Massachusetts Institute of Technology

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NOTES FOR A COURSE IN
ALGEBRAIC GEOMETRY

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PREFACE

These are notes for an algebraic geometry course at MIT. I had thought of developing such a course for quite a while, motivated partly by the fact that MIT didn't have many courses that were suitable for students who had taken the standard theoretical math classes. I got around to thinking about this seriously twelve years ago, and have now taught the class seven times. Since I wanted to get to cohomology of $\mathcal{O}$-modules (aka coherent sheaves) in one semester, without presupposing a knowledge of sheaf theory or of much commutative algebra, it has been a challenge. Fortunately, MIT has many outstanding students who are interested in mathematics. The students and I have made some progress, but much remains to be done. One would like the development to be so natural as to seem obvious, and this has yet to be achieved. There are too many pages for my taste but, to paraphrase Pascal, we haven't had time to make it shorter.

To cut the material down, I decided to work exclusively with varieties over the complex numbers, and to use this restriction freely. Schemes are not discussed. Some may disagree with these decisions, but I feel that absorbing schemes and general ground fields won't be too difficult for someone who is familiar with complex varieties. Also, I don't go out of my way to state and prove things in their most general form.

Great thanks are due to the students who have been in my classes. Many of you contributed to these notes by commenting on the drafts or by creating figures. Though I remember you well, I'm not naming you individually because I'm sure that I'd overlook someone important. I hope you will understand.

A Note for the Student

The prerequisites are standard undergraduate courses in algebra, analysis, and topology, and the definitions of category and functor. I also suppose a familiarity with the implicit function theorem for complex variables. But don't worry too much about the prerequisites. You can look them up as needed, and many points are reviewed briefly in the notes as they come up.

Proofs of some lemmas and propositions are omitted. I do this when the proof is simple enough that including it would just clutter up the text or, occasionally, when I feel that it is important for the reader to supply a proof.

As with any mathematics course, working exercises and writing up the solutions carefully is, by far, the best way to learn the material well.

Chapter 1 PLANE CURVES

planecurves

We begin with a chapter on plane curves, because they the first algebraic varieties to be studied. Chapters 2 - 7 are about varieties of arbitrary dimension. We will see in Chapter 5 how curves control higher dimensional varieties, and we will come back to study curves in Chapter 8.

1.1 The Affine Plane

affine-plane

The n-dimensional *affine space* $\mathbb{A}^n$ is the space of n-tuples of complex numbers. The *affine plane* $\mathbb{A}^2$ is the two-dimensional affine space.

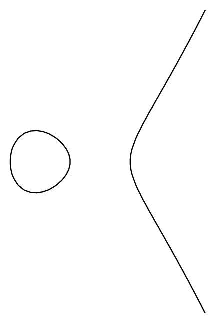
Let $f(x_1, x_2)$ be an irreducible polynomial in two variables with complex coefficients. The set of points of the affine plane at which f vanishes, the *locus of zeros* of f, is called a *plane affine curve*. Let's denote that locus by X. Writing x for the vector (x_1, x_2),

affcurve (1.1.1)

$$X = \{x \,|\, f(x) = 0\}$$

When it seems unlikely to cause confusion, we may, as here, abbreviate the notation for an indexed set, using a single letter.
The *degree* of the curve X is the degree of its irreducible defining polynomial f.

goober8 **1.1.2.**



The Cubic Curve $y^2 = x^3 - x$ (real locus)

1.1.3. Note. In contrast with complex polynomials in one variable, most polynomials in two or more variables are *irreducible* — they cannot be factored. This can be shown by a method called "counting constants". For instance, quadratic polynomials in x_1, x_2 depend on the six coefficients of the monomials $1, x_1, x_2, x_1^2, x_1x_2, x_2^2$ of degree at most two. Linear polynomials ax_1+bx_2+c depend on three coefficients, but the product of two linear polynomials depends on only five parameters, because a scalar factor can be moved from one of the linear polynomials to the other. So the quadratic polynomials cannot all be written as products of linear polynomials. This reasoning is fairly convincing. It can be justified formally in terms of *dimension*, which will be discussed in Chapter 5. □ polyirred

About figures. In algebraic geometry, dimensions are too big to allow realistic figures. Even with an affine plane curve, one is dealing with a locus in the space $\mathbb{A}^2$, whose topological dimension is four. In some cases, such as in Figure 1.1.2 above, depicting the real locus can be helpful, but in most cases, even the real locus is too big, and one must make do with a schematic diagram. □

We will get an understanding of the geometry of a plane curve as we go along, and we mention just one point here. A plane curve X is called a *curve* because it is defined by one equation in two variables. Its *algebraic* dimension is one: The only proper subsets of X that can be defined by polynomial equations are the finite sets (see Proposition 1.3.12). But because our scalars are complex numbers, the affine plane $\mathbb{A}^2$ is a real space of dimension four, and X will be a surface in that space. This is analogous to the fact that the *affine line* $\mathbb{A}^1$ is the plane of complex numbers.

One can see that a plane curve X is a surface by inspecting its projection to a line. To do this, one may write the defining polynomial as a polynomial in x_2, whose coefficients c_i are polynomials in x_1:

$$f(x_1, x_2) = c_0x_2^d + c_1x_2^{d-1} + \cdots + c_d$$

Let's suppose that d is positive, i.e., that f isn't a polynomial in x_1 alone.

The *fibre* of a map $V \to U$ over a point p of U is the inverse image of p — the set of points of V that map to p. The fibre of the projection $X \to \mathbb{A}^1$ over the point $x_1 = a$ is the set of points (a, b) such that b is a root of the one-variable polynomial

$$f(a, x_2) = \overline{c}_0x_2^d + \overline{c}_1x_2^{d-1} + \cdots + \overline{c}_d$$

with $\overline{c}_i = c_i(a)$. There will be finitely many points in this fibre, and it won't be empty unless $f(a, x_2)$ is a constant. So the plane curve X covers most of the x_1-line, a complex plane, finitely often.

(1.1.4) changing coordinates planeco-ords

We allow linear changes of variable and translations in the affine plane $\mathbb{A}^2$. When a point x is written as the column vector: $x = (x_1, x_2)^t$, the coordinates $x' = (x'_1, x'_2)^t$ after such a change of variable will be related to x by the formula

(1.1.5) $$x = Qx' + a$$ chgcoord

where Q is an invertible 2×2 matrix with complex coefficients and $a = (a_1, a_2)^t$ is a complex translation vector. This changes a polynomial equation $f(x) = 0$, to $f(Qx' + a) = 0$. One may also multiply a polynomial f by a nonzero complex scalar without changing the locus $\{f = 0\}$. Using these operations, all *lines*, plane curves of degree 1, become equivalent.

An *affine conic* is a plane affine curve of degree two. Every affine conic is equivalent to one of the loci

(1.1.6) $$x_1^2 - x_2^2 = 1 \quad \text{or} \quad x_2 = x_1^2$$ affconic

The proof of this is similar to the one used to classify real conics. They might be called a complex 'hyperbola' and 'parabola', respectively. The complex 'ellipse' $x_1^2 + x_2^2 = 1$ becomes the 'hyperbola' when one multiplies x_2 by i.

On the other hand, there are infinitely many inequivalent cubic curves. Cubic polynomials in two variables depend on the coefficients of the ten monomials $1,\ x_1, x_2,\ x_1^2, x_1x_2, x_2^2,\ x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$ of degree at most 3 in x. Linear changes of variable, translations, and scalar multiplication give us only seven scalars to work with, leaving three essential parameters.

1.2 The Projective Plane

projplane

The n-dimensional *projective space* $\mathbb{P}^n$ is the set of equivalence classes of *nonzero* vectors $x = (x_0, x_1, ..., x_n)$, the equivalence relation being

equivrel (1.2.1)
$$(x_0', ..., x_n') \sim (x_0, ..., x_n) \quad \text{if} \quad (x_0', ..., x_n') = (\lambda x_0, ..., \lambda x_n) \qquad \big(\text{or} \quad x' = \lambda x\big)$$

for some nonzero complex number λ. The equivalence classes are the *points* of $\mathbb{P}^n$, and one often refers to a point by a particular vector in its class.

Points of $\mathbb{P}^n$ correspond bijectively to one-dimensional subspaces of the complex vector space $\mathbb{C}^{n+1}$. When x is a nonzero vector, the one-dimensional subspace of $\mathbb{C}^{n+1}$ spanned by x consists of the vectors λx, together with the zero vector.

projline **(1.2.2) the projective line**

Points of the *projective line* $\mathbb{P}^1$ are equivalence classes of nonzero vectors $x = (x_0, x_1)$.

If the first coordinate x_0 of a vector x isn't zero, we may multiply by $\lambda = x_0^{-1}$ to normalize the first entry to 1, and to write the point that x represents in a unique way as $(1, u)$, with $u = x_1/x_0$. There is one remaining point, the point represented by the vector $(0, 1)$. The projective line $\mathbb{P}^1$ can be obtained by adding this point, called the *point at infinity*, to the affine u-line, which is a complex plane. Topologically, $\mathbb{P}^1$ is a two-dimensional sphere.

projpl **(1.2.3) lines in projective space**

Let p and q be vectors that represent distinct points of projective space $\mathbb{P}^n$. There is a unique *line* L in $\mathbb{P}^n$ that contains those points, the set of points $L = \{rp + sq\}$, with r, s in $\mathbb{C}$ not both zero. The points of L correspond bijectively to points of the projective line $\mathbb{P}^1$, by

pline (1.2.4)
$$rp + sq \quad \longleftrightarrow \quad (r, s)$$

A line in the projective plane $\mathbb{P}^2$ can also be described as the locus of solutions of a homogeneous linear equation

eqline (1.2.5)
$$s_0x_0 + s_1x_1 + s_2x_2 = 0$$

linesmeet **1.2.6. Lemma.** *In the projective plane, two distinct lines have exactly one point in common, and in a projective space of any dimension, a pair of distinct points is contained in exactly one line.* □

standcov **(1.2.7) the standard covering of $\mathbb{P}^2$**

The *projective plane* $\mathbb{P}^2$ is the two-dimensional projective space. Its points are equivalence classes of nonzero vectors (x_0, x_1, x_2).

If the first entry x_0 of a point $p = (x_0, x_1, x_2)$ of the projective plane isn't zero, we may normalize it to 1 without changing the point: $(x_0, x_1, x_2) \sim (1, u_1, u_2)$, where $u_i = x_i/x_0$. We did the analogous thing for $\mathbb{P}^1$ above. The representative vector $(1, u_1, u_2)$ is uniquely determined by p, so points with $x_0 \neq 0$ correspond bijectively to points of an affine plane $\mathbb{A}^2$ with coordinates (u_1, u_2):

$$(x_0, x_1, x_2) \sim (1, u_1, u_2) \quad \longleftrightarrow \quad (u_1, u_2)$$

We regard the affine u-plane as a subset of $\mathbb{P}^2$ by this correspondence, and we denote that subset by $\mathbb{U}^0$. The points of $\mathbb{U}^0$, those with $x_0 \neq 0$, are the *points at finite distance*. The *points at infinity* of $\mathbb{P}^2$ are those of the form $(0, x_1, x_2)$. They are on the *line at infinity* L^0, the locus $\{x_0 = 0\}$ in $\mathbb{P}^2$. The projective plane is the union of the two sets $\mathbb{U}^0$ and L^0. When a point is given by a coordinate vector, we can assume that the first coordinate is either 1 or 0.

We may write a point (x_0, x_1, x_2) of $\mathbb{U}^0$ as $(1, u_1, u_2)$ with $u_i = x_i/x_0$ as above, or we may simply assume that its first coordinate is 1 and write the point as $(1, x_1, x_2)$. The notation $u_i = x_i/x_0$ is important only when the coordinate vector (x_0, x_1, x_2) has been given.

There is an analogous correspondence between points $(x_0, 1, x_2)$ and points of an affine plane $\mathbb{A}^2$, and between points $(x_0, x_1, 1)$ and points of an affine plane. We denote the subsets $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$ by $\mathbb{U}^1$ and $\mathbb{U}^2$, respectively. The three sets $\mathbb{U}^0, \mathbb{U}^1, \mathbb{U}^2$ form the *standard covering* of $\mathbb{P}^2$ by three *standard affine open sets*. Since the vector $(0, 0, 0)$ has been ruled out, every point of $\mathbb{P}^2$ lies in at least one of the three standard affine open sets. Points whose three coordinates are nonzero lie in all of them.

