Chapter 3  PROJECTIVE ALGEBRAIC GEOMETRY

3.1 Projective Varieties

The projective space $\mathbb{P}^n$ of dimension $n$ was defined in Chapter ??, Its points are equivalence classes of nonzero vectors $(x_0, ..., x_n)$, the equivalence relation being that, for any nonzero complex number $\lambda$,

$$ (x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n). $$

A subset of $\mathbb{P}^n$ is Zariski closed if it is the set of common zeros of a family of homogeneous polynomials $f_1, ..., f_k$ in the coordinate variables $x_0, ..., x_n$, or if it is the set of zeros of the ideal $I$ generated by such a family. Homogeneity is required because the vectors $(x)$ and $(\lambda x)$ represent the same point of $\mathbb{P}^n$. As explained in (??), $f(\lambda x) = 0$ for all $\lambda$ if and only if $f$ is homogeneous. The Zariski closed sets are the closed sets in the Zariski topology on $\mathbb{P}^n$. We usually refer to them simply as closed sets.

Because the polynomial ring $\mathbb{C}[x_0, ..., x_n]$ is noetherian, $\mathbb{P}^n$ is a noetherian space: Every strictly increasing family of ideals of $\mathbb{C}[x]$ is finite, and every strictly decreasing family of closed subsets of $\mathbb{P}^n$ is finite. Therefore every closed subset of $\mathbb{P}^n$ is a finite union of irreducible closed sets s(??). The irreducible closed sets are the projective varieties, the closed subvarieties of $\mathbb{P}^n$.

Thus a projective variety $X$ is an irreducible closed subset of some projective space. We will also want to know when two projective varieties are isomorphic. This will be explained in Section 3.4 when morphisms are defined.

The Zariski topology on a projective variety $X$ is induced from the topology on the projective space that contains it. Since a projective variety $X$ is closed in $\mathbb{P}^n$, a subset of $X$ is closed in $\mathbb{P}^n$ if it is closed in $\mathbb{P}^n$.

3.1.2. Lemma. The one-point sets in projective space are closed.

proof. This simple proof illustrates a general method. Let $p$ be the point $(a_0, ..., a_n)$. The first guess might be that the one-point set $\{p\}$ is defined by the equations $x_i = a_i$, but the polynomials $x_i - a_i$ aren’t homogeneous in $x$. This is reflected in the fact that, for any $\lambda \neq 0$, the vector $(\lambda a_0, ..., \lambda a_n)$ represents the same point, though it won’t satisfy those equations. The equations that define the set $\{p\}$ are

$$ a_i x_j = a_j x_i, $$

for $i, j = 0, ..., n$, which imply that the ratios $a_i/a_j$ and $x_i/x_j$ are equal. □

3.1.4. Lemma. The proper closed subsets of the projective line are the nonempty finite subsets, and the proper closed subsets of the projective plane are finite unions of points and curves. □

Though affine varieties are important, most of algebraic geometry concerns projective varieties. It won’t be clear why this is so, but one property of projective space gives a hint of its importance: With its classical topology, projective space is compact.
A topological space is compact if it has these properties:

**Hausdorff property:** Distinct points \( p, q \) of \( X \) have disjoint open neighborhoods, and

**quasicompactness:** If \( X \) is covered by a family \( \{ U^i \} \) of open sets, then a finite subfamily covers \( X \).

By the way, when we say that the sets \( \{ U^i \} \) cover a topological space \( X \), we mean that \( X \) is the union \( \bigcup U^i \). We don’t allow \( U^i \) to contain elements that aren’t in \( X \), though that would be a customary English usage.

In the classical topology, affine space \( \mathbb{A}^n \) isn’t quasicompact, and therefore it isn’t compact. The **Heine-Borel Theorem** asserts that a subset of \( \mathbb{A}^n \) is compact in the classical topology if and only if it is closed and bounded.

We’ll show that \( \mathbb{P}^n \) is compact, assuming that the Hausdorff property has been verified. The \( 2n+1 \)-dimensional sphere \( S \) of unit length vectors in \( \mathbb{A}^{n+1} \) is a bounded set, and because it is the zero locus of the equation \( \sum_{i=0}^n x_i = 1 \), it is closed. The Heine-Borel Theorem tells us that \( S \) is compact. The map \( S \to \mathbb{P}^n \) that sends a vector \( (x_0, \ldots, x_n) \) to the point of projective space with that coordinate vector is continuous and surjective. The next lemma of topology shows that \( \mathbb{P}^n \) is compact.

**3.1.5. Lemma.** Let \( Y \stackrel{f}{\longrightarrow} X \) be a continuous map. Suppose that \( Y \) is compact and that \( X \) is a Hausdorff space. Then the image \( Z = f(Y) \) is a closed and compact subset of \( X \). □

The rest of this section contains a few examples of projective varieties.

**3.1.6 linear subspaces**

If \( W \) is a subspace of dimension \( r+1 \) of the vector space \( V \), the points of \( \mathbb{P}^n \) that are represented by the nonzero vectors in \( W \) form a linear subspace \( L \) of \( \mathbb{P}^n \), of dimension \( r \). If \( (w_0, \ldots, w_r) \) is a basis of \( W \), the linear subspace \( L \) corresponds bijectively to a projective space of dimension \( r \), by

\[
e_0 w_0 + \cdots + e_r w_r \leftrightarrow (e_0, \ldots, e_r)
\]

For example, the set of points \((x_0, \ldots, x_r, 0, \ldots, 0)\) is a linear subspace of dimension \( r \). □

**3.1.7 a quadric surface**

A **quadric** in \( \mathbb{P}^3 \) is the locus of zeros of an irreducible homogeneous quadratic equation in four variables.

We describe a bijective map from the product \( \mathbb{P}^1 \times \mathbb{P}^1 \) of projective lines to a quadric. Let coordinates in the two copies of \( \mathbb{P}^1 \) be \((x_0, x_1)\) and \((y_0, y_1)\), respectively, and let the four coordinates in \( \mathbb{P}^3 \) be \( w_{ij} \), with \( 0 \leq i, j \leq 1 \). The map is defined by \( w_{ij} = x_i y_j \). Its image is the quadric \( Q \) whose equation is

\[
w_{00} w_{11} = w_{01} w_{10}
\]

Let’s check that the map \( \mathbb{P}^1 \times \mathbb{P}^1 \to Q \) is bijective. If \( w \) is a point of \( Q \), one of the coordinates, say \( w_{00} \), will be nonzero. Then if \((x, y)\) is a point of \( \mathbb{P}^1 \times \mathbb{P}^1 \) whose image is \( w \), so that \( w_{ij} = x_i y_j \), the coordinates \( x_0 \) and \( y_0 \) must be nonzero. When we normalize \( w_{00}, x_0, \) and \( y_0 \) to 1, there is a unique solution for \( x \) and \( y \) such that \( w_{ij} = x_i y_j \), namely \( x_1 = w_{10} \) and \( y_1 = w_{01} \).

The quadric with the equation \((3.1.8)\) contains two families of lines (one dimensional linear subspaces), the images of the subsets \( x \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times y \) of \( \mathbb{P} \times \mathbb{P} \).

**Note.** Equation \((3.1.8)\) can be diagonalized by the substitution \( w_{00} = s + t, w_{11} = s - t, w_{01} = u + v, w_{10} = u - v \). This substitution changes the equation \((3.1.8)\) to \( s^2 - t^2 = u^2 - v^2 \). When we look at the affine open set \( \{ u = 1 \} \), the equation becomes \( s^2 + v^2 - t^2 = 1 \). The real locus of this equation is a one-sheeted hyperboloid in \( \mathbb{R}^3 \), and the two families of complex lines in the quadric correspond to the familiar rulings of this hyperboloid by real lines.
A hypersurface in projective space \( \mathbb{P}^n \) is the locus of zeros of an irreducible homogeneous polynomial \( f(x_0, \ldots, x_n) \). The degree of \( f \) is the degree of \( Y \).

Plane projective curves and quadric surfaces are hypersurfaces.

### Segre embedding of a product

The product \( \mathbb{P}^m \times \mathbb{P}^n \) of projective spaces can be embedded by its Segre embedding into a projective space \( \mathbb{P}^N \) that has coordinates \( w_{ij} \), with \( i = 0, \ldots, m \) and \( j = 0, \ldots, n \). So \( N = (m+1)(n+1) - 1 \). The Segre embedding is defined by

\[
 w_{ij} = x_i y_j. 
\]

We call the coordinates \( w_{ij} \) the Segre variables.

The map from \( \mathbb{P}^m \times \mathbb{P}^n \) to \( \mathbb{P}^{N} \) that was described in (3.1.7) is the simplest case of a Segre embedding.

### Proposition

The Segre embedding maps the product \( \mathbb{P}^m \times \mathbb{P}^n \) bijectively to the locus \( S \) of the Segre equations

\[
 w_{ij} w_{k\ell} - w_{i\ell} w_{j\ell} = 0. 
\]

**proof.** The proof is the same as the one given above, in (3.1.7). When one substitutes (3.1.11) into the Segre equations, one obtains equations in \( \{x_i, y_j\} \) that are true. So the image of the Segre embedding is contained in \( S \).

Say that we have a point \( p \) of the locus \( S \), that is the image of a point \( (x, y) \) of \( \mathbb{P}^m \times \mathbb{P}^n \). Some coordinate of \( p \), say \( w_{i0} \), will be nonzero, and then \( x_0 \) and \( y_0 \) are also nonzero. We normalize \( w_{i0}, x_0, \) and \( y_0 \) to 1. Then \( w_{ij} = w_{i0} w_{0j} \) for all \( i, j \). The unique solution of the Segre equations is \( x_i = w_{i0} \) and \( y_j = w_{0j} \).

The Segre embedding is important because it makes the product of projective spaces into a projective variety, the closed subvariety of \( \mathbb{P}^N \) defined by the Segre equations. However, to show that the product is a variety, we need to show that the locus \( S \) of the Segre equations is irreducible. This is less obvious than one might expect, so we defer the discussion to Section 3.3 (see Proposition 3.3.1).

### Veronese embedding of projective space

Let the coordinates in \( \mathbb{P}^n \) be \( x_i \), and let those in \( \mathbb{P}^N \) be \( v_{ij} \), with \( 0 \leq i \leq n \). Then \( N = \binom{n+2}{2} - 1 \). The Veronese embedding is the map \( \mathbb{P}^n \xrightarrow{f} \mathbb{P}^N \) defined by \( v_{ij} = x_i x_j \). The Veronese embedding resembles the Segre embedding, but in the Segre embedding, there are distinct sets of coordinates \( x \) and \( y \), and there is no requirement that \( i \leq j \).

The proof of the next proposition is similar to the proof of (3.1.12), once one has untangled the inequalities.

### Proposition

The Veronese embedding \( f \) maps \( \mathbb{P}^n \) bijectively to the locus \( X \) in \( \mathbb{P}^N \) of the equations

\[
 v_{ij} v_{k\ell} = v_{i\ell} v_{kj} \quad \text{for} \quad 0 \leq i \leq k \leq j \leq \ell \leq n 
\]

For example, the Veronese embedding maps \( \mathbb{P}^1 \) bijectively to the conic \( v_{00} v_{11} = v_{01}^2 \) in \( \mathbb{P}^2 \).
There are higher order Veronese embeddings, defined in an analogous way by the monomials of some degree \( d > 2 \). The first example is the embedding of \( \mathbb{P}^1 \) by the cubic monomials in two variables, which maps \( \mathbb{P}^1 \) to \( \mathbb{P}^3 \). Let the coordinates in \( \mathbb{P}^3 \) be \( v_0, v_1, v_2, v_3 \). The cubic Veronese embedding is defined by
\[
\begin{align*}
v_0 &= x_0^3, & v_1 &= x_0^2x_1, & v_2 &= x_0x_1^2, & v_3 &= x_1^3
\end{align*}
\]
Its image is a twisted cubic in \( \mathbb{P}^3 \), the locus \( (v_0, v_1, v_2, v_3) = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3) \). This loci is the set of common zeros of the three polynomials
\[
(3.1.17) \quad v_0v_2 - v_1^2, \quad v_1v_2 - v_0v_3, \quad v_1v_3 - v_2^2
\]
which are the \( 2 \times 2 \) minors of the \( 2 \times 3 \) matrix
\[
\begin{pmatrix}
v_0 & v_1 & v_2 \\
v_1 & v_2 & v_3
\end{pmatrix}
\]
A \( 2 \times 3 \) matrix has rank \( \leq 1 \) if and only if its \( 2 \times 2 \) minors are zero. So a point \( (v_0, v_1, v_2, v_3) \) lies on the twisted cubic if \( (3.1.18) \) has rank one. This means that the vectors \( (v_0, v_1, v_2) \) and \( (v_1, v_2, v_3) \), if both are nonzero, represent the same point of \( \mathbb{P}^2 \), and also a zero locus in the affine space.

