct roots. Suppose that C has an ordinary triple point at p and no other singularity. Determine the genus of the normalization $C^{\#}$.

(iii) Do the same for an ordinary four-fold point.

xpolirred **8.10.30.** U Factor the polynomial $x^3y^2 - x^3z^2 + y^3z^2$.

xcpltint **8.10.31.** Let f and g be irreducible, homogeneous polynomials in $\mathbb{C}[x_0,x_1,x_2,x_3]$, of degrees d and e respectively, and suppose that g is not a scalar multiple of f. Let X be the locus of common zeros of f and g in the projective space $\mathbb{P}^3$, and let i be the inclusion $X \to \mathbb{P}$.

(i) Construct an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-d-e) \to \mathcal{O}_{\mathbb{P}}(-d) \oplus \mathcal{O}_{\mathbb{P}}(-e) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_X \to 0$$

(ii) Determine the cohomology of $\mathcal{O}_X$.

(iii) Prove that X is connected, i.e., that it is not the union of two proper disjoint Zariski-closed subsets of $\mathbb{P}$.

xtwothree **8.10.32.** Let

$$N = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}$$

be a 3×2 matrix whose entries are homogeneous polynomials of degree d in $R = \mathbb{C}[x_0,x_1,x_2]$, and let $M = (m_1,m_2,m_3)$ be the 1×3 matrix of minors: $m_1 = y_{21}y_{32} - y_{22}y_{31},\ m_2 = -y_{11}y_{32} + y_{12}y_{31},\ m_3 = y_{11}y_{22} - y_{12}y_{21}$. Let I be the ideal of R generated by the minors.

(i) Suppose that I is the unit ideal of R. Prove that this sequence is exact:

$$0 \leftarrow R \overset{M}{\leftarrow} R^3 \overset{N}{\leftarrow} R^2 \leftarrow 0$$

(ii) Let $X = \mathbb{P}^2$, and suppose that the locus Y of zeros of I in X has dimension zero. Prove that this sequence is exact:

$$0 \leftarrow R/I \leftarrow R \overset{M}{\leftarrow} R^3 \overset{N}{\leftarrow} R^2 \leftarrow 0$$

(iii) The sequence in **(ii)** corresponds to the following sequence, in which the terms R are replaced by twisting modules:

$$0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_X \overset{M}{\leftarrow} \mathcal{O}_X(-2d)^3 \overset{N}{\leftarrow} \mathcal{O}_X(-3d)^2 \leftarrow 0$$

Use this sequence to determine $h^0(Y,\mathcal{O}_Y)$. Check your work in an example in which y_{ij} are homogeneous linear polynomials.

Chapter 9 BACKGROUND

back

9.1 Rings and Modules

ringreview

A (commutative) ring B that contains another ring A as subring is called an A-*algebra*. Rings that contain the complex numbers, $\mathbb{C}$-algebras, occur frequently, so we refer to them simply as *algebras*.

An element a of a ring R is a *zero divisor* if it is nonzero, and if there is another nonzero element b of R such that the product ab is zero. A *domain* is a nonzero ring with no zero divisors.

Let F be a field. We use the customary notation $F[\alpha_1, ..., \alpha_n]$ or $F[\alpha]$ for the F-algebra generated by a set of elements $\alpha = \{\alpha_1, ..., \alpha_n\}$, and we may denote the field of fractions of $F[\alpha_1, ..., \alpha_n]$ by $F(\alpha_1, ..., \alpha_n)$ or by $F(\alpha)$.

A set of elements $\alpha = \{\alpha_1, ..., \alpha_n\}$ *generates* A if every element of A can be expressed, usually not uniquely, as a polynomial in $\alpha_1, ..., \alpha_n$, with complex coefficients.

Another way to state this is that the set α generates an algebra A if the homomorphism $\mathbb{C}[x_1, ..., x_n] \xrightarrow{\tau} A$ that evaluates a polynomial at $x = \alpha$ is surjective. If α generates A, then A will be isomorphic to a quotient $\mathbb{C}[x]/I$ of the polynomial algebra $\mathbb{C}[x]$, where I is the kernel of τ. A *finite-type algebra* is an algebra that can be generated by a finite set of elements.

We usually regard a module M over a ring R as a *left module*, writing the scalar product of an element m of M by an element a of R as am. However, it is sometimes convenient to view M as a right module, writing ma instead of am, and defining $ma = am$. This is permissible when the ring is commutative.

A homomorphism of modules $M \to N$ over a ring R may also be called an R-*linear map*. When we say that a map is linear without mentioning a ring, we mean a $\mathbb{C}$-linear map, a homomorphism of vector spaces.

A set $(m_1, ..., m_k)$ of elements it generates an R-module M if every element of M can be obtained as a combination $r_1m_1 + \cdots + r_km_k$ with coefficients r_i in R, or if the homomorphism from the free R-module R^k to M that sends a vector $(r_1, ..., r_k)$ to the combination $r_1m_1 + \cdots + r_km_k$ is surjective. A *finite* module M is one that is spanned, or generated, by some finite set of elements.

A set $(m_1, ..., m_k)$ that generates a module M is an R-*basis* if every element of M is a combination in a unique way, if $r_1m_1 + \cdots + r_km_k = 0$ only when $r_1 = \cdots = r_k = 0$. A module M that has a basis of order k is a *free* R-*module* of rank k. Most modules have no basis.

Let I be an ideal of a ring R. A set of elements $(\alpha_1, ..., \alpha_k)$ generates I if it generates I as R-module — if every element of I can be written as a combination $r_1\alpha_1 + \cdots + r_k\alpha_k$, with r_i in R. A ideal I is *finitely generated* if it is a finite module.

Try not to confuse the concept of a finite module or ideal with that of a finite-type algebra. An R-module M is a finite module if every element of M can be written as a *(linear) combination* $r_1m_1 + \cdots + r_km_k$ of some finite set of elements of M, with coefficients in R. A finite-type algebra A is an algebra in which every element can be written as a *polynomial* $f(\alpha_1, ..., \alpha_k)$ with complex coefficients, in a finite set of elements of A.

If I and J are ideals of a ring R, the sum $I + J$ is an ideal. The *product ideal*, which is denoted by IJ, is the ideal whose elements are finite sums of products $\sum a_ib_i$, with $a_i \in I$ and $b_i \in J$. The product ideal is usually different from the product *set* whose elements are products ab. The product set may not be an ideal.

The *power* I^k of I is the product of k copies of I, the ideal generated by products of k elements of I. The intersection $I \cap J$ of two ideals is an ideal, and

(9.1.1) $$(I\cap J)^2 \subset IJ \subset I\cap J$$ intersect-product

An ideal M of a ring R is a *maximal ideal* if M isn't the unit ideal, $M < R$, and if there is no ideal I with $M < I < R$. An ideal M is a maximal ideal if and only if the quotient ring R/M is a field. An ideal P of R is a *prime ideal* if the quotient R/P is a domain. A maximal ideal is a prime ideal.

9.1.2. Lemma. *Let P be an ideal of a ring R, not the unit ideal. The following conditions are equivalent.* *defprime*
(i) *P is a prime ideal.*
(ii) *If a and b are elements of R, and if the product ab is in P, then $a \in P$ or $b \in P$.*
(iii) *If A and B are ideals of R, and if the product ideal AB is contained in P, then $A \subset P$ or $B \subset P$.*

The following equivalent version of **(iii)** is sometimes convenient:

(iii') *If A and B are ideals that contain P, and if the product ideal AB is contained in P, then $A = P$ or $B = P$.* □

9.1.3. Proposition. *Let $R \xrightarrow{\varphi} S$ be a ring homomorphism. The inverse image of a prime ideal of S is a prime ideal of R.* □ *invim-prime*

9.1.4. localizing a module locmod

Let A be a domain, let M be an A-module, and let's regard M as a right module here. A *torsion element* of M is an element that is annihilated by some nonzero element s of A: $ms = 0$. A nonzero element m such that $ms = 0$ is an *s-torsion* element. The set of torsion elements of M is the *torsion submodule* of M, and a module whose torsion submodule is zero is *torsion-free*.