1.2.8. Note. Which points of $\mathbb{P}^2$ are at infinity depends on which of the standard affine open sets is taken to be the one at finite distance. When the coordinates are (x_0, x_1, x_2), I like to normalize x_0 to 1, as above. Then the points at infinity are those of the form $(0, x_1, x_2)$. But when coordinates are (x, y, z), I may normalize z to 1. Then the points at infinity are the points $(x, y, 0)$. I hope this won't cause too much confusion. $\square$ pointatin-finity

(1.2.9) digression: the real projective plane realproj-plane

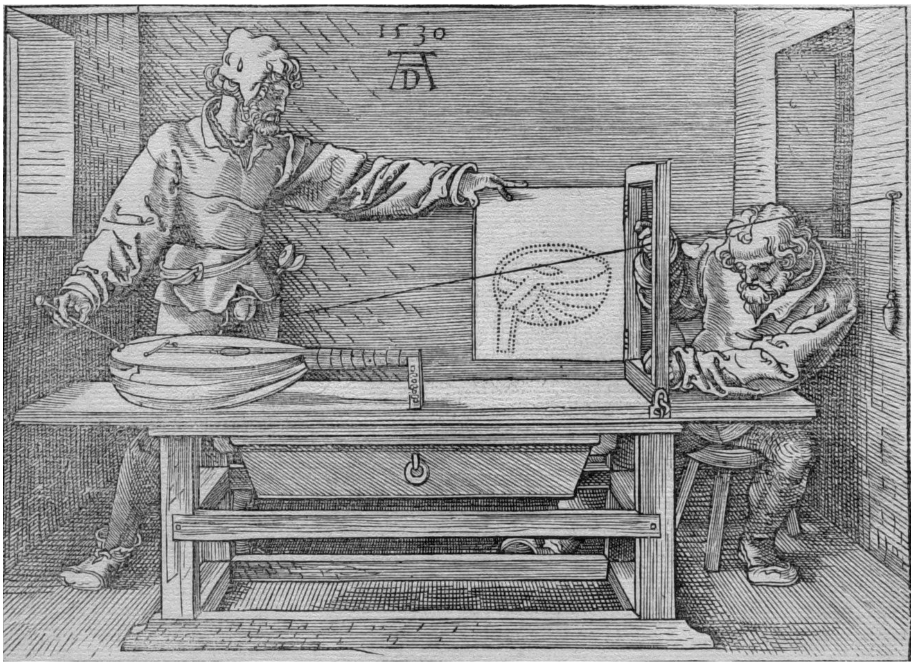
Points of the *real projective plane* $\mathbb{RP}^2$ are equivalence classes of nonzero real vectors $x = (x_0, x_1, x_2)$, the equivalence relation being $x' \sim x$ if $x' = \lambda x$ for some nonzero real number λ. The real projective plane can also be thought of as the set of one-dimensional subspaces of the real vector space $\mathbb{R}^3$.

Let's denote $\mathbb{R}^3$ by V. The plane $U : \{x_0 = 1\}$ in V is analogous to the standard affine open subset $\mathbb{U}^0$ in the complex projective plane $\mathbb{P}^2$. We can project V from the origin $p_0 = (0, 0, 0)$ to U, sending a point $x = (x_0, x_1, x_2)$ of V to the point $(1, u_1, u_2)$, with $u_i = x_i/x_0$. The fibres of this projection are the lines through p_0 and x, with p_0 omitted.

The projection to U is undefined at the points $(0, x_1, x_2)$, which are orthogonal to the x_0-axis. The line connecting such a point to p_0 doesn't meet U. The points $(0, x_1, x_2)$ correspond to the points at infinity of $\mathbb{RP}^2$.

Looking from the origin, U becomes a "picture plane".

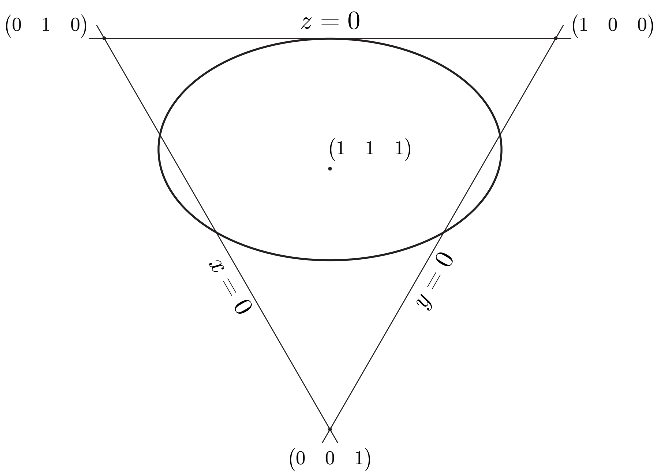
durerfig **1.2.10.**



This is an illustration from a book on perspective by Albrecht Dürer

The projection from 3-space to a picture plane goes back to the the 16th century, the time of Desargues and Dürer, but projective coordinates were introduced 200 years later, by Möbius.

goober4 **1.2.11.**



The Real Projective Plane

This figure shows the plane W: $x{+}y{+}z = 1$ in the real vector space $\mathbb{R}^3$, together with its coordinate lines and a conic. The one-dimensional subspace spanned by a nonzero vector (x_0, y_0, z_0) in $\mathbb{R}^3$ will meet W in a single point unless that vector is on the line $L:\ x{+}y{+}z = 0$. So W is a faithful representation of most of $\mathbb{RP}^2$. It contains all points except those on L.

chgco-ordssec **(1.2.12) changing coordinates in the projective plane**

An invertible 3×3 matrix P determines a linear change of coordinates in $\mathbb{P}^2$. With $x = (x_0, x_1, x_2)^t$ and

$x' = (x'_0, x'_1, x'_2)^t$ represented as column vectors, the coordinate change is given by

(1.2.13) $$Px' = x$$ chg

As the next proposition shows, four special points, the points

$$e_0 = (1,0,0)^t, e_1 = (0,1,0)^t, e_2 = (0,0,1)^t \quad \text{and} \quad \epsilon = (1,1,1)^t$$

determine the coordinates.

1.2.14. Proposition. *Let p_0, p_1, p_2, q be four points of $\mathbb{P}^2$, no three of which lie on a line. There is, up to scalar factor, a unique linear coordinate change $Px' = x$ such that $Pp_i = e_i$ and $Pq = \epsilon$.* *fourpoints*

proof. The hypothesis that the points p_0, p_1, p_2 don't lie on a line tells us that the vectors that represent those points are independent. They span $\mathbb{C}^3$. So q will be a combination $q = c_0p_0 + c_1p_1 + c_2p_2$, and because no three of the four points lie on a line, the coefficients c_i will be nonzero. We can *scale* the vectors p_i (multiply them by nonzero scalars) to make $q = p_0+p_1+p_2$ without changing the points. Next, the columns of P can be an arbitrary set of independent vectors. We let them be p_0, p_1, p_2. Then $Pe_i = p_i$, and $P\epsilon = q$. The matrix P is unique up to scalar factor. $\square$

(1.2.15) conics projconics

A polynomial $f(x_0, x_1, x_2)$ is *homogeneous of degree* d, if all monomials that appear with nonzero coefficient have (total) degree d. For example, $x_0^3 + x_1^3 - x_0x_1x_2$ is a homogeneous cubic polynomial.

A homogeneous quadratic polynomal is a combination of the six monomials

$$x_0^2,\ x_1^2,\ x_2^2,\ x_0x_1,\ x_1x_2,\ x_0x_2$$

A *conic* is the locus of zeros of an irreducible homogeneous quadratic polynomial.

1.2.16. Proposition. *For any conic C, there is a choice of coordinates so that C becomes the locus* *classify-conic*

$$x_0x_1 + x_0x_2 + x_1x_2 = 0$$

proof. Since the conic C isn't a line, it will contain three points that aren't colinear. Let's leave the verification of this fact as an exercise. We choose three non-colinear points on C, and adjust coordinates so that they become the points e_0, e_1, e_2. Let f be the quadratic polynomial in those coordinates whose zero locus is C. Because e_0 is a point of C, $f(1,0,0) = 0$, and therefore the coefficient of x_0^2 in f is zero. Similarly, the coefficients of x_1^2 and x_2^2 are zero. So f has the form

$$f = ax_0x_1 + bx_0x_2 + cx_1x_2$$

Since f is irreducible, a, b, c aren't zero. By scaling appropriately (adjusting the variables by scalar factors), we can make $a = b = c = 1$. We will be left with the polynomial $x_0x_1 + x_0x_2 + x_1x_2$. $\square$

1.3 Plane Projective Curves

projcurve

The loci in projective space that are studied in algebraic geometry are those that can be defined by systems of *homogeneous* polynomial equations.

The reason that we use homogeneous equations is this: To say that a polynomial $f(x_0, ..., x_n)$ vanishes at a point p of projective space $\mathbb{P}^n$ means that if the vector $a = (a_0, ..., a_n)$ represents the point p, then $f(a) = 0$. Perhaps this is obvious. Now, if a represents p, the other vectors that represent p are the vectors λa $(\lambda \neq 0)$. When f vanishes at p, $f(\lambda a)$ must also be zero. Then if $a = (a_0, ..., a_n)$ represents the point p, a polynomial $f(x)$ vanishes at p if and only if $f(\lambda a) = 0$ for every λ.

We write a polynomial $f(x_0, ..., x_n)$ as a sum of its *homogeneous parts*:

(1.3.1) $$f = f_0 + f_1 + \cdots + f_d$$ homparts

where f_0 is the constant term, f_1 is the linear part, etc., and d is the degree of f.

hompart-szero **1.3.2. Lemma.** *Let $f = f + 0 + \cdots + f_d$ be a polynomial of degree d in $x_0, ..., x_n$, and let $a = (a_0, ..., a_n)$ be a nonzero vector. Then $f(\lambda a) = 0$ for every nonzero complex number λ if and only if $f_i(a)$ is zero for every $i = 0, ..., d$.*

As this lemma shows, we may as well work with homogeneous equations.

proof of the lemma. We substitute into (1.3.1): $f(\lambda x) = f_0 + \lambda f_1(x) + \lambda^2 f_2(x) + \cdot + \lambda^d f_d(x)$. Evaluating at $x = a$, $f(\lambda a) = f_0 + \lambda f_1(a) + \lambda^2 f_2(a) + \cdot + \lambda^d f_d(a)$. The right side of this equation is a polynomial of degree at most d in λ. Since a nonzero polynomial of degree at most d has at most d roots, $f(\lambda a)$ won't be zero for *every* λ unless that polynomial is zero — unless $f_i(x)$ is zero for every i. □

fequalsgh **1.3.3. Lemma. (i)** *If a homogeneous polynomial $f(x_0, ..., x_n)$ is a product of polynomials, $f = gh$, then g and h are homogeneous.*
(ii) *If $f = gh$, the zero locus $\{f = 0\}$ in projective space is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.*
(iii) *In affine space, the zero locus of a product gh of nonhomogeneous polynomials is the union of the two loci $\{g = 0\}$ and $\{h = 0\}$.* □

It is also true that relatively prime homogeneous polynomials f and g in three variables have only finitely many common zeros. This will be proved below, in Proposition 1.3.12.

locipone
(1.3.4) loci in the projective line

Before going to plane projective curves, we describe the zero locus of a homogeneous polynomial in two variables in the projective line $\mathbb{P}^1$.

factorhompoly **1.3.5. Lemma.** *Every nonzero homogeneous polynomial $f(x, y) = a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d$ with complex coefficients is a product of homogeneous linear polynomials that are unique up to scalar factor.*

To prove this, one uses the fact that the field of complex numbers is algebraically closed. A one-variable complex polynomial factors into linear factors in the polynomial ring $\mathbb{C}[y]$. To factor $f(x, y)$, one may factor the one-variable polynomial $f(1, y)$ into linear factors, substitute y/x for y, and multiply the result by x^d. When one adjusts scalar factors, one will obtain the expected factorization of $f(x, y)$. For instance, to factor $f(x, y) = x^2 - 3xy + 2y^2$, we substitute $x = 1$: $2y^2 - 3y + 1 = 2(y - 1)(y - \frac{1}{2})$. Substituting $y = y/x$ and multiplying by x^2, $f(x, y) = 2(y - x)(y - \frac{1}{2}x)$. The scalar 2 can be distributed arbitrarily among the linear factors. □

When a homogeneous polynomial f is a product of linear factors, we can collect the ones differing by scalar factors together, and adjust by scalars to make those factors equal, so that f has the form

factorpolytwo
$$f(x, y) = c(v_1 x - u_1 y)^{r_1} \cdots (v_k x - u_k y)^{r_k} \tag{1.3.6}$$

where no factor $v_i x - u_i y$ is a constant multiple of another, c is a scalar, and $r_1 + \cdots + r_k$ is the degree of f. The exponent r_i is the *multiplicity* of the linear factor $v_i x - u_i y$.