### 3.2 Homogeneous Ideals

We denote the polynomial algebra \( \mathbb{C}[x_0, \ldots, x_n] \) by \( R \) here.

#### 3.2.1. Lemma
Let \( \mathcal{I} \) be an ideal of \( R \). The following conditions are equivalent.

(i) \( \mathcal{I} \) can be generated by homogeneous polynomials.

(ii) A polynomial \( f \) is in \( \mathcal{I} \) if and only if its homogeneous parts are in \( \mathcal{I} \).

An ideal \( \mathcal{I} \) of \( R \) that satisfies these conditions is a homogeneous ideal.

#### 3.2.2. Lemma
The radical \( \sqrt{\mathcal{I}} \) of a homogeneous ideal \( \mathcal{I} \) is homogeneous.

*Proof.* Let \( \mathcal{I} \) be a homogeneous ideal, and let \( f \) be an element of its radical \( \sqrt{\mathcal{I}} \). So \( f^r \) is in \( \mathcal{I} \) for some \( r \). When \( f \) is written as a sum \( f_0 + \cdots + f_d \) of its homogeneous parts, the highest degree part of \( f^r \) is \( (f_d)^r \). Since \( \mathcal{I} \) is homogeneous, \( (f_d)^r \) is in \( \mathcal{I} \) and \( f_d \) is in \( \sqrt{\mathcal{I}} \). Then \( f_0 + \cdots + f_{d-1} \) is also in \( \sqrt{\mathcal{I}} \). By induction on \( d \), all of the homogeneous parts \( f_0, \ldots, f_d \) are in \( \sqrt{\mathcal{I}} \).

If \( f \) is a set of homogeneous polynomials, the set of its zeros in \( \mathbb{P}^n \) may be denoted by \( V(f) \) or \( V_{\mathbb{P}^n}(f) \), and the set of zeros of a homogeneous ideal \( \mathcal{I} \) by \( V(\mathcal{I}) \) or \( V_{\mathbb{P}^n}(\mathcal{I}) \). This is the same notation as is used for closed subsets of affine space.

The complement of the origin in the affine space \( \mathbb{A}^{n+1} \) is mapped to the projective space \( \mathbb{P}^n \) by sending a vector \( (x_0, \ldots, x_n) \) to the point of \( \mathbb{P}^n \) it defines. This map can be useful when one studies projective space.

A homogeneous ideal \( \mathcal{I} \) has a zero locus in projective space \( \mathbb{P}^n \) and also a zero locus in the affine space \( \mathbb{A}^{n+1} \). We can’t use the \( V(\mathcal{I}) \) notation for both of them here, so let’s denote these two loci by \( V \) and \( W \), respectively. Unless \( \mathcal{I} \) is the unit ideal, the origin \( x = 0 \) will be a point of \( W \), and the complement of the origin will map surjectively to \( V \). If a point \( x \) other than the origin is in \( W \), then every point of the one-dimensional subspace of \( \mathbb{A}^{n+1} \) spanned by \( x \) is in \( W \), because a homogeneous polynomial \( f \) vanishes at \( x \) if and only if it vanishes at \( Ax \). An affine variety that is the union of such lines through the origin is called an affine cone. If the locus \( W \) contains a point \( x \) other than the origin, it is an affine cone.

The familiar locus \( x_0^3 + x_1^2 - x_2^2 = 0 \) is a cone in \( \mathbb{A}^3 \). The zero locus of the polynomial \( x_0^3 + x_1^2 - x_2^2 \) is also called a cone.

**Note.** The real locus \( x_0^3 + x_1^2 - x_2^2 = 0 \) in \( \mathbb{R}^3 \) decomposes into two parts when the origin is removed. Because of this, it is sometimes called a “double cone”. However, the complex locus doesn’t decompose.

#### 3.2.3 the irrelevant ideal

In the polynomial algebra \( R = \mathbb{C}[x_0, \ldots, x_n] \), the maximal ideal \( \mathcal{M} = (x_0, \ldots, x_n) \) that is generated by the variables is called the irrelevant ideal because its zero locus in projective space is empty.
3.2.4. Proposition. The zero locus in $\mathbb{P}^n$ of a homogeneous ideal $I$ of $R$ is empty if and only if $I$ contains a power of the irrelevant ideal.

Another way to say this is that the zero locus $V(I)$ in projective space of a homogeneous ideal $I$ is empty if and only if either $I$ is the unit ideal $R$, or its radical is the irrelevant ideal.

proof of Proposition 3.2.4. Let $Z$ be the zero locus of $I$ in $\mathbb{P}^n$. If $I$ contains a power of $M$, it contains a power of each variable. Powers of the variables have no common zeros in projective space, so $Z$ is empty.

Suppose that $Z$ is empty, and let $W$ be the locus of zeros of $I$ in the affine space $\mathbb{A}^{n+1}$ with the same coordinates $x_0, \ldots, x_n$. Since the complement of the origin in $W$ maps to the empty locus $Z$, it is empty. The origin is the only point that might be in $W$. If $W$ is the one point space consisting of the origin, then $\text{rad } I$ is the irrelevant ideal $M$. If $W$ is empty, $I$ is the unit ideal. □

3.2.5. Lemma. Let $P$ be a homogeneous ideal in the polynomial algebra $R$, not the unit ideal. The following conditions are equivalent:

(i) $P$ is a prime ideal.
(ii) If $f$ and $g$ are homogeneous polynomials, and if $fg \in P$, then $f \in P$ or $g \in P$.
(iii) If $A$ and $B$ are homogeneous ideals, and if $AB \subset P$, then $A \subset P$ or $B \subset P$.

In other words, a homogeneous ideal is a prime ideal if the usual conditions for a prime ideal are satisfied when the polynomials or ideals are homogeneous.

proof of the lemma. When the word homogeneous is omitted, (ii) and (iii) become the definition of a prime ideal. So (i) implies (ii) and (iii). The fact that (iii) $\Rightarrow$ (ii) is proved by considering the principal ideals generated by $f$ and $g$.

(ii) $\Rightarrow$ (i) Suppose that a homogeneous ideal $P$ satisfies the condition (ii), and that the product $fg$ of two polynomials, not necessarily homogeneous, is in $P$. If $f$ has degree $d$ and $g$ has degree $e$, the highest degree part of $fg$ is the product $f_{d,e}$ of the homogeneous parts of $f$ and $g$ of maximal degree. Since $P$ is a homogeneous ideal, it contains $f_{d,e}$. Therefore one of the factors, say $f_d$, is in $P$. Let $h = f - f_d$. Then $hg$ is in $P$, and it has lower degree than $fg$. By induction on the degree of $fg$, $h$ or $g$ is in $P$, and if $h$ is in $P$, so is $f$. □

3.2.6. Proposition. Let $Y$ be the zero locus in $\mathbb{P}^n$ of a homogeneous radical ideal $I$ that isn’t the irrelevant ideal. Then $Y$ is a projective variety (an irreducible closed subset of $\mathbb{P}^n$) if and only if $I$ is a prime ideal. Thus a subset $Y$ of $\mathbb{P}^n$ is a projective variety if and only if it is the zero locus of a homogeneous prime ideal that isn’t the irrelevant ideal.

proof. The locus $W$ of zeros of $I$ in the affine space $\mathbb{A}^{n+1}$ is irreducible if and only if $Y$ is irreducible. This is easy to see. Proposition 3.2.4 tells us that $W$ is irreducible if and only if the radical ideal $I$ is a prime ideal. □

3.2.7. Strong Nullstellensatz, projective version.

(i) Let $g$ be a nonconstant homogeneous polynomial in $x_0, \ldots, x_n$, and let $I$ be a homogeneous ideal of $\mathbb{C}[x]$. If $g$ vanishes at every point of the zero locus $V(I)$ in $\mathbb{P}^n$, then $I$ contains a power of $g$.
(ii) Let $f$ and $g$ be homogeneous polynomials. If $f$ is irreducible and if $V(f) \subset V(g)$, then $f$ divides $g$.
(iii) Let $I$ and $J$ be homogeneous ideals, and suppose that $\text{rad } I$ isn’t the irrelevant ideal or the unit ideal. Then $V(I) = V(J)$ if and only if $\text{rad } I = \text{rad } J$.

proof. (i) Let $W$ be the locus of zeros of $I$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, \ldots, x_n$. The homogeneous polynomial $g$ vanishes at every point of $W$ different from the origin, and since $g$ isn’t a constant, it vanishes at the origin too. So the affine Strong Nullstellensatz applies. □

3.2.8. Quasiprojective varieties

We will also want to consider nonempty open subsets of a projective variety. We will call any such subset a variety. Any open subset of a variety is called a variety.

For example, the complement of a point in a projective variety is a variety. An affine variety $X = \text{Spec } A$ may be embedded as a closed subvariety into the standard affine space $\mathbb{A}^0 : \{x_0 \neq 0\}$. It becomes an open
subset of its closure in \( \mathbb{P}^n \), which is a projective variety (Lemma 3.2.9 (ii)). And of course, a projective variety is a variety.

In more usual terminology such a variety is called a \emph{quasiprojective variety}, but we drop the adjective ‘quasiprojective’. There are varieties that cannot be embedded into any projective space, but they aren’t very important. We will not study them, and in fact, it is hard enough to find convincing examples of such varieties that we won’t try to give one here. So the adjective ‘quasiprojective’ is superfluous as well as ugly.

The topology on a (quasiprojective) variety is induced from the topology on projective space.

### Lemma. The topology on the affine open subset \( \mathbb{U}^0 : x_0 \neq 0 \) of \( \mathbb{P}^n \) that is induced from the Zariski topology on \( \mathbb{P}^n \) is the Zariski topology obtained by viewing \( \mathbb{U}^0 \) as the affine space \( \text{Spec} \mathbb{C}[u_1, \ldots, u_n] \), \( u_i = x_i/x_0 \).

### Product Varieties

The properties of products of varieties seem intuitive, but some of the proofs aren’t obvious. The reason is that the (Zariski) topology on a product of varieties isn’t the product topology.

The \emph{product topology} on the product \( X \times Y \) of topological spaces is the coarsest topology such that the projection maps \( X \times Y \to X \) and \( X \times Y \to Y \) are continuous. So if \( C \) and \( D \) are closed subset of \( X \) and \( Y \), then \( C \times D \) is a closed subset of \( X \times Y \) in the product topology. Every closed set in the product topology is a finite union of such subsets.

Products of the form \( C \times D \), where \( C \) is a closed subset of \( \mathbb{P}^n \) and \( D \) is a closed subset of \( \mathbb{P}^n \), are the first examples of closed subsets of \( \mathbb{P}^n \times \mathbb{P}^n \). But the product topology on \( \mathbb{P}^n \times \mathbb{P}^n \) is much coarser than the Zariski topology. For example, the proper (Zariski) closed subsets of \( \mathbb{P}^1 \) are the nonempty finite subsets. In the product topology, the proper closed subsets of \( \mathbb{P}^1 \times \mathbb{P}^1 \) are finite unions of points and sets of the form \( \mathbb{P}^1 \times \{ q \} \) and \( \{ p \} \times \mathbb{P}^1 \) (“horizontal” and “vertical” lines). Most Zariski closed subsets of \( \mathbb{P}^1 \times \mathbb{P}^1 \) aren’t of this form. The diagonal \( \Delta = \{(p, p) \mid p \in \mathbb{P}^1\} \) is a simple example.

#### 3.3.1. Proposition. Let \( X \) and \( Y \) be irreducible topological spaces, and suppose that a topology is given on the product \( \Pi = X \times Y \), with the following properties:

- The projections \( \Pi \overset{\pi_1}{\to} X \) and \( \Pi \overset{\pi_2}{\to} Y \) are continuous.
- For all \( x \) in \( X \) and all \( y \) in \( Y \), the fibres \( x \times Y \) and \( X \times y \), with topologies induced from \( \Pi \), are homeomorphic to \( Y \) and \( X \), respectively.

Then \( \Pi \) is an irreducible topological space.

The first condition means that the topology on \( X \times Y \) is at least as fine as the product topology, and the second one assures us that the topology isn’t too fine. (We don’t want the discrete topology on \( \Pi \), for example.)

The product of varieties has the two properties mentioned in the proposition.

#### 3.3.2. Lemma. Let \( X, Y \), and \( \Pi \) be as in the proposition. If \( W \) is an open subset of \( \Pi \), its image \( U \) via the projection \( \Pi \to Y \) is an open subset of \( Y \).

**proof.** The intersection \( \pi_1^{-1}(W) = W \cap (x \times Y) \) is an open subset of the fibre \( x \times Y \), and its image \( x \pi_1(U) \) in the homeomorphic space \( Y \) is open too. Since \( W \) is the union of the sets \( \pi_1^{-1}(W) \), \( U \) is the union of the open sets \( x \pi_1(U) \).

So \( U \) is open.