Let s be a nonzero element of A. The *localization* M_s of M is defined in the natural way, as the A_s-module whose elements are equivalence classes of fractions $m/s^r = ms^{-r}$, with m in M and $r \geq 0$. An alternate notation for the localization M_s is $M[s^{-1}]$.

The only complication comes from the fact that M may contain s-torsion elements. If $ms = 0$, then m must map to zero in M_s, because in M_s, we will have $mss^{-1} = m$. To define M_s, one must to modify the equivalence relation, as follows: Two fractions $m_1s^{-r_1}$ and $m_2s^{-r_2}$ are defined to be equivalent if $m_1s^{r_2+n} = m_2s^{r_1+n}$ when n is sufficiently large. This takes care of torsion, and M_s becomes an A_s-module. There is a homomorphism $M \to M_s$ that sends an element m to the fraction $m/1$. If M is an s-torsion module, then $M_s = 0$.

In this definition of localization, it isn't necessary to assume that $s \neq 0$. But if $s = 0$, then $M_s = 0$.

9.1.5. the Mapping Property of quotients. *mapprop*
(i) *Let K be an ideal of a ring R, let $R \xrightarrow{\tau} \overline{R}$ denote the canonical map from R to the quotient ring $\overline{R} = R/K$, and let S be another ring. Ring homomorphisms $\overline{R} \xrightarrow{\overline{\varphi}} S$ correspond bijectively to ring homomorphisms $R \xrightarrow{\varphi} S$ whose kernels contain K, the correspondence being $\varphi = \overline{\varphi} \circ \tau$:*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\tau}\big\downarrow & & \big\| \\ \overline{R} & \xrightarrow{\overline{\varphi}} & S \end{array}$$

If $\ker \varphi = I$, *then* $\ker \overline{\varphi} = I/K$.
(ii) *Let M and N be modules over a ring R, let L be a submodule of M, and let $M \xrightarrow{\tau} \overline{M}$ denote the canonical map from M to the quotient module $\overline{M} = M/L$. Homomorphisms of modules $\overline{M} \xrightarrow{\overline{\varphi}} N$ correspond bijectively to homomorphisms $M \xrightarrow{\varphi} N$ whose kernels contain L, the correspondence being $\varphi = \overline{\varphi} \circ \tau$. If* $\ker \varphi = L$, *then* $\ker \overline{\varphi} = L/L$. □

The word *canonical* that appears here is used often, to mean a construction that is the natural one in the given context. Exactly what this means in a particular case is usually left unspecified.

corrthm **9.1.6. Correspondence Theorem.**

(i) *Let $R \xrightarrow{\varphi} S$ be a surjective ring homomorphism with kernel K. (For instance, φ might be the canonical map from R to the quotient ring R/K. In any case, S will be isomorphic to R/K.) There is a bijective correspondence*

$$\{\textit{ideals of } R \textit{ that contain } K\} \quad \longleftrightarrow \quad \{\textit{ideals of } S\}$$

This correspondence associates an ideal I of R that contains K with its image $\varphi(I)$ in S and it associates an ideal J of S with its inverse image $\varphi^{-1}(J)$ in R.

If an ideal I of R that contains K corresponds to the ideal J of S, then φ induces an isomorphism of quotient rings $R/I \to S/J$. If one of the ideals, I or J, is prime or maximal, they both are.

(ii) *Let R be a ring, and let $M \xrightarrow{\varphi} N$ be a surjective homomorphism of R-modules with kernel L. There is a bijective correspondence*

$$\{\textit{submodules of } M \textit{ that contain } L\} \quad \longleftrightarrow \quad \{\textit{submodules of } N\}$$

This correspondence associates a submodule V of M that contains L with its image $\varphi(V)$ in N and it associates a submodule W of N with its inverse image $\varphi^{-1}(W)$ in M. $\square$

Ideals $I_1, ..., I_k$ of a ring R are said to be *comaximal* if the sum of any two of them is the unit ideal.

comax **9.1.7. Chinese Remainder Theorem.** *Let $I_1, ..., I_k$ be comaximal ideals of a ring R.*

(i) *The product ideal $I_1 \cdots I_k$ is equal to the intersection $I_1 \cap \cdots \cap I_k$.*

(ii) *The map $R \longrightarrow R/I_1 \times \cdots \times R/I_k$ that sends an element a of R to the vector of its residues in R/I_ν is a surjective homomorphism, and its kernel is $I_1 \cdots I_k$.*

(iii) *Let M be an R-module, and let $M_\nu = M/I_\nu$. The canonical homomorphism $M \to M_1 \times \cdots \times M_k$ is surjective.* $\square$

quotof-prod **9.1.8. Proposition.** *Let R be a product of rings, $R = R_1 \times \cdots \times R_k$, and let K be an ideal of R. There are ideals K_j of R_j such that $K = K_1 \times \cdots \times K_k$ and $R/K = R_1/K_1 \times \cdots \times R_k/K_k$.*

proof. A decomposition of R as a product is determined by a set of *idempotents* elements e_i such that

$$e^2 = e, \qquad e_i e_j = 0 \ \text{ when } \ i \neq j \ , \quad \text{and} \ \ e_1 + e_2 + \cdots + e_k = 1$$

In R, e_i is the unit element of R_i, and $R_i = e_i R e_i$. The residues of e_i in R/K form the system of idempotents that defines the product deomposition of R/K. $\square$

noethr **9.1.9. Noetherian rings**

A ring R is *noetherian* if all of its ideals are finitely generated. The ring $\mathbb{Z}$ of integers is noetherian. Fields are notherian. If I is an ideal of a noetherian ring R, the quotient ring R/I is noetherian.

basisthm **9.1.10. The Hilbert Basis Theorem.** *Let R be a noetherian ring. The ring $R[x_1, ..., x_n]$ of polynomials with coefficients in R is noetherian.* $\square$

Thus $\mathbb{Z}[x_1, ..., x_n]$ is noetherian, and if F is a field, the polynomial ring $F[x_1, ..., x_n]$ is noetherian.

qnoeth **9.1.11. Corollary.** *Every finite-type algebra is noetherian.* $\square$

ascchcond **9.1.12. the ascending chain condition**

The condition that a ring R be noetherian can be rewritten in several ways that we review here.

Our convention is that, if X' and X are sets, the notation $X' \subset X$ means that X' is a subset of X, while $X' < X$ means that X' is a subset that is distinct from X. A *proper subset* X' of a set X is a nonempty subset distinct from X, a set such that $\emptyset < X' < X$.

A sequence $X_1, X_2, ...,$, finite or infinite, of subsets of a set Z forms an *increasing chain* if $X_n \subset X_{n+1}$ for all n, equality $X_n = X_{n+1}$ being permitted. If $X_n < X_{n+1}$ for all n, the chain is *strictly increasing*.

member **9.1.13.** Let $\mathcal{S}$ be a set whose elements are subsets of a set Z. We may refer to an element of $\mathcal{S}$ as a *member* of $\mathcal{S}$ to avoid confusion with the elements of Z. A member of $\mathcal{S}$ is a subset of Z. The words 'member' and 'element' are synonymous.