A linear polynomial $vx - uy$ determines a point (u, v) in the projective line $\mathbb{P}^1$, the unique *zero* of that polynomial, and changing the polynomial by a scalar factor doesn't change its zero. Thus the linear factors of the homogeneous polynomial (1.3.6) determine points of $\mathbb{P}^1$, the *zeros* of f. The points (u_i, v_i) are zeros of multiplicity r_i. The total number of those points, counted with multiplicity, will be the degree of f.

zerosandroots **1.3.7.** The zero (u_i, v_i) of f corresponds to a root $x = u_i/v_i$ of multiplicity r_i of the one-variable polynomial $f(x, 1)$, except when the zero is the point $(1, 0)$. This happens when the coefficient a_0 of f is zero, and y is a factor of f. One could say that $f(x, y)$ has a zero at infinity in that case.

This sums up the information contained in the locus of a homogeneous polynomial $f(x, y)$ in the projective line. It will be a finite set of points with multiplicities.

intersectline
(1.3.8) intersections with a line

Let Z be the zero locus of a homogeneous polynomial $f(x_0,...,x_n)$ of degree d in projective space $\mathbb{P}^n$, and let L be a line in $\mathbb{P}^n$ (1.2.4). Say that L is the set of points $rp+sq$, where $p=(a_0,...,a_n)$ and $q=(b_0,...,b_n)$ are represented by specific vectors. So L corresponds to the projective line $\mathbb{P}^1$, by $rp+sq \leftrightarrow (r,s)$. Let's also assume that L isn't entirely contained in the zero locus Z. The intersection $Z \cap L$ corresponds to the zero locus in $\mathbb{P}^1$ of the polynomial in r,s obtained by substituting $rp+sq$ into f. This substitution yields a homogeneous polynomial $\overline{f}(r,s)$ in r,s, of degree d. For example, let $f=x_0x_1+x_0x_2+x_1x_2$. Then with $p=(a_o,a_1,a_2)$ and $q=(b_0,b_1,b_2)$, $\overline{f}$ is the following quadratic polynomial in r,s:

$$\overline{f}(r,s)=f(rp+sq)=(ra_0+sb_0)(ra_1+sb_1)+(ra_0+sb_0)(ra_2+sb_2)+(ra_1+sb_1)(ra_2+sb_2)$$
$$=(a_0a_1+a_0a_2+a_1a_2)r^2+\big(\textstyle\sum_{i\neq j}a_ib_j\big)rs+(b_0b_1+b_0b_2+b_1b_2)s^2$$

The zeros of $\overline{f}$ in $\mathbb{P}^1$ correspond to the points of $Z \cap L$. If f has degree d, there will be d zeros, when counted with multiplicity.

1.3.9. Definition. With notation as above, the *intersection multiplicity* of Z and L at a point p is the multiplicity of zero of the polynomial $\overline{f}$. $\square$ intersect-linetwo

1.3.10. Corollary. *Let Z be the zero locus in $\mathbb{P}^n$ of a homogeneous polynomial f, and let L be a line in $\mathbb{P}^n$ that isn't contained in Z. The number of intersections of Z and L, counted with multiplicity, is equal to the degree of f.* $\square$ *XcapL*

(1.3.11) loci in the projective plane lociptwo

1.3.12. Proposition. *Let $f_1,...,f_r$ be homogeneous polynomials in three variables, with no common factor, If $r>1$, these polynomials have finitely many common zeros in $\mathbb{P}^2$.* *fgzerofinite*

The proof of this proposition is below.

The locus of zeros of a single irreducible homogeneous polynomial equation is called a *plane projective curve*. As the proposition shows, plane projective curves are most interesting loci in the projective plane. The *degree* of a plane projective curve is the degree of its irreducible defining polynomial.

1.3.13. Note. Suppose that a homogeneous polynomial is reducible, say $f=g_1\cdots g_k$, that g_i are irreducible, and that g_i and g_j don't differ by a scalar factor when $i\neq j$. Then the zero locus C of f is the union of the zero loci V_i of the factors g_i. In this case, C may be called a *reducible curve*. redcurve

When there are multiple factors, say $f=g_1^{e_1}\cdots g_k^{e_k}$ and some e_i are greater than 1, it is still true that the locus $C:\{f=0\}$ will be the union of the loci $V_i:\{g_i=0\}$, but the connection between the geometry of C and the algebra is weakened. In this situation, the structure of a *scheme* becomes useful. We won't discuss schemes. The only situation in which we will need to keep track of multiple factors is when counting intersections with another curve D. For this purpose, one can use the *divisor* of f, which is defined to be the integer combination $e_1V_1+\cdots+e_kV_k$. $\square$

A *rational function* in variables $x,y,...$ is a fraction of polynomials in those variables. The polynomial ring $\mathbb{C}[x,y]$ embeds into its field of fractions, the field F of rational functions in x,y, which is often denoted by $\mathbb{C}(x,y)$. The polynomial ring $\mathbb{C}[x,y,z]$ is a subring of the one-variable polynomial ring $F[z]$. It can be useful to begin by studying a problem in $F[z]$, which is a principal ideal domain. Its algebra is simpler. The polynomial rings $\mathbb{C}[x,y]$ and $\mathbb{C}[x,y,z]$ are unique factorization domains, but not principal ideal domains.

1.3.14. Lemma. *Let $F=\mathbb{C}(x,y)$ be the field of rational functions in x,y.* *relprime*

(i) *If $f_1,...,f_k$ are homogeneous polynomials in x,y,z with no common factor, their greatest common divisor in $F[z]$ is 1, and therefore they generate the unit ideal of $F[z]$. There is an equation of the form $\sum g_i'f_i=1$ with g_i' in $F[z]$. (The unit ideal of a ring R is the ring R itself.)*

(ii) *Let f be an irreducible polynomial in $\mathbb{C}[x,y,z]$ of positive degree in z. Then f is also an irreducible element of $F[z]$.*

proof of the lemma. **(i)** Let h' be an element of $F[z]$ that isn't a unit of $F[z]$, i.e., that isn't an element of F. Suppose that h' divides f_i in $F[z]$ for every i, say $f_i = u_i'h'$. The coefficients of h' and u_i' have denominators that are polynomials in x, y. We clear denominators from the coefficients, to obtain elements of $\mathbb{C}[x, y, z]$. This will give us equations of the form $d_if_i = u_ih$, where d_i are polynomials in x, y and u_i and h are polynomials in x, y, z. Since h' isn't in F, neither is h. So h will have positive degree in z. Let g be an irreducible factor of h of positive degree in z. Then g divides d_if_i but doesn't divide d_i which has degree zero in z. So g divides f_i, and this is true for every i. This contradicts the hypothesis that $f_1, ..., f_k$ have no common factor.

(ii) Say that $f(x, y, z)$ factors in $F[z]$, $f = g'h'$, where g' and h' are polynomials of positive degree in z with coefficients in F. When we clear denominators from g' and h', we obtain an equation of the form $df = gh$, where g and h are polynomials in x, y, z of positive degree in z and d is a polynomial in x, y. Neither g nor h divides d, so f must be reducible. □

proof of Proposition 1.3.12. We are to show that homogeneous polynomials $f_1, ..., f_r$ in x, y, z with no common factor have finitely many common zeros. Lemma 1.3.14 tells us that we may write $\sum g_i'f_i = 1$, with g_i' in $F[z]$. Clearing denominators from g_i' gives us an equation of the form

$$\sum g_if_i = d$$

where g_i are polynomials in x, y, z and d is a polynomial in x, y. Taking suitable homogeneous parts of g_i and d produces an equation $\sum g_if_i = d$ in which all terms are homogeneous.

Lemma 1.3.5 asserts that $d(x, y)$ is a product of linear polynomials, say $d = \ell_1 \cdots \ell_r$. A common zero of $f_1, ..., f_k$ is also a zero of d, and therefore it is a zero of ℓ_j for some j. It suffices to show that, for every j, $f_1, ..., f_r$ and ℓ_j have finitely many common zeros.

Since $f_1, ..., f_k$ have no common factor, there is at least one f_i that isn't divisible by ℓ_j. Then Corollary 1.3.10 shows that f_i and ℓ_j have finitely many common zeros. □

pointscurves **1.3.15. Corollary.** *Every locus in the projective plane $\mathbb{P}^2$ that can be defined by a system of homogeneous polynomial equations is a finite union of points and curves.* □

The next corollary is a special case of the Strong Nullstellensatz, which will be proved in the next chapter.

idealprincipal **1.3.16. Corollary.** *Let $f(x, y, z)$ be an irreducible homogeneous polynomial that vanishes on an infinite set S of points of $\mathbb{P}^2$. If another homogeneous polynomial $g(x, y, z)$ vanishes on S, then f divides g. Therefore, if an irreducible polynomial vanishes on an infinite set S, that polynomial is unique up to scalar factor.*

proof. If the irreducible polynomial f doesn't divide g, then f and g have no common factor, and therefore they have finitely many common zeros. □

classicaltopology

(1.3.17) the classical topology

The usual topology on the affine space $\mathbb{A}^n$ will be called the *classical topology*. A subset U of $\mathbb{A}^n$ is open in the classical topology if, whenever U contains a point p, it contains all points sufficiently near to p. We call this the classical topology to distinguish it from another topology, the *Zariski topology*, which will be discussed in the next chapter.

The projective space $\mathbb{P}^n$ also has a classical topology. A subset U of $\mathbb{P}^n$ is open if, whenever a point p of U is represented by a vector $(x_0, ..., x_n)$, all vectors $x' = (x_0', ..., x_n')$ sufficiently near to x represent points of U.

isopts

(1.3.18) isolated points

A point p of a topological space X is *isolated* if the set $\{p\}$ is both open and closed, or if both $\{p\}$ and its complement $X - \{p\}$ are closed sets. If X is a subset of $\mathbb{A}^n$ or $\mathbb{P}^n$, a point p of X is isolated in the classical topology if X doesn't contain points p' distinct from p, but arbitrarily close to p.

1.3.19. Proposition *Let n be an integer greater than one. In the classical topology, the zero locus of a polynomial in $\mathbb{A}^n$ or in $\mathbb{P}^n$ contains no isolated points.* *noisolatedpoint*

1.3.20. Lemma. *Let f be a polynomial of degree d in n variables. After a suitable coordinate change, $f(x)$ will become a scalar multiple of a monic polynomial of degree d in the variable x_n.* *polymonic*

proof. We write $f = f_0 + f_1 + \cdots + f_d$, where f_i is the homogeneous part of f of degree i, and we choose a point p of $\mathbb{A}^n$ at which f_d isn't zero. We change variables so that p becomes the point $(0, ..., 0, 1)$ (see (**1.1.4**)). We call the new variables $x_1, .., .x_n$ and the new polynomial f. Then $f_d(0, ..., 0, x_n)$ will be equal to cx_n^d for some nonzero constant c, and f/c will be a monic polynomial. □

proof of Proposition 1.3.19. The proposition is true for loci in affine space and also for loci in projective space. We look at the affine case. Let $f(x_1, ..., x_n)$ be a polynomial, and let p be a point of its zero locus Z. If f factors, $f = gh$, then Z is the union of the zero loci $Z_1 : \{g = 0\}$ and $Z_2 : \{h = 0\}$. An isolated point p of Z will be an isolated point of Z_1 or Z_2. So it suffices to prove that Z_1 and Z_2 have no isolated points. Therefore we may assume that f is irreducible.

We adjust coordinates so that p is the origin $(0, ..., 0)$ and f is monic in x_n. We relabel x_n as y, and write f as a polynomial in y:

$$\widetilde{f}(y) = f(x, y) = y^d + c_{d-1}(x)y^{d-1} + \cdots + c_0(x)$$

where c_i is a polynomial in $x_1, ..., x_{n-1}$. For fixed x, $c_0(x)$ is the product of the roots of $\widetilde{f}(y)$, and since f is irreducible, $c_0(x) \neq 0$. Since p is the origin and $f(p) = 0$, $c_0(0) = 0$. So $c_0(x)$ will tend to zero with x. When $c_0(x)$ is small, at least one root y of $\widetilde{f}(y)$ will be small. So there are points of Z that are dstinct from p, but arbitrarily close to p. □

1.3.21. Corollary. *Let C' be the complement of a finite set of points in a plane curve C. In the classical topology, a continuous function g on C that is zero at every point of C' is identically zero.* □ *functioniszero*

1.4 Tangent Lines

tanlines

(1.4.1) notation for working locally standnot

We will often want to inspect a plane projective curve $C : \{f(x_0, x_1, x_2) = 0\}$ in a neighborhood of a particular point p. To do this we may adjust coordinates so that p becomes the point $(1, 0, 0)$, and work with points $(1, x_1, x_2)$ in the standard affine open set $\mathbb{U}^0 : \{x_0 \neq 0\}$. When we identify $\mathbb{U}^0$ with the affine x_1, x_2-plane, p becomes the origin, and C becomes the zero locus of the nonhomogeneous polynomial $f(1, x_1, x_2)$.