**proof of Proposition 3.3.1.** Let \( C \) and \( C' \) be closed subsets of the product \( \Pi \). Suppose that \( C \subset \Pi \) and \( C' \subset \Pi \), and let \( W = \Pi - C \) and \( W' = \Pi - C' \) be the open complements of \( C \) and \( C' \) in \( \Pi \). To show that \( \Pi \) is irreducible, we must show that \( C \cup C' \subset \Pi \). We do this by showing that \( W \cap W' \) isn’t the empty set.

Since \( C \subset \Pi \), \( W \) isn’t empty, and similarly, \( W' \) isn’t empty. The lemma tells us that the images \( U \) and \( U' \) of \( W \) and \( W' \) via projection to \( Y \) are nonempty open subsets of \( Y \). Since \( Y \) is irreducible, \( U \cap U' \) is nonempty. Let \( y \) be a point of \( U \cap U' \). The intersection \( W_y = W \cap (x \times y) \) is an open subset of \( X \times y \), and since its image \( U \) contains \( y \), \( W_y \) contains a point of the form \( p = (x, y) \), it is nonempty. Similarly, \( W'_y = W' \cap (x \times y) \) is a nonempty open subset of \( X \times y \). Since \( X \times y \) is homeomorphic to the irreducible space \( X \), it is irreducible. So \( W_y \cap W'_y \) is nonempty, and therefore \( W \cap W' \) is nonempty, as was to be shown.

### 3.3.9. Lemma. The topology on the affine open subset \( \mathbb{U}^0 : x_0 \neq 0 \) of \( \mathbb{P}^n \) that is induced from the Zariski topology on \( \mathbb{P}^n \) is the Zariski topology obtained by viewing \( \mathbb{U}^0 \) as the affine space \( \text{Spec} \mathbb{C}[u_1, \ldots, u_n] \), \( u_i = x_i/x_0 \).
(3.3.3) products of affine varieties

We inspect the product \( X \times Y \) of the affine varieties \( X = \text{Spec} A \) and \( Y = \text{Spec} B \). Say that \( X \) is embedded as a closed subvariety of \( \mathbb{A}^m \), so that \( A = \mathbb{C}[x_1, ..., x_m]/P \) for some prime ideal \( P \), and that \( Y \) is embedded similarly into \( \mathbb{A}^n \), and \( B = \mathbb{C}[y_1, ..., y_n]/Q \). Then in affine \( x, y \)-space \( \mathbb{A}^{m+n} \), \( X \times Y \) is the locus of the equations \( f(x) = 0 \) and \( g(y) = 0 \) with \( f \in P \) and \( g \in Q \). Proposition 3.3.4 shows that \( X \times Y \) is irreducible. Therefore it is a variety.

Let \( X \) and \( Y \) be subvarieties of \( \mathbb{A}^m \) and \( \mathbb{A}^n \) defined by prime ideals \( P \) and \( Q \) of \( \mathbb{C}[x] \) and \( \mathbb{C}[y] \), respectively. Let \( P' \) be the ideal of \( \mathbb{C}[x, y] \) generated by the elements of \( P \). So \( P' \) consists of sums of products of elements of \( P \) with polynomials in \( x, y \). Let \( Q' \) be defined analogously using \( Q \), and let \( I = P' + Q' \).

3.3.4. Proposition. The ideal \( I = P' + Q' \) consists all elements of \( \mathbb{C}[x, y] \) that vanish on the variety \( X \times Y \). Therefore it is a prime ideal.

The fact that \( X \times Y \) is a variety tells us only that the radical of \( I \) is a prime ideal.

proof of Proposition 3.3.4 Let \( A = \mathbb{C}[x]/P \), \( B = \mathbb{C}[y]/Q \), and \( R = \mathbb{C}[x, y]/I \). Any polynomial in \( x, y \) can be written, in many ways, as a sum, each of whose terms is a product of a polynomial in \( x \) and a polynomial in \( y \): \( p(x, y) = \sum a_i(x)b_i(y) \). Therefore any element \( p \) of \( R \) can be written as a finite sum

\[
p = \sum_{i=1}^{k} a_i b_i
\]

with \( a_i \) in \( A \) and \( b_i \) in \( B \). We show that if \( p \) vanishes identically on \( X \times Y \), then \( p = 0 \). To do this, we show that the same element \( p \) can also be written as a sum of \( k - 1 \) products.

Suppose that \( p = 0 \). If \( a_k = 0 \), then \( p = \sum_{i=1}^{k-1} a_i b_i \). If \( a_k \neq 0 \), the function defined by \( a_k \) isn’t identically zero on \( X \). We choose a point \( x^0 \) of \( X \) such that \( a_k(x^0) \neq 0 \). Let \( a_i = a_i(x^0) \) and \( p^0(y) = p(x^0, y) \). So \( p^0(y) = \sum_{i=1}^{k} a_i^0 b_i \). Since \( p \) vanishes on \( X \times Y \), \( p^0 \) vanishes on \( Y \), and therefore \( p^0 = 0 \). Since \( a_i^0 \neq 0 \), we can solve the equation \( \sum_{i=1}^{k} a_i^0 b_i = 0 \) for \( b_k \): \( b_k = \sum_{i=1}^{k-1} c_i b_i \), where \( c_i = -a_i^0/a_k^0 \). Substituting into \( p \) gives us an expression for \( p \) as a sum of \( k - 1 \) terms. Finally, when \( k = 1 \), \( a_1^0 b_1 = 0 \). Then \( b_1 = 0 \), and \( p = 0 \).

(3.3.5) the Zariski topology on \( \mathbb{P}^m \times \mathbb{P}^n \)

As mentioned above (3.1.10), the product of projective spaces \( \mathbb{P}^m \times \mathbb{P}^n \) is made into a projective variety by identifying it with its Segre image, the locus of the Segre equations \( w_{ij} w_{kl} = w_{ik} w_{lj} \). Since \( \mathbb{P}^m \times \mathbb{P}^n \), with its Segre embedding, is a projective variety, we don’t really need a separate definition of its Zariski topology. Its closed subsets are the zero sets of families of homogeneous polynomials in the Segre variables \( w_{ij} \) that include the Segre equations.

One can also describe the closed subsets of \( \mathbb{P}^m \times \mathbb{P}^n \) directly, in terms of bihomogeneous polynomials. A polynomial \( f(x, y) \) is bihomogeneous if it is homogeneous in the variables \( x \) and also in the variables \( y \). For example, the polynomial \( x_2 y_0 + x_0 x_1 y_1 \) is bihomogeneous, of degree 2 in \( x \) and degree 1 in \( y \).

Because \( x, y \) and \( (\lambda x, \mu y) \) represent the same point of \( \mathbb{P}^m \) for all nonzero \( \lambda \) and \( \mu \), we want to know that \( f(x, y) = 0 \) if and only if \( f(\lambda x, \mu y) = 0 \). This is true for all nonzero \( \lambda \) and \( \mu \) if and only if \( f \) is bihomogeneous.

3.3.7. Lemma. The (Zariski) topology on \( \mathbb{P}^m \times \mathbb{P}^n \) has the properties listed in Proposition 3.3.1.

- The projections \( \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m \) and \( \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n \) are continuous maps.
- For all \( y \) in \( \mathbb{P}^n \), the fibre \( \mathbb{P}^m \times y \), with the topology induced from \( \mathbb{P}^m \times \mathbb{P}^n \), is homeomorphic to \( \mathbb{P}^m \), and the analogous statement is true for the fibre \( x \times \mathbb{P}^n \).

proof. We look at the projection \( \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m \). If \( X \) is the closed subset of \( \mathbb{P}^m \) defined by a system of homogeneous polynomials \( f_i(x) \), its inverse image in \( \mathbb{P}^m \times \mathbb{P}^n \) is the zero set of the same system, considered as a family of bihomogeneous polynomials of degree zero in \( y \). So the inverse image is closed.
For the second property, because the projection \( \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m \) is continuous, it suffices to show that the inclusion map \( \mathbb{P}^m \to \mathbb{P}^m \times \mathbb{P}^n \) that sends \( \mathbb{P}^m \) to \( \mathbb{P}^m \times \mathbb{P}^n \) is continuous. If \( f(x, y) \) is a bihomogeneous polynomial and \( y^0 \) is a point of \( \mathbb{P}^n \), the zero set of \( f \) in \( \mathbb{P}^m \times \mathbb{P}^n \) is the zero set of \( f(x, y^0) \). This polynomial also defines a closed subset of \( \mathbb{P}^m \).

3.3.8. Proposition. 
(i) A subset of \( \mathbb{P}^m \times \mathbb{P}^n \) is closed if and only if it is the locus of zeros of a family of bihomogeneous polynomials.
(ii) If \( X \) and \( Y \) are closed subsets of \( \mathbb{P}^m \) and \( \mathbb{P}^n \), respectively, then \( X \times Y \) is a closed subset of \( \mathbb{P}^m \times \mathbb{P}^n \).

proof. (i) We denote the Segre image of \( \mathbb{P}^m \times \mathbb{P}^n \) by \( \Pi \) for the proof. Let \( f(w) \) be a homogeneous polynomial in the Segre variables \( w_{ij} \). When we substitute \( w_{ij} = x_{i}y_{j} \) into \( f \), we obtain a polynomial \( f(x, y_j) \) that is bihomogeneous and that has the same degree as \( f \) in \( x \) and in \( y \). Let’s denote that bihomogeneous polynomial by \( f(x, y) \). The inverse image of the zero set of \( f \) in \( \Pi \) is the zero set of \( f \) in \( \mathbb{P}^m \times \mathbb{P}^n \). Therefore the inverse image of a closed subset of \( \Pi \) is the zero set of a family of bihomogeneous polynomials in \( \mathbb{P}^m \times \mathbb{P}^n \).

Conversely, let \( g(x, y) \) be a bihomogeneous polynomial, say of degree \( r \) and \( s \) in \( y \). If \( r = s \), we may collect variables that appear in \( g \) in pairs \( x_{i}y_{j} \) and replace each pair \( x_{i}y_{j} \) by \( w_{ij} \). We will obtain a homogeneous polynomial \( G \) in \( w \) such that \( G(w) = g(x, y) \) when \( w_{ij} = x_{i}y_{j} \). The zero set of \( G \) in \( \Pi \) is the image of the zero set of \( g \) in \( \mathbb{P}^m \times \mathbb{P}^n \).

Suppose that \( r \geq s \), and let \( k = r - s \). Because the variables \( y \) cannot all be zero at any point of \( \mathbb{P}^n \), the equation \( g = 0 \) on \( \mathbb{P}^m \times \mathbb{P}^n \) is equivalent with the system of equations \( g y_{0}^{k} = g y_{1}^{k} = \cdots = g y_{n}^{k} = 0 \). The polynomials \( g y_{i}^{k} \) are bihomogeneous, of same degree in \( x \) and in \( y \).

(ii) A polynomial \( f(x) \) can be viewed as a bihomogeneous polynomial of degree zero in \( y \), and a polynomial \( g(y) \) as a bihomogeneous polynomial of degree zero in \( x \). So \( X \times Y \), which is the locus \( f = g = 0 \) in \( \mathbb{P}^m \times \mathbb{P}^n \), is closed in \( \mathbb{P}^m \times \mathbb{P}^n \).

3.3.9. Corollary. Let \( X \) and \( Y \) be projective varieties, and let \( \Pi \) denote the product \( X \times Y \). This is a closed subset of \( \mathbb{P}^m \times \mathbb{P}^n \).

(i) The projections \( \Pi \to X \) and \( \Pi \to Y \) are continuous.
(ii) For all \( x \) in \( X \) and all \( y \) in \( Y \), the fibres \( x \times Y \) and \( X \times y \), with topologies induced from \( \Pi \), are homeomorphic to \( X \) and \( Y \), respectively.

3.3.10. Corollary. If \( X \) and \( Y \) are projective varieties, so is \( X \times Y \).

We will come back to products in Chapter ??.

3.4 Morphisms and Isomorphisms

3.4.1. the function field

Let \( X \) be a projective variety, and let \( \mathcal{O}^i \) be its intersection with the standard affine open subset \( \mathbb{U}^{i} \) of the projective space with coordinates \( x_0, \ldots, x_n \). If nonempty, \( X^i \) will be an affine variety – an irreducible closed subset of \( \mathbb{U}^{i} \). Let’s omit the indices for which \( X^i \) is empty. Then the intersection \( X^V = X^i \cap X^j \) will be a localization of \( X^i \) and also a localization of \( X^j \). If \( X^i = \text{Spec} \, A_i \) and \( u_{ij} = x_j/x_i \), then \( X^{ij} = \text{Spec} \, A_{ij} \), where \( A_{ij} = A_i[u_{ij}]^{-1} = A_j[u_{ji}]^{-1} \). So the fraction fields of the coordinate algebras \( A_i \) are equal for all \( i \) such that \( X^i \) isn’t empty.