A member M of $\mathcal{S}$ is a *maximal member* if it isn't properly contained in another member — if there is no member M' of $\mathcal{S}$ such that such that $M < M'$. For example, the set of proper subsets of a set of five elements contains five maximal members, the subsets of order four. The set of finite subsets of the set of integers contains no maximal member. A maximal ideal of a ring R is a maximal member of the set of ideals of R different from the unit ideal.

9.1.14. Proposition. *The following conditions on a ring R are equivalent:* *noether-conds*

(i) *Every ideal of R is finitely generated.*

(ii) *The* ***ascending chain condition****: Every strictly increasing chain $I_1 < I_2 < \cdots$ of ideals of R is finite.*

(iii) *Every nonempty set of ideals of R contains a maximal member.* $\square$

It is customary bad grammar to say that a ring has the ascending chain condition if it has no infinite, strictly increasing sequence of ideals.

The next corollaries follow from the ascending chain condition, though the conclusions are true whether or not R is noetherian.

9.1.15. Corollary. *Let R be a noetherian ring.* *idealin-maximal*

(i) *Every ideal of R except the unit ideal is contained in a maximal ideal.*

(ii) *A nonzero ring R contains at least one maximal ideal.*

(iii) *An element of R that isn't in any maximal ideal is a unit — an invertible element of R.* $\square$

9.1.16. Corollary. *Let $s_1, ..., s_k$ be elements that generate the unit ideal of a ring R. For any positive integer n, the powers $s_1^n, ..., s_k^n$ generate the unit ideal.* $\square$ *powers-generate*

9.1.17. Proposition. *Let R be a noetherian ring, and let M be a finite R-module.* *noetheri-anmodule*

(i) *Every submodule of M is a finite module.*

(ii) *The set of submodules of M satisfies the ascending chain condition.*

(iii) *Every nonempty set of submodules of M contains a maximal member.* $\square$

9.1.18. exact sequences exactseq

An *exact sequence*

$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

of modules over a ring R is a sequence of homomorphisms, finite or infinite, such that for all k, the image of d^{k-1} is equal to the kernel of d^k. For instance, a sequence $0 \to V \xrightarrow{d} V'$ is exact if d is injective, and a sequence $V \xrightarrow{d} V' \to 0$ is exact, if d is surjective.

A *short exact sequence* is an exact sequence of the form

(9.1.19) $$0 \to V^0 \xrightarrow{a} V^1 \xrightarrow{b} V^2 \to 0$$ sexseq

To say that this sequence is exact means that the map a is injective, and that b induces an isomorphism from the quotient module V^1/aV^0 to V^2.

The short exact sequence (9.1.19) *splits* if there is a map $V^1 \xleftarrow{s} V^2$ such that bs is the identity on V^2. If the sequence splits, V^1 will be isomorphic to the direct sum $V^0 \oplus V^2$.

9.1.20. Lemma. *Let $0 \to V^0 \to V^1 \to V^2 \to \cdots \to V^n \to 0$ be an exact sequence, and let N be the kernel of the map $V^! \to V^2$. The given sequence breaks up into the two exact sequences* *splitseq*

$$0 \to V^0 \to V^1 \to N \to 0 \ \ \textit{and} \ \ 0 \to N \to V^2 \to \cdots \to V^n \to 0$$

$\square$

Let $V \xrightarrow{d} V'$ be a homomorphism of R-modules, and let W be the image of d. The *cokernel* of d is the module $C = V'/W \quad (= V'/dV)$. The homomorphism d embeds into an exact sequence

(9.1.21) $$0 \to K \to V \xrightarrow{d} V' \to C \to 0$$ kercok-erseq where K and C are the kernel and cokernel of d, respectively.

A module homomorphism $V' \xrightarrow{f} M$ with cokernel C induces a homomorphism $C \to M$ if and only if the composed homomorphism fd is zero. This follows from the mapping property Corollary 9.1.5**(ii)**.

snake **9.1.22. Proposition.** *(functorial property of the kernel and cokernel) Suppose given a diagram of R-modules*

$$\begin{array}{ccccccc} & & V & \xrightarrow{u} & V' & \longrightarrow & V'' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & \\ 0 & \longrightarrow & W & \longrightarrow & W' & \xrightarrow[v]{} & W'' & & \end{array}$$

whose rows are exact sequences. Let K, K', K'' and C, C', C'' denote the kernels and cokernels of f, f', and f'', respectively.

(i) (kernel is left exact) *The kernels form an exact sequence $K \to K' \to K''$. If u is injective, the sequence $0 \to K \to K' \to K''$ is exact.*

(ii) (cokernel is right exact) *The cokernels form an exact sequence $C \to C' \to C''$. If v is surjective, the sequence $C \to C' \to C'' \to 0$ is exact.*

(iii) (Snake Lemma) *There is a canonical homomorphism $K'' \xrightarrow{d} C$ that combines with the sequences* **(i),(i)** *to form an exact sequence*

$$K \to K' \to K'' \xrightarrow{d} C \to C' \to C''$$

If u is injective and/or v is surjective, the sequence remains exact with zeros at the appropriate ends. □

dmod **9.1.23. the dual module**

Let V be a finite-dimensional $\mathbb{C}$-module (a vector space). The *dual module* V^* is the module of linear maps (homomorphisms of $\mathbb{C}$-modules) $V \to \mathbb{C}$. When $V \xrightarrow{d} V'$ is a homomorphism of vector spaces, there is a canonical dual homomorphism $V^* \xleftarrow{d^*} V'^*$. The dual of the sequence (9.1.21) is an exact sequence

$$0 \leftarrow K^* \leftarrow V^* \xleftarrow{d^*} V'^* \leftarrow C^* \leftarrow 0$$

So the dual of the kernel K is the cokernel K^* of d^*, and the dual of the cokernel C is the kernel C^* of d^*. This is the reason for the term “cokernel”.

present-module **9.1.24. presenting a module**

Let R be a ring. A *presentation* of an R-module M is an exact sequence of modules of the form

$$R^\ell \to R^k \to M \to 0$$

The map $R^\ell \to R^k$ will be given by an $\ell \times k$ matrix with entries in R, an *R-matrix*, and that matrix determines the module M up to isomorphism as the cokernel of that map.

Every finite module over a noetherian ring R has a presentation. To obtain a presentation, one chooses a finite set $m = (m_1, ..., m_k)$ of generators for the finite module M, so that multiplication by m defines a surjective map $R^k \to M$. Let N be the kernel of that map. Because R is noetherian, N is a finite module. Next, one chooses a finits set of generators of N, which gives us a surjective map $R^\ell \to N$. Composition of that map with the inclusion $N \subset R^\ell$ produces an exact sequence $R^\ell \to R^k \to M \to 0$.

sumprod **9.1.25. direct sum and direct product**

Let M and N be modules over a ring R. The *product module* $M \times N$ is the product set, whose elements are pairs (m, n), with m in M and n in N. The laws of composition are the same as the laws for vectors: $(m_1, n_1)+(m_2, n_2) = (m_1+m_2, n_1+n_2)$ and $r(m, n) = (rm, rn)$. There are homomorphisms $M \xrightarrow{i_1} M\times N$ and $M\times N \xrightarrow{\pi_1} M$, defined by $i_1(m) = (m, 0)$ and $\pi_1(m, n) = m$, and similar homomorphisms $N \xrightarrow{i_2} M\times N$ and $M \times N \xrightarrow{\pi_2} N$. So i_1 and i_2 are inclusions, and π_1 and π_2 are projections.