The loci $f(x_0, x_1, x_2) = 0$ and $f(1, x_1, x_2) = 0$ are the same on the subset $\mathbb{U}^0$S Since x_0 is invertible on $\mathbb{U}^0$, $f(x_0, .x_1, x_2) = 0$ if and only if $f(1, u_1, u_2) = 0$, $u_i = x_i/x_0$.

This will be a standard notation for working locally. Of course, it doesn't matter which variable we set to 1. If the variables are x, y, z, we may prefer to take for p the point $(0, 0, 1)$ and work with the polynomial $f(x, y, 1)$.

1.4.2. Lemma. *A homogeneous polynomial $f(x_0, x_1, x_2)$ that isn't divisible by x_0 is irreducible if and only if $f(1, x_1, x_2)$ is irreducible.* □ *foneirred*

(1.4.3) homogenizing and dehomogenizing homdehomone

If $f(x_0, x_1, ..., x_n)$ is a homogeneous polynomial, the polynomial $f(1, x_1, ..., x_n)$ is called the *dehomogenization* of f with respect to the variable x_0. A simple procedure, *homogenization*, inverts this dehomogenization. Suppose given a nonhomogeneous polynomial $F(x_1, x_2)$ of degree d. To homogenize F, we replace the variables x_i, $i = 1, 2$, by $u_i = x_i/x_0$. Then since u_i have degree zero in x, so does $F(u_1, ..., u_n)$. When we multiply by x_0^d, the result will be a homogeneous polynomial of degree d in $x_0, ..., x_n$ that isn't divisible by x_0.

(1.4.4) smooth points and singular points smsingpts

Let C be the plane curve defined by an irreducible homogeneous polynomial $f(x_0,x_1,x_2)$, and let f_i denote the partial derivative $\frac{\partial f}{\partial x_i}$, computed by the usual calculus formula. A point of C at which the partial derivatives f_i aren't all zero is a *smooth point* of C, and a point at which all partial derivatives are zero is a *singular point*. A curve is *smooth*, or *nonsingular*, if it contains no singular point; otherwise it is a *singular curve*.

fermatcurve

The *Fermat curve*

(1.4.5)
$$x_0^d + x_1^d + x_2^d = 0$$

is smooth because the only common zero of the partial derivatives $dx_0^{d-1}, dx_1^{d-1}, dx_2^{d-1}$, which is $(0,0,0)$, doesn't represent a point of $\mathbb{P}^2$. The cubic curve $x_0^3 + x_1^3 - x_0x_1x_2 = 0$ is singular at the point $(0,0,1)$.

The Implicit Function Theorem explains the meaning of smoothness. Suppose that $p = (1,0,0)$ is a point of C. We set $x_0 = 1$ and inspect the locus $f(1,x_1,x_2) = 0$ in the standard affine open set $\mathbb{U}^0$. If $f_2 = \frac{\partial f}{\partial x_2}$ isn't zero at p, the Implicit Function Theorem tells us that we can solve the equation $f(1,x_1,x_2) = 0$ for x_2 locally (i.e., for small x_1), as an analytic function φ of x_1, with $\varphi(0) = 0$. Sending x_1 to $(1,x_1,\varphi(x_1))$ inverts the projection from C to the affine x_1-line locally. So at a smooth point, C is locally homeomorphic to the affine line.

eulerformula

1.4.6. Euler's Formula. Let f be a homogeneous polynomial of degree d in the variables $x_0,...,x_n$. Then

$$\sum_i x_i \frac{\partial f}{\partial x_i} = d\,f$$

It suffices to check this formula when f is a monomial. As an example, let f be the monomial x^2y^3z, then

$$xf_x + yf_y + zf_z = x(2xy^3z) + y(3x^2y^2z) + z(x^2y^3) = 6x^2y^3z = 6\,f \qquad \square$$

singpointoncurve

1.4.7. Corollary. **(i)** *If all partial derivatives of an irreducible homogeneous polynomial f are zero at a point p of $\mathbb{P}^2$, then f is zero at p, and therefore p is a singular point of the curve $\{f = 0\}$.*

(ii) *At a smooth point of a plane curve, at least two partial derivatives will be nonzero.*

(iii) *The partial derivatives of an irreducible polynomial have no common (nonconstant) factor.*

(iv) *A plane curve has finitely many singular points.* $\square$

tangent

(1.4.8) tangent lines and flex points

Let C be the plane projective curve defined by an irreducible homogeneous polynomial $f(x_0,x_1,x_2)$. A line L is *tangent* to C at a smooth point p if the intersection multiplicity of C and L at p is at least 2. (See (1.3.9).) A smooth point p of C is a *flex point* if the intersection multiplicity of C and its tangent line at p is at least 3, and p is an *ordinary* flex point if the intersection multiplicity is equal to 3.

Let L be a line through a point p and let q be a point of L distinct from p. We represent p and q by specific vectors (p_0,p_1,p_2) and (q_0,q_1,q_2), to write a variable point of L as $p+tq$, and we expand the restriction of f to L in a Taylor's series. (The Taylor expansion carries over to complex polynomials because it is an identity.) Let $f_i = \frac{\partial f}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Taylor's formula is

taylor

(1.4.9)
$$f(p+tq) \;=\; f(p) \;+\; \Big(\sum_i f_i(p)\,q_i\Big)t \;+\; \tfrac{1}{2}\Big(\sum_{i,j} q_i\,f_{ij}(p)\,q_j\Big)t^2 \;+\; O(3)$$

where the symbol $O(3)$ stands for a polynomial in which all terms have degree at least 3 in t. The point q is missing from this parametrization of L, but this won't be important.

We write the equation in terms of the *gradient vector* $\nabla = (f_0, f_1, f_2)$ and *Hessian matrix* H of f, the matrix of second partial derivatives:

(1.4.10) $$H \;=\; \begin{pmatrix} f_{00} & f_{01} & f_{02} \\ f_{10} & f_{11} & f_{12} \\ f_{20} & f_{21} & f_{22} \end{pmatrix}$$ hessian-matrix

Let ∇_p and H_p be the evaluations of ∇ and H, respectively, at p.

Regarding p and q as column vectors, Equation 1.4.9 can be written as

(1.4.11) $$f(p+tq) = \; f(p) \;+\; \big(\nabla_p\, q\big)t \;+\; \tfrac{1}{2}(q^t H_p\, q)t^2 \;+\; O(3)$$ texp

in which q^t is the transpose of the column vector q, and $\nabla_p q$ and $q^t H_p q$ are computed as matrix products.

The intersection multiplicity of C and L at p is the lowest power of t that has nonzero coefficient in $f(p + tq)$ (1.3.9). The intersection multiplicity is at least 1 if p lies on C, i.e., if $f(p) = 0$, and p is a smooth point of C if $f(p) = 0$ and $\nabla_p \neq 0$.

If p is a smooth point of C, then L is tangent to C at p if the coefficient $(\nabla_p q)$ of t is zero, and p is a flex point if $(\nabla_p q)$ and $(q^t H_p q)$ are both zero.

The equation of the tangent line L at a smooth point p is $\nabla_p x = 0$, or

(1.4.12) $$f_0(p)x_0 + f_1(p)x_1 + f_2(p)x_2 = 0$$ tanlineeq

which tells us that a point q lies on L if the linear term in t of (1.4.11) is zero. So a line L is a tangent line at a smooth point p if it is orthogonal to the gradient ∇_p. There is a unique tangent line at a smooth point.

Note. Taylor's formula shows that the restriction of f to every line through a singular point has a multiple zero. However, we will speak of tangent lines only at smooth points of the curve.

The next lemma is obtained by applying Euler's Formula to the entries of H_p and ∇_p.

1.4.13. Lemma. $\nabla_p\, p = d\, f(p)$ *and* $p^t H_p = (d-1)\nabla_p$. $\square$ *applyeuler*

We rewrite Equation 1.4.9 one more time, using the notation $\langle u, v\rangle$ to represent the symmetric bilinear form $u^t H_p\, v$ on $V = \mathbb{C}^3$. It makes sense to say that this form vanishes on a pair of points of $\mathbb{P}^2$, because the condition $\langle u, v\rangle = 0$ doesn't change when u or v is multiplied by a nonzero scalar λ.

1.4.14. Proposition. *With notation as above,* *linewith-form*
(i) *Equation (1.4.9) can be written as*

$$f(p+tq) \;=\; \tfrac{1}{d(d-1)}\langle p, p\rangle \;+\; \tfrac{1}{d-1}\langle p, q\rangle t \;+\; \tfrac{1}{2}\langle q, q\rangle t^2 \;+\; O(3)$$

(ii) *A point p is a smooth point of C if and only if $\langle p, p\rangle = 0$ but $\langle p, v\rangle$ isn't identically zero.*

proof. **(i)** This follows from Lemma 1.4.13.

(ii) At a smooth point p, $\langle p, v\rangle = (d-1)\nabla_p v$ isn't identically zero because ∇_p isn't zero. $\square$

1.4.15. Corollary. *Let p be a smooth point of C, let q be a point of $\mathbb{P}^2$ distinct from p, and let L be the line through p and q. Then* *bilinform*
(i) *L is tangent to C at p if and only if $\langle p, p\rangle = \langle p, q\rangle = 0$, and*
(ii) *p is a flex point of C with tangent line L if and only if $\langle p, p\rangle = \langle p, q\rangle = \langle q, q\rangle = 0$.* $\square$

1.4.16. Theorem. *A smooth point p of the curve C is a flex point if and only if the* **Hessian determinant** $\det H_p$ *at p is zero.* *tangent-line*

proof. Let p be a smooth point of C, so that $\langle p, p\rangle = 0$. If $\det H_p = 0$, the form $\langle u, v\rangle$ is degenerate, and there is a nonzero null vector q. Then $\langle p, q\rangle = \langle q, q\rangle = 0$. But p isn't a null vector, because $\langle p, v\rangle$ isn't identically zero. So q is distinct from p. Therefore p is a flex point.

Conversely, suppose that p is a flex point and let q be a point on the tangent line at p and distinct from p, so that $\langle p, p\rangle = \langle p, q\rangle = \langle q, q\rangle = 0$. The restriction of the form to the two-dimensional space W spanned by p

and q is zero, and this implies that the form is degenerate. If (p, q, v) is a basis of V with p, q in W, the matrix of the form will look like this:

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$$

□

hess-notzero **1.4.17. Proposition.**
(i) *Let $f(x, y, z)$ be an irreducible homogeneous polynomial of degree at least two. The Hessian determinant* $\det H$ *isn't divisible by f. In particular, it isn't identically zero.*
(ii) *A curve that isn't a line has finitely many flex points.*

proof. **(i)** Let C be the curve defined by f. If f divides the Hessian determinant, every smooth point of C will be a flex point. We set $z = 1$ and look on the standard affine $\mathbb{U}^2$, choosing coordinates so that the origin p is a smooth point of C, and so that $\frac{\partial f}{\partial y} \neq 0$ at p. The Implicit Function Theorem tells us that we can solve the equation $f(x, y, 1) = 0$ for y locally, say $y = \varphi(x)$, where φ is an analytic function. The graph $\Gamma : \{y = \varphi(x)\}$ will be equal to C in a neighborhood of p. (See the review below.) A point of Γ is a flex point if and only if $\frac{d^2\varphi}{dx^2}$ is zero there. If this is true for all points near to p, then $\frac{d^2\varphi}{dx^2}$ will be identically zero. This implies that φ is linear, and since $\varphi(0) = 0$, that $y = ax$. Then $y = ax$ solves $f = 0$, and therefore $y - ax$ divides $f(x, y, 1)$. But $f(x, y, z)$ is irreducible, and so is $f(x, y, 1)$. Therefore $f(x, y, 1)$ and $f(x, y, z)$ are linear (1.4.2), contrary to hypothesis.

(ii) This follows from **(i)** and (1.3.12). The irreducible polynomial f and the Hessian determinant $\det H$ have finitely many common zeros. □

ifthm **1.4.18. Review.** (*about the Implicit Function Theorem*)

By *analytic function* $\varphi(x_1, ..., x_k)$, in one or more variables, we mean a complex-valued function that can be represented as a convergent power series for small x. For us, φ will often be a function of one variable.

Let $f(x, y)$ be a polynomial of two variables, such that $f(0, 0) = 0$ and $\frac{df}{dy}(0, 0) \neq 0$. The Implicit Function Theorem asserts that there is a unique analytic function $\varphi(x)$ such that $\varphi(0) = 0$ and $f(x, \varphi(x))$ is identically zero.