3.4.2. Definition. The function field \( K \) of a projective variety \( X \) is the field of fractions of the coordinate algebra \( A_i \) of any one of its nonempty affine open subsets \( X^i = X \cap \mathbb{U}^i \). If \( X^i \) is an open subvariety of a projective variety \( X \), the function field of \( X^i \) is the function field of \( X \).

Thus all open subvarieties of a variety have the same function field. In particular, suppose that we regard an affine variety \( X = \text{Spec} \, A \) as and as a closed subvariety of \( \mathbb{U}^0 \). The function field of \( X \) will be the field of fractions of \( A \).

3.4.3. Definition. A rational function on a variety \( X^i \) is an element of the function field \( K \) of \( X^i \).
A rational function can be evaluated at some points of $X'$, but probably not all of them. Suppose that $X'$ is an open subvariety of a projective variety $X$, and that $p$ is a point of the affine open set $X^i = X \cap U^i = \text{Spec } A_i$, as above.

A rational function $\alpha$ on $X$ is regular at $p$ if it is a regular function at $p$ on one of the open sets $X^i$. This means that one can write $\alpha$ as a fraction $a/b$ of elements of $A_i$, with $b(p) \neq 0$. Then the value of $\alpha$ at $p$ is $\alpha(p) = a(p)/b(p)$.

**Lemma.** The regularity of a rational function at $p$ doesn’t depend on the choice of the open set $X^i$ that contains $p$. $\square$

**Sublemma.** Let $A$ be a finite-type algebra. The intersection of the maximal ideals of $A$ is the zero ideal.

**Proof.** We present $A$ as $\mathbb{C}[x_1, \ldots, x_n]/P$ where $P$ is a prime ideal. Then $U$ becomes a closed subset of $\mathbb{K}^n$. Let $g(x)$ be a polynomial that represents $\alpha$. The function defined by $\alpha$ is restriction of the polynomial function $g$ to $U$. The Strong Nullstellensatz tells us that if $g$ vanishes on $U$, it is in the prime ideal $P$. Then $\alpha = 0$. $\square$

**Lemma.** a rational function $\alpha$ that is regular on an open subset $U$ of $X$ is determined by the function it defines on $U$.

**Proof.** Show that if the function is identically zero, then $\alpha = 0$. For some $i$, the intersection $U \cap X^i$ will be nonempty. We may replace $U$ by a localization $X^i$ of $X^i$, so we may assume that $U$ is affine, say $U = \text{Spec } A$, where $A$ is a finite-type domain. Then what is to be proved is that if an element $\alpha$ of $A$ defines the zero function, it is zero.

**Sublemma.** Let $A$ be a finite-type algebra. The intersection of the maximal ideals of $A$ is the zero ideal.

**Proof.** We present $A$ as $\mathbb{C}[x_1, \ldots, x_n]/P$ where $P$ is a prime ideal. Then $U$ becomes a closed subset of $\mathbb{K}^n$. Let $g(x)$ be a polynomial that represents $\alpha$. The function defined by $\alpha$ is restriction of the polynomial function $g$ to $U$. The Strong Nullstellensatz tells us that if $g$ vanishes on $U$, it is in the prime ideal $P$. Then $\alpha = 0$. $\square$

**Points with values in a field**

Let $K$ be a field that contains the complex numbers. A point of projective space $\mathbb{P}^n$ with values in $K$ is an equivalence class of nonzero vectors $\alpha = (\alpha_0, \ldots, \alpha_n)$ with $\alpha_i$ in $K$, the equivalence relation being analogous to the one for ordinary points: $\alpha \sim \alpha'$ if $\alpha' = \lambda \alpha$ for some $\lambda$ in $K$.

If $K$ is the function field of a variety $X$, the embedding of $X$ into projective space $\mathbb{P}^n$ defines a point $(\alpha_0, \ldots, \alpha_n)$ of $X$ with values in $K$. To get this point, one may choose a standard affine open set, say $U^i$, of $\mathbb{P}^n$ such that $X^i = X \cap U^i$ isn’t empty. Then $X^i$ is affine, say $X^i = \text{Spec } A$. The embedding of $X^i$ into the affine space $U^i$ is defined by a homomorphism $\mathbb{C}[u_1, \ldots, u_n] \rightarrow A$, and the images $\alpha_i$ of the variables $u_i$ in $A$ give us a point $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ with values in $K$, and with $\alpha_0 = 1$. This is the point.

** Morphisms to projective space**

For the rest of this section, it will be helpful to have a separate notation for the point with values in the function field $K$ of a variety $X$ determined by a nonzero vector $\alpha = (\alpha_0, \ldots, \alpha_n)$, with $\alpha_i \in K$. We’ll denote that point by $\overline{\alpha}$. So $\overline{\alpha} = \overline{\alpha'}$ if $\alpha' = \lambda \alpha$ for some nonzero $\lambda$ in $K$. We’ll drop this notation later.

We define a morphism from a variety $X$ to projective space using a point $\overline{\alpha}$ of $\mathbb{P}^n$ with values in the function field $K$ of $X$. When doing this, we must keep in mind that the points of projective space are equivalence classes of vectors, not the vectors themselves. As we will see, this complication turns out to be useful.

We begin with a simple example.
3.4.10. Example. Let $C$ be the conic in the projective plane $\mathbb{P}^2$ defined by $f(x_0, x_1, x_2) = x_0x_1 + x_0x_2 + x_1x_2$. We project $C$ to the line $L_0 : \{x_0 = 0\}$, defining $\mathbb{P}^2 \to \mathbb{P}^1$ by $f(x_0, x_1, x_2) = (x_1, x_2)$. The formula for $\pi$ is undefined at the point $q = (1, 0, 0)$, though the map extends to the whole conic $C$.

Let's write this projection using a point with values in the function field $K$ of $C$. The affine open set $\{x_0 \neq 0\}$ of $\mathbb{P}^2$ is the polynomial algebra $\mathbb{C}[u_1, u_2]$, with $u_1 = x_1/x_0$ and $u_2 = x_2/x_0$. We also denote by $u_i$ the restriction of the function $u_i$ to $C^0 = C \cap \mathbb{A}$.

3.4.15. Example. identmap

The identity map $\alpha$ is defined by a good point $\alpha$.

\[\alpha'(q) = (\alpha'(q), \alpha'(q)) \quad (\alpha' = \alpha(q))\]

If there exists a vector $\alpha'$ such that $\alpha'(q)$ exists for every point $q$ of $Y$, we call $\alpha$ a good point.

3.4.12. Lemma. A point $\alpha$ of $\mathbb{P}^n$ with values in the function field $K_Y$ of $Y$ is a good point if either one of the two following conditions holds for every point $q$ of $Y$:

- There is an element $\lambda$ in $K_Y$ such that the rational functions $\alpha'_i = \lambda \alpha_i$, $i = 0, ..., n$, are regular and not all zero at $q$.
- There is an index $j$, $0 \leq j \leq n$, such that the rational functions $\alpha_i/\alpha_j$, $j = 0, ..., n$, are regular at $q$.

\[\alpha' = \lambda \alpha \quad \alpha'' = \mu \alpha \quad \alpha' = \lambda \alpha \quad \alpha'' = \mu \alpha\]

Conversely, suppose that $\alpha'_i = \lambda \alpha_1$ are all regular at $q$ and that $\alpha'_i$ isn’t zero there. Then $\alpha'_i = 1$ is a regular function at $q$, so the rational functions $\alpha'_i/\alpha'_j$, which are equal to $\alpha_i/\alpha_j$, are regular for $q$ at all $i$.

3.4.13. Lemma. With notation as in \ref{3.4.11}, the point $\alpha(q)$ is independent of the choice of the vector $\alpha'$.

\[\alpha(q) = (\alpha(q), \alpha(q)) \quad (\alpha = \alpha(q))\]

Suppose that $\alpha_i/\alpha_j$ are regular at $q$ for all $i$. Let $\lambda = \alpha^{-1}_j$, and let $\alpha'_i = \lambda \alpha_i = \alpha_i/\alpha_j$. The rational functions $\alpha'_i$ are regular at $q$, and they aren’t all zero there because $\alpha'_i = 1$.

Conversely, suppose that $\alpha'_i = \lambda \alpha_1$ are all regular at $q$ and that $\alpha'_i$ isn’t zero there. Then $\alpha'_i = 1$ is a regular function at $q$, so the rational functions $\alpha'_i/\alpha'_j$, which are equal to $\alpha_i/\alpha_j$, are regular for $q$ at all $i$.

3.4.14. Definition. Let $Y$ be a variety with function field $K$. A morphism from $Y$ to projective space $\mathbb{P}^n$ is a map that is defined by a good point $\alpha$ with values in $K$. We denote the morphism defined by a good point by $\alpha$ too.

3.4.15. Example. The identity map $\mathbb{P}^1 \to \mathbb{P}^1$.

Let $X = \mathbb{P}^1$, and let $(x_0, x_1)$ be coordinates in $X$. The function field of $X$ is the field $K = \mathbb{C}(t)$ of rational functions in the variable $t = x_1/x_0$. The identity map $X \to X$ is the map $\alpha$ defined by the point $\alpha = (1, t)$ with values in $K$. For every point $p$ of $X$ except the point $(1, 0)$, $\alpha(p)$ is defined an not zero, so $\alpha(p) = \alpha(p)$.

At the point $q = (0, 1)$, $\alpha' = (t^{-1}, 1) = t^{-1} \alpha$ defines $\alpha$: $\alpha(q) = (0, 1)$. \hfill $\square$
3.4.16. **Morphisms to Quasiprojective Varieties**

Let $Y$ be a variety, and let $X$ be a subvariety of a projective space $\mathbb{P}^n$. A morphism of varieties $Y \xrightarrow{\alpha} X$ is the restriction of a morphism $Y \xrightarrow{\alpha} \mathbb{P}^n$ whose image is contained in $X$.

If a projective variety $X$ is the locus of zeros of a family $f$ of homogeneous polynomials, a morphism $Y \xrightarrow{\alpha} \mathbb{P}^n$ defines a morphism $Y \to X$ if $f(\alpha) = 0$.

A word of caution: A morphism $Y \xrightarrow{\alpha} X$ won’t define a map of function fields $K_X \to K_Y$ unless the image of $Y$ is dense in $X$.

3.4.17. **Definition.** Let $Y$ be an open or a closed subvariety of $X$. Then $f$ is a morphism if and only if its restriction $Y \cap X$ is a morphism.

3.4.18. **Proposition.** A morphism of varieties $Y \xrightarrow{\alpha} X$ is a continuous map in the Zariski topology, and also a continuous map in the classical topology.

**Proof.** See Section 7. Since $Y$ can be covered by affine open sets such as $Y^i$, $\alpha$ is continuous. Continuity for a morphism to a subvariety $Y$ of $\mathbb{P}^m$ follows, because the topology on $X$ is the induced topology.

3.4.19. **Proposition.** Let $X$, $Y$, and $Z$ be varieties and let $Z \xrightarrow{\beta} Y$ and $Y \xrightarrow{\alpha} X$ be morphisms. The composed map $Z \xrightarrow{\alpha \circ \beta} X$ is a morphism.

**Proof.** The proof is easy. Say that $X$ is a subvariety of $\mathbb{P}^m$. The morphism $\alpha$ is the restriction of a morphism $Y \to \mathbb{P}^n$ whose image is in $X$, and is defined by a good point $\alpha = (\alpha_0, ..., \alpha_m)$ of $\mathbb{P}^m$ with values in the function field $K_Y$ of $Y$. Similarly, if $Y$ is a subvariety of $\mathbb{P}^n$, the morphism $\beta$ is the restriction of a morphism $Z \to \mathbb{P}^n$ whose image is contained in $Y$, and is defined by a good point $\beta = (\beta_0, ..., \beta_n)$ of $\mathbb{P}^n$ with values in the function field $K_Z$ of $Z$.

Let $z$ be a point (an ordinary point) of $Z$. Since $\beta$ is a good point, we may adjust $\beta$ by a factor in $K_\beta$ so that the rational functions $\beta_i$ are regular and not all zero at $z$. Then $\beta(z)$ is the point $(\beta_0(z), ..., \beta_n(z))$. Let’s denote that point by $q = (q_0, ..., q_n)$. So $q_i = \beta_i(z)$. The elements $\alpha_j$ are rational functions on $Y$. We may adjust $\alpha$ by a factor in $K_Y$, so that they are regular and not all zero at $q$. Then $[\alpha \beta](z) = \alpha(q) = (\alpha_0(q), ..., \alpha_m(q)), \alpha_j(q) = \alpha_j(\beta_0(z), ..., \beta_n(z)) = \alpha_j(\beta(z))$ are not all zero. When these adjustments have been made, the point of $\mathbb{P}^m$ with values in $K_Y$ that defines $\alpha \beta$ is $(\alpha_0(\beta(z)), ..., \alpha_m(\beta(z)))$.