The product module is characterized by this mapping property:

• Let T be an R-module. Homomorphisms $T \xrightarrow{\varphi} M\times N$ correspond bijectively to pairs of homomorphisms $T \xrightarrow{\alpha} M$ and $T \xrightarrow{\beta} N$. The homomorphism φ that corresponds to the pair α, β is $\varphi(m,n) = (\alpha m, \beta n)$, and when φ is given, the homomorphisms to M are $\alpha = \pi_1\varphi$ and $\beta = \pi_2\varphi$.

A different product, the *tensor product* is defined below.

The product module $M\times N$ is isomorphic to the *direct sum* $M \oplus N$. Elements of $M \oplus N$ may be written either as $m + n$, or with product notation, as (m,n).

The direct sum $M \oplus N$ is characterized by this mapping property:

• Let S be an R-module. Homomorphisms $M \oplus N \xrightarrow{\psi} S$ correspond bijectively to pairs u, v of homomorphisms $M \xrightarrow{u} S$ and $N \xrightarrow{v} S$. The homomorphism ψ that corresponds to the pair u, v is $\psi(m,n) = um + vn$, and when ψ is given, $u = \psi i_1$ and $v = \psi i_2$.

We use the direct product and direct sum notations interchangeably, but we note that the direct sum of an infinite set of modules isn't the same as their product.

9.1.26. tensor products tensprod

Let U and V be modules over a ring R. The *tensor product* $U \otimes_R V$ is an R-module generated by elements $u\otimes v$ called *tensors*, one for each u in U and each v in V. The elements of the tensor product are combinations $\sum_1^k r_i(u_i \otimes v_i)$ of tensors with coefficients in R.

The defining relations among the tensors are the *bilinear relations*:

(9.1.27)
$$(u_1+u_2) \otimes v = u_1 \otimes v + u_2 \otimes v \qquad u \otimes (v_1+v_2) = u \otimes v_1 + u \otimes v_2$$
bilinrels

and
$$r(u \otimes v) = (ru) \otimes v = u \otimes (rv)$$

for all u in U, v in V, and r in R. The symbol $\otimes$ is used as a reminder that tensors are manipulated using those relations.

One can absorb a coefficient from R into the element on the left (or on the right) in a tensor, so every element of $U \otimes_R V$ can be written as a finite sum $\sum u_i \otimes v_i$ with u_i in U and v_i in V.

There is an obvious map of sets

(9.1.28)
$$U\times V \xrightarrow{\beta} U \otimes_R V$$
bilin

from the product *set* to the tensor product, that sends a pair (u,v) to the tensor $u \otimes v$. This map isn't a homomorphism of R-modules. The defining relations (9.1.27) show that β is *R-bilinear*, not R-linear.

An *R-bilinear map* of R-modules $U\times V \xrightarrow{f} W$ is a map that has properties analogous to (9.1.27):

$$f(u_1 + u_2) = f(u_1,v) + f(u_2,v) \qquad f(u, v_1 + v_2) = f(u,v_1) + f(u,v_2)$$

and
$$rf(u,v) = f(ru,v) = f(u,rv)$$

The next corollary follows from the defining relations of the tensor product.

9.1.29. Corollary. *Let U, V, and W be R-modules. Homomorphisms of R-modules $U \otimes_R V \to W$ correspond bijectively to R-bilinear maps $U\times V \to W$.* □ *tensorbilin*

Thus the map $U\times V \xrightarrow{\beta} U \otimes_R V$ is a *universal* bilinear map. Any R-bilinear map $U\times V \xrightarrow{f} W$ to a module W can be obtained from a module homomorphism $U \otimes_R V \xrightarrow{\widetilde{f}} W$ by composition with the bilinear map β: $U\times V \xrightarrow{\beta} U \otimes_R V \xrightarrow{\widetilde{f}} W$.

canonisom **9.1.30. Proposition.** *There are canonical isomorphisms*

- $R \otimes_R U \approx U$, *defined by* $r \otimes u \leftrightsquigarrow ru$
- $(U \oplus U') \otimes_R V \approx (U \otimes_R V) \oplus (U' \otimes_R V)$, *defined by* $(u_1 + u_2) \otimes v \leftrightsquigarrow u_1 \otimes v + u_2 \otimes v$
- $U \otimes_R V \approx V \otimes_R U$, *defined by* $u \otimes v \leftrightsquigarrow v \otimes u$
- $(U \otimes_R V) \otimes_R W \approx U \otimes_R (V \otimes_R W)$, *defined by* $(u \otimes v) \otimes w \leftrightsquigarrow u \otimes (v \otimes w)$

The verification of these isomorphisms isn't difficult. We verify the second one as an example. The map $(U \oplus U') \times V \to (U \otimes V) \oplus (U' \otimes V)$ defined by $(u + u', v) \to (u \otimes v) + (u' \otimes v)$ is bilinear. It corresponds to a linear map $(U \oplus U') \otimes V \to (U \otimes V) \oplus (U' \otimes V)$. There is also a bilinear map $(U \times V) \times (U' \times V) \to (U \oplus U') \otimes V$ defined by $((u, v_1), (u', v_2)) \to u \otimes v_1 + u' \otimes v_2$. It induces the linear inverse map. $\square$

colxrow **9.1.31. Examples. (i)** If U is the space of m dimensional (complex) column vectors, and V is the space of n-dimensional row vectors, then $U \otimes_{\mathbb{C}} V$ identifies naturally with the space of $m \times n$-matrices.

(ii) If U and V are free R-modules of ranks m and n, with bases $\{u_1, ..., u_m\}$ and $\{v_1, ..., v_n\}$, respectively, the tensor product $U \otimes_R V$ is a free R-module of rank mn, with basis $\{u_i \otimes v_j\}$. In contrast, the product module $U \times V$ is a free module of rank $m + n$, with basis $\{(u_i, 0)\} \cup \{(0, v_j)\}$.

(iii) If V is a free module of rank n, then for any module U, $U \otimes_R V$ is isomorphic to U^n.

The tensor product of general modules can be complicated, and right exactness is the main tool for describing it.

rexacttensor **9.1.32. Proposition.** *Tensor product is right exact. If* $U \xrightarrow{f} U' \xrightarrow{g} U'' \to 0$ *is an exact sequence of* R*-modules and* V *is another* R*-module, the sequence*

$$U \otimes_R V \xrightarrow{f \otimes 1} U' \otimes_R V \xrightarrow{g \otimes 1} U'' \otimes_R V \to 0$$

in which $[f \otimes 1](u \otimes v) = f(u) \otimes v$, *is exact.* $\square$

Tensor product isn't left exact. For example, if $R = \mathbb{C}[x]$, then $R/xR \approx \mathbb{C}$. There is an exact sequence $0 \to R \xrightarrow{x} R \to \mathbb{C} \to 0$. When we tensor with $\mathbb{C}$ we get a sequence $0 \to \mathbb{C} \xrightarrow{x} \mathbb{C} \to \mathbb{C} \to 0$, in which multiplication by x is the zero map.

proof of Proposition 9.1.32. We suppose that an exact sequence of R-modules $U \xrightarrow{f} U' \xrightarrow{g} U'' \to 0$ and another R-module V are given. We are to prove that the sequence $U \otimes_R V \xrightarrow{f \otimes 1} U' \otimes_R V \xrightarrow{g \otimes 1} U'' \otimes_R V \to 0$ is exact. It is evident that the composition $(g \otimes 1)(f \otimes 1)$ is zero, and that $g \otimes 1$ is surjective. We must prove that $U'' \otimes_R V$ is isomorphic to the cokernel of $f \otimes 1$.