Let $\mathcal{R}$ be the ring af analytic functions in x. In the polynomial ring $\mathcal{R}[y]$, the polynomial $y - \varphi(x)$, which is monic in y, divides $f(x, y)$. To see this, we do division with remainder of f by $y - \varphi(x)$:

divrem (1.4.19)
$$f(x, y) = (y - \varphi(x))q(x, y) + r(x)$$

The quotient q and remainder r are in $\mathcal{R}[y]$, and $r(x)$ has degree zero in y, so it is in $\mathcal{R}$. Setting $y = \varphi(x)$ in the equation, one sees that $r(x) = 0$.

Let Γ be the graph of φ in a suitable neighborhood U of the origin in x, y-space. Since $f(x, y) = (y - \varphi(x))q(x, y)$, the locus $f(x, y) = 0$ in U has the form $\Gamma \cup \Delta$, where Γ is the zero locus of $y - \varphi(x)$ and Δ is the zero locus of $q(x, y)$. Differentiating, we find that $\frac{\partial f}{\partial y}(0, 0) = q(0, 0)$. So $q(0, 0) \neq 0$. Then Δ doesn't contain the origin, while Γ does. This implies that Δ is disjoint from Γ, locally. A sufficiently small neighborhood U of the origin won't contain any points of Δ. In such a neighborhood, the locus of zeros of f will be Γ.

If $\frac{\partial f}{\partial x}(0, 0)$ is also nonzero, one can also solve for x as an analytic function of y. The solution will be a local inverse function of φ. □

1.5 digression: Transcendence Degree

transcdeg

Let $F \subset K$ be a field extension. A set $\alpha = \{\alpha_1, ..., \alpha_n\}$ of elements of K is *algebraically dependent over F* if there is a nonzero polynomial $f(x_1, ..., x_n)$ with coefficients in F, such that $f(\alpha) = 0$. If there is no such polynomial, the set α is *algebraically independent over F*.

An infinite set is called algebraically independent if every finite subset is algebraically independent — if there is no polynomial relation among any finite set of its elements.

The set $\{\alpha_1\}$ consisting of a single element of K is algebraically dependent if α_1 is algebraic over F. Otherwise, it is algebraically independent. Then α_1 is said to be *transcendental* over F.

A finite algebraically independent set $\alpha = \{\alpha_1, ..., \alpha_n\}$ that isn't contained in a larger algebraically independent set is a *transcendence basis* for K over F. If there is a finite transcendence basis, its order is the *transcendence degree* of the field extension K of F. Lemma 1.5.3 below shows that all transcendence bases for K over F have the same order, so the transcendence degree is well-defined. If there is no finite transcendence basis, the transcendence degree of K over F is said to be infinite.

For example, let $K = F(x_1, ..., x_n)$ be the field of rational functions in n variables. The variables x_i form a transcendence basis of K over F, and the transcendence degree of K over F is n.

A *domain* is a nonzero ring with no zero divisors, and a domain that contains a field F as a subring is an F*-algebra*. Because $\mathbb{C}$-algebras occur frequently, we will refer to them simply as *algebras*. (Nonzero elements a and b of a ring R are called *zero divisors* if their product ab is zero.)

1.5.1. Proposition. *Let A be domain that is an F-algebra, and let K be its field of fractions. If K has transcendence degree n over F, then every algebaically independent set of elements of A is contained in an algebraically independent subset of A of order n.* *trdegalg*

As this proposition shows, one can speak of the transcendence degree of an F-algebra A that is a domain. It will be equal to the transcendence degree of its field of fractions.

We use the customary notation $F[\alpha_1, ..., \alpha_n]$ or $F[\alpha]$ for the F-algebra generated by a set $\alpha = \{\alpha_1, ..., \alpha_n\}$, and we may denote its field of fractions by $F(\alpha_1, ..., \alpha_n)$ or by $F(\alpha)$.

proof of Proposition 1.5.1. Let $a_1, ..., a_k$ be a maximal algebraically independent set of elements of A. Then every element of A will be algebraic over the field $F = \mathbb{C}(a_1, ..., a_k)$, and therefore K will be algebraic over F, so $k = n$. □

A set $\{\alpha_1, ..., \alpha_n\}$ is algebraically independent over F if and only if the surjective map from the polynomial algebra $F[x_1, ..., x_n]$ to the F-algebra $F[\alpha_1, ..., \alpha_n]$ that sends x_i to α_i is injective.

1.5.2. Lemma. *Let K/F be a field extension, let $\alpha = \{\alpha_1, ..., \alpha_n\}$ be a set of elements of K that is algebraically independent over F, and let $F(\alpha)$ be the field of fractions of $F[\alpha]$.* *algindtrivialities*

(i) *If β is another element of K, the set $\{\alpha_1, ..., \alpha_n, \beta\}$ is algebraically dependent if and only if β is algebraic over $F(\alpha)$.*

(ii) *The algebraically independent set α is a transcendence basis if and only if every element of K is algebraic over $F(\alpha)$.*

proof. **(i)** We can write an element z of $F(\alpha)$ as a fraction $p/q = p(\alpha)/q(\alpha)$, where $p(x)$ and $q(x)$ are relatively prime polynomials. Suppose that z satisfies a nontrivial polynomial relation $c_0z^n + c_1z^{n-1} + \cdots + c_n = 0$ with c_i in F. We may assume that $c_0 = 1$. Substituting $z = p/q$ and multiplying by q^n gives us the equation

$$p^n = -q\big(c_1p^{n-1} + \cdots + c_nq^{n-1}\big)$$

By hypothesis, α is an algebraically independent set, so this equation is equivalent with a polynomial equation in $F[x_1, ..., x_n]$. It shows that q divides p^n, which contradicts the hypothesis that p and q are relatively prime. So z satisfies no polynomial relation, and therefore it is transcendental over F. □

1.5.3. Lemma. *Let K/F be a field extension. If K has a finite transcendence basis, then all algebraically independent subsets of K are finite, and all transcendence bases have the same order.* *trdeg*

proof. Let $\alpha = \{\alpha_1, ..., \alpha_r\}$ and $\beta = \{\beta_1, ..., \beta_s\}$ be subsets of K. Assume that K is algebraic over $F(\alpha)$ and that the set β is algebraically independent over F. We show that $s \le r$. The fact that all transcendence bases have the same order will follow: If both α and β are transcendence bases, then $s \le r$, and since we can interchange α and β, $r \le s$.

The proof that $s \le r$ proceeds by reducing to the trivial case that β is a subset of α. Suppose that some element of β, say β_s, isn't in the set α. The set $\beta' = \{\beta_1, ..., \beta_{s-1}\}$ is algebraically independent, but it isn't a transcendence basis. So K isn't algebraic over $F(\beta')$. Since K is algebraic over $F(\alpha)$, there is at least one element of α, say α_r, that isn't algebraic over $F(\beta')$. Then $\gamma = \beta' \cup \{\alpha_s\}$ will be an algebraically independent set of order s, and it contains more elements of the set α than β does. Induction shows that $s \le r$. □

sametrdeg **1.5.4. Corollary.** *Let $L \supset K \supset F$ be fields. and If the degree $[L:K]$ of the field extension L of K is finite, then L and K have the same transcendence degree over F.* $\square$

1.6 The Dual Curve

dualcurve

dualplane-sect **(1.6.1) the dual plane**

Let $\mathbb{P}$ denote the projective plane with coordinates x_0, x_1, x_2, let s_0, s_1, s_2 be scalars, not all zero, and let L be the line in $\mathbb{P}$ with the equation

lineequa-tion (1.6.2)
$$s_0x_0 + s_1x_1 + s_2x_2 = 0$$

The solutions (x_0, x_1, x_2) of this equation determine the coefficients s_i only up to a common nonzero scalar factor, so L determines a point (s_0, s_1, s_2) in another projective plane $\mathbb{P}^*$ called the *dual plane*. We denote that point by L^*. Moreover, a point $p = (x_0, x_1, x_2)$ in $\mathbb{P}$ determines a line in the dual plane, the line with the equation (1.6.2), when s_i are regarded as the variables and x_i as the scalar coefficients. We denote that line by p^*. The equation exhibits a duality between $\mathbb{P}$ and $\mathbb{P}^*$. A point p of $\mathbb{P}$ lies on the line L if and only if the equation is satisfied, and this means that, in $\mathbb{P}^*$, the point L^* lies on the line p^*.

As this duality shows, the dual $\mathbb{P}^{**}$ of the dual plane $\mathbb{P}^*$ is the plane $\mathbb{P}$.

dual-curvetwo **(1.6.3) the dual curve**

Let C be the plane projective curve defined by an irreducible homogeneous polyomial of degree at least two, and let U be the set of its smooth points. Corollary 1.4.7 tells us that U is the complement of a finite set in C. We define a map
$$U \xrightarrow{t} \mathbb{P}^*$$
as follows: Let p be a point of U and let L be the tangent line to C at p. The definition of t is that $t(p) = L^*$, where L^* is the point of $\mathbb{P}^*$ that corresponds to the tangent line L. Thus thee image $t(U)$ is the locus of tangent lines to C at smooth points.

We assume that C has degree at least two because, if C were a line, the image $t(U)$ of U would be a point. Since the partial derivatives have no common factor, the tangent lines aren't constant when the degree is greater than one.

Using vector notation $x = (x_0, x_1, x_2)$, $s = (s_0, s_1, s_2)$, we denote the gradient of f by $\nabla f = (f_0, f_1, f_2)$, with $f_i = \frac{\partial f}{\partial x_i}$ as before. The tangent line L at a smooth point $p = (x_0, x_1, x_2)$ of C has the equation $f_0x_0 + f_1x_1 + f_2x_2 = 0$. Therefore L^* is the point

ellstare-quation (1.6.4)
$$(s_0, s_1, s_2) \sim \big(f_0(x), f_1(x), f_2(x)\big) = \nabla f(x)$$

phize-rogzero **1.6.5. Lemma.** *Let $\varphi(s_0, s_1, s_2)$ be a homogeneous polynomial of degree r, and let $g(x_0, x_1, x_2) = \varphi(\nabla f(x))$. Then $\varphi(s)$ is identically zero on the image $t(U)$ of U if and only if $g(x)$ is identically zero on U. This is true if and only if f divides g.*

proof. The point s is in $t(U)$ if and only if $\lambda s = \nabla f(x)$ for some x in U and some λ. Then $g(x) = \varphi(\nabla f(x)) = \varphi(\lambda s) = \lambda^r\varphi(s)$. So $g(x) = 0$ if and only if $\varphi(s) = 0$. $\square$

dual-curvethm **1.6.6. Theorem.** *Let C be the plane curve defined by an irreducible homogeneous polynomial f of degree at least two. With notation as above, the image $t(U)$ is contained in a curve C^* in the dual plane $\mathbb{P}^*$.*

The curve C^* referred to in the theorem is the *dual curve*.

proof of Theorem 1.6.6. If an irreducible homogeneous polynomial $\varphi(s)$ vanishes on $t(U)$, it will be unique up to scalar factor (Corollary 1.3.16). We show first that there is a nonzero polynomial $\varphi(s)$, not necessarily irreducible or homogeneous, that vanishes on $t(U)$. The field $\mathbb{C}(x_0, x_1, x_2)$ has transcendence degree three over $\mathbb{C}$. Therefore the four polynomials f_0, f_1, f_2, and f are algebraically dependent. There is a nonzero polynomial $\psi(s_0, s_1, s_2, t)$ such that $\psi(f_0(x), f_1(x), f_2(x), f(x)) = \psi(\nabla f(x), f(x))$ is the zero polynomial. We can cancel factors of t, so we may assume that ψ isn't divisible by t. Let $\varphi(s) = \psi(s_0, s_1, s_2, 0)$. This isn't the zero polynomial when t doesn't divide ψ. If $x = (x_1, x_2, x_3)$ is a vector that represents a point of U, then $f(x) = 0$, and therefore

$$\psi(f_0(x), f_1(x), f_2(x), f(x)) = \psi(\nabla f(x), 0) = \varphi(\nabla f(x))$$

Since $\psi(\nabla f(x), f(x))$ is identically zero, $\varphi(\nabla f(x)) = 0$ for all x in U.

Next, since f has degree d, the partial derivatives f_i have degree $d-1$. Therefore $\nabla f(\lambda x) = \lambda^{d-1}\nabla f(x)$ for all λ, and because the vectors x and λx represent the same point of U, $\varphi(\nabla f(\lambda x)) = \varphi(\lambda^{d-1}\nabla f(x))) = 0$ for all λ, when x is in U. Writing $\nabla f(x) = s$, $\varphi(\lambda^{d-1}s) = 0$ for all λ when x is in U. Since λ^{d-1} can be any complex number, Lemma 1.3.2 tells us that the homogeneous parts of $\varphi(s)$ vanish at s, when $s = \nabla f(x)$ and x is in U. So the homogeneous parts of $\varphi(s)$ vanish on $t(U)$. This shows that there is a homogeneous polynomial $\varphi(s)$ that vanishes on $t(U)$. We choose such a polynomial $\varphi(s)$. Let its degree be r.

