Chapter 1 Exercises

1.0.1. Let \( f(x, y, z) \) be an irreducible homogeneous polynomial of degree greater than one. Prove that the locus \( f = 0 \) in \( \mathbb{P}^2 \) contains three points that do not lie on a line.

1.0.2. Prove that a plane curve contains infinitely many points.

1.0.3. (a) Classify conics in \( \mathbb{P}^2 \) by writing an irreducible quadratic polynomial in three variables in the form \( X^tAX \) where \( A \) is symmetric, and diagonalizing the quadratic form.
(b) Quadrics in \( \mathbb{P}^3 \) are zero sets of irreducible homogeneous quadratic polynomials in four variables. Classify quadrics in \( \mathbb{P}^3 \).

1.0.4. Prove that the path \( x(t) = t, y(t) = \sin t \) doesn’t lie on any plane algebraic curve in \( \mathbb{A}^2 \).

1.0.5. By counting constants, prove that most nonhomogeneous polynomials in two or more variables are irreducible.

1.0.6. Prove that the elementary symmetric functions \( s_1 = x_1 + \cdots + x_n, \ldots, s_n = x_1 \cdots x_n \) are algebraically independent.

1.0.7. Let \( L \supset K \supset F \) be fields, and let \( \text{tr}(K/F) \) denote the transcendence degree of a field extension \( K/F \). Prove that \( \text{tr}(L/F) = \text{tr}(L/K) + \text{tr}(L/F) \).

1.0.8. Let \( C \) be the plane curve defined by a homogeneous polynomial \( f(x_0, x_1, x_2) \) of degree \( d \). Use the following method to prove that the image of the set of smooth points in the dual plane is contained in a curve \( C^* \): Let \( N_r(k) \) be the dimension of the space of polynomials of degree \( \leq k \) in \( r \) variables. Determine \( N_r(k) \) for \( r = 3 \) and 4. Show that \( N_3(k) > N_3(kd) \) if \( k \) is large enough. Conclude that there has to be a polynomial \( G \) that maps to zero by the substitution of \( f_i \) for \( x_i \).

1.0.9. Compute the resultant of the polynomials \( x^m \) and \( x^n - 1 \).

1.0.10. Let \( C \) be a cubic curve with a node. Determine the degree of \( C^* \), and the numbers of flexes, bitangents, nodes, and cusps of \( C \) and of \( C^* \).

1.0.11. (i) Let \( f(t, y) = ty^2 - 4y + t \). Solve \( f = 0 \) for \( y \) by the quadratic formula, and sketch the real locus \( f = 0 \) in the \( t, y \) plane.
(ii) What does Hensel’s Lemma say tell us? Factor \( d \), modulo \( t^4 \).

1.0.12. Factor \( f(t, x) = x^3 + 2tx^2 + t^2x + x + t \), modulo \( t^2 \).

1.0.13. Prove that all affine conics can be put into one of the forms ?? by linear change of variable, translation, and scalar multiplication.

1.0.14. Let \( f \) and \( g \) be irreducible homogeneous polynomials in \( x, y, z \). Prove that if the loci \( \{ f = 0 \} \) and \( \{ g = 0 \} \) are equal, then \( g = cf \).

1.0.15. Compute \( \prod_{i \neq j} (\zeta^i - \zeta^j) \) when \( \zeta = e^{2\pi i/n} \).

1.0.16. Let \( F(x) = \prod (x - \alpha_i) \). Then \( \text{Discr}(F) = \pm \prod_{i < j} (\alpha_i - \alpha_j)^2 \). Determine the sign.

1.0.17. Let \( C \) be a smooth cubic curve in \( \mathbb{P}^2 \), and let \( p \) be a flex point of \( C \). Choose coordinates so that \( p \) is the point \( (0, 1, 0) \) and the tangent line to \( C \) at \( p \) is the line \( \{ z = 0 \} \).
(a) Show that the coefficients of \( x^2y, xy^2, \) and \( y^3 \) in the defining polynomial \( f \) of \( C \) are zero.
(b) Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form \( f = y^2 z + x^3 + a x z^2 + b z^4 \), and that \( x^3 + a z + b \) will be a polynomial with distinct roots.

(c) Show that one of the coefficients \( a \) or \( b \) can be eliminated, and therefore that smooth cubic curves depend on just one parameter.

1.0.18. Let \( p(t, x) = x^3 + x^2 + t \). Then \( p(0, x) = x^2(x + 1) \). Since \( x^2 \) and \( x + 1 \) are relatively prime, Hensel’s Lemma predicts that \( p = f g \), where \( g \) and \( q \) are polynomials in \( x \) whose coefficients are analytic functions in \( t \), and \( f \) is monic, \( f(0, x) = x^2 \), and \( g(0, x) = x + 1 \). Determine this factorization up to degree 3 in \( t \). Do the same for the polynomial \( t x^4 + x^3 + x^2 + t \).

1.0.19. Let \( f \), \( g \), and \( h \) be polynomials. Prove that

(i) \( \text{Res}(f, gh) = \text{Res}(f, g) \text{Res}(f, h) \).

(ii) If the degree of \( gh \) is less than or equal to the degree of \( f \), then \( \text{Res}(f, g) = \text{Res}(f + gh, g) \).

1.0.20. Let \( f = a_0 x^m + a_1 x^{m-1} + \cdots + a_m \) and \( g = b_0 x^n + b_1 x^{n-1} + \cdots + b_n \), and let \( R = \text{Res}(f, g) \) be the resultant of these polynomials. Prove that \( R \) is a polynomial that is homogeneous in each of the sets of variables \( a \) and \( b \), and determine the degrees. Prove also that, if one assigns weighted degree \( i \) to the coefficients \( a_i \) and \( b_i \), then \( R \) is homogeneous, of weighted degree \( mn \).

1.0.21. Prove that a generic curve is ordinary.

1.0.22. Let coordinates in \( \mathbb{A}^4 \) be \( x, y, z, w \), let \( Y \) be the variety defined by \( z^2 = x^2 - y^2 \), \( w(z - x) = 1 \), and let \( \pi \) denote projection from \( Y \) to \( (x, y) \)-space. Describe the fibres and the image of \( \pi \).

1.0.23. Prove that a plane curve \( X \) of degree 4 can have at most three singular points by showing that there is a conic \( C \) that passes through any five points of \( X \).

1.0.24. By parametrizing a conic \( C \), show that \( C \) meets a plane curve \( X \) of degree \( d \) and distinct from \( C \) in \( 2d \) points, when counted with multiplicity.

1.0.25. Prove that a smooth point of a curve is a flex point if and only if the Hessian determinant is zero, in the following way: Given a smooth point \( p \) of \( X \), choose coordinates so that \( p = (0, 0, 1) \) and the tangent line \( \ell \) is the line \( \{ x_1 = 0 \} \). Then compute the Hessian.

1.0.26. Consider a change of coordinates of the form \((x, y)t = P(x', y')t\), where \( P \) is an invertible \( 2 \times 2 \) matrix. Let \( f' \) denote the homogeneous polynomial in \((x', y')t\) obtained by substitution into \( f(x, y) \). Determine \( \text{Res}(f', g') \) in terms of \( \text{Res}(f, g) \). Do the same for \( \text{Discr}(f') \).