Let C be the cokernel of $f \otimes 1$. The mapping property (9.1.5)**(ii)** gives us a canonical map $C \xrightarrow{\varphi} U'' \otimes_R V$ that we want to show is an isomorphism. To show this, we construct the inverse of φ. We choose an element v of V, and form a diagram of R-modules

$$\begin{array}{ccccccc}
U \times v & \xrightarrow{f_v} & U' \times v & \xrightarrow{g_v} & U'' \times v & \longrightarrow & 0 \\
{\scriptstyle \beta_v}\downarrow & & {\scriptstyle \beta'_v}\downarrow & & \downarrow{\scriptstyle \gamma_v} & & \\
U \otimes_R V & \xrightarrow{f \otimes 1} & U' \otimes_R V & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

in which $U \times v$ denotes the module of pairs (u, v) with u in U. The top row is isomorphic to the given sequence $U \to U' \to U'' \to 0$. So $U'' \times v$ is the cokernel of f_v. The rows in the diagram are exact sequences. The vertical arrows β_v and β'_v are obtained by restriction from the canonical bilinear maps (9.1.28). They are R-linear because v is held constant. The map γ_v is determined by the definition of the cokernel $U'' \times v$. Putting the maps γ_v together for all v in V gives us a bilinear map $U \times V \to C$. That bilinear map induces the linear map $U \otimes_R V \to C$ which is the inverse of φ. $\square$

tensorquotient **9.1.33. Corollary.** *Let* I *be an ideal of a domain* R, *let* $\overline{R} = R/I$, *and let* M *be an* R*-module. Then* $\overline{R} \otimes_R M$ *is isomorphic to* M/IM.

proof. We tense the exact sequence $I \to R \to \overline{R} \to 0$ with M and form a diagram

$$\begin{tikzcd} & I \otimes M \arrow[r] \arrow[d, "a"'] & R \otimes M \arrow[r] \arrow[d, "b"'] & \overline{R} \otimes M \arrow[r] \arrow[d, "c"'] & 0 \\ 0 \arrow[r] & IM \arrow[r] & M \arrow[r] & \overline{M} \arrow[r] & 0 \end{tikzcd}$$

in which $\overline{M} = M/IM$, and b is the isomorphism defined by $r \otimes m \to rm$ (9.1.30). It maps the image of $I \otimes M$ surjectively to IM. The bottom row is exact, and the Snake Lemma shows that c is an isomorphism. $\square$

9.1.34. Corollary. *Let M and N be modules over a domain R and let s be a nonzero element of R. Let R_s be the localization of R.* *locistensor*
(i) *The localization M_s is isomorphic to $R_s \otimes_R M$.*
(ii) *Tensor products are compatible with localization: $M_s \otimes_{R_s} N_s \approx (M \otimes_R N)_s$*

proof. **(i)** There is a bilinear map $R_s \times_R M \to M_s$ defined by $(s^{-k}a, m) \to s^{-k}am$. This gies us the map $R_s \otimes M \to M_s$. In the other direction, we have a map $M \to R_s \otimes M$ defined by $m \to 1 \otimes m$. Since s is invertible in $R_s \otimes M$, this map extends to the inverse map $M_s \to R_s \otimes M$.

(ii) The composition of the canonical maps $M \times N \to M_s \times N_s \to M_s \otimes_{R_s} N_s$ is R-bilinear. It defines an R-linear map $M \otimes_R N \to M_s \otimes_{R_s} N_s$. Since s is invertible in $M_s \otimes_{R_s} N_s$, this map extends to an R_s-linear map $(M \otimes_R N)_s \to M_s \otimes_{R_s} N_s$. Next, we define an R_s-bilinear map $M_s \times N_s \to (M \otimes_R N)_s$ by mapping a pair (us^{-m}, vs^{-n}) to $(u \otimes v)s^{-m+n}$. This bilinear map induces the inverse map $M_s \otimes_{R_s} N_s \to (M \otimes_R N)_s$.$\square$

9.1.35. direct limits dirlim

A directed set $M_\bullet$ of modules over a ring R is a sequence of homomorphisms $M_0 \to M_1 \to M_2 \to \cdots$. Its *limit* $\varinjlim M_\bullet$, also called the *direct limit*, is the R-module whose elements are equivalence classes of the elements of the union $\bigcup M_k$. The equivalence relation is that elements m in M_i and m' in M_j are equivalent if they have the same image in M_n when n is sufficiently large. So an element of $\varinjlim M_\bullet$ will be represented by an element of M_i for some i.

A map of directed sets of modules $M_\bullet \to N_\bullet$ is a diagram

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & \cdots \\ N_1 \arrow[r] & N_2 \arrow[r] & \cdots \end{tikzcd}$$

Such a map induces a homomorphism $\varinjlim M_\bullet \to \varinjlim N_\bullet$. The limit operation is functorial.

A sequence $M_\bullet \to N_\bullet \to P_\bullet$ of maps of directed sets is exact if the sequences $M_i \to N_i \to P_i$ are exact for every i.

9.1.36. Lemma. **(i)** *The limit operation is exact. If $M_\bullet \to N_\bullet \to P_\bullet$ is an exact sequence of directed sets, the limits form an exact sequence.* *jstarlimit*
(ii) *Let $M_\bullet$ be a directed set of R-modules, let P be another R-module. Homomorphisms $\varinjlim M_\bullet \to P$ correspond bijectively to sequences of homomorphisms $\{M_i \to P\}$ that are compatible, i.e., such that the map $M_i \to P$ is the composition of the maps $M_i \to M_{i+1} \to P$ for all i.*

proof. **(ii)** When we replace P by the constant directed set $P = P = \cdots$, this follows from the fact that the limit operation is a functor. $\square$

9.1.37. Examples. If M_i is an increasing family of submodules of a module, $M_0 \subset M_1 \subset \cdots$, then $\varinjlim M_\bullet$ is their union. If the maps $M_i \to M_{i+1}$ are surjective (or bijective) for all i, the maps $M_i \to \varinjlim M_\bullet$ are surjective (or bijetcive). $\square$ limexample

9.1.38. Lemma. *Tensor products are compatible with limits: If $M_\bullet$ is a directed set of R-modules and N is another R-module, then $\varinjlim\,[M_\bullet \otimes_R N]$ and $[\varinjlim M_\bullet] \otimes_R N$ are isomorphic.* *tensorlimit* *proof.* An element of either of these limits can be reprsented by an element of $M_i \otimes N$ for large i. Two such elements represent the same element of either limit if their images in $M_j \otimes N$ are equal for some $j > i$. $\square$

extend-scalars

9.1.39. extension of scalars

Let $A \xrightarrow{\rho} B$ be a ring homomorphism. Let's write scalar multiplication on the right. So M will be a right A-module. Then $M \otimes_A B$ becomes a right B-module. Scalar multiplication by $b \in B$ is defined by $(m \otimes b')b = m \otimes (b'b)$. This gives the functor

$$A{-}modules \xrightarrow{\otimes B} B{-}modules$$

called the *extension of scalars* from A to B.

resscal

9.1.40. restriction of scalars

If $A \xrightarrow{\rho} B$ is a ring homomorphism, a (left) B-module M can be made into an A-module by *restriction of scalars*. Scalar multiplication by an element a of A is defined by the formula

restrscal

$$am = \rho(a)m \qquad (9.1.41)$$

It is customary to denote a module and the one obtained by restriction of scalars by the same symbol. But when it seems advisable, one can denote a B-module M and the A-module obtained from M by restriction of scalars by ${}_BM$ and ${}_AM$, respectively. The additive groups of ${}_BM$ and ${}_AM$ are the same.