If f has degree d, the polynomial $g(x) = \varphi(\nabla f(x))$ will be homogeneous, of degree $r(d-1)$. It will vanish on U, and therefore on C (1.3.21). So f will divide g. Finally, if $\varphi(s)$ factors, then $g(x)$ factors accordingly, and because f is irreducible, it will divide one of the factors of g. The corresponding factor of φ will vanish on $t(U)$ (1.6.5). So we may replace the homogeneous polynomial φ by one of its irreducible factors. □

In principle, the proof of Theorem 1.6.6 gives a method for finding a polynomial that vanishes on the dual curve. That method is to find a polynomial relation among f_x, f_y, f_z, f, and set $f = 0$. But it is usually painful to determine the defining polynomial of C^* explicitly. Most often, the degrees of C and C^* will be different, and several points of the dual curve C^* may correspond to a singular point of C, and vice versa.

We give two examples in which the computation is easy.

1.6.7. Examples. exampled-ualone

(i) (*the dual of a conic*) Let $f = x_0x_1 + x_0x_2 + x_1x_2$ and let C be the conic $f = 0$. Let $(s_0, s_1, s_2) = (f_0, f_1, f_2) = (x_1{+}x_2,\ x_0{+}x_2,\ x_0{+}x_1)$. Then

(1.6.8) $$s_0^2 + s_1^2 + s_2^2 - 2(x_0^2 + x_1^2 + x_2^2) \ = 2f \qquad \text{and} \qquad s_0s_1 + s_1s_2 + s_0s_2 - (x_0^2 + x_1^2 + x_2^2) \ = 3f$$ exampled-ualtwo

We eliminate $(x_0^2 + x_1^2 + x_2^2)$ from the two equations.

(1.6.9) $$(s_0^2 + s_1^2 + s_2^2) - 2(s_0s_1 + s_1s_2 + s_0s_2) \ = \ -4f$$ equationd-ualthree

Setting $f = 0$ gives us the equation of the dual curve. It is another conic.

(ii) (*the dual of a cuspidal cubic*) The dual of a smooth cubic is a curve of degree 6. It is too much work to compute that dual here. We compute the dual of a singular cubic instead. The curve C defined by the irreducible polynomial $f = y^2z + x^3$ has a cusp at $(0, 0, 1)$. The Hessian matrix of f is

$$H \ = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2z & 2y \\ 0 & 2y & 0 \end{pmatrix}$$

and the Hessian determinant $\det H$ is $h = -24xy^2$. The common zeros of f and h are the singular point $(0, 0, 1)$ and a single flex point $(0, 1, 0)$.

We scale the partial derivatives of f to simplify notation. Let $u = f_x/3 = x^2$, $v = f_y/2 = yz$, and $w = f_z = y^2$. Then

$$v^2w - u^3 = y^4z^2 - x^6 = (y^2z + x^3)(y^2z - x^3) = f(y^2z - x^3)$$

The zero locus of the irreducible polynomial $v^2w - u^3$ is the dual curve. It is another singular cubic. □

(1.6.10) a local equation for the dual curve equa-tionofcstar

We label the coordinates in $\mathbb{P}$ and $\mathbb{P}^*$ as x, y, z and u, v, w, respectively, and we work in a neighborhood of a smooth point p_0 of the curve C that is defined by a homogeneous polynomial $f(x, y, z)$. We choose coordinates so that $p_0 = (0,0,1)$, and that the tangent line at p_0 is the line $L_0 : \{y = 0\}$. The image of p_0 in the dual curve C^* is $L_0^* : (u,v,w) = (0,1,0)$.

Let $\widetilde{f}(x,y) = f(x,y,1)$. In the affine x,y-plane, the point p_0 becomes the origin $p_0 = (0,0)$. So $\widetilde{f}(p_0) = 0$, and since the tangent line is L_0, $\frac{\partial \widetilde{f}}{\partial x}(p_0) = 0$, while $\frac{\partial \widetilde{f}}{\partial y}(p_0) \neq 0$. We solve the equation $\widetilde{f} = 0$ for y as an analytic function $y(x)$, with $y(0) = 0$. Let $y'(x)$ denote the derivative $\frac{dy}{dx}$. Differentiating the equation $f(x, y(x)) = 0$ shows that $y'(0) = 0$.

Let $\widetilde{p}_1 = (x_1, y_1)$ be a point of C_0 near to $\widetilde{p}_0$, so that $y_1 = y(x_1)$, and let $y_1' = y'(x_1)$. The tangent line L_1 at $\widetilde{p}_1$ has the equation

localtangent (1.6.11)
$$y - y_1 = y_1'(x - x_1)$$

Putting z back, the homogeneous equation of the tangent line L_1 at the point $p_1 = (x_1, y_1, 1)$ is

$$-y_1' x + y + (y_1' x_1 - y_1) z = 0$$

The point L_1^* of the dual plane that corresponds to L_1 is $(-y_1', 1, y_1' x_1 - y_1)$.

As x varies, the image in C^* of the point p_1 is the point L_1^*:

projlocaltangent (1.6.12)
$$(u_1, v_1, w_1) = (-y_1', 1, y_1' x_1 - y_1)$$

bidualone **(1.6.13) the bidual**

The *bidual* C^{**} of C is the dual of the curve C^*. It is a curve in the space $\mathbb{P}^{**}$, which is $\mathbb{P}$.

bidualC **1.6.14. Theorem.** *A plane curve C of degree greater than one is equal to its bidual C^{**}.*

We use the following notation for the proof:

- U is the set of smooth points of a curve C, and U^* is the set of smooth points of the dual curve C^*.
- $U^* \xrightarrow{t^*} \mathbb{P}^{**} = \mathbb{P}$ is the map analogous to the map $U \xrightarrow{t} \mathbb{P}^*$.
- V is the set of points p of C such that p is a smooth point of C and also $t(p)$ is a smooth point of C^*.

Then $V \subset U \subset C$ and $t(V) \subset U^* \subset C^*$.

vopen **1.6.15. Lemma.**

(i) *V is the complement of a finite set in C.*

(ii) *Let p_1 be a point near to a smooth point p of a curve C, let L_1 and L be the tangent line to C at p_1 and p, respectively, and let q be intersection point $L_1 \cap L$. Then $\lim_{p_1 \to p_0} q = p$.*

(iii) *If L is the tangent line to C at a point p of V, then p^* is the tangent line to C^* at the point L^*, and $t^*(L^*) = p$.*

proof. **(i)** Let S and S^* denote the finite sets of singular points of C, and C^*, respectively. Then V is obtained from U by deleting points of S and points in the inverse image of S^*. The fibre of t over a point L^* of C^* is the set of smooth points of C whose tangent line is L. Since L meets C in finitely many points, the fibre is finite. So the inverse image of the finite set S^* is a finite set.

(ii) We work analytically in a neighborhood of p, choosing coordinates so that $p = (0,0,1)$ and that L is the line $\{y = 0\}$. Let $(x_q, y_q, 1)$ be the coordinates of the intersection point q of L and L_1. Since q is a point of L, $y_q = 0$. The coordinate x_q can be obtained by substituting $x = x_q$ and $y = 0$ into the local equation (1.6.11) for L_1: $x_q = x_1 - y_1/y_1'$.

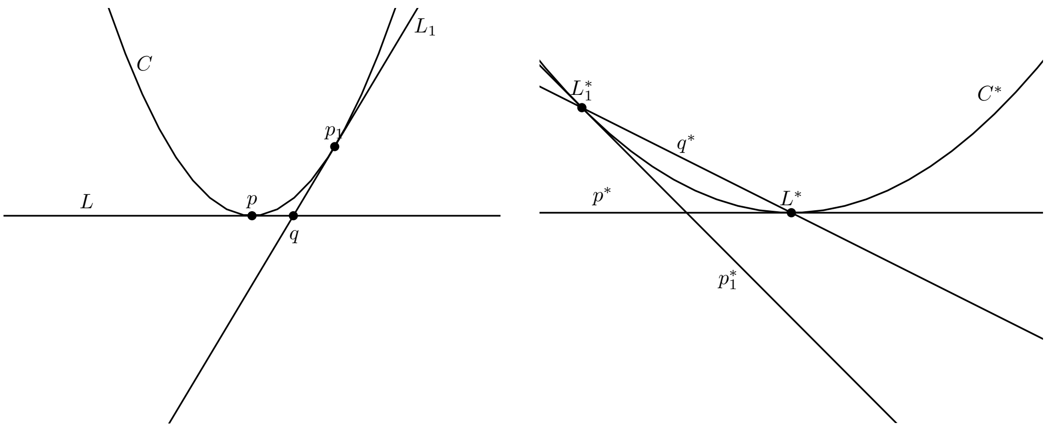
Now, when a function has an nth order zero at the point $x = 0$, i.e, when it has the form $y = x^n h(x)$, where $n > 0$ and $h(0) \neq 0$, the order of zero of its derivative at that point is $n-1$. This is verified by differentiating $x^n h(x)$. Since the function $y(x)$ has a zero of positive order at p, $\lim_{p_1 \to p_0} y_1/y_1' = 0$. We also have $\lim_{p_1 \to p_0} x_1 = 0$. Therefore $\lim_{p_1 \to p_0} x_q = 0$, and $\lim_{p_1 \to p_0} q = \lim_{p_1 \to p_0} (x_q, y_q, 1) = (0,0,1) = p$.

(iii) Let p_1 be a point of C near to p, and let L_1 be the tangent line to C at p_1. The image of p_1 is $L_1^* = (f_0(p_1), f_1(p_1), f_2(p_1))$. Because the partial derivatives f_i are continuous,

$$\lim_{p_1 \to p_0} L_1^* = (f_0(p), f_1(p), f_2(p)) = L^*$$

With $q = L \cap L_1$ as above, q^* is the line through the points L^* and L_1^*. As p_1 approaches p, L_1^* approaches L^*, and therefore q^* approaches the tangent line to C^* at L^*. On the other hand, it follows from **(ii)** that q^* approaches p^*. Therefore the tangent line to C^* at L^* is p^*. By definition, $t^*(L^*)$ is the point of C that corresponds to the tangent line p^* at L^*. So $t^*(L^*) = p^{**} = p$. □

1.6.16. curveanddual



A Curve and its Dual

In this figure, the curve C on the left is the parabola $y = x^2$. We used the local equation (1.6.11) to obtain an equation $u^2 = 4w$ of the dual curve C^*.

proof of theorem 1.6.14. Let p be a point of V, and let L be the tangent line at p. The map t^* is defined at L^*, and $t^*(L^*) = p$. Thus, since $L^* = t(p)$, $\;t^*t(p) = p$. It follows that the restriction of t to V is injective, and that it defines a bijective map from V to its image $t(V)$, whose inverse function is t^*. So V is contained in the bidual C^{**}. Since V is dense in C and since C^{**} is a closed set, C is contained in C^{**}. Since C and C^{**} are curves, $C = C^{**}$. □

1.6.17. Corollary. **(i)** *Let U be the set of smooth points of a plane curve C, and let t denote the map from U to the dual curve C^*. The image $t(U)$ of U is the complement of a finite subset of C^*.* *cstarisimage*

(ii) *If C is a smooth curve, the map $C \xrightarrow{t} C^*$, is defined at all points of C, and it is a surjective map..*

(iii) *Suppose that C is smooth, and that the tangent line L_0 at a point p_0 of C isn't tangent to C at another point (i.e., that L_0 isn't a bitangent). Then the path defined by the local equation (1.6.12) traces out the dual curve C^* near to $L_0^* = (0,1,0)$.*

proof. **(i)** With U, U^*, and V as above, $V = t^*t(V) \subset t^*(U^*) \subset C^{**} = C$. Since V is the complement of a finite subset of C, $\;t^*(U^*)$ is a finite subset of C too. The assertion to be proved follows when we switch C and C^*.

(ii) The map t is continuous, so its image $t(C)$ is a compact subset of C^*, and by **(i)**, its complement S is a finite set. Therefore S is both open and closed. It consists of isolated points of C^*. Since a plane curve has no isolated point, S is empty.