1.0.27. Let \( p \) be a cusp of the curve \( C \) defined by a homogeneous polynomial \( f \). Prove that there is just one line \( L \) through \( p \) such that the restriction of \( f \) to \( L \) has as zero of order \( > 2 \) at \( p \), and that the order of zero for that line is precisely 3.

1.0.28. Let \( f = x^3 + x z + y z \) and \( g = x^2 + y^2 \). Compute the resultant \( \text{Res}_{x}(f, g) \) with respect to the variable \( x \).

1.0.29. Let \( C \) be a smooth cubic curve in the plane \( \mathbb{P}^2 \), and let \( q \) be a generic point of \( \mathbb{P}^2 \). How many lines through \( q \) are tangent lines to \( C \)?

1.0.30. Let \( f(x, y) = a_0(x) y^n + \cdots + a_0(x) \) be an irreducible homogeneous polynomial, where \( a_i \) are polynomials in \( x \). Prove that for most values \( x = x^0 \), the polynomial \( f(x^0, y) \) has distinct roots.

1.0.31. Let \( f(x, y) = y^n + a_1(x) y^{n-1} + \cdots + a_n(x) \) be an irreducible monic polynomial. Show that the roots of the one-variable polynomial \( f(x_0, y) \) remain bounded as \( x_0 \) tends to 0.

1.0.32. Let \( C \) be the plane projective curve defined by the equation \( x_0 x_1 + x_1 x_2 + x_2 x_0 = 0 \), and let \( p \) be the point \((-1, 2, 2)\). What is the equation of the tangent line to \( C \) at \( p \)?
Chapter 2 Exercises

2.0.1. Prove that if \( f(x_0, x_1, x_2) \) is an irreducible homogeneous polynomial, not \( x_0 \), then its dehomogenization \( f(1, x_1, x_2) \) is also irreducible.

2.0.2. Prove that if a noetherian ring contains just one prime ideal, then that ideal is nilpotent.

2.0.3. Describe all prime ideals of the two-variable polynomial ring \( \mathbb{C}[x, y] \).

2.0.4. Derive version 1 of the Nullstellensatz from the Strong Nullstellensatz.

2.0.5. Let \( B \) be a finite type domain, and let \( p \) and \( q \) be points of the affine variety \( Y = \text{Spec} \ B \). Let \( A \) be the set of elements \( f \in B \) such that \( f(p) = f(q) \). Prove
(a) \( A \) is a finite type domain.
(b) \( B \) is a finite \( A \)-module.
(c) Let \( \varphi : \text{Spec} \ B \rightarrow \text{Spec} \ A \) be the morphism obtained from the inclusion \( A \subseteq B \). Show that \( \varphi(p) = \varphi(q) \), and that \( \varphi \) is bijective everywhere else.

2.0.6. The cyclic group \( G = \langle \sigma \rangle \) of order \( n \) operates on the polynomial algebra \( A = \mathbb{C}[x, y] \) by \( \sigma(x) = \zeta x \) and \( \sigma(y) = \zeta y \), where \( \zeta = e^{2\pi i/n} \).
(a) Describe the invariant ring \( A^G \) by exhibiting generators and defining relations.
(b) Prove that the inverse of \( \sigma \) is bijective everywhere else.
(c) Prove directly that the morphism \( \text{Spec} \ A = \mathbb{A}^2 \rightarrow \text{Spec} \ B \) defined by the inclusion \( B \subseteq A \) is surjective, and that its fibres are the \( G \)-orbits.

2.0.7. Let \( I_1, \ldots, I_k \) and \( J \) ideals of a finite-type domain, such that \( J \not\subseteq I_j \) if \( j \). Prove that there is an element \( x \in J \) that isn’t contained in \( I_j \) for any \( j \).

2.0.8. The equation \( y^2 = x^3 \) defines a plane curve \( X \) with a cusp at the origin, the spectrum of the algebra \( A = \mathbb{C}[x, y]/(y^2 - x^3) \). There is a homomorphism \( A \rightarrow \mathbb{C}[t] \), with \( \varphi(x) = t^2 \) and \( \varphi(y) = t^3 \), and the associated morphism \( \mathbb{A}^1 \rightarrow X \) sends a point \( t \) of \( \mathbb{A}^1 \) to the point \( (x, y) = (t^2, t^3) \) of \( X \). Prove that \( u \) is a homeomorphism the Zariski topology and also in the classical topology.

2.0.9. Prove that, if an algebra \( A \) is a complex vector space of dimension \( d \), it contains at most \( d \) maximal ideals.

2.0.10. Let \( T \) denote the ring \( \mathbb{C}[\epsilon] \), with \( \epsilon^2 = 0 \). If \( A \) is the coordinate ring of an affine variety \( X \), a tangent vector to \( X \) is, by definition, given by an algebra homomorphism \( \varphi : A \rightarrow T \).
(a) Show that such a homomorphism can be written in the form \( \varphi(a) = f(a) + d(a) \epsilon \), where \( f \) and \( d \) are functions \( A \rightarrow \mathbb{C} \). Show that \( f \) is an algebra homomorphism, and that \( d \) is an \( f \)-derivation, a linear map that satisfies the identity \( d(ab) = f(a)d(b) + d(a)f(b) \).
(b) Show that, when \( A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \) the tangent vectors are defined by the equations \( \nabla_f(p)x = 0 \).

2.0.11. The homomorphism \( \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, z] \) by \( y \mapsto xz \) defines a morphism \( \mathbb{A}^2 \rightarrow \mathbb{A}^2 \). Describe the fibres of \( \pi \). (The inverse of \( \pi \) is called a blowing up, though it isn’t defined everywhere.)

2.0.12. Let \( A \) be a noetherian ring. Prove that a radical ideal \( I \) of \( A \) is the intersection of finitely many prime ideals.

2.0.13. A minimal prime ideal is an ideal that doesn’t properly contain any other prime ideal. Prove that a nonzero, finite-type algebra \( A \) (not necessarily a domain) contains at least one and only finitely many minimal prime ideals. Try to find a proof that doesn’t require much work.
2.0.14. Explain what a morphism $\text{Spec } B \to \text{Spec } A$ means in terms of polynomials, when $A = \mathbb{C}[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ and $B = \mathbb{C}[y_1, \ldots, y_n]/(g_1, \ldots, g_k)$.

2.0.15. Let $A = \mathbb{C}[x_1, \ldots, x_2]$, and let $B = A[\alpha]$, where $\alpha$ is an element of the fraction field $\mathbb{C}(x)$ of $A$. Describe the fibres of the morphism $Y = \text{Spec } B \to \text{Spec } A = X$.

2.0.16. Let $X$ be the plane curve $y^2 = x(x - 1)^2$, let $A = \mathbb{C}[x, y]/(y^2 - x(x - 1)^2)$ be its coordinate algebra, and let $x, y$ denote the residues of those elements in $A$ too.
(a) Points of the curve can be parametrized by a variable $t$. Use the lines $y = t(x - 1)$ to determine such a parametrization.
(b) Let $B = \mathbb{C}[t]$ and let $T$ be the affine line $\text{Spec } \mathbb{C}[t]$. The parametrization gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.

2.0.17. Let coordinates in $\mathbb{A}^4$ be $x, y, z, w$, let $Y$ be the variety defined by $z^2 = x^2 - y^2$, $w(z - x) = 1$, and let $\pi$ denote projection to $(x, y)$-space. Describe the fibres and the image of $\pi$.