This next is a lemma of topology.

3.4.20. **Lemma.** Let $\{X^i\}$ be a covering of a topological space $X$ by open sets. A subset $Y$ of $X$ is open (or closed) if and only if $Y \cap X^i$ is open (or closed) in $X^i$ for every $i$. In particular, if $\{U^i\}$ is the standard affine cover of $\mathbb{P}^n$, a subset $Y$ of $\mathbb{P}^n$ is open (or closed) if and only if $Y \cap U^i$ is open (or closed) in $U^i$ for every $i$.

3.4.21. **Lemma.**

(i) The inclusion of an open or a closed subvariety $Y$ into a variety $X$ is a morphism.

(ii) Let $Y \xrightarrow{f} X$ be a map whose image lies in an open or a closed subvariety $Z$ of $X$. Then $f$ is a morphism if and only if its restriction $Y \to Z$ is a morphism.

(iii) Let $\{Y^i\}$ be an open covering of a variety $Y$, and let $Y^i \xrightarrow{f^i} X$ be morphisms. If the restrictions of $f^i$ and $f^j$ to the intersections $Y^i \cap Y^j$ are equal for all $i, j$, there is a unique morphism $f$ whose restriction to $Y^i$ is $f^i$.

We omit the proof, noting only that (iii) is trivial because the points with values in $K$ are all the same.
A bijective morphism $Y \overset{u}{\longrightarrow} X$ of quasiprojective varieties whose inverse function is also a morphism is an isomorphism. Isomorphisms are important because they allow us to identify different incarnations of the “same” variety, i.e., to describe an isomorphism class of varieties. For example, the projective line $\mathbb{P}^1$, a conic in $\mathbb{P}^2$, and the twisted cubic in $\mathbb{P}^3$ are isomorphic.

3.4.23. Example.

Let $Y$ denote the projective line, with coordinates $y_0, y_1$. As before, the function field of $Y$ is the field $K = \mathbb{C}(t)$ of rational functions in $t = y_1/y_0$. The degree 3 Veronese map $Y \rightarrow \mathbb{P}^3$ (3.1.16) defines an isomorphism of $Y$ to its image, a twisted cubic $X$. The Veronese map is defined by the point $\alpha = (1, t, t^2, t^3)$ of $\mathbb{P}^3$ with values in $K$. On the open set $\{y_1 \neq 0\}$ of $Y$, the rational functions $1, t, t^2, t^3$ are regular and not all zero. Let $\lambda = t^{-3}$ and $\alpha' = \lambda \alpha = (t^{-3}, t^{-2}, t^{-1}, 1)$. The functions $t^{-k}$ are regular on the open set $\{y_1 \neq 0\}$. So $\alpha$ is a good point that defines a morphism $Y \overset{\Delta}{\rightarrow} X$.

The twisted cubic $X$ is the locus of zeros of the equations (3.1.17).

$$v_0v_2 = v_1^2, \quad v_2v_1 = v_0v_3, \quad v_1v_3 = v_2^2$$

To identify the function field $F$ of $X$, we put $v_0 = 1$, obtaining relations $v_2 = v_1^2, v_3 = v_2^3$. Then $F$ is the field $\mathbb{C}(v_1)$. The point of $Y = \mathbb{P}^1$ with values in $F$ that defines the inverse of the morphism $\alpha$ is $\beta = (1, v_1)$. □

3.4.24. Lemma. Let $Y \overset{f}{\rightarrow} X$ be a morphism of varieties, let $\{X^i\}$ be an open covering of $X$, and let $Y^i = f^{-1}X^i$. If the restrictions $Y^i \overset{f^i}{\rightarrow} X^i$ of $f$ are isomorphisms, then $f$ is an isomorphism.

proof. Let $g^i$ denote the inverse of the morphism $f^i$. Then $g^i = g^j$ on $X^i \cap X^j$ because $f^i = f^j$ on $Y^i \cap Y^j$. By (3.4.21) (iii), there is a unique morphism $X \overset{g}{\rightarrow} Y$ whose restriction to $Y^i$ is $g^i$. That morphism is the inverse of $f$. □

(3.4.25) the diagonal

Let $X$ be a variety. The diagonal $X_\Delta$ is the set of points $(p, p)$ in $X \times X$. It is an example of a subset of $X \times X$ that is closed in the Zariski topology, but not in the product topology.

3.4.26. Proposition. Let $X$ be a variety. The diagonal $X_\Delta$ is a closed subvariety of the product variety $X \times X$.

proof. Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$ that contains $X$, and let $x_0, ..., x_n$ and $y_0, ..., y_n$ be coordinates in the two factors of $\mathbb{P} \times \mathbb{P}$. The diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$ is the closed subvariety defined by the bilinear equations $x_iy_j = x_jy_i$, or in the Segre variables, by the equations $w_{ij} = w_{ji}$, which show that the ratios $x_i/x_j$ and $y_i/y_j$ are equal.

Next, suppose that $X$ is the closed subvariety of $\mathbb{P}$ defined by a system of homogeneous equations $f(x) = 0$. The diagonal $X_\Delta$ can be identified as the intersection of the product $X \times X$ with the diagonal $\mathbb{P}_\Delta$ in $\mathbb{P} \times \mathbb{P}$, so it is a closed subvariety of $X \times X$. As a closed subvariety of $\mathbb{P} \times \mathbb{P}$, the diagonal $X_\Delta$ is defined by the equations

\[(3.4.27) \quad x_iy_j = x_jy_i \quad \text{and} \quad f(x) = 0 \quad \text{$X_\Delta$}\]

The equations $f(y) = 0$ are redundant. Finally, $X_\Delta$ is irreducible because it is homeomorphic to $X$. □

It is interesting to compare Proposition (3.4.26) with the Hausdorff condition for a topological space. The proof of the next lemma is often assigned as an exercise in topology.

3.4.28. Lemma. A topological space $X$ is a Hausdorff space if and only if, when $X \times X$ is given the product topology, the diagonal $X_\Delta$ is a closed subset of $X \times X$. □

Though a variety $X$ with its Zariski topology isn’t a Hausdorff space unless it is a point, Lemma (3.4.28) doesn’t contradict Proposition (3.4.26) because the Zariski topology on $X \times X$ is finer than the product topology.
Let \( Y \rightarrow X \) be a morphism of varieties. The **graph** \( \Gamma \) of \( f \) is the subset of \( Y \times X \) of pairs \((q, p)\) such that \( p = f(q)\).

**3.4.30. Proposition.** The graph \( \Gamma_f \) of a morphism \( Y \rightarrow X \) is a closed subvariety of \( Y \times X \), that is isomorphic to \( Y \).

**proof.** We form a diagram of morphisms

\[
\begin{array}{ccc}
\Gamma_f & \rightarrow & Y \times X \\
v \downarrow & & \downarrow f \circ id \\
X_\Delta & \rightarrow & X \times X
\end{array}
\]

where \( v \) sends a point \((q, p)\) of \( \Gamma_f \) with \( f(q) = p \) to \((p, p)\). The graph \( \Gamma_f \) is the inverse image in \( Y \times X \) of the diagonal \( X_\Delta \). Since the diagonal is closed in \( X \times X \), \( \Gamma_f \) is closed in \( Y \times X \).

Let \( \pi_1 \) denote the projection from \( X \times Y \) to \( Y \). The composition of the morphisms \( Y \xrightarrow{(id, f)} Y \times X \xrightarrow{\pi_1} Y \) is the identity map on \( Y \), and the image of the map \((id, f)\) is the graph \( \Gamma_f \). Therefore \( Y \) maps bijectively to \( \Gamma_f \). The two maps \( Y \rightarrow \Gamma_f \) and \( \Gamma_f \rightarrow Y \) are inverses, so \( \Gamma_f \) is isomorphic to \( Y \). \( \square \)

**3.4.32. Definition**

The map

\[
\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^{n-1}
\]

that drops the last coordinate of a point: \( \pi(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1}) \) is called a *projection*. It is defined at all points of \( \mathbb{P}^n \) except at the point \( q = (0, \ldots, 0, 1) \), which is called the center of projection. So \( \pi \) is a morphism from the complement \( U = \mathbb{P}^n - \{q\} \) to \( \mathbb{P}^{n-1} \).

Let the coordinates in \( \mathbb{P}^n \) and \( \mathbb{P}^{n-1} \) be \( x = x_0, \ldots, x_n \) and \( y = y_0, \ldots, y_{n-1} \), respectively. The fibre \( \pi^{-1}(y) \) over a point \((y_0, \ldots, y_{n-1})\) is the set of points \((x_0, \ldots, x_n)\) such that \((x_0, \ldots, x_{n-1}) = (\lambda y_0, \ldots, \lambda y_{n-1})\), while \( x_n \) is arbitrary. It is the line in \( \mathbb{P}^n \) through the points \((y_1, \ldots, y_{n-1}, 0)\) and \( q = (0, \ldots, 0, 1) \), with the center of projection \( q \) omitted.

In Segre coordinates, the graph \( \Gamma \) of \( \pi \) in \( U \times \mathbb{P}^{n-1} \) is the locus of solutions of the equations \( w_{ij} = w_{ji} \) for \( 0 \leq i, j \leq n-1 \), which imply that the vectors \((x_0, \ldots, x_{n-1})\) and \((y_0, \ldots, y_{n-1})\) are proportional.

**3.4.34. Proposition.** In \( \mathbb{P}^n \times \mathbb{P}^{n-1} \), the locus \( \Gamma \) of the equations \( x_iy_j = x_jy_i \) or \( w_{ij} = w_{ji} \), with \( 0 \leq i, j \leq n-1 \), is the closure of the graph \( \Gamma \) of \( \pi \).

**proof.** The equations are true at points \((x, y)\) of \( \Gamma \) at which \( x \neq q \), and also at all points \((q, y)\). So the locus \( \Gamma \), a closed set, is the union of the graph \( \Gamma \) and the set \( q \times \mathbb{P}^{n-1} \). We must show that a homogeneous polynomial \( g(w) \) that vanishes on \( \Gamma \) vanishes at all points of \( q \times \mathbb{P}^{n-1} \). Given \( y \), let \( x = (t y_0, \ldots, t y_{n-1}, 1) \). For all \( t \neq 0 \), the point \((x, y)\) is in \( \Gamma \) and therefore \( g(x, y) = 0 \). Since \( g \) is a continuous function, \( g(x, y) \) approaches \( g(q, y) \) as \( t \to 0 \). So \( g(q, y) = 0 \). \( \square \)

The projection \( \Gamma \to \mathbb{P}^n \) that sends a point \((x, y)\) to \( x \) is bijective except when \( x = q \). The fibre over \( q \), which is \( q \times \mathbb{P}^{n-1} \), is a projective space of dimension \( n-1 \). Because the point \( q \) of \( \mathbb{P}^n \) is replaced by a projective space in \( \Gamma \), the map \( \Gamma \to \mathbb{P}^n \) is called a blowup of the point \( q \).
proof. Let $P$ and $q$ be points of $X$ and $Y$, respectively. We may assume that $\alpha_i$ are regular and not all zero at $p$ and that $\beta_j$ are regular and not all zero at $q$. Then, in the Segre coordinates $w_{ij}$, $\alpha_i(p)\beta_j(q)$ is the point $w_{ij} = \alpha_i(p)\beta_j(q)$. We must show that $\alpha_i,\beta_j$ are all regular at $(p,q)$ and are not all zero there. This follows from the analogous properties of $\alpha_i$ and $\beta_j$. \hfill \Box

When defining morphisms varieties, one must keep in mind that points of projective space are equivalence classes of vectors, not the vectors themselves. This complication turns out to be very useful.

Some morphisms are sufficiently obvious that they don’t require discussion. They include the projection from a product variety $X \times Y$ to $X$, the inclusion of $X$ into the product $X \times Y$ as the set $X \times y$ for some point $y$ of $Y$, the morphism of products $X \times Y \rightarrow X' \times Y'$ when a morphism $X \rightarrow X'$ is given, and of course, the analogous maps when $Y$ replaces $X$.

If $X$ and $Y$ are subvarieties of projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively, a morphism $Y \rightarrow X$ will be determined by a morphism from $Y$ to $\mathbb{P}^m$ whose image is contained in $X$. However, it is important to note that a morphism $Y \xrightarrow{f} X$ needn’t be the restriction of a morphism from $\mathbb{P}^m$ to $\mathbb{P}^n$. There will often be no way to extend the morphism from $Y$ to $\mathbb{P}^n$. It may not be possible to define $f$ using polynomials in the coordinate variables of $\mathbb{P}^n$.