(iii) Because C is smooth, the continuous map $C \xrightarrow{t} C^*$ is defined at all points, and when L isn't a bitangent, the only point that maps to L^* is p. Then t will map a small disc D around p bijectively to its image D^*. That map is the one given by the formula (1.6.12). The complement W^* of D^* in C^* is a compact space that doesn't contain L^*. So a small neighborhood Z of L^* in $\mathbb{P}^*$ won't contain any point of W^*. Then $Z \cap C^*$ will be D^*. □

1.7 Resultants and Discriminants

resultant

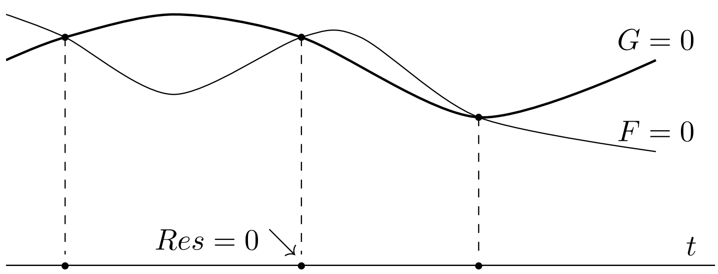
Let

polys (1.7.1) $$F(x) = x^m + a_1x^{m-1} + \cdots + a_m \quad \text{and} \quad G(x) = x^n + b_1x^{n-1} + \cdots + b_n$$

be monic polynomials in with variable coefficients a_i, b_j. The *resultant* $\mathrm{Res}(F, G)$ of F and G is a certain polynomial in the coefficients. Its important property is that, when the coefficients of F and G are given values in a field, the resultant becomes zero if and only if F and G have a common factor.

For instance, suppose that $F(x) = x + a$ and $G(x) = x^2 + b_1x + b_2$. The root $-a$ of F is a root of G if $G(a) = a^2 - b_1a + b_2$ is zero. The resultant of F and G is $a^2 - b_1a + b_2$.

restwovar **1.7.2. Example.** Suppose that the coefficients a_i and b_j in (1.7.1) are polynomials in t, so that F and G become polynomials in two variables. Let C and D be (possibly reducible) curves $F = 0$ and $G = 0$ in the affine plane $\mathbb{A}^2_{t,x}$, and let S be the set of intersections $C \cap D$. The resultant $r = \mathrm{Res}(F, G)$, computed regarding x as the variable, will be a polynomial in t whose roots are the t-coordinates of the elements of S.



The analogous statement is true when there are more variables. If F and G are relatively prime polynomials in x, y, z, the loci $C : \{F = 0\}$ and $D : \{G = 0\}$ in $\mathbb{A}^3$ will be surfaces, and $S = C \cap D$ will be a curve. The resultant $\mathrm{Res}_z(F, G)$, computed regarding z as the variable, is a polynomial in x, y whose zero locus in the x, y-plane is the projection of S to that plane. □

The formula for the resultant is nicest when one allows leading coefficients different from 1. We work with homogeneous polynomials in two variables to prevent the degrees from dropping when a leading coefficient happens to be zero. The common zeros of two homogeneous polynomials $f(x, y)$ and $g(x, y)$ correspond to the common roots of the polynomials $F(x) = f(x, 1)$ and $G(x) = g(x, 1)$, except when the zero is the point $(0, 1)$.

Let

hompolys (1.7.3) $$f(x, y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m, \quad \text{and} \quad g(x, y) = b_0x^n + b_1x^{n-1}y + \cdots + b_ny^n$$

be homogeneous polynomials in x and y, of degrees m and n, respectively, and with complex coefficients. If these polynomials have a common zero $(x, y) = (u, v)$ in $\mathbb{P}^1_{xy}$, then $vx - uy$ divides both g and f (see (1.3.6)). So the polynomial $h = fg/(vx - uy)$ will be divisible by f and by g, say $h = pf = qg$, where p and q are homogeneous polynomials of degrees n–1 and m–1, respectively, and h has degree $m+n-1$. Then h will be the linear combination p of the polynomials x^iy^jf, with $i+j = n-1$ and it will also be the linear combination q of the polynomials $x^ky^\ell g$, with $k+\ell = m-1$. The fact that these two combinations are equal tells us that the $r+1$ polynomials of degree $r = m+n-1$,

mplus-npolys (1.7.4) $$x^{n-1}f,\ x^{n-2}yf, ..., \ y^{n-1}f \ \ ; \ \ x^{m-1}g,\ x^{m-2}yg, ..., \ y^{m-1}g$$

will be (linearly) dependent. For example, suppose that f has degree 3 and g has degree 2. If f and g have a common zero, the polynomials

$$xf = \quad a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3$$

$$\begin{aligned} yf &= \qquad\quad a_0x^3y + a_1x^2y^2 + a_2xy^3 + a_3y^4 \\ x^2g &= b_0x^4 + b_1x^3y + b_2x^2y^2 \\ xyg &= \qquad\quad b_0x^3y + b_1x^2y^2 + b_2xy^3 \\ y^2g &= \qquad\qquad\qquad bx^2y^2 + b_1xy^3 + b_2y^4 \end{aligned}$$

will be dependent. Conversely, if the polynomials (1.7.4) are dependent, there will be an equation of the form $pf - qg = 0$, with p of degree $n-1$ and q of degree $m-1$. Then at least one zero of g must also be a zero of f.

The polynomials (1.7.4) have degree r. We form a square $(r+1)\times(r+1)$ matrix $\mathcal{R}$, the *resultant matrix*, whose columns are indexed by the monomials $x^r, x^{r-1}y, ..., y^r$ of degree r, and whose rows list the coefficients of the polynomials (1.7.4). The matrix is illustrated below for the cases $m, n = 3, 2$ and $m, n = 1, 2$, with dots representing entries that are zero:

(1.7.5) $$\mathcal{R} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdot \\ \cdot & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & \cdot & \cdot \\ \cdot & b_0 & b_1 & b_2 & \cdot \\ \cdot & \cdot & b_0 & b_1 & b_2 \end{pmatrix} \quad \text{or} \quad \mathcal{R} = \begin{pmatrix} a_0 & a_1 & \cdot \\ \cdot & a_0 & a_1 \\ b_0 & b_1 & b_2 \end{pmatrix}$$ resmatrix

The *resultant* of f and g is defined to be the determinant of $\mathcal{R}$.

(1.7.6) $$\operatorname{Res}(f, g) = \det \mathcal{R}$$ resequals-det

In this definition, the coefficients of f and g can be in any ring.

The resultant $\operatorname{Res}(F, G)$ of the monic, one-variable polynomials $F(x) = x^m + a_1x^{m-1} + \cdots + a_m$ and $G(x) = x^n + b_1x^{n-1} + \cdots + b_n$ is the determinant of the matrix $\mathcal{R}$, with $a_0 = b_0 = 1$.

1.7.7. Corollary. *Let f and g be homogeneous polynomials in two variables or monic polynomials in one variable, of degrees m and n, respectively, and with coefficients in a field. The resultant $\operatorname{Res}(f, g)$ is zero if and only if f and g have a common factor. If so, there will be polynomials p and q of degrees $n-1$ and $m-1$ respectively, such that $pf = qg$. If the coefficients are complex numbers, the resultant is zero if and only if f and g have a common zero.* $\square$ *homogre-sult*

When the leading coefficients a_0 and b_0 of f and g are both zero, the point $(1, 0)$ of $\mathbb{P}^1$ will be a zero of f and of g. One could say that f and g have a common zero at infinity in this case.

(1.7.8) weighted degree

weights

When defining the degree of a polynomial, one may assign an integer called a *weight* to each variable. If one assigns weight w_i to the variable x_i, the monomial $x_1^{e_1} \cdots x_n^{e_n}$ gets the *weighted degree*

$$e_1w_1 + \cdots + e_nw_n$$

For instance, it is natural to assign weight k to the coefficient a_k of the polynomial $f(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots \pm a_n$ because, if f factors into linear factors, $f(x) = (x - \alpha_1)\cdots(x - \alpha_n)$, then a_k will be the kth elementary symmetric function in $\alpha_1, ..., \alpha_n$. When written as a polynomial in those symmetric functions, the degree of a_k will be k.

1.7.9. Lemma. *Let $f(x, y)$ and $g(x, y)$ be homogeneous polynomials of degrees m and n respectively, with variable coefficients a_i and b_i, as in (1.7.3). When one assigns weight i to a_i and to b_i, the resultant $\operatorname{Res}(f, g)$ becomes a weighted homogeneous polynomial of degree mn in the variables $\{a_i, b_j\}$.* $\square$ *degresult*

1.7.10. Proposition. *Let F and G be products of monic linear polynomials, say $F = \prod_i (x - \alpha_i)$ and $G = \prod_j (x - \beta_j)$. Then* *resroots*

$$\operatorname{Res}(F, G) \quad = \quad \prod_{i,j} (\alpha_i - \beta_j) \quad = \quad \prod_i G(\alpha_i)$$

proof. The equality of the second and third terms is obtained by substituting α_i for x into the formula $G = \prod(x-\beta_j)$. We prove that the first term is equal to the second one.

We suppose that the polynomials F and G have variable roots α_i, and β_j. Let R denote the resultant $\mathrm{Res}(F,G)$ and let Π denote the product $\prod_{i.j}(\alpha_i-\beta_j)$. When we write the coefficients of F and G as symmetric functions in the roots, α_i and β_j, R will be homogeneous. Its (unweighted) degree in $\{\alpha_i,\beta_j\}$ will be mn, the same as the degree of Π. To show that $R=\Pi$, we choose i and j. Viewing R as a polynomial in the variable α_i, we divide by $\alpha_i-\beta_j$, which is monic in α_i:

$$R = (\alpha_i-\beta_j)q + r$$

where r has degree zero in α_i. The coefficients of F and G are in the field of rational functions in $\{\alpha_i,\beta_j\}$, so Corollary 1.7.7 tells us that the resultant R vanishes when we make the substitution $\alpha_i=\beta_j$. Looking at the above equation, we see that the remainder r also vanishes when $\alpha_i=\beta_j$. On the other hand, the remainder is independent of α_i. It doesn't change when we make that substitution. Therefore the remainder is zero, and $\alpha_i-\beta_j$ divides R. This is true for all i and j, so Π divides R, and since these two polynomials have the same degree, $R=c\Pi$ for some scalar c. To show that $c=1$, one needs to compute R and Π for some particular polynomials. We suggest making the computation with $F=x^m$ and $G=x^n-1$. □

restriviali-ties

1.7.11. Corollary. *Let $F,G,$ and H be monic polynomials and let c be a scalar. Then*
(i) $\mathrm{Res}(F,GH)=\mathrm{Res}(F,G)\,\mathrm{Res}(F,H)$, *and*
(ii) $\mathrm{Res}(F(x-c),G(x-c))=\mathrm{Res}(F(x),G(x))$. □

discrim-sect

(1.7.12) the discriminant

The *discriminant* $\mathrm{Discr}(F)$ of a polynomial $F=a_0x^m+a_1x^{n-1}+\cdots a_m$ is the resultant of F and its derivative F':

discrdef

(1.7.13)
$$\mathrm{Discr}(F)=\mathrm{Res}(F,F')$$

It is computed using the formula for the resultant of a polynomial of degree m, and it will be a weighted polynomial of degree $m(m-1)$. The definition makes sense when the leading coefficient a_0 is zero, but the discriminant will be zero in that case.

When F is a polynomial of degree n with complex coefficients, the discriminant is zero if and only if F has a multiple root, which happens when F and F' have a common factor.

Note. The formula for the discriminant is often normalized by a scalar factor. We won't make this normalization, so our formula is slightly different from the usual one.

The discriminant of the quadratic polynomial $F(x)=ax^2+bx+c$ is

discrquadr

(1.7.14)
$$\det\begin{pmatrix} a & b & c\\ 2a & b & \cdot\\ \cdot & 2a & b\end{pmatrix} = -a(b^2-4ac)$$

and the discriminant of a monic cubic x^3+px+q whose quadratic coefficient is zero is

discrcubic

(1.7.15)
$$\det\begin{pmatrix} 1 & \cdot & p & q & \cdot\\ \cdot & 1 & \cdot & p & q\\ 3 & \cdot & p & \cdot & \cdot\\ \cdot & 3 & \cdot & p & \cdot\\ \cdot & \cdot & 3 & \cdot & p\end{pmatrix} = 4p^3+27q^2$$

As mentioned, these formulas differ from the usual ones by a scalar factor. The usual formula for the discriminant of the quadratic ax^2+bx+c is b^2-4ac, and the discriminant of the cubic $\mathrm{y}x^3+px+q$ is usually written as $-4p^3-27q^2$.

Though it conflicts with our definition, we'll follow tradition and continue writing the discriminant of the quadratic as b^2-4ac.