2.0.18. Classify algebras that are complex vector spaces of dimensions two and three.

2.0.19. Prove that, in the ring $\mathbb{C}[x_1, \ldots, x_n]$ of formal power series, an element whose constant term is nonzero is invertible.

2.0.20. Show that the algebra $A = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is isomorphic to the Laurent Polynomial Ring $\mathbb{C}[t, t^{-1}]$, but that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not isomorphic to $\mathbb{R}[t, t^{-1}]$.

2.0.21. Prove that there are varieties in the affine plane $\mathbb{A}^2$ are points, curves, and the affine plane $\mathbb{A}^2$ itself.

2.0.22. Find generators for the ideal of $\mathbb{C}[x, y]$ that vanish on the three points $(0, 0), (0, 1), (1, 0)$.

2.0.23. Let $C$ and $D$ be closed subsets of an affine variety $X = \text{Spec } A$. Suppose that no component of $D$ is contained in $C$. Prove that there is a regular function $f$ that vanishes on $C$ and isn't identically zero on any component of $D$.

2.0.24. Let $K$ be a field and let $R = K[x_1, \ldots, x_n]$ with $n > 0$. Prove that the field of fractions of $R$ is not a finitely generated $K$-algebra.

2.0.25. Let $A = \mathbb{C}[u, v]/(v^2 - u(1 - u))$ and $B = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$, and let $X = \text{Spec } A, Y = \text{Spec } B$. Show that the substitution $u = x^2, v = xy$ defines a morphism $Y \to X$.

2.0.26. Let $X$ be the affine line $\text{Spec } \mathbb{C}[x]$. Considering $P = \text{Spec } \mathbb{C}[x_1, x_2]$ as the product $X \times X$, determine all morphisms $P \to X$ that define group laws on $X$. 

4
Chapter 3  Exercises

3.0.1. Prove that every finite subset $S$ of a projective variety $X$ is contained in an affine open subset.

3.0.2. A pair $f_0, f_1$ of homogeneous polynomials in $x_0, x_1$ of the same degree $d$ can be used to define a morphism $\mathbb{P}^1 \to \mathbb{P}^1$. At a point $q$, the morphism evaluates $(1, f_1/f_0)$ or $(f_0/f_1, 1)$ at $q$.
(a) The degree of such a morphism is the number of points in a generic fibre. Determine the degree.
(b) Describe the group of automorphisms of $\mathbb{P}^1$.

3.0.3. (a) What are the conditions that a triple of $f = (f_0, f_1, f_2)$ homogeneous polynomials in $x_0, x_1, x_2$ of the same degree $d$ must satisfy in order to define a morphism $\mathbb{P}^2 \to \mathbb{P}^2$?
(b) If $f$ does define a morphism, what is its degree?

3.0.4. Prove that relatively prime polynomials in $F, G$ two variables $x, y$, not necessarily homogeneous, have finitely many common zeros in $\mathbb{A}^2$.

3.0.5. Let $C$ be the projective plane curve $x^3 - y^2z = 0$.
(a) Show that the function field $K$ of $C$ is the field $\mathbb{C}(t)$ of rational functions in $t = y/x$.
(b) Show that the point $(t^2 - 1, t^3 - 1)$ of $\mathbb{P}^1$ with values in $K$ defines a morphism $C \to \mathbb{P}^1$.

3.0.6. Let $Y$ and $Z$ be the zero sets in $\mathbb{P}$ of relatively prime homogeneous polynomials $g$ and $h$ of the same degree $r$. Prove that the rational function $\alpha = g/h$ will tend to infinity as one approaches a point of $Z$ that isn’t also a point of $Y$ and that, at intersections of $Y$ and $Z$, $\alpha$ is indeterminate in the sense that the limit depends on the path.

3.0.7. Describe the ideals that define closed subsets of $\mathbb{A}^m \times \mathbb{P}^n$.

3.0.8. Let $V$ be a vector space of dimension 5, let $G$ denote the Grassmanian $G(2, 5)$ of lines in $\mathbb{P}^4$, let $W = \bigwedge^2 V$, and let $D$ denote the subset of decomposable vectors in the projective space $\mathbb{P}(W)$ of one-dimensional subspaces of $W$. Prove that there is a bijective correspondence between two-dimensional subspaces $U$ of $V$ and the points of $D$, and that a vector $w$ in $\bigwedge^2 V$ is decomposable if and only if $ww = 0$. Exhibit defining equations for $G$ in the space $\mathbb{P}(W)$.

3.0.9. Let $\mathbb{P} = \mathbb{P}^3$. The space of planes in $\mathbb{P}$ is the dual projective space $\mathbb{P}^*$. The variety $F$ that parametrizes triples $(p, l, H)$ consisting of a point $p$, a line $l$, and a plane $H$ in $\mathbb{P}$, with $p \in l \subset H$, is called a flag variety. Exhibit defining equations for $F$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^*$. The equations should be homogeneous in each of 3 sets of variables.

3.0.10. Describe all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$.

3.0.11. Let $f$ a homogeneous polynomial in $x, y, z$, not divisible by $z$. Prove that $f$ is irreducible if and only if $f(x, y, 1)$ is irreducible.

3.0.12. (blowing up a point in $\mathbb{P}^2$) Consider the Veronese embedding of $\mathbb{P}^2_{yz} \to \mathbb{P}^5_{x^2}$ by monomials of degree 2 defined by $(u_0, u_1, u_2, u_3, u_4, u_5) = (s^2, y^2, x^2, yz, xz, xy)$. If we drop the coordinate $u_0$, we obtain a map $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^4$. $\varphi(x, y, z) = (y^2, x^2, yz, xz, xy)$ that is defined at all points except the point $q = (0, 0, 1)$. Find defining equations for the closure of the image $X$. Prove that the inverse map $X \xrightarrow{\varphi^{-1}} \mathbb{P}^2$ is everywhere defined, and that the fibre of $\varphi^{-1}$ over $q$ is a projective line.

3.0.13. Let $K = \mathbb{C}(x)$ and $L = \mathbb{C}(y)$ be the fields of rational functions in one variable. Compare the tensor product $R = K \otimes_{\mathbb{C}} L$ with the field of fractions $\mathbb{C}(x, y)$ of $\mathbb{C}[x, y]$. 
3.0.14. Let $U$ be a nonempty open subset of $\mathbb{P}^n$. Prove that if a rational function is bounded on $U$, it is a constant.

3.0.15. Let $Y$ be the cusp curve $\text{Spec} B$, where $B = \mathbb{C}[x, y]/(y^2 - x^3)$. This algebra embeds as subring into $\mathbb{C}[t]$, by $x = t^2$. $y = t^3$. Show that the two vectors $v_0 = (x-1, y-1)$ and $v_1 = (t+1, t^2 + t + 1)$ define the same point of $\mathbb{P}^1$ with values in the fraction field $K$ of $B$, and that they define morphisms from $Y$ to $\mathbb{P}^1$ wherever the entries are regular functions on $Y$. Prove that the two morphisms they define piece together to give a morphism $Y \to \mathbb{P}^1$.

3.0.16. Let $\mathcal{P}$ be a homogeneous ideal in $\mathbb{C}[x_0, \ldots, x_n]$ whose dehomogenization $P$ is a prime ideal. Is $\mathcal{P}$ a prime ideal?

3.0.17. Let $f = x_0^2 - x_1 x_2$. Determine generators and defining relations for the ring $\mathcal{R}_f$ of homogeneous fractions of degree zero whose denominator is a power of $f$.