For example, a module over the prime field $\mathbb{F}_p$ beomes a $\mathbb{Z}$-module by restriction of scalars. If $\overline{n}$ denotes the residue of an integer n in $\mathbb{F}_p$ and V is an $\mathbb{F}_p$-module, scalar multiplication in ${}_{\mathbb{Z}}V$ is defined in the obvious way, by $nv = \overline{n}v$.

xtscaltens

9.1.42. Lemma. *(extension and restriction of scalars are adjoint operators)*

Let $A \xrightarrow{\rho} B$ be a ring homomorphism, let M be an A-module, and let N be an B-module. Homomorphisms of A-modules $M \xrightarrow{\varphi} {}_AN$ correspond bijectively to homomorphisms of B-modules $M \otimes_A B \xrightarrow{\psi} {}_BN$. $\square$

9.2 The Implicit Function Theorem

ifthm

An *analytic function* $\varphi(x_1, ..., x_k)$ in one or more variables is a complex-valued function, defined for small x, that can be represented as a convergent power series.

IFT

9.2.1. Implicit Function Theorem *Let $f(x, y)$ be a polynomial or an analytic function of two variables. If $f(0, 0) = 0$ and $\frac{\partial f}{\partial y}(0, 0) \neq 0$, there is a unique analytic function $\varphi(x)$ with $\varphi(0) = 0$ such that the series $f(x, \varphi(x))$ is identically zero.*

proof. The constant term of f is zero, so f has the form $ax + by + O(2)$, where $O(2)$ denotes a polynomial or power series, all of whose terms have total degree at least 2. Then $\frac{\partial f}{\partial y}(0, 0) = b$. So $b \neq 0$. We normalize b to 1, so that $f(x, y) = y + ax + O(2)$.

We construct the power series solution $\varphi(x)$ inductively, beginning with $\varphi_0 = 0$. Suppose that we have found a polynomial $\varphi_{k-1} = c_1x + \cdots + c_{k-1}x^{k-1}$ of degree $k-1$, with $f(0, 0) = 0$ and $f(x, \varphi_{k-1}) \equiv 0$ modulo x^k, and suppose that φ_{k-1} is the unique such polynomial. We use Newton's Method to construct a unique polynomial $\varphi_k = \varphi_{k-1} + c_kx^k$ of degree k, such that $f(x, \varphi_k) \equiv 0$ modulo x^{k+1}. By hypothesis, $f(x, \varphi_{k-1}) = dx^k + O(x^{k+1})$ for some scalar d. We substitute $\varphi_k = \varphi_{k-1} + c_kx^k$ with undetermined coefficient c_k into f. Since $\frac{\partial f}{\partial y}(0, 0) = 1$,

$$f(x, \varphi_k) = f(x, \varphi_{k-1}) + c_kx^k + O(k{+}1) = dx^k + c_kx^k + O(k{+}1)$$

The unique solution for φ_k is obtained with $c_k = -d$.

One needs to show that the power series $\varphi(x)$ constructed in this way converges for small x, but the fact that $\varphi(x)$ is unique makes its convergence automatic. So it is permissible to omit that verification, because

9.2.2. The same method works with more variables. Let $f_1, ..., f_k$ be polynomials or analytic functions in the variables $x, y_1, ..., y_k$, with $f_i(0,0) = 0$, and let $f_{ij} = \frac{\partial f_i}{\partial y_j}$. If the Jacobian matrix $J = (f_{ij})$, evaluated at $(x,y) = (0,0)$, is invertible, there are unique analytic functions $\varphi_j(x)$ such that $\varphi_j(0) = 0$ and $f_i(x, \varphi_1, ..., \varphi_k) = 0$. The proof is analogous. $\square$ iftmorevar

Let $\mathcal{R}$ be the ring af analytic functions in x. With f and φ as above, the polynomial $y–\varphi(x)$ divides $f(x,y)$ in the ring $\mathcal{R}[y]$ of polynomials with coefficients in $\mathcal{R}$. To see this, we do division with remainder of f by the monic polynoial $y - \varphi(x)$ in y:

(9.2.3) $$f(x,y) = (y - \varphi(x))q(x,y) + r(x)$$ divrem

The quotient q and remainder r are in $\mathcal{R}[y]$, and $r(x)$ has degree zero in y, so it is in $\mathcal{R}$. Setting $y = \varphi(x)$ in the equation, one sees that $r(x) = 0$.

Let Γ be the graph of φ in a suitable neighborhood U of the origin in x, y-space. Since $f(x,y) = (y - \varphi(x))q(x,y)$, the locus $f(x,y) = 0$ in U has the form $\Gamma \cup \Delta$, where Γ is the zero locus of $y - \varphi(x)$ and Δ is the zero locus of $q(x,y)$. Differentiating, we find that $\frac{\partial f}{\partial y}(0,0) = q(0,0)$. So if $f(0,0) \neq 0$ then $q(0,0) \neq 0$. Then Δ doesn't contain the origin. A small neighborhood U of the origin won't contain any points of Δ. In such a neighborhood, the locus of zeros of f will be Γ. So Δ is disjoint from Γ, locally.

9.3 Transcendence Degree

transcdeg

Let $F \subset K$ be a field extension. A set $\alpha = \{\alpha_1, ..., \alpha_n\}$ of elements of K is *algebraically dependent* over F if there is a nonzero polynomial $f(x_1, ..., x_n)$ with coefficients in F, such that $f(\alpha) = 0$. If there is no such polynomial, the set α is *algebraically independent* over F.

A set $\{\alpha_1, ..., \alpha_n\}$ is algebraically independent over F if and only if the surjective map from the polynomial algebra $F[x_1, ..., x_n]$ to $F[\alpha_1, ..., \alpha_n]$ that sends x_i to α_i is bijective. If so, we may refer to the ring $F[\alpha_1, ..., \alpha_n]$ as a polynomial algebra too.

An infinite set is called algebraically independent over F if every finite subset is algebraically independent over F — if there is no polynomial relation among any finite set of its elements.

The set $\{\alpha_1\}$ consisting of a single element of K is algebraically dependent if α_1 is algebraic over F. Otherwise, it is algebraically independent, and then α_1 is said to be *transcendental* over F.

A *transcendence basis* for K over F is a finite algebraically independent set $\alpha = \{\alpha_1, ..., \alpha_n\}$ that isn't contained in a larger algebraically independent set. If there is a transcendence basis, its order is the *transcendence degree* of the field extension K. As Proposition 9.3.3 below shows, all transcendence bases for K over F have the same order. If there is no (finite) transcendence basis, the transcendence degree of K over F is said to be infinite.

When $K = F(x_1, ..., x_n)$ is the field of rational functions in n variables, the variables form a transcendence basis of K over F, and the transcendence degree of K over F is n. The elementary symmetric functions $s_1 = x_1 + \cdots + x_n$, ... form another transcendence basis of K over F.

9.3.1. Lemma. *Let F be a field, let A be an F-algebra that is a domain, and let K be the field of fractions of A. If K has transcendence degree n over F, then every algebaically independent set of elements of A is contained in an algebraically independent set of elementsof A of order n.* $\square$ *trdegalg*

9.3.2. Lemma. *Let K/F be a field extension, let $\alpha = \{\alpha_1, ..., \alpha_n\}$ be a set of elements of K that is algebraically independent over F, and let $F(\alpha)$ be the field of fractions of $F[\alpha]$.* *algindtrivialities*

(i) *Let β be another element of K. The set $\{\alpha_1, ..., \alpha_n, \beta\}$ is algebraically dependent if and only if β is algebraic over $F(\alpha)$.*

(ii) *The algebraically independent set α is a transcendence basis if and only if every element of K is algebraic over $F(\alpha)$.* $\square$

9.3.3. Proposition. *Let K/F be a field extension. If K has a finite transcendence basis, then all algebraically independent subsets of K are finite, and all transcendence bases have the same order.* *trdeg*

proof. Let $\alpha = \{\alpha_1, ..., \alpha_r\}$ and $\beta = \{\beta_1, ..., \beta_s\}$ be subsets of K. Suppose that K is algebraic over $F(\alpha)$ and that the set β is algebraically independent over F. We show that $s \leq r$. The fact that all transcendence bases have the same order will follow: If both α and β are transcendence bases, then we can interchange α and β, so $r \le s$.