For example, the Veronese map from the projective line $\mathbb{P}^1$ to $\mathbb{P}^2$, defined by $(x_0,x_1) \mapsto (x_0^2, x_0x_1, x_1^2)$, is an obvious morphism. Its image is the conic $C$ : $v_{00}v_{11} - v_{01}^2 = 0$ in the projective plane $\mathbb{P}^2$. The Veronese defines a bijective morphism $\mathbb{P}^1 \xrightarrow{\phi} C$ whose inverse function sends a point $(v_{00}, v_{01}, v_{11})$ of $C$ with $v_{00} \neq 0$ to the point $(x_0,x_1) = (v_{01}, v_{11})$. There is no way to extend the inverse function $f^{-1}$ to $\mathbb{P}^2$, though it is a morphism. In fact, there is no nonconstant morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$.

In order to have a definition that includes all cases, we will define morphisms using points with values in a field.

(3.4.36) the function field of a product

To define the function field of a product $X \times Y$ of projective varieties, one can use the Segre embedding $\mathbb{P}^m_x \times \mathbb{P}^n_y \rightarrow \mathbb{P}^N_x$. We use notation as in (3.1.10), and let’s denote the product $X \times Y$ by $\Pi$. So $x_i$, $y_j$, and $w_{ij}$ are coordinates in the three projective spaces. The Segre map is defined by $w_{ij} = x_iy_j$. Let $\mathcal{U}^i$, $\mathcal{V}^j$, and $\mathcal{W}^{ij}$ be the standard affine open sets $x_i \neq 0$, $y_j \neq 0$ and $w_{ij} \neq 0$, respectively. The function field will be the field of fractions of the nonempty intersections $\Pi \cap \mathcal{U}^i \cap \mathcal{V}^j \cap \mathcal{W}^{ij} = \Pi^{ij}$, and $\Pi^{ij} \approx X^i \times Y^j$, where $X^i = X \cap \mathcal{U}^i$ and $Y^j = Y \cap \mathcal{V}^j$. Since $\Pi^{ij}$, $X^i$, and $Y^j$ are affine varieties, the function field of the product $\Pi = X \times Y$ is the field of fractions of any one of the nonempty affine open sets $\Pi^{ij}$.

Since $\Pi^{ij} = X^i \cap Y^j$, all that remains to do is to describe the field of fractions of a product of affine varieties $\Pi = X \times Y$, when $X = \text{Spec} A$ and $Y = \text{Spec} B$. If $A = \mathbb{C}[x]/P$ and $B = \mathbb{C}[y]/Q$, the coordinate algebra of $\Pi$ is the algebra $\mathbb{C}[x,y]/(P,Q)$. This is the tensor product algebra $A \otimes B$. We don’t need to know much about the tensor product algebra here, but let’s use the tensor notation.

The function field $K_X$ of $X$ is the field of fractions of the coordinate algebra $A$. Similarly, $K_Y$ is the field of fractions of $B$ and $K_{X \times Y}$ is the field of fractions of $A \otimes B$. The one important fact to note is that $K_{X \times Y}$ isn’t generated by $K_X$ and $K_Y$. For example, if $A = \mathbb{C}[x]$ and $B = \mathbb{C}[y]$ (one $x$ and one $y$), then $K_{X \times Y}$ is the field of rational functions in two variables $\mathbb{C}(x,y)$. The algebra generated by the fraction fields $\mathbb{C}(x)$ and $\mathbb{C}(y)$ consists of the rational functions $p(x,y)/q(x,y)$ in which $q(x,y)$ is a product of $f(x)y^d$ of polynomials in $x$ and a polynomial in $y$. Most rational functions, $1/(x + y)$ for example, aren’t of this type.

But, $K_{X \times Y}$ is the fraction field of $A \otimes B$.

(3.4.37) interlude: rational functions on projective space

###Do we use this? Perhaps make into exercise.### Let $R$ denote the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$. If $f$ is a homogeneous polynomial of positive degree $d$, it makes sense to say that $f$ vanishes at a point of $\mathbb{P}^n$, because $f(x_0) = x_0^d f(x)$. But $f$ doesn’t define a function on $\mathbb{P}^n$. On the other hand, a fraction $g/h$ of homogeneous polynomials of the same degree $d$ does define a function wherever $h$ isn’t zero, because

$$g(\lambda x)/h(\lambda x) = \lambda^d g(x)/\lambda^d h(x) = g(x)/h(x)$$
A homogeneous fraction $f$ is a fraction of homogeneous polynomials. The degree of a homogeneous fraction $f = g/h$ is the difference of degrees: $\deg f = \deg g - \deg h$.

A homogeneous fraction $f$ is regular at a point $p$ of $\mathbb{P}^n$ if, when it is written as a fraction $g/h$ of relatively prime homogeneous polynomials, the denominator $h$ isn’t zero at $p$, and $f$ is regular on a subset $U$ if it is regular at every point of $U$. This definition agrees with the one given above, in Definition 3.4.2.

3.4.38. Lemma. (i) Let $h$ be a homogeneous polynomial of positive degree $d$, and let $V$ be the open subset of $\mathbb{P}^n$, of points at which $h$ isn’t zero. The nonzero rational functions that are regular on $V$ are those of the form $g/h^k$, where $k \geq 0$ and $g$ is a homogeneous polynomial of degree $dk$.

(ii) The only rational functions that are regular at every point of $\mathbb{P}^n$ are the constant functions. The proof of this will be given later (see Corollary ??). When studying projective varieties, the constant functions are useless. One has to look at regular functions on open subsets. One way that affine varieties appear in projective algebraic geometry is as open subsets on which there are enough regular functions.

3.5 Affine Varieties

We have used the term ‘affine variety’ in several contexts:

A closed subset of affine space $\mathbb{A}^n_x$ is an affine variety, the set of zeros of a prime ideal $P$ of $\mathbb{C}[x]$. Its coordinate algebra is $A = \mathbb{C}[x]/P$.

The spectrum $\text{Spec} A$ of a finite type domain $A$ is an affine variety that becomes a closed subvariety of affine space when one chooses a presentation $A = \mathbb{C}[x]/P$.

An affine variety becomes a variety in projective space when the ambient affine space $\mathbb{A}^n_x$ is identified with the standard open subset $\mathbb{U}^0$.

We combine these definitions now: An affine variety $X$ is a variety that is isomorphic to a variety of the form $\text{Spec} A$.

If $X = \text{Spec} A$ is an affine variety with function field $K$, its coordinate algebra $A$ will be the subalgebra of $K$ consisting of the regular functions on $X$. So $A$ and therefore $\text{Spec} A$, are determined uniquely by $X$, and the isomorphism $\text{Spec} A \rightarrow X$ is determined uniquely too. When $X$ is affine, it seems permissible to identify $X$ with $\text{Spec} A$.

(3.5.1) regular functions on affine varieties

Let $X = \text{Spec} A$ be an affine variety. Its function field $K$ is the field of fractions of the coordinate algebra $A$. A rational function $\alpha$ is regular at a point $p$ of $X$ if it can be written as a fraction $a/s$ where $a, s$ are in $A$ and $s(p) \neq 0$, and $\alpha$ is regular on $X$ if it is regular at every point of $X$. On the other hand, in Chapter ?? (??), $\alpha$ is defined to be a regular function on $X$ if and only if it is an element of $A$. The next lemma shows that the two conditions are equivalent.
3.5.2. Lemma. The regular functions on an affine variety \( X = \text{Spec} \ A \), as defined in \( \text{(3.4.2)} \), are the elements of its coordinate algebra \( A \).

\[ \text{proof.} \] Let \( \alpha \) be a regular function on \( X \), as defined above. For every point \( p \) of \( X \), there is a localization \( X_p = \text{Spec} \ A_p \) that contains \( p \), such that \( \alpha \) is an element of \( A_p \). Because \( X \) is quasicompact, a finite set of these localizations, say \( X_{s_1}, \ldots, X_{s_k} \), will cover \( X \). Then \( s_1, \ldots, s_k \) have no common zeros on \( X \). They generate the unit ideal of \( A \). Since \( \alpha \) is in \( A_s \), we can write \( \alpha = s_i^{-1}b_i \) with \( b_i \) in \( A \), and we can use the same exponent \( n \) for all \( i \). Since the elements \( s_i \) generate the unit ideal of \( A \), so do the powers \( s_i^n \). Say that \( \sum s_i^n a_i = 1 \), with \( a_i \) in \( A \). Then \( \alpha = \sum s_i^n a_i \alpha = \sum a_i b_i \) is in \( A \). \( \square \)

3.5.3. Lemma. \( X \)

(i) Let \( R \) be the algebra of regular functions on a variety \( Y \), and let \( A \) be a finite-type domain. A homomorphism \( A \to R \) defines a morphism \( Y \to \text{Spec} \ A \).

(ii) When \( X \) and \( Y \) are affine varieties, say \( X = \text{Spec} \ A \) and \( Y = \text{Spec} \ B \), morphisms \( Y \to X \), as defined in \( \text{(3.4.17)} \), correspond bijectively to algebra homomorphisms \( A \to B \), as in Definition ??.

Note. Since \( Y \) isn’t affine, all that we know about the algebra \( R \) is that its elements are the rational functions that are regular on \( Y \).

\[ \text{proof of Lemma 3.5.3} \] (i) Let \( \{ Y^i \} \) be an affine open covering of \( Y \), and let \( R_i \) be the coordinate algebra of \( Y^i \). The inclusions \( A \subset R \subset R_i \) define morphisms \( R^i = \text{Spec} \ R_i \to \text{Spec} \ A \). It is true that \( f^i = f^j \) on \( Y^i \cap Y^j \), so Lemma \( \text{[3.4.2]} \)\( \text{(iii)} \) applies.

(ii) We choose a presentation of \( A \), to embed \( X \) as a closed subvariety of affine space, and we identify that affine space with the standard affine open set \( \mathbb{A}^n \) of \( \mathbb{P}^n \). Let \( K \) be the function field of \( Y \) – the field of fractions of \( B \). A morphism \( Y \to X \) is determined by a good point \( \alpha \) with values in \( K \), for which \( \alpha(t) \neq 0 \). We may suppose that this point has the form \( \alpha = (1, \alpha_1, \ldots, \alpha_n) \). Then the rational functions \( \alpha_i \) will be regular at every point of \( Y \). They are elements of \( B \). The coordinate algebra \( A \) of \( X \) is generated by the residues of the coordinate variables \( x_1, \ldots, x_n \), with \( x_0 = 1 \). Sending \( x_i \to \alpha_i \) defines a homomorphism \( A \to B \). Conversely, if \( \varphi \) is a such a homomorphism, the good point that defines the morphism \( Y \to X \) is \( (1, \varphi(x_1), \ldots, \varphi(x_n)) \). \( \square \)

3.5.4) *affine open subsets*

An affine open subset of a variety \( X \) is an open subset that is an affine variety. If \( V \) is a nonempty open subset of \( X \) and \( R \) is the algebra of rational functions that are regular on \( V \), then \( V \) is an affine open subset if and only if \( R \) is a finite-type domain and \( V \) is isomorphic to \( \text{Spec} \ R \).

3.5.5. Proposition. The complement of a hypersurface is an affine open subvariety of \( \mathbb{P}^n \).

\[ \text{proof.} \] Let \( V \) be the complement of the hypersurface \( \{ f = 0 \} \), where \( f \) is an irreducible homogeneous polynomial of degree \( d \), let \( R \) be the algebra of regular functions on \( V \), and let \( K \) be its fraction field, the field of rational functions on \( V \).

The regular functions on \( V \) are the homogeneous fractions of degree zero of the form \( g/f^k \) \( \text{(3.4.37)} \), and the fractions \( m/f \), where \( m \) is a monomial of degree \( d \), generate \( R \). Since there are finitely many monomials of degree \( d \), \( R \) is a finite-type domain. Let \( w \) be an arbitrary monomial of degree \( d - 1 \), and let \( s_i = x_i w/f \). The point \( (x_0, \ldots, x_n) \) of \( V \) can also be written as \( (s_0, \ldots, s_n) \), and the fractions \( s_i \) are among the generators for \( R \). Let \( W = \text{Spec} \ R \). Then \( (s_0, \ldots, s_n) \) is a point of \( W \) with values in \( K \) that defines a morphism \( W \to V \). We show that \( \iota \) is an isomorphism.

3.5.6. Lemma. Let \( \mathbb{U}^i \) be the standard affine open subset of \( \mathbb{P}^n \). With \( s_i \) as above, the intersection \( V^i = V \cap \mathbb{U}^i \) is isomorphic to the localization \( W_{s_i} / W \).

\[ \text{proof.} \] We work with the index \( i = 0 \), as usual. Let \( s = x_0^d/f \) and \( t = s^{-1} = f/x_0^d \). Also, let \( P \) be the coordinate algebra of \( \mathbb{U}^0 \). Then \( V^0 = V \cap \mathbb{U}^0 \) is the set of points of \( \mathbb{U}^0 \) at which \( t \) isn’t zero. Its coordinate algebra is the localization \( P_t \), and \( V^0 \) is the affine variety \( \text{Spec} \ P_t \).
Applying Lemma 3.4.24 one sees that the lemma will follow if we show that $P_i$ is the localization $R_s$ of $R$. With coordinates $u_j = x_j/x_0$ for $U^0$, a fraction $m/f$, where $m$ is the monomial $x_{j_1} \ldots x_{j_d}$, can also be written as $u_{j_1} \ldots u_{j_d}/t$. These fractions generate $R$, so $R \subset P_i$, and since $s^{-1} t = t$ in $P_i = P_i = P_i = P_i$. For the other inclusion, we write $u_j = (x_j x_0)^{-1}/f$ $s^{-1}$. Because $x_j x_0^{-1}/f$ is in $R$, $u_j$ is in $R_s$. Therefore $P \subset R_s$, and $P_i \subset R_s$, as claimed.