3.0.18. Let $f$ be an irreducible polynomial in $\mathbb{C}[x_1, \ldots, x_n]$, and let $A$ finite-type domain. Prove that $f$ an irreducible element of $A[x]$.

3.0.19. Let $f$ and $g$ be irreducible homogeneous polynomials in $x, y, z$. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ are equal, then $g$ is a constant multiple of $f$.

3.0.20. Let $C$ be the curve defined by a homogeneous polynomial $f$ of degree $d$. To prove that the images in the dual plane of the smooth points of $C$ lie on a curve $C^*$, we used transcendence degree to conclude that there is a polynomial $G(t, s_0, s_1, s_2)$ such that $G(f, f_0, f_1, f_2)$ is identically zero. Use the following method to give an alternate proof: Determine the dimensions $N_r(k)$ of the spaces of polynomials of degree $\leq k$ in $r$ variables, for $r = 3$ and $r = 4$. Show that $N_r(k) > N_3(kd)$ if $k$ is large enough. Use counting constants to show that there has to be a polynomial $G$ that maps to zero by the substitution.

Note: This method doesn’t give a good bound for the degree of $C^*$. One reason may be that $f$ and its derivatives are related by Euler’s Formula. It is tempting try using Euler’s Formula to help compute the equation of $C^*$, but I haven’t succeeded in getting anywhere that way. If you have an idea, please let me know.

3.0.21. Show that the conic $C$ in $\mathbb{P}^2$ defined by the polynomial $y_0^2 + y_1^2 + y_2^2 = 0$ and the twisted cubic $V$ in $\mathbb{P}^3$, the zero locus of the polynomials $v_0 v_2 - v_1^2, v_0 v_3 - v_1 v_2, v_1 v_3 - v_2^2$ are isomorphic by exhibiting inverse morphisms between them.

3.0.22. Let $C$ be a cubic curve, the locus of a homogeneous cubic polynomial $f(x, y, z)$ in $\mathbb{P}^2$. Suppose that $(001)$ and $(010)$ are flex points of $C$, that the tangent line to $C$ at $(001)$ is the line $\{y = 0\}$, and the tangent line at $(010)$ is the line $\{z = 0\}$. What are the possible polynomials $f$? Disregard the question of whether $f$ is irreducible.

3.0.23. How many real flex points can a real cubic curve have?

3.0.24. Let $X$ be the affine plane with coordinates $(x, y)$. Given a pair of polynomials $u(x, y), v(x, y)$ in $x, y$, one may try to define a morphism $f : X \to \mathbb{P}^1$ by $f(x, y) = (u, v)$. Under what circumstances is $f$ a morphism?

3.0.25. Let $X$ be the affine surface in $\mathbb{A}^3$ defined by the equation $x^3 + x_1 x_2 x_3 + x_1 x_3 + x_2^2 + x_3 = 0$, and let $\overline{X}$ be its closure in $\mathbb{P}^3$. Describe the intersection of $\overline{X}$ with the plane at infinity in $\mathbb{P}^3$.

3.0.26. With coordinates $x_0, x_1, x_2$ in the plane $\mathbb{P}$ and $s_0, s_1, s_2$ in the dual plane $\mathbb{P}^*$, let $C$ be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where $f$ is an irreducible homogeneous polynomial in $x$. Let $\Gamma$ be the locus of pairs $(x, s)$ of $\mathbb{P} \times \mathbb{P}^*$ such that the line $s_0 x_0 + s_1 x_1 + s_2 x_2 = 0$ is the tangent line to $C$ at $x$. Prove that $\Gamma$ is a Zariski closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

3.0.27. Let $x_0, x_1, x_2$ be the coordinate variables in the projective plane $X$. The function field $K$ of $X$ is the field of rational functions in the variables $u_1, u_2, u_3 = x_3/x_0$. Let $f(u_1, u_2)$ and $g(u_1, u_2)$ be polynomials. Under what circumstances does the point $(1, f, g)$ with values in $K$ define a morphism $X \to \mathbb{P}^2$?
Chapter 4  Exercises

4.0.1. Prove that the Fermat curve \( C : \{x^d + y^d + z^d = 0\} \) is connected by studying its projection to \( \mathbb{P}^1 \) from the point \((0, 0, 1)\).

4.0.2. Prove that \( \mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0 \) is a maximal chain of closed subsets of \( \mathbb{P}^n \).

4.0.3. Let \( A \subset B \) be noetherian domains and suppose that \( B \) is a finite \( A \)-module. Prove that \( A \) is a field if and only if \( B \) is a field.

4.0.4. Use Noether Normalization to prove this alternate form of the Nullstellensatz: Let \( k \) be a field, and let \( B \) be a domain that is a finitely generated \( k \)-algebra. If \( B \) is a field, then \([B : k] < \infty\).

4.0.5. Let \( A \) be a domain with fraction field \( K \), and let \( \alpha \) and \( \beta \) be elements of \( K \) such that \( \alpha \beta = 1 \). Prove that if \( \alpha \) is integral over \( A[\beta] \), then it is an element of \( A[\beta] \), and it is integral over \( A \).

4.0.6. Prove every nonconstant morphism \( \mathbb{P}^2 \to \mathbb{P}^2 \) is a finite morphism.

4.0.7. A ring \( A \) is said to have the descending chain condition (dcc) if every strictly decreasing chain of ideals is finite. Let \( A \) be a finite type \( \mathbb{C} \)-algebra. Prove:
(a) \( A \) has dcc if and only if it is a finite dimensional complex vector space.
(b) If \( A \) has dcc, then it has finitely many maximal ideals, and every prime ideal is maximal.
(c) If a finite-type algebra \( A \) has finitely many maximal ideals, then \( A \) has dcc.
(d) (Strong Nakayama) Suppose that \( A \) has dcc, let \( M \) be an arbitrary \( A \)-module, and let \( I \) denote the intersection of the maximal ideals of \( A \). If \( IM = M \), then \( M = 0 \). (The usual Nakayama lemma requires that \( M \) be finitely generated.)

4.0.8. Let \( A \subset B \) be finite type domains with fraction fields \( K \subset L \), and let \( Y \to X \) be the corresponding morphism of affine varieties. Prove the following:
(a) There is a nonzero element \( s \in A \) such that \( A_s \) is integrally closed.
(b) There is a nonzero element \( s \in A \) such that \( B_s \) is a finite module over a polynomial ring \( A_s[y_1, \ldots, y_d] \).
(c) Suppose that \( L \) is a finite extension of \( K \) of degree \( d \). There is a nonzero element \( s \in A \) such that all fibres of the morphism \( Y \to X \) consist of \( d \) points.

4.0.9. Let \( A \) be a finite type domain, \( R = \mathbb{C}[t] \), \( X = \text{Spec} A \), and \( Y = \text{Spec} R \). Let \( \varphi : A \to R \) be a homomorphism whose image is not \( \mathbb{C} \), and let \( \pi : Y \to X \) be the corresponding morphism.
(a) Show that \( R \) is a finite \( A \)-module.
(b) Show that the image of \( \pi \) is a closed subset of \( X \).