The proof that $s \le r$ proceeds by reducing to the trivial case that β is a subset of α. Suppose that some element of β, say β_s, isn't in the set α. The set $\beta' = \{\beta_1, ..., \beta_{s-1}\}$ is algebraically independent, but it isn't a transcendence basis. So K isn't algebraic over $F(\beta')$. Since K is algebraic over $F(\alpha)$, there is at least one element of α, say α_r, that isn't algebraic over $F(\beta')$. Then $\beta' \cup \{\alpha_r\}$ will be an algebraically independent set of order s that contains more elements of the set α than β does. Induction shows that $s \le r$. □

sametrdeg **9.3.4. Corollary.** *Let $L \supset K \supset F$ be fields. If the degree $[L:K]$ of the field extension L/K is finite, then L and K have the same transcendence degree over F.*

This follows from Lemma 9.3.2**(ii)**. □

GLOSSARY

algebra: a ring that contains the complex numbers.

algebraically independent Elements that satisfy no polynomial relation are algebraically independent. See Section 9.3.

analytic function: A function that can be represented by a convergent power series (9.2).

annihilator: The annihilator of an element m of an R-module is the ideal of elements a of R such that $am = 0$ (6.6).

arithmetic genus: The arithmetic genus of a smooth projective curve is $p_a = 1 - \mathbf{C}^1\mathcal{O}$ (7.6.2).

basis for a topology: A basis $\mathcal{B}$ for a topology is a set of open subsets such that every open subset if a union of members of $\mathcal{B}$ (2.6.2).

bitangent: a line that is tangent to a curve at two points (1.7.18).

branch point: a point at which the ramification index of a branched covering is greater than 1. (8.5).

branched covering: a finite morphism of curves (1.7.16), (8.5).

canonical: A mathematical construction is called canonical if it is the natural one in the context.

canonical divisor: a divisor K on a smooth projective curve X such that $\mathcal{O}(K)$ is isomorphic to Ω_X (8.8.7).

canonical map: the map fom a curve to projective space defined by the regular differentials (8.8.25).

classical topology: the usual topology (1.3.17).

closure: The closure of a subset S of a topological space is the smallest closed subset that contains S.

coarser topology: A topology T' on a set X is coarser than another topology T if T' contains fewer closed subsets than T.

cohomological functor: A sequence of functors $H^0, H^1, H^2, ...$ to vector spaces such that a short exact sequence produces a long cohomology sequence (7.1.4).

cokernel: The cokernel of a homomorphism $M \to N$ is the quotient $N/\operatorname{im} M$ (9.1.18).

commutative diagram: A diagram of maps is commutative if all maps from A to B that can be obtaind by composition of the ones in the diagram are equal (2.3.6).

complement: The complement of a subset S of a set X is the set of elements of X that are not in S.

complex: A complex of vector spaces is a sequence $\cdots \to V^{n-1} \xrightarrow{d^n} V^n \to \cdots$ of vector spaces such that $\ker d^n \subset \operatorname{im} d^{n-1}$ (7.2).

constructible set: a finite union of locally closed sets (5.3).

counting constants: An informal method of determining dimension.

cusp: a certain type of singular point of a curve (1.7.10).

dimension: The dimension of a variety is the transcendence degree of its field of rational functions, or the length of a maximal chain of closed subvarieties (4.5).

discriminant: a polynomial in the coefficients of a polynomial f that vanishes if and only if f has a double root (1.6.13).

divisor: A divisor on a smooth curve is an integer combination $a_1q_1 + \cdots + a_kq_k$ of points (8.1).

domain: A nonzero ring with no zero divisors.

dual curve: The dual curve of a smooth plane curve is the locus of tangent lines as smooth points (1.5.4).

dual plane: the projective plane whose points correspond to lines in the given plane (1.5.1).

elliptic curve: a smooth projective curve of genus 1 (8.8.11).

Euler characteristic: The Euler characteristic of an $\mathcal{O}$-module is the alternating sum of the dimensions of its cohomology (7.7.7).

exact sequence: a sequence $\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \to \cdots$ is exact if $\ker d_n = \operatorname{im} d^{n-1}$ (9.1.18).

exterior algebra: the graded algebra generated by the elements of a vector space, with the relations $vv = 0$ (3.7.2).

Fermat Curve: one of the plane curves $x_o^k + x_1^k + x_2^k = 0$.

fibre: The fibre of a map $Y \xrightarrow{\pi} X$ over a point x is the number of points in its inverse image.

finer topology: A topology T' on a set X is finer than another topology T if T' contains more closed subsets than T.

finite module: a module that can be generated by finitiely many of its elements (9.1).

finite-type algebra: an algebra that can be generated, as algbera, by finitely many elements (9.1).

generic, general position: not in a special, or 'bad' position (1.7.18).

genus: the genus of a compact two-dimensional manifold is the number of its holes (1.7.23).

Grassmanian: a variety that parametrizes subspaces of a given dimension of a vector space (3.7).

Hessian matrix: the matrix of second partial derivatives (1.4.10).

homogeneous parts: the homogeneous part of degree k of a polynomial is the sum of terms of degree k (1.3.1).

hyperelliptic curve: a curve of genus at least two, that can be represented as a double cover of $\mathbb{P}^1$ (8.8.21).

hypersurface: a subvariety of projective space that is defined by one equation (2.2.3).

increasing sequence: a sequence S_n of sets is increasing if $S_n \subset S_{n+1}$ for all n, and it is strictly increasing if if $S_n < S_{n+1}$ for all n (9.1.12).

integral morphism: a morphism $Y \to X$ such that $\mathcal{O}_Y$ becomes a finite $\mathcal{O}_X$-module (4.2.1).

invertible module: a locally free module of rank 1 (8.1.16).

irreducible polynomial: a polynomial of positive degree that isn't the product of two polynomials of positive degree.

irreducible space: a topological space that isn't the union of two proper closed subsets (2.1.12).

isolated point: a point p of topological space such that both p and its complement are closed (1.3.18).

line at infinity: The line at infinity in the projective plane $\mathbb{P}^2$ is the locus $\{x_0 = 0\}$ (1.2.7).

local property: a property that is true in an open neighborhood of any point (5.1.12).

localization: the process of adjoining inverses.

locally closed set: the intersection of a closed set and an open set (5.3).

member: When a set is made up of subsets of another set, we call an element of that set a member to avoid confusion (9.1.12).

module homomorphism: a homomorphism from an R-module M to an R'-module M' is defined in Section (6.2).

morphism: one of the allowed maps between varieties (2.5),(3.5).

nilradical: the radical of the zero ideal (2.4.13).

node: a point at which two branches of a curve met transverslly (1.7.10).

noetherian space: a space that satisfies the descending chain condition on closed sets (2.1.9).

normal domain: an integrally closed domain (4.3).

Nullstellensatz: the theorem that identifies points with maximal ideals (2.3).

ordinary: a plane curve is ordinary if all flexes and bitangents are ordinary, and there are no accidents (1.9.11).