We go back to the proof of Proposition 3.5.5. The sets $V^i = V \cap U^i$ for $i = 0, \ldots, n$ cover $V$, and the morphism $W \to V$ restricts to an isomorphism $V^i \to \text{Spec } R_s$. So $z$ is an isomorphism (3.4.24).

affinesbasis

3.5.7. Lemma. The affine open subsets of a variety $X$ form a basis for the topology on $X$.

proof. See Proposition ??.

intersectaffine

3.5.8. Theorem Let $U$ and $V$ be affine open subvarieties of a variety $X$, say $U \approx \text{Spec } A$ and $V \approx \text{Spec } B$. The intersection $U \cap V$ is an affine open subvariety whose coordinate algebra is generated by the two algebras $A$ and $B$.

proof. We denote the algebra generated by two subalgebras $A$ and $B$ of the function field $K$ of $X$ by $[A, B]$. The elements of $[A, B]$ are finite sums of products of elements of $A$ and $B$. If $A = \mathbb{C}[a_1, \ldots, a_r]$, and $B = \mathbb{C}[b_1, \ldots, b_s]$, then $[A, B]$ is the finite-type subalgebra of $K$ generated by the set $\{a_i \cup b_j\}$, and let $W = \text{Spec } R$, where $R = [A, B]$. We are to show that $W$ is isomorphic to $U \cap V$. The varieties $U, V, W$ have the same function field $K$ as $X$. The inclusions of coordinate algebras $A \to R$ and $B \to R$ give us morphisms $W \to U$ and $W \to V$. We also have inclusions $U \subset X$ and $V \subset X$, and $X$ is a subvariety of a projective space $\mathbb{P}^n$. Let $\alpha$ be the $\mathbb{P}^n$ with values in $K$ that defines the projective embedding $X \to \mathbb{P}^n$. This point also defines morphisms $U \to \mathbb{P}^n$ and $V \to \mathbb{P}^n$ and $W \to \mathbb{P}^n$. The morphisms $\alpha$ and $\beta$ are the restrictions of $\varphi$ to the open subsets $U$ and $V$ of $X$, respectively.

The embedding $\varphi$ of $W$ can be obtained as the composition of the morphisms $W \to U \subset X \to \mathbb{P}^n$, and also as the analogous composition, in which $V$ replaces $U$. Therefore the image of $W$ in $X$ is contained in $U \cap V$. (I suggest this slightly confusing point as an exercise.) Thus we obtain a morphism $W \to U \cap V$.

We show that $\epsilon$ is an isomorphism.

Let $p$ be a point of $U \cap V$. We choose an affine open subset $Z$ of $U \cap V$ that is a localization of $U$ and of $V$, and that contains $p$. Let $S$ be the coordinate ring of $Z$. So $S = A_s$ for some nonzero $s$ in $A$ and also $S = B_t$ for some nonzero $t$ in $B$.

Then $R_s = [A, B]_s = [A_s, B] = [S, B] = S$.

So $\epsilon$ maps the localization $W_s = \text{Spec } R_s$ of $W$ isomorphically to the open subset $Z$ of $U \cap V$. Since we can cover $U \cap V$ by open sets such as $Z$, Lemma 3.4.21 (ii) shows that $\epsilon$ is an isomorphism.

3.6 Lines in Projective Three-Space

The Grassmanian $G(m, n)$ is a variety whose points correspond to subspaces of dimension $m$ of the vector space $\mathbb{C}^n$, and to linear subspaces of dimension $m - 1$ of $\mathbb{P}^{n-1}$. One says that $G(m, n)$ parametrizes those subspaces. For example, the Grassmanian $G(1, n+1)$ is the projective space $\mathbb{P}^n$. Points of $\mathbb{P}^n$ parametrize one-dimensional subspaces of $\mathbb{C}^{n+1}$.

The Grassmanian $G(2, 4)$ parametrizes two-dimensional subspaces of $\mathbb{C}^4$, and lines in $\mathbb{P}^3$. In this section we describe that Grassmanian, denoting it by $G$. The point of $G$ that corresponds to a line $\ell$ in $\mathbb{P}^3$ will be denoted by $[\ell]$.

One can get some insight into the structure of $G$ using row reduction. Let $V = \mathbb{C}^4$, let $u_1, u_2$ be a basis of a two-dimensional subspace $U$ of $V$ and let $M$ be the $2 \times 4$ matrix whose rows are $u_1, u_2$. The rows of the matrix $M'$ obtained from $M$ by row reduction span the same space $U$, and the row-reduced matrix $M'$ is uniquely determined by $U$. Provided that the left hand $2 \times 2$ submatrix of $M$ is invertible, $M'$ will have the form

$$
M' = \begin{pmatrix}
1 & 0 & \ast & \ast \\
0 & 1 & \ast & \ast 
\end{pmatrix}
$$

So the Grassmanian $G$ contains, as an open subset, a four-dimensional affine space whose coordinates are the variable entries of $M'$.
In any $2 \times 4$ matrix $M$ with independent rows, some pair of columns will be independent. Those columns can be used in place of the first two in a row reduction. So $G$ is covered by six four-dimensional affine spaces that we denote by $\mathbb{W}^j$, $1 \leq i < j \leq 4$, $\mathbb{W}^j$ being the space of $2 \times 4$ matrices such that $column_i = (1, 0)^t$ and $column_j = (0, 1)^t$. Since $\mathbb{P}^4$ and the Grassmanian are both covered by affine spaces of dimension four, they may seem similar, but they aren’t the same.

(3.6.2) the exterior algebra

Let $V$ be a complex vector space. The exterior algebra $\bigwedge V$ (read ‘wedge $V$’) is a noncommutative ring that contains the complex numbers and is generated by the elements of $V$, with the relations

\begin{equation}
vw = -wv \quad \text{for all } v, w \in V.
\end{equation}

(3.6.3)

3.6.4. Lemma. The condition \(3.6.3\) is equivalent with: $vw = 0$ for all $v$ in $V$.

**proof.** To get $vw = 0$ from \(3.6.3\), one sets $w = v$. Suppose that $vw = 0$ for all $v$ in $V$. Then $(v+w)(v+w) = vv + vw + vw + ww$, $vw + vw = 0$. \( \square \)

To familiarize yourself with computation in $\bigwedge V$, verify that $v_2v_3v_4 = v_1v_2v_3v_4$ and that $v_2v_3v_4v_1 = -v_1v_2v_3v_4$.

Let $\bigwedge^ r V$ denote the subspace of $\bigwedge V$ spanned by products of length $r$ of elements of $V$. The exterior algebra $\bigwedge V$ is the direct sum of the subspaces $\bigwedge^ r V$. An algebra $A$ that is a direct sum of subspaces $A^i$, and such that multiplication maps $A^i \times A^j$ to $A^{i+j}$ is called a graded algebra. Since its multiplication law isn’t commutative, the exterior algebra is a noncommutative graded algebra.

3.6.5. Proposition. If $(v_1, \ldots, v_n)$ is a basis for $V$, the products $v_1 \cdots v_r$, of length $r$ with increasing indices $1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^ r V$.

The proof is at the end of the section.

3.6.6. Corollary. Let $v_1, \ldots, v_r$ be elements of $V$. The product $v_1 \cdots v_r$ is zero in $\bigwedge V$ if and only if the set $(v_1, \ldots, v_r)$ is dependent. \( \square \)

For the rest of the section, we let $V$ be a vector space of dimension four with basis $(v_1, \ldots, v_4)$. Proposition \(3.6.5\) tells us that

\begin{align*}
\bigwedge^0 V &= \mathbb{C} \text{ is a space of dimension 1, with basis } \{1\} \\
\bigwedge^1 V &= V \text{ is a space of dimension 4, with basis } \{v_1, v_2, v_3, v_4\} \\
\bigwedge^2 V &= V \text{ is a space of dimension 6, with basis } \{v_iv_j \mid i < j\} = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\} \\
\bigwedge^3 V &= V \text{ is a space of dimension 4, with basis } \{v_iv_jv_k \mid i < j < k\} = \{v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4\} \\
\bigwedge^4 V &= V = 0 \text{ when } q > 4.
\end{align*}

The elements of $\bigwedge^2 V$ are combinations

\begin{equation}
w = \sum_{i < j} a_{ij}v_iv_j
\end{equation}

We regard $\bigwedge^2 V$ as an affine space of dimension 6, identifying the combination $w$ with the vector whose coordinates are the six coefficients $a_{ij}$ ($i < j$). We use the same symbol $w$ to denote the point of the projective space $\mathbb{P}^5$ with those coordinates: $w = (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$.

3.6.9. Definition. An element $w$ of $\bigwedge^2 V$ is decomposable if it is a product of two elements of $V$.

3.6.10. Proposition. The decomposable elements of $\bigwedge^2 V$ are those such that $ww = 0$, and the relation $ww = 0$ is given by the following equation in $v$ the coefficients $a_{ij}$ of $w = \sum_{i < j} a_{ij}v_iv_j$:

\begin{equation}
a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0
\end{equation}

\(3.6.11\)
proof. If \( w \) is decomposable, say \( w = u_1 u_2 \), then \( w^2 = u_1 u_2 u_1 u_2 = -u_1^2 u_2^2 \) is zero because \( u_1^2 = 0 \). For the converse, we compute \( w^2 \) when \( w = \sum_{i<j} a_{ij} v_i v_j \). The answer is

\[
ww = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})v_1 v_2 v_3 v_4
\]

To show that \( w \) is decomposable if \( w^2 = 0 \), it seems simplest to factor \( w \) explicitly. Since the assertion is trivial when \( w = 0 \), we may suppose that some coefficient of \( w \), say \( a_{12} \), is nonzero. Then if \( w^2 = 0 \), \( w \) is the product

\[
(3.6.12) \quad w = \frac{1}{a_{12}}(a_{12}v_2 + a_{13}v_3 + a_{14}v_4)(-a_{12}v_1 + a_{23}v_3 + a_{24}v_4)
\]

\[
\square
\]

3.6.13. Corollary. (i) Let \( w \) be a nonzero decomposable element of \( \wedge^2 V \), say \( w = u_1 u_2 \), with \( u_1 \) in \( V \). Then \( (u_1, u_2) \) is a basis for a two-dimensional subspace of \( V \).

(ii) If \( (u_1, u_2) \) and \( (u_1', u_2') \) are bases for the same subspace \( U \) of \( V \), then \( w = u_1 u_2 \) and \( w' = u_1' u_2' \) differ by a scalar factor. Their coefficients represent the same point of \( \mathbb{P}^5 \).

(iii) Let \( u_1, u_2 \) be a basis for a two-dimensional subspace \( U \) of \( V \), and let \( w = u_1 u_2 \). The rule \( \epsilon(U) = w \) defines a bijection \( \epsilon \) from \( G \) to the quadric \( Q \) in \( \mathbb{P}^5 \) whose equation is \( \epsilon(u_1, u_2) = 0 \).

Thus \( G \) can be represented as the quadric \( \epsilon(u_1, u_2) = 0 \).

proof. (i) If an element \( w \) of \( \wedge^2 V \) is decomposable, say \( w = u_1 u_2 \), and if \( w \) is nonzero, then \( u_1 \) and \( u_2 \) must be independent \( (3.6.6) \). They span a two-dimensional subspace.

(ii) When we write the second basis in terms of the first one, say \( (u_1', u_2') = (au_1 + bu_2, cu_2 + du_2) \), the product \( u_1' u_2' \) becomes \( (ad - bc)u_1 u_2 \), and \( ad - bc \neq 0 \).

(iii) In view of (i) and (ii), all that remains to show is that, if \( (u_1, u_2) \) and \( (u_1', u_2') \) are bases for distinct two-dimensional subspaces \( U \) and \( U' \), then in \( \wedge^2 V \), \( u_1 u_2 \neq u_1' u_2' \).

Since \( U \neq U' \), the intersection \( W = U \cap U' \) has dimension at most 1. At least three of the vectors \( u_1, u_2, u_1', u_2' \) will be independent. Therefore \( u_1 u_2 \neq u_1' u_2' \).

\[
\square
\]

For the rest of this section, we use the algebraic dimension of a variety, a concept that will be studied in the next chapter. We refer to the algebraic dimension simply as the dimension. The dimension of a variety \( X \) can be defined as the length \( d \) of the longest chain \( C_0 > C_1 > \cdots > C_d \) of closed subvarieties of \( X \).