4.0.10. A module \( M \) over a ring \( B \) is faithful if, for every nonzero element \( b \) of \( B \), scalar multiplication by \( b \) isn’t the zero operation on \( M \). Let \( A \) be a domain, let \( z \) be an element of its field of fractions, and let \( B \) be the ring generated by \( z \) over \( A \). Suppose there is a faithful \( B \)-module \( M \) that is finitely generated as an \( A \)-module. Prove that \( z \) is integral over \( A \).

4.0.11. Let \( A \subset B \) be an integral extension of finite-type algebras, and suppose that \( A \) and \( B \) are domains with fraction fields \( K \) and \( L \), respectively, and let \( P \) be a prime ideal of \( A \). Prove that the number of prime ideals of \( B \) that lie over \( P \) is at most equal to the degree \([L : K]\) of the field extension.

4.0.12. Let \( Y \) be a projective double plane with branch locus \( \Delta \) in \( X = \mathbb{P}^2 \), let \( C \) be a curve in \( X \), and let \( D \) be a curve in \( Y \) that lies over \( C \). The curve \( C \) will split in the double plane unless \( D \) is symmetric with respect
to the automorphism – unless $D = D\sigma$. Most curves in $Y$ won’t be symmetric, so their images will split. On the other hand, most curves $C$ in $X$ intersect $\Delta$ transversally, and therefore they don’t split. Try to explain how this is possible.

4.0.13. Let $K$ be the field of fractions of a normal finite-type domain $A$, let $L$ be a Galois extension of $K$ with Galois group $G$, and let $B$ be the integral closure of $A$ in $L$, a finite $A$-module. Let $X = \text{Spec } A$ and $Y = \text{Spec } B$. Show that $A$ is the algebra of invariants $B^G$.

4.0.14. Let $Y$ be a closed subvariety of projective space $\mathbb{P}^n$ with coordinates $y_0, ..., y_n$, let $d$ be a positive integer, and let $w_0, ..., w_k$ be homogeneous polynomials in $y$ of degree $d$ that have no common zeros on $Y$. Prove that sending a point $q$ of $Y$ to $(w_0(q), ..., w_k(q))$ defines a finite morphism $Y \rightarrow \mathbb{P}^k$. Consider the case that $w = u_1, ..., u_k$ first.

4.0.15. Let $Y \rightarrow X$ be a finite morphism of curves, and let $K$ and $L$ be the function fields of $X$ and $Y$, respectively, and suppose $[L : K] = n$. Prove that all fibres have order at most $n$, and all but finitely many fibres of $Y$ over $X$ have order equal to $n$.

4.0.16. Let $A$ be a finite type algebra that satisfies the descending chain condition: Every strictly decreasing chain $I_1 > I_2 > \cdots$ of ideals of $A$ is finite. Prove that $A$ has finite dimension as a complex vector space.

4.0.17. Let $\alpha$ be an element of a domain $A$, and let $\beta = \alpha^{-1}$. Prove that if $\beta$ is integral over $A$, then $\beta$ is an element of $A$. 

8
Chapter 5  Exercises

5.0.1. Prove that the ring $k[[x, y]]$ of formal power series with coefficients in a field $k$ is a local ring.

5.0.2. Prove that, if a variety $X$ is covered by countably many constructible sets, a finite number of those sets will cover $X$.

5.0.3. Prove that a variety that is quasiprojective and proper is projective.

5.0.4. Prove that a proper curve is projective.

5.0.5. Let $X$ be the subset obtained by deleting the origin from $k^2$. Prove that there is no injective morphism from an affine variety $Y$ to $k^2$ whose image is $X$.

5.0.6. Show that if $f(x, y)$ is polynomial and if $d$ divides $f_x$ and $f_y$, then $f$ is constant on the locus $d = 0$.

5.0.7. Let $S$ be a multiplicative system in a finite-type domain $R$, and let $A$ and $B$ be finite-type domains that contain $R$ as subring. Let $R', A', B'$ be the rings of $S$-fractions of $R, A, B$, respectively. Prove: (i) If some elements $\alpha_1, ..., \alpha_k$ generate $A$ as $R$-algebra, they also generate $A'$ as $R'$-algebra.

(ii) Let $A' \xrightarrow{\varphi'} B'$ be a homomorphism. For suitable $s$ in $S$, there is a homomorphism $A_s \xrightarrow{\varphi_s} B_s$ whose localization is $\varphi'$. If $\varphi'$ is injective, so is $\varphi_s$. If $\varphi'$ is surjective or bijective, there will be an $s$ such that $\varphi_s$ is surjective or bijective.

(iii) If $A'$ is contained in $B'$ and if $B'$ is a finite $A'$-module, then for suitable $s$ in $S$, $A_s$ is contained in $B_s$, and $B_s$ is a finite $A_s$-module.

5.0.8. Let $Y \xrightarrow{m} X$ be a surjective morphism, and let $K$ and $L$ be the function fields of $X$ and $Y$, respectively. Show that if $\dim Y = \dim X$, there is a nonempty open subset $X'$ of $X$ such that all fibres over points of $X'$ have the same order $n$, and that $n = [L : K]$.

5.0.9. Prove that fibre dimension is a semicontinuous function. I recommend this outline, but you may use any method you like.

(a) We may assume that $Y$ and $X$ are affine, $Y = \text{Spec } B$ and $X = \text{Spec } A$.

(b) The theorem is true when $A \subset B$ and $B$ is an integral extension of a polynomial subring $A[y_1, ..., y_d]$.

(c) The fibre dimension is a constructible function.

(d) The theorem is true when $X$ is a smooth curve.

(e) The theorem is true for all $X$. 
Chapter 6  Exercises

6.0.1. Let $R = \mathbb{C}[x, y]$. Determine the limit of the directed set
\[ R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots. \]

6.0.2. Let $V$ be the complement of a point in projective space $\mathbb{P}^n$. Prove that if $n > 1$, then $\mathcal{O}_p(V) = \mathbb{C}$.

6.0.3. Prove that if a variety is covered by two affine open sets, its cohomology is zero in dimensions greater than one.

6.0.4. Give an example of a finite $\mathcal{O}$-module $\mathcal{M}$ and an open set $U$ such that $\mathcal{M}(U)$ isn’t a finite $\mathcal{O}(U)$-module. Hint: The reason that this might occur is that there might not be rational functions that are regular on $X$, though $\mathcal{M}$ has global sections.

6.0.5. Let $s$ be an element of a domain $A$, and let $M$ be an $A$-module. Prove that the limit of the directed set
\[ M \xrightarrow{s} M \xrightarrow{s} \cdots \] is isomorphic to the localization $M_s$.

6.0.6. Prove that, to define an $\mathcal{O}$-module $\mathcal{M}$ on $\mathbb{P}^1$, it is enough to give modules $M_0$, $M_1$, and $M_{01}$ over the rings $\mathbb{C}[u]$, $\mathbb{C}[u^{-1}]$, and $\mathbb{C}[u, u^{-1}]$, respectively, together with isomorphisms $M_0[u^{-1}] \approx M_{01} \approx M_1[u]$.

6.0.7. Describe the kernel and cokernel of multiplication by a homogeneous polynomial $f$ of degree $d$:
\[ \mathcal{M}(k) \xrightarrow{f} \mathcal{M}(k + d) \]

6.0.8. What are the sections of $\mathcal{O}(nH)$ on an open set $V$?

6.0.9. Prove that a simple module over a finite type $\mathbb{C}$-algebra has dimension $1$.

6.0.10. Let $X = \mathbb{P}^2$. What are the sections of the twisting module $\mathcal{O}_X(n)$ on the open complement of the line $\{x_1 + x_2 = 0\}$?