Plücker formulas: the formulas that count flexes, bitangents, nodes and cusps of an ordinary curve (1.10).

polynomial algebra: an algebra that is isomorphic to an algebra of polynomials.

quadric: the locus of an irreducible homogeneous quadratic equation in projective space (3.1.6).

quasicompact: A topological space is quasicompact if every open covering has a finite subcovering.

radical of an ideal: the radical of an ideal I is the set of elements such that some power is in I (2.1.20).

reducible curve: a union of finitely many irreducible curves.

resultant: a polynomial in the coefficients of two polynomials f and g that vanishes if and only if they have a common root (1.6).

scalar: a complex number.

scaling: adjusting by scalar factors.

Segre embedding. a map that embeds a product of projective varieties into projective space (3.1.9).

Serre dual. The Serre dual $\mathcal{M}^S$ of a locally free $\mathcal{O}$-module $\mathcal{M}$ is the module $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M}, \Omega)$ (8.7.1).

smooth point, singular point: a point p of a plane curve $\{f = 0\}$ is a smooth point if at least one partial derivative is nonzero at p. Otherwise it is a singular point (1.4.4).

Special line: A line L through a singular point of a curve whose intersection multiplicity with C is greater than the multiplicity of C. (1.7.6).

spectrum: the spectrum of a finite-tpe domain is the set of its maximal ideals (2.4).

structure sheaf: its sections on an open set are the regular functions on that set (6.1).

tensor algebra: the graded algebra T such that T^n is the nth tensor power of a vector space V.

torsion: an element m of am R-module is a torsion element if there is a nonzero element r in R such that $rm = 0$ (9.1.4).

transcendence basis: a maximal algebraically independent set of elements (9.3).

transcendence degree: the number of elements in a transcendnce basis (9.3).

transversal intersection: two curves intersect transvesally at a point p if they are smooth at p and their tangent lines there are distinct (1.8.12).

trigonal curve: a curve that can be represented as a overing of $\mathbb{P}^1$ of degree 3 (8.8.28).

twisted cubic: the locus of points $(x_0^3, x_0^2x_1, x_1^2x_2, x_1^2)$ in $\mathbb{P}^3$ (3.1.15).

unit ideal: the unit ideal of a ring R is R.

valuation: a surjective homomorphism from the multiplicative group $K^\times$ of nonzero elements of a field K to the additive group $\mathbb{Z}^+$ of integers (5.1.3).

valuation ring: the set of elements with value greater than zero, together with the identity element.

variety: an irreducible subspace of affine or projective space (2.1.17),(3),(3.2.11).

Veronese embedding: the embeding of a projective space using the monomials of given degree (3.1.13).

weight: a variable may be assigned an integer called a *weight* (4.7.7).

weighted projective space: projective space when the variables have weights (4.7.7).

Zariski topology: the closed sets are the zero sets of families of polynomial equations (2.1).

INDEX OF NOTATION

$\mathbb{A}^n$ affine space (1.1).
(affines) the category of affine open sets, morphisms being localizations (6.1).
ann annihilator (6.6).
C^* the dual of the curve C (1.5.7).
$\mathbf{C}^q$ the cohomology of a complex (7.2).
$\mathrm{Discr}(F)$ the discriminant of f (1.6.14).
Δ the diagonal, or the branch locus (3.5.19), (4.7.1).
e the Euler characteristic, or the ramification index (1.7.22),(7.7.7),(8.5).
g often, the genus of a curve. (1.7.20)
H ; the Hessian matrix of second partial derivatives. (1.4.10)
H_p the evaluation of the Hessian matrix H at the point p.
H^q cohomology (7.1).
$\mathbf{h}^q$ the dimension of H^q.
$K^\times$ the multiplicative group of nonzero elements of the field K.
$k(p)$ the residue field at a point (2.2.1),(2.4).
L^* the point of $\mathbb{P}^*$ that corresponds to the line L in the plane $\mathbb{P}$ (1.5.1).
$\mathcal{M}$ an $\mathcal{O}$-module (6.2.1).
$\mathcal{M}^S$ the Serre dual of $\mathcal{M}$ (8.7.1).
$\mathfrak{m}$ a maximal ideal (2.2.1),(2.4).
$\mathcal{O}$ the structure sheaf on a variety (6.1.1).
${}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$, ${}_X(\mathcal{M},\mathcal{N})$ abbreviated notations for the $\mathcal{O}_X$-module of homomorphisms $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$.
(opens) the category whose objects are open subsets (6.1).
p^* the line in the dual plane $\mathbb{P}^*$ that corresponds to the point p of the plane $\mathbb{P}$ (1.5.1).
p_a the arithmetic genus (8.7.4),(8.8.2).
$\mathbb{P}$, $\mathbb{P}^n$ projective space (1.2).
$\mathbb{P}^*$ the dual of the plane $\mathbb{P}$ (1.5.1).
π_p the homomorphism to the residue field $k(p)$ (2.2.1),(2.4).
$\mathrm{rad}\, I$ the radical of the ideal I (2.1.20).
$\mathrm{Res}(f,g)$ the resultant of f and g (1.6).
$\mathrm{Spec}\, A$ the set of maximal ideals of a finite-type algebra A (2.4).
$\mathbb{U}^i$ a standard affine open subset $\{x_i \neq 0\}$ of projective space (1.2.7).
$V(f)$ the locus of zeros of f (2.1),(3.2.4).
$\bigwedge V$ the exterior algebra (3.7.2).
∇ the gradient vector of partial derivatives (1.4.10.
∇_p the evaluation of ∇ at the point p.
$\approx$ an isomorphism.
$\otimes$ tensor product (9.1.26).
$\bigcap$ intersection.
$<$ When S, T are sets, $S < T$ means that the set S is a subset of T and is not equal to T (9.1.12).
$A^\#$ denotes the normalization of the algebra A, and $X^\#$ denotes the normalization of the variety X.
[ ] For clarity, square brackets are sometimes used in place of parentheses (2.5).

BIBLIOGRAPHY

Two classic works:

Fulton, W., *Algebraic Curves*, 2008,
available on the web www.math.lsa.umich.edu/ wfulton/CurveBook.pdf

Mumford, D. *Algebraic Geometry I, Complex Projectie Varieties* Grundlehren, Springer, 1976.

An introduction based on many specific examples:

Harris, J. *Algebraic Geometry, A First Course* Grad. Texts in Math. Springer, 1992.

Books for a more formal introduction:

Cutkosky, D. *Introduction to Algebraic Geometry* Grad. Studies in Math., AMS, 2018.

Hartshorne, R. *Algebraic Geometry* Grad. Texts in Math. Springer, 1977.

Books with an analytic point of view:

Griffiths, P., Harris, J. *Principles of Algebraic Geometry* Wiley, 2014.

Miranda, R. *Algebraic Curves and Riemann Surfaces* Grad. Studies in Math., AMS, 1979.

A computational approach to algebraic geometry:

Cox, D., Little, J., O'Shea, D. *Using Algebraic Geometry* 2nd ed Grad. Texts in Math., Springer, 2004.

Books for the background on algebra:

Cohn, P. M. *Algebra* vol, 2 Wiley, 1977.

Lang, S. *Algebra* 3rd ed Grad. Texts in Math., Springer, 2002.

Books for further reading:

Beauville, A. *Complex algebraic surfaces* London Mathmatical Society Cambridge, 1996.

Serre, J.-P. *Algebraic Groups and Class Fields* Springer, 1997.

Mumford, D., Fogarty, J. C. *Geometric Invariant Thery* Springer, 1982.

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