As was mentioned in Chapter ??, the topological dimension of \( X \) its dimension in the classical topology, is always twice the algebraic dimension. Because the Grassmanian \( G \) is covered by affine spaces of dimension 4, its algebraic dimension is 4 and its topological dimension is 8.

3.6.14. Proposition. Let \( \mathbb{P}^3 \) be the projective space associated to a four dimensional vector space \( V \). In the product \( \mathbb{P}^3 \times G \), the locus \( \Gamma \) of pairs \( p, [\ell] \) such that the point \( p \) of \( \mathbb{P}^3 \) lies on the line \( \ell \) is a closed subset of dimension 5.

proof. Let \( \ell \) be the line in \( \mathbb{P}^3 \) that corresponds to the subspace \( U \) with basis \( (u_1, u_2) \), and say that \( p \) represented by the vector \( x \) in \( V \). Let \( w = u_1 u_2 \). Then \( p \in \ell \) means \( x \in U \), which is true if and only if \( (x, u_1, u_2) \) is a dependent set, and this happens if and only if \( x w = 0 \) \( (3.6.5) \). So \( \Gamma \) is the closed subset of points \( (x, w) \) of \( \mathbb{P}^3 \times \mathbb{P}^5 \) defined by the bihomogeneous equations \( w^2 = 0 \) and \( x w = 0 \).

When we project \( \Gamma \) to \( G \), The fibre over a point \( [\ell] \) of \( G \) is the set of points \( p, [\ell] \) such that \( p \) is a point of \( \ell \), which maps bijectively to the line \( \ell \). Thus \( \Gamma \) can be viewed as a family of lines, parametrized by the four-dimensional variety \( G \) of \( X \). Its dimension is \( \dim \ell + \dim G = 1 + 4 = 5 \).

\[
\square
\]

3.6.15 lines on a surface

One may ask whether or not a given surface in \( \mathbb{P}^3 \) contains a line. One surface that contains lines is the quadric \( Q \) in \( \mathbb{P}^3 \) with equation \( w_0 w_1 w_0 w_1 = u_0 w u_1 \), the image of the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) \( (3.1.7) \). It contains two families of lines, corresponding to the two “rulings” \( p \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times q \) of \( \mathbb{P}^1 \times \mathbb{P}^1 \). There are surfaces of arbitrary degree that contain lines, but, that a generic surface of degree four or more doesn’t contain any line.
We use coordinates $x_i$, with $i = 1, 2, 3, 4$ for $\mathbb{P}^3$ here. There are $N = \binom{d+3}{3}$ monomials of degree $d$ in four variables, so homogeneous polynomials of degree $d$ are parametrized by an affine space of dimension $N$, and surfaces of degree $d$ in $\mathbb{P}^3$ by a projective space of dimension $N-1$. Let $S$ denote that projective space, and let $[S]$ denote the point of $S$ that corresponds to a surface $S$. The coordinates of $[S]$ are the coefficients of the monomials in the defining polynomial $f$ of $S$. Speaking informally, we say that a point of $S$ “is” a surface of degree $d$ in $\mathbb{P}^3$. (When $f$ is reducible, its zero locus isn’t a variety. Let’s not worry about this.)

Consider the line $\ell_0$ defined by $x_3 = x_4 = 0$. Its points are those of the form $(x_1, x_2, 0, 0)$, so a surface $S : \{ f = 0 \}$ will contain $\ell_0$ if and only if $f(x_1, x_2, 0, 0) = 0$ for all $x_1, x_2$. Substituting $x_3 = x_4 = 0$ into $f$ leaves us with a polynomial in two variables:

$$(3.6.16) \quad f(x_1, x_2, 0, 0) = c_0 x_1^d + c_1 x_1^{d-1} x_2 + \cdots + c_d x_2^d$$

where the coefficients $c_i$ are among the coefficients of the polynomial $f$. If $f(x_1, x_2, 0, 0)$ is identically zero, all of those coefficients must be zero. So the surfaces that contain $\ell_0$ correspond to the lines of the Grassmanian $G$. The line $\ell$ is zero for all $r, s$.

Let $[S]$ denote the set of pairs $\ell, [S]$ such that $\ell \subset S$ is a closed subset.

**Lemma.** In the product variety $\mathbb{G} \times S$, the set $\Gamma$ of pairs $[\ell], [S]$ such that $\ell \subset S$ is a closed subset.

**Proof.** Let $\mathbb{W}^j$, $1 \leq i < j \leq 4$ denote the six affine spaces that cover the Grassmanian, as at the beginning of this section. It suffices to show that the intersection $\Gamma^{ij} = \Gamma \cap (\mathbb{W}^j \times S)$ is closed in $\mathbb{W}^j \times S$. (3.4.20). We inspect the case $i, j = 1, 2$.

A line $\ell$ such that $[\ell]$ is in $\mathbb{W}^{12}$ corresponds to a subspace of $\mathbb{C}^2$ with basis of the form $u_1 = (1, a_2, a_3)$, $u_2 = (0, 1, b_2, b_3)$, and $\ell$ is the line whose points are combinations $ru_1 + su_2$ of $u_1$, $u_2$. Let $f(x_1, x_2, x_3, x_4)$ be the polynomial that defines a surface $S$. The line $\ell$ is contained in $S$ if and only if $f(r, s, x_2, x_3, x_4)$ is zero for all $r$ and $s$. This is a homogeneous polynomial of degree $d$ in $r, s$. Let’s call it $f(r, s)$. If we write $f(r, s) = z_0 r^d + z_1 r^{d-1} s + \cdots + z_d s^d$, the coefficients $z_i$ will be polynomials in $a_i, b_i$ and in the coefficients of $f$. The locus $z_0 + \cdots + z_d = 0$ is the closed set $\Gamma^{12}$ of $\mathbb{W}^{12} \times S$.

The set of surfaces that contain our special line $\ell_0$ corresponds to the linear space $L_0$ of $S$ of dimension $N-d-2$, and $\ell_0$ can be carried to any other line $\ell$ by a linear map $\mathbb{P}^3 \rightarrow \mathbb{P}^3$. So the surfaces that contain another line $\ell$ also form a linear subspace of $S$ of dimension $N-d-2$. They are the fibres of $\Gamma$ over $S$. The dimension of the Grassmanian $G$ is 4. Therefore the dimension of $\Gamma$ is $\text{dim } \Gamma = \text{dim } L_0 + \text{dim } G = (N-d-2) + 4$. Since $S$ has dimension $N-1$,

$$(3.6.18) \quad \text{dim } \Gamma = \text{dim } S - d + 3$$

dim-space_lines

We project the product $\mathbb{G} \times S$ and its subvariety $\Gamma$ to $S$. The fibre of $\Gamma$ over a point $[S]$ is the set of pairs $[\ell], [S]$ such that $\ell$ is contained in $S$ – the set of lines in $S$.

When the degree $d$ of the surfaces we are studying is 1, $\text{dim } \Gamma = \text{dim } S + 2$. Every fibre of $\Gamma$ over $S$ will have dimension at least 2. In fact, every fibre has dimension equal to 2. Surfaces of degree 1 are planes, and the lines in a plane form a two-dimensional family.

When $d = 2$, $\text{dim } \Gamma = \text{dim } S + 1$. We can expect that most fibres of $\Gamma$ over $S$ will have dimension 1. This is true: A smooth quadric contains two one-dimensional families of lines. (All smooth quadrics are equivalent with the quadric $(3.1.8)$.) But if a quadratic polynomial $f(x_1, x_2, x_3, x_4)$ is the product of linear polynomials, its locus of zeros will be a union of planes. It will contain two-dimensional families of lines. Some fibres have dimension 2.

When $d \geq 4$, $\text{dim } \Gamma < \text{dim } S$. The projection $\Gamma \rightarrow S$ cannot be surjective. Most surfaces of degree 4 or more contain no lines.

The most interesting case is that $d = 3$. In this case, $\text{dim } \Gamma = \text{dim } S$. Most fibres will have dimension zero. They will be finite sets. In fact, a generic cubic surface contains 27 lines. We have to wait to see why the number is precisely 27 (see Theorem ??).
Our conclusions are intuitively plausible, but to be sure about them, we need to study dimension carefully. We do this in the next chapters.

proof of Proposition 3.6.5 Let $v = (v_1, ..., v_q)$ be a basis of the vector space $V$. The proposition asserts that the products $v_1 \cdots v_r$ of length $r$ with increasing indices $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$.

To prove this, we need to be more precise about the definition of the exterior algebra $\bigwedge V$. We start with the algebra $T(V)$ of noncommutative polynomials in the basis $v$, which is also called the tensor algebra on $V$. The part $T^n(V)$ of $T(V)$ of degree $r$ has as basis the $n^r$ noncommutative monomials of degree $r$, products $v_1 \cdots v_r$ of length $r$ of elements of the basis $v$. Its dimension is $n^r$. When $n = r = 2$, $T^2(V)$ has the basis $(x_1^2, x_1 x_2, x_2 x_1, x_2^2)$.

The exterior algebra $\bigwedge V$ is the quotient of $T(V)$ obtained by forcing the relations $vw + wv = 0$ (3.6.3). Using the distributive law, one sees that the relations $v_1 v_j + v_j v_1 = 0$, $1 \leq i, j \leq n$, are sufficient to define this quotient. The relations $v_i v_i = 0$ are included when $i = j$.

To obtain $\bigwedge V$, we multiply the relations $v_i v_j + v_j v_i$ on left and right by arbitrary noncommutative monomials $p(v)$ and $q(v)$ in $v_1, ..., v_n$ whose degrees add to $r - 2$. The noncommutative polynomials

$$p((v_i v_j + v_j v_i) q)$$

span the kernel of the linear map $T^r(V) \to \bigwedge^r V$. So in $\bigwedge^r V$, $p(v_i v_j) q = -p(v_j v_i) q$. Using these relations, any product $v_1 \cdots v_r$ in $\bigwedge^r V$ is, up to sign, equal to a product in which the elements $v_i$ are listed in increasing order. Thus the products with indices in increasing order span $\bigwedge^r V$, and because $v_i v_i = 0$, such a product will be zero unless the indices are strictly increasing.

We go to the proof now. Let $v = (v_1, ..., v_n)$ be a basis for $V$. We show first that the product $w = v_1 \cdots v_n$ in increasing order of the basis elements of $V$ is a basis of $\bigwedge^n V$. We have shown that this product spans $\bigwedge^n V$, and it remains to show that $w \neq 0$, or that $\bigwedge^n V \neq 0$.

Let’s use multi-index notation: $(i) = (i_1, ..., i_r)$, and $v(i) = v_{i_1} \cdots v_{i_r}$. We define a surjective linear map $T^n(V) \to \mathbb{C}$ on the basis of $T^n(V)$ of products $v(i) = (v_{i_1} \cdots v_{i_n})$ of length $n$. If there is no repetition among the indices $i_1, ..., i_n$, then $(i)$ will be a permutation of the indices $1, ..., n$. In that case, we set $\varphi(v(i)) = \varphi(v_{i_1} \cdots v_{i_n}) = \text{sign}(i)$. If there is a repetition, we set $\varphi(v(i)) = 0$.

Let $p$ and $q$ be noncommutative monomials whose degrees add to $n - 2$. If the product $p(v_i v_j) q$ has no repeated index, the indices in $p(v_i v_j) q$ and $p(v_j v_i) q$ will be permutations of $1, ..., n$, and those permutations will have opposite signs. Then $p(v_i v_j + v_j v_i) q$ will be in the kernel of $\varphi$. Since these elements span the space of relations, $\varphi$ defines a surjective linear map $\bigwedge^n V \to \mathbb{C}$. Therefore $\bigwedge^n V \neq 0$.

To prove (3.6.5), we must show that for $r \leq n$, the products $v_1 \cdots v_r$ with $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$, and we know that those products span $\bigwedge^r V$. We must show that they are independent. Suppose that a combination $z = \sum c_i v(i)$ is zero, the sum being over sets of strictly increasing indices. We choose a set $(j_1, ..., j_r)$ of $r$ strictly increasing indices, and we let $(k) = (k_1, ..., k_{n-r})$ be the set of $n - r$ indices that don’t occur in $(j)$, listed in arbitrary order. Then all terms in the sum $z v(k) = \sum c_i v(i) v(k)$ will be zero except the term with $(i) = (j)$. On the other hand, since $z = 0$, $z v(k) = 0$. Therefore $c_j v(j) v(k) = 0$, and since $v(j) v(k)$ differs by sign from $v_1 \cdots v_n$, it isn’t zero. It follows that $c_j = 0$. This is true for all $(j)$, so $z = 0$. \qed