6.0.11. Let $U$ be the complement of a finite set in $\mathbb{P}^2$. Prove that $H^0(U, \mathcal{O}_U) = \mathbb{C}$.
Chapter 7 Exercises

7.0.1. Prove that $m_1^{e_1} \cdot \cdots \cdot m_k^{e_k}$ is isomorphic to the tensor product $m_1^{\otimes e_1} \otimes \cdots \otimes m_k^{\otimes e_k}$.

7.0.2. The Cousin problem: Let $\{U_i\}$ be an open covering of projective space. Suppose that rational functions $f_i$ are given such that $f_i - f_j$ is a regular function on $U_i \cap U_j$. The problem is to find a rational function $f$ such that $f - f_i$ is a regular function on $U_i$ for every $i$. Analyze this problem making use of the exact sequence $0 \to O \to F \to Q \to 0$, where $F$ is the constant function field module and $Q$ is the quotient $F/O$.

7.0.3. Let $Y \rightarrow \mathbb{P}^d$ be a finite morphism of varieties. Prove that $Y$ is a projective variety. Do this by showing that the global sections of $\mathcal{O}_Y(nH) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(nH)$ define a map to projective space whose image is isomorphic to $Y$.

7.0.4. Prove that cohomology compatible with products
(a) of two modules
(b) of an arbitrary family of modules.

7.0.5. A hexagon in $\mathbb{P}^2$ is any collection $\ell_0, ..., \ell_5$ of six distinct lines. Let $p_i = \ell_i \cap \ell_{i+1}$ for $i = 1, ..., 5$ and $p_0 = \ell_5 \cap \ell_0$. Assuming that these intersections are distinct, they are the vertices of the hexagon. Pascal’s Theorem asserts that if the vertices of a hexagon lie on a conic $C$, then the three points $q_1 = \ell_1 \cap \ell_4$, $q_2 = \ell_2 \cap \ell_5$ and $q_3 = \ell_3 \cap \ell_0$ lie on a line. Let $L$ be the line through $q_1$ and $q_2$. Let $f_1, f_2, g$ be homogeneous cubics whose zero loci are $\ell_1 + \ell_3 + \ell_5$, $\ell_0 + \ell_2 + \ell_4$, and $C + L$, respectively, and let $h = f_1 + g$. Prove Pascal’s Theorem by applying Bézout’s theorem to the divisors of zeros of $f_1, f_2, g$ and $h$.

7.0.6. Let $N$ be a $3 \times 2$ matrix with variable entries $(n_{ij})$, and let $M = (m_1, m_2, m_3)$ be the $1 \times 3$ matrix of $2 \times 2$ minors of $N$:

$$m_1 = n_{21}n_{32} - n_{22}n_{31}, \quad m_2 = -n_{11}n_{32} + n_{12}n_{31}, \quad m_3 = n_{11}n_{22} - n_{12}n_{21}.$$ 

Let $I$ be the ideal of the polynomial ring $P = \mathbb{C}[[n_{ij}]]$ generated by the minors.
(a) Show that the locus $V(I)$ in $\mathbb{A}^6$ is irreducible, and that it has dimension 4.
(b) Assume that the locus $X = V(I)$ in $Y = \text{Spec } P$ has scodimension at least 2. Prove that this sequence is exact:

$$0 \to B^2 \xrightarrow{N} B^3 \xrightarrow{M} B \to B/I \to 0.$$ 

(c) Suppose that the entries of $N$ are homogeneous polynomials in $x_0, x_1, x_2$; that for some integers $d_i$, $i = 1, 2, 3$, the entries in row $i$ have degree $d_i$. Suppose also that the locus $V(I)$ in $\mathbb{P}^2$ has dimension zero. Construct an exact sequence that allows you to bound the number of points of $V(I)$.
(d) Check your answer in a particular case, when $d_i = 1$.

7.0.7. Let $A, B$ be $2 \times 2$ variable matrices, let $P$ be the polynomial ring $\mathbb{C}[a_{ij}, b_{ij}]$, and let $R$ be the algebra $P/(AB = BA)$. Show that $R$ has a resolution as $P$-module of the form $0 \to P^2 \to P^3 \to P \to R \to 0$.
(Hint: Write the equations in terms of $a_{11} - a_{22}$ and $b_{11} - b_{22}$.)

7.0.8. (algebraic version of Bézout’s Theorem) Let $f$ and $g$ be homogeneous polynomials of degrees $m$ and $n$, respectively, in $x, y, z$. The algebra $A = \mathbb{C}[x, y, z]/(f, g)$ inherits a grading by degree: $A = A_0 \oplus A_1 \oplus \cdots$, where $A_n$ is the image of the space of homogeneous polynomials of degree $n$, together with 0. Prove that $\dim A_k = mn$ for all sufficiently large $k$. 
7.0.9. Let \( f(x, y, z) \) and \( g(x, y, z) \) be homogeneous polynomials of degrees \( m \) and \( n \), and with no common factor. Let \( R \) be the polynomial ring \( \mathbb{C}[x, y, z] \), and let \( A = R/(f, g) \). Show that the sequence

\[
0 \to R \xrightarrow{(-g, f)} R^2 \xrightarrow{(f, g)} R \to A \to 0
\]

is exact.

7.0.10. Extend the Bézout theorem to count the intersections of three surfaces in \( \mathbb{P}^3 \). (The problem here comes down to finding a resolution of \( O/(f, g, h)/O \). When you have guessed one, you may be able to use the snake lemma to prove that it is exact.

7.0.11. Let \( f(x_0, x_1, x_2) \) be an irreducible homogeneous polynomial of degree \( 2d \), and let \( Y \) be the projective double plane \( y^2 = f(x_0, x_1, x_2) \). Compute the cohomology \( H^i(Y, \mathcal{O}_Y) \).

7.0.12. Let \( Y \) be an affine variety with integrally closed coordinate ring \( B \). Let \( I \) be an ideal of \( B \) generated by two elements \( u, v \), and let \( X \) be the locus \( V(I) \) in \( Y \). Suppose that \( \dim X \leq \dim Y - 2 \). Use the fact that \( B = \bigcup B_Q \) where \( Q \) ranges over prime ideals of codimension 1 to prove that this sequence is exact:

\[
0 \to B \xrightarrow{(v, -u)} B^2 \xrightarrow{(a, v)} B \to B/I \to 0.
\]

7.0.13. Let \( I \) be the ideal of \( \mathbb{C}[x_0, x_1, x_2, x_3] \) generated by two homogeneous polynomials \( f, g \), of dimensions \( d, e \) respectively, and assume that the locus \( X = V(I) \) in \( \mathbb{P}^3 \) has dimension 1. Let \( O = \mathcal{O}_Y \).

(a) Construct an exact sequence

\[
0 \to O(-d - e) \to O(-d) \oplus O(-e) \to O \to O_X \to 0.
\]

(b) Show that \( X \) is a connected subset of \( \mathbb{P}^3 \) for the Zariski topology, i.e., that it is not the union of two proper disjoint Zariski-closed subsets.

(c) Determine the genus, assuming that \( X \) is a smooth algebraic curve.

7.0.14. A curve \( Y \) in \( \mathbb{P}^3 \) is a complete intersection if the homogeneous prime ideal of \( \mathbb{C}[y_0, y_1, y_2, y_3] \) that defines \( Y \) is generated by two elements, say \( P = (f, g) \). Suppose that this is the case, and that the homogeneous polynomials \( f \) and \( g \) have degrees \( r \) and \( s \), respectively.

(a) Construct an exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-r - s) \to \mathcal{O}_{\mathbb{P}^3}(-r) \oplus \mathcal{O}_{\mathbb{P}^3}(-s) \to \mathcal{O}_{\mathbb{P}^3} \to i_* \mathcal{O}_Y \to 0
\]

(b) Determine the genus of \( Y \).

7.0.15. Let \( Y \) be an affine variety with integrally closed coordinate ring \( B \). Let \( I \) be an ideal of \( B \) generated by two elements \( u, v \), and let \( X \) be the locus \( V(I) \) in \( Y \). Suppose that \( \dim X \leq \dim Y - 2 \). Use the fact that \( B = \bigcup B_Q \) where \( Q \) ranges over prime ideals of codimension 1 to prove that this sequence is exact:

\[
0 \to B \xrightarrow{(v, -u)} B^2 \xrightarrow{(a, v)} B \to B/I \to 0.
\]

7.0.16. Let \( I \) be the ideal of \( \mathbb{C}[x_0, x_1, x_2, x_3] \) generated by two homogeneous polynomials \( f, g \), of degrees \( d, e \) respectively, and assume that the locus \( X = V(I) \) in \( \mathbb{P}^3 \) has dimension 1. Let \( O = \mathcal{O}_Y \).

(a) Construct an exact sequence

\[
0 \to O(-d - e) \to O(-d) \oplus O(-e) \to O \to O_X \to 0.
\]

(b) Show that \( X \) is a connected subset of \( \mathbb{P}^3 \) for the Zariski topology, i.e., that it is not the union of two proper Zariski-closed subsets.

(c) Determine the genus, when \( X \) is a smooth algebraic curve.
7.0.17. Let
\[ N = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix} \]
be a 3 × 2 matrix whose entries are homogeneous polynomials of degree \( d \) in \( R = \mathbb{C}[x_0, x_1, x_2] \), and let \( M = (m_1, m_2, m_3) \) be the 1 × 3 matrix of minors
\[ m_1 = y_{21}y_{32} - y_{22}y_{31}, \quad m_2 = -y_{11}y_{32} + y_{12}y_{31}, \quad m_3 = y_{11}y_{22} - y_{12}y_{21}. \]
Let \( I \) be the ideal of \( R \) generated by the minors \( m_1, m_2, m_3 \).

(a) By counting dimensions, prove that if \( I \) is the unit ideal of \( R \), the sequence
\[ 0 \twoheadrightarrow R \xrightarrow{M} R^3 \xrightarrow{N} R^2 \xrightarrow{} 0 \]
is exact.

(b) Let \( X = \mathbb{P}^2 \), and suppose that the locus \( Y \) of zeros of \( I \) in \( X \) has dimension zero. Prove that the sequence
\[ 0 \twoheadrightarrow R/I \xrightarrow{} R \xrightarrow{M} R^3 \xrightarrow{N} R^2 \xrightarrow{} 0 \]
is exact.

(c) The sequence in (b) corresponds to the following sequence, in which the terms \( R \) are replaced by twisting modules:
\[ 0 \twoheadrightarrow \mathcal{O}_Y \leftarrow \mathcal{O}_X \xrightarrow{M} \mathcal{O}_X(-2d)^3 \xrightarrow{N} \mathcal{O}_X(-3d)^2 \leftarrow 0. \]
Use this sequence to determine \( h^0(Y, \mathcal{O}_Y) \). Check your work by counting points in some example in which \( y_{ij} \) are homogeneous linear polynomials.

7.0.18. Prove that a variety of any dimension contains no isolated point.

7.0.19. Prove that a regular function on a projective variety is constant.

7.0.20. Prove morphism from a curve \( Y \) to \( \mathbb{P}^1 \) is a finite morphism without appealing to Chevalley’s Theorem.

7.0.21. Let \( Y \) be the surface in \( \mathbb{P}^3 \) defined by an irreducible polynomial of degree 5. Determine the dimensions of the cohomology groups \( H^q(Y, \mathcal{O}_Y) \).
Chapter 8  Exercises

8.0.1. Determine the number of points of order 2 on an elliptic curve.

8.0.2. Let $C$ be a smooth plane cubic curve. Show if origin is a flex point the other the flexes of $C$ are the points of order 3. Therefore there are eight points of order 3.

8.0.3. Let $Y$ be a smooth curve of genus $g > 0$, and let $E$ be a divisor of degree $2g - 1$ on $Y$.

(i) Prove that $h^1(O(E)) = g$ and $h^1(O(D)) = 0$, and that if $D$ is the divisor of degree $g$ obtained by subtracting $g - 1$ generic points from $E$, then $h^0(O(D)) = 1$ and $h^1(O(D)) = 0$.

(ii) A basis $(1, y)$ of $h^0(O(D))$ defines a map $Y \to X = \mathbb{P}^1$ of degree $g$, and the direct image of $O_Y$ becomes an $O_X$-module of rank $g$. Show that the direct image $O_Y$ has the form $O_X \oplus \mathcal{L}$ for some locally free module $\mathcal{L}$ of rank $g - 1$.

(iii) Prove that $\mathcal{L} \cong \mathcal{O}_X(-1)^{g-1}$.

8.0.4. (group law on an elliptic curve) Let $o, p$, and $q$ be points of an elliptic curve $Y$. Show that $O_Y(p + q - o)$ has a nonzero global section that is unique up to scalar factor, and that has a unique zero. That zero is defined to be $p \oplus q$. Prove that, with the law of composition $\oplus$, $Y$ becomes a commutative group.

8.0.5. Let $C$ be a smooth curve of genus 1, with a chosen point $o$. The global sections of $O_C(2o)$ define a morphism $\pi$ of degree 2 from $C$ to $\mathbb{P}^1$. Describe all other morphisms $C \to \mathbb{P}^1$ of degree 2 in terms of $\pi$ and the group law.

8.0.6. Let $C$ and $D$ be conics in $\mathbb{P}^2$ that meet in four distinct points, and let $D^*$ be the dual conic of tangent lines to $D$. Let $E$ be the locus of points $(p, \ell^*)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\ell^* \in D^*$ and $p \in \ell$.

(a) Prove that $E$ is a smooth elliptic curve.

(b) Show that, for most $p \in C$, there will be two tangent lines $\ell$ to $D$ such that $(p, \ell^*)$ is in $E$, and that, for most $\ell^* \in D^*$, there will be two points $p$ such that $(p, \ell^*)$ is in $E$. Identify the exceptional points.

(c) If $(p_1, \ell_1)$ is given, let $p_2$ denote the second intersection of $C$ with $\ell_1^*$, and let $\ell_2$ denote the second tangent to $D$ that contains $p_2$. Define a map, where possible, by sending $(p_1, \ell_1^*) \to (p_2, \ell_2^*) \to (p_2, \ell_2^*)$. Show that this map extends to a morphism $E \to E$ on $E$, and that this morphism is a translation $p \to p \oplus a$, for some point $a$ of $E$.

(d) It might happen that for some point $p$ of $C$ and some $n$, $\gamma^n(p) = p$. Show that if this occurs, the same is true for every point of $C$. For example, if $\gamma^3(p) = p$, the lines $\ell_1, \ell_2, \ell_3$ will form a triangle, and this will be true for all points $p$ of $C$. This is Poncelet’s Theorem.

8.0.7. Let $Y \to X$ be a branched covering, and let $p$ be a point of $X$ whose inverse image in $Y$ consists of one point $q$. Use a basis to prove the main theorem on the trace map for differentials locally at $p$.

8.0.8. Let $D$ be a divisor of degree $d$ on a smooth projective curve $Y$. Show that $h^0(O(D)) \leq d + 1$, and if $h^0(O(D)) = d + 1$, then $Y$ is a smooth rational curve, isomorphic to $\mathbb{P}^1$.

8.0.9. Prove that if $D$ is a divisor on a smooth curve $Y$, then $O(D)$ is an $O$-module.

8.0.10. Let $A$ be a finite-type domain.

(a) Let $B = A[x]$ be the ring of polynomials in one variable with coefficients in $A$. Describe the module $\Omega_B$ in terms of $\Omega_A$.

(b) Let $s$ be a nonzero element of $A$ and let $A'$ be the localization $A[x]/(sx - 1)$. Describe the module $\Omega_{A'}$.

8.0.11. Prove that a projective curve $Y$ such that $h^1(O_Y) = 0$, smooth or not, is isomorphic to the projective line $\mathbb{P}^1$. 17
8.0.12. The projective line $X = \mathbb{P}^1$ with coordinates $x_0, x_1$ is covered by the two standard affine open sets $U_0 = \text{Spec } R_0$ and $U_1 = \text{Spec } R_1$, $R_0 = \mathbb{C}[u]$ with $u = x_1/x_0$, and $R_1 = \mathbb{C}[v]$ with $v = x_0/x_1 = u^{-1}$. The intersection $U_{01}$ is the spectrum of the Laurent polynomial ring $R_{01} = \mathbb{C}[u,v] = \mathbb{C}[u,u^{-1}]$. The units of $R_{01}$ are the monomials $cu^k$, where $k$ can be any integer.

(a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible $R_{01}$-matrix. Prove that there is an invertible $R_0$-matrix $Q$ and there is an invertible $R_1$-matrix $P$ such that $Q^{-1}AP$ is diagonal.

(b) Use part (a) to prove the Birkhoff-Grothendieck Theorem for torsion-free $O_X$-modules of rank 2.

8.0.13. Let $Y$ be a smooth projective curve of genus 2.

(a) Determine the possible dimensions of $H^q(Y, O(D))$, when $D$ is an effective divisor of degree $n$.

(b) Let $K$ be an effective canonical divisor. Then 1 is a global section of $O(K)$, and there is also a nonconstant global section $x$. Prove that the pair of functions $(1, x)$ defines a morphism $Y \to \mathbb{P}^1$ that represents $Y$ as a double cover of the projective line.

8.0.14. Suppose that $g = 3$, an let $K$ be an effective canonical divisor.

(a) Let $(1, x, y)$ be a basis for $H^0(Y, O(K))$. Use Riemann-Roch for multiples of $K$ to show that $x, y$ satisfy a polynomial relation of degree at most 4.

(b) Let $f$ be the morphism from $Y$ to $\mathbb{P}^2$ defined by the rational functions $(1, x, y)$. Show that the image $C$ of $f$ is a plane curve of degree at most 4, and that if its degree is 4, then $C$ is a smooth curve.

8.0.15. Let $C$ be a plane projective curve of degree $d$, with $d$ genus of curve with $\delta$ nodes and $\kappa$ cusps, and let $C'$ be the normalization of $C$. Determine the Genus of $C'$.

8.0.16. Let $M$ be a module over a finite-type domain $A$, and let $\alpha$ be an element of $A$. Prove that for all but finitely many complex numbers $c$, scalar multiplication by $s = \alpha - c$ is an injective map $M \xrightarrow{\alpha-c} M$.

8.0.17. Do Euler characteristic of complex.

8.0.18. On $\mathbb{P}^1$, when is $O(m) \oplus O(n)$ isomorphic to $O(r) \oplus O(s)$?

8.0.19. Prove that, on a smooth curve, any finite module is the direct sum of its torsion submodule and a locally free module.

8.0.20. Let $Y$ be a curve of genus two, and let $p$ be a point $p$ of $Y$.

(a) Prove that there are two cases: Either $h^0(O_Y(2p)) = 1$ and $H^1(O_Y(2p)) = 0$, or else $h^0(O_Y(2p)) = 2$ and $h^1(O_Y(2p)) = 1$.

(b) Suppose we are in the first case. Show that then $h^0(O_Y(rp)) = r - 1$ and $H^1(O_Y(rp)) = 0$, for all $r \geq 2$.

(c) Show that when $r = 4$, there is a basis of global sections of $O_Y(4p)$ of the form $(1, x, y)$, where $x$ and $y$ have poles of orders 3 and 4 at $p$. This basis defines a morphism $Y \to \mathbb{P}^2$ whose image is a curve $Y'$ of degree 4.

(d) Prove that $Y'$ is a singular curve.

8.0.21. Prove that a finite $O$-module on a smooth curve is a direct sum of a torsion module and a locally free module.

8.0.22. Let $Y$ be a smooth curve of genus 1. Use version 1 of Riemann-Roch to prove that, if $r \geq 1$, then $\dim H^0(Y, O_Y(rp)) = r$ and $H^1(Y, O_Y(rp)) = 0$.

8.0.23. Let $Y = \text{Spec } B$ a smooth affine curve, $y \in B$. At what points does $dy$ generate $\Omega_Y$ locally?

8.0.24. Let $Y$ be a smooth projective curve of genus two.

(a) Determine the possible dimensions of $H^q(Y, O(D))$, when $D$ is an effective divisor of some given degree $n$.

(b) Let $K$ be an effective canonical divisor. Then 1 is a global section of $O(K)$, and there is also a nonconstant global section $x$. Prove that the pair of functions $(1, x)$ defines a morphism $Y \to \mathbb{P}^1$ that represents $Y$ as a double cover of the projective line.

8.0.25. Let $Y$ be a smooth projective curve of genus three.
(a) Let \((1, x, y)\) be a basis for \(H^0(Y, \mathcal{O}(K))\). Use Riemann-Roch for multiples of \(K\) to show that \(x, y\) satisfy a polynomial relation of degree at most 4.

(b) Let \(f\) be the morphism from \(Y\) to \(\mathbb{P}^2\) defined by the rational functions \((1, x, y)\). Show that the image \(C\) of \(f\) is a plane curve of degree at most 4, and that if its degree is 4, then \(C\) is a smooth curve.

8.0.26. Let \(X = \mathbb{P}^1\). Are the \(\mathcal{O}_X\)-modules \(\mathcal{O} \oplus \mathcal{O}\) and \(\mathcal{O}(-1) \oplus \mathcal{O}(1)\) isomorphic?

=======================================

• A a domain, \(P_1, \ldots, P_r\) and \(Q_1, \ldots, Q_s\) prime ideals of \(A\). Assume that for all \(i, j\), \(P_i \not\supset Q_j\). There exists an element \(a \in A\) such that \(a \in P_i\) for all \(i\) but \(a \not\in Q_j\) for all \(j\).

• equations defining the degree part of \(k[x, y, z]_{z^2 = xy}\), and proof that \(\mathcal{O}(1)\) is not free there.