1.7.16. Example. Suppose that the coefficients a_i of F are polynomials in t, so that F becomes a polynomial, let's say irreducible, in two variables t, x. Let C be the curve $F = 0$ in the t, x-plane. The discriminant $\mathrm{Discr}_x(F)$, computed regarding x as the variable, will be a polynomial in t. At a root t_0 of the discriminant, the line $L_0 : \{t = t_0\}$ is tangent to C, or passes though a singular point of C. □ discrt-wovar

1.7.17. Proposition. *Let K be a field of characteristic zero, and let F be an irreducible polynomial in $K[x]$. The discriminant of F isn't zero, and therefore F has no multiple root.* *dis-crnotzero*

proof. When F is irreducible, it cannot have a factor in common with the derivative F', which has lower degree. □

This proposition is false when the characteristic of K isn't zero. In characteristic p, the derivative F' might be the zero polynomial.

1.7.18. Proposition. *Let $F = \prod(x - \alpha_i)$ be a product of monic linear factors. Then* *discrfor-mulas*

$$\mathrm{Discr}(F) = \prod_i F'(\alpha_i) = \prod_{i \neq j}(\alpha_i - \alpha_j) = \pm \prod_{i<j}(\alpha_i - \alpha_j)^2$$

proof. The fact that $\mathrm{Discr}(F) = \prod F'(\alpha_i)$ follows from (1.7.10). We show that $F'(\alpha_i) = \prod_{j, j \neq i}(\alpha_i - \alpha_j)$. By the product rule for differentiation,

$$F'(x) = \sum_k (x - \alpha_1) \cdots (\widehat{x - \alpha_k}) \cdots (x - \alpha_n)$$

where the hat $\widehat{\ }$ indicates that that term is deleted. When we substitute $x = \alpha_i$, all terms in this sum, except the one with $k = i$, become zero. □

1.7.19. Corollary. $\mathrm{Discr}(F(x)) = \mathrm{Discr}(F(x - c))$. □ *translate-discr*

1.7.20. Proposition. *Let $F(x)$ and $G(x)$ be monic polynomials. Then* *discrprop*

$$\mathrm{Discr}(FG) = \pm \,\mathrm{Discr}(F)\,\mathrm{Discr}(G)\mathrm{Res}(F, G)^2$$

proof. This proposition follows from Propositions 1.7.10 and 1.7.18 for polynomials with complex coefficients. It is true for polynomials with coefficients in any ring because it is an identity. For the same reason, Corollary 1.7.11 remains true with coefficients in any ring. □

When f and g are polynomials in several variables, one of which is z, $\mathrm{Res}_z(f, g)$ and $\mathrm{Discr}_z(f)$ will denote the resultant and the discriminant, computed regarding f, g as polynomials in z. They will be polynomials in the other variables.

1.7.21. Lemma. *Let f be an irreducible polynomial in $\mathbb{C}[x, y, z]$ of positive degree in z, and not divisible by z. The discriminant $\mathrm{Discr}_z(f)$ of f with respect to the variable z is a nonzero polynomial in x, y.* *firroverK*

proof. This follows from Lemma 1.3.14 **(ii)** and Proposition 1.7.17. □

1.8 Nodes and Cusps

nodes

(1.8.1) the multiplicity of a singular point

singmult

Let C be the projective curve defined by an irreducible homogeneous polynomial $f(x, y, z)$ of degree d, and let p be a point of C. We choose coordinates so that $p = (0, 0, 1)$, and we set $z = 1$. This gives us an affine curve C_0 in $\mathbb{A}^2_{x,y}$, the zero set of the polynomial $\widetilde{f}(x, y) = f(x, y, 1)$, and p becomes the origin $(0, 0)$. We write

(1.8.2) $$\widetilde{f}(x, y) = f_0 + f_1 + f_2 + \cdots + f_d$$ seriesf where f_i is the homogeneous part of $\widetilde{f}$ of degree i, which is also the coefficient of z^{d-i} in $f(x,y,z)$. If the origin p is a point of C_0, the constant term f_0 will be zero, and the linear term f_1 will define the tangent direction to C_0 at p, If f_0 and f_1 are both zero, p will be a singular point of C.

It seems permissible to drop the tilde and the subscript 0 in what follows, denoting $f(x,y,1)$ by $f(x,y)$, and C_0 by C.

We use analogous notation for an analytic function $f(x,y)$ (see 1.4.18). Let f_i denote the homogeneous part of degree i of the series f:

multr (1.8.3)
$$f(x,y) = f_0 + f_1 + \cdots$$

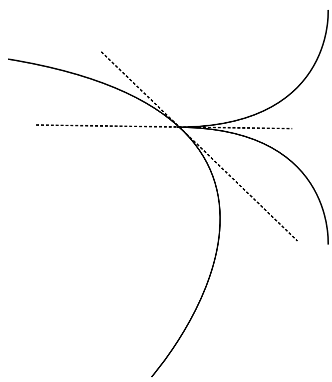
and let C denote the locus of zeros of f in a neighborhood of the origin p.

To describe the singularity of C at the origin, we look at the part of f of lowest degree. The smallest integer r such that $f_r(x,y)$ isn't zero is the *multiplicity* of C at p. When the multiplicity is r, f will have the form $f_r + f_{r+1} + \cdots$.

Let L be the line $\{vx = uy\}$ through p, and suppose that $u \neq 0$. The intersection multiplicity (1.3.9) of C and L at p is the order of zero of the series in x obtained by substituting $y = vx/u$ into f. The intersection multiplicity will be r unless $f_r(u,v)$ is zero. If $f_r(u,v) = 0$, the intersection multiplicity will be greater than r.

A line L through p whose intersection multiplicity with C at p is greater than the multiplicity of C at p will be called a *special line*. The special lines correspond to the zeros of f_r in $\mathbb{P}^1$. Because f_r has degree r, there will be at most r special lines.

goober14 **1.8.4.**



a Singular Point, with its Special Lines (real locus)

dpt **(1.8.5) double points**

To analyze a singularity at the origin, one may blow up the plane. The *blowup* is the map $W \xrightarrow{\pi} X$ from the (x,w)-plane W to the (x,y)-plane X defined by $\pi(x,w) = (x,xw)$. It is called a "blowup" because the fibre over the origin in X is the w-axis $\{x=0\}$ in W : $\pi(0,w) = (0,0)$ for all w. The blowup is bijective at points at which $x \neq 0$, and points $(x,0)$ of X with $x \neq 0$ aren't in its image. (It might seem more appropriate to call the inverse of π the blowup, but the inverse isn't a map.)

Suppose that the origin p is a *double point*, a point of multiplicity 2, and let the quadratic part of f be

quadrat-icterm
$$f_2 = ax^2 + bxy + cy^2$$

We adjust coordinates so that c isn't zero, and we normalize c to 1. Writing

$$f(x,y) = ax^2 + bxy + y^2 + dx^3 + \cdots$$

we make the substitution $y = xw$ and cancel x^2. This gives us a polynomial

$$g(x,w) \;=\; f(x,xw)/x^2 \;=\; a + bw + w^2 + dx + \cdots$$

in which all of the terms represented by $\cdots$ are divisible by x. Let D be the locus $\{g=0\}$ in W. The map π restricts to a map $D \xrightarrow{\overline{\pi}} C$. Since π is bijective at points at which $x \neq 0$, so is $\overline{\pi}$.

Suppose first that the quadratic polynomial $y^2 + by + a$ has distinct roots α, β, so that $ax^2 + bxy + y^2 = (y - \alpha x)(y - \beta x)$ and $g(x,w) = (w-\alpha)(w-\beta) + dx + \cdots$. In this case, the fibre of D over the origin in X consists of the two points $p_1 = (0,\alpha)$ and $p_2 = (0,\beta)$. The partial derivative $g_w = \frac{\partial g}{\partial w}$ isn't zero at p_1 or p_2, so those are smooth points of D. At each of those points, we can solve $g(x,w) = 0$ for w as analytic functions of x, say $w = u(x)$ and $w = v(x)$, with $u(0) = \alpha$ and $v(0) = \beta$. The image $\pi(D)$ is C, so C has two analytic branches $y = xu(x)$ and $y = xv(x)$ through the origin, with distinct tangent directions α and β. In this case, the singularity of C at p is called a *node*. A node is the simplest singularity that a curve can have.

When the discriminant $b^2 - 4ac$ is zero, f_2 will be a square, and f will have the form

(1.8.6) $$f(x,y) = (y - \alpha x)^2 + dx^3 + \cdots$$ cuspeq

In this case, the blowup substitution $y = xw$ gives

$$g(x,w) = (w - \alpha)^2 + dx + \cdots$$

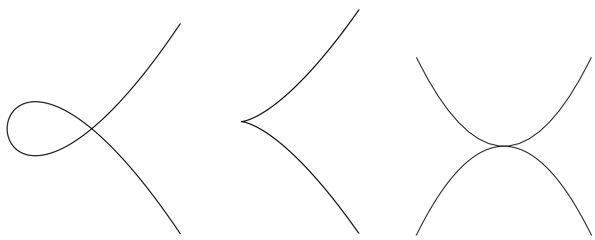
Here the fibre over $(x,y) = (0,0)$ is the point $(x,w) = (0,\alpha)$, and $g_w(0,\alpha) = 0$. However, $g_x(0,\alpha) \neq 0$, provided that $d \neq 0$. If so, D is smooth at $(0,0)$, and the singularity at the origin is called a *cusp*. The equation of C will have the form $(y - \alpha x)^2 = dx^3 + \cdots$.

The *standard cusp* is the locus $y^2 = x^3$. All cusps are analytically equivalent with the standard cusp.

1.8.7. Corollary. *A double point p of a curve C is a node or a cusp if and only if the blowup of C is smooth at the points that lie over p.* □ *nodeor-cusp*

The simplest example of a double point that isn't a node or cusp is a *tacnode*, a point at which two smooth branches of a curve intersect with the same tangent direction.

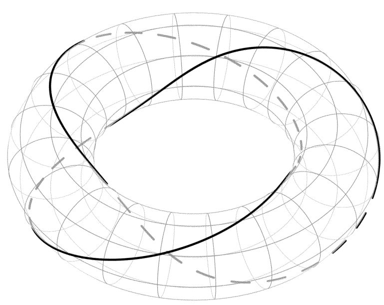
goober23



1.8.8. **a Node, a Cusp, and a Tacnode** (real locus)

Cusps have an interesting geometry. The intersection of the standard cusp $X : \{y^2 = x^3\}$, with a small 3-sphere $S : \{\overline{x}x + \overline{y}y = \epsilon\}$ in $\mathbb{C}^2$ is a *trefoil knot*, as is illustrated below.

1.8.9. trefoil



Intersection of a Cusp Curve with a Three-Sphere

This nice figure was made by Jason Chen and Andrew Lin. The standard cusp, the locus $y^2 = x^3$, can be parametrized as $(x, y) = (t^2, t^3)$. The points of X of absolute value $\sqrt{2}$ are $(x, y) = (e^{2i\theta}, e^{3i\theta})$. This locus is embedded into the product of a unit x-circle and a unit y-circle in $\mathbb{C}^2$, a torus T_1. The figure depicts T_1 as the usual torus T_0 in $\mathbb{R}^3$, though the mapping $T_1 \to T_0$ distorts T_1. The circumference of T_0 represents the x-coordinate, and the loop through the hole represents y. As θ runs from 0 to 2π, the point (x, y) goes around the circumference twice, and it loops through the hole three times, as is illustrated. $\square$

tracecusp **1.8.10. Proposition.** *Let $x(t) = t^2 + \cdots$ and $y(t) = t^3 + \cdots$ be analytic functions of t (1.4.18), whose orders of vanishing are 2 and 3, as indicated. For small t, the path $(x, y) = (x(t), y(t))$ in the x, y-plane traces out a curve with a cusp at the origin.*

proof. We show that there are analytic functions $b(x) = b_2x^2 + \cdots$ and $c(x) = x^3 + \cdots$ that vanish to orders 2 and 3 at $x = 0$, such that $x(t)$ and $y(t)$ solve the equation $y^2 + b(x)y + c(x) = 0$. The locus of this equation has a cusp at the origin.

We solve for b and c: Since $x = t^2 + \cdots = t^2(1 + \cdots)$, x has an analytic square root of the form $z = t + \cdots$. This follows from the Implicit Function Theorem, which also tells us that t can be written as an analytic function of z. So the function z is a coordinate equivalent to t, and we may replace t by z. Then we will have $x = t^2$, and y still has a zero of order 3, $y = t^3 + \cdots$, though the series is changed.

Let call the *even part* of a series $\sum a_n t^n$ the sum of the terms $a_n t^n$ with n even, and the *odd part* the sum of terms with n odd. We write $y(t) = u(t) + v(t)$, where u an v are the even and the odd parts of y, respectively. The convergent series $y(t)$ is absolutely convergent its radius of convergence. So $u(t)$ and $v(t)$ are also convergent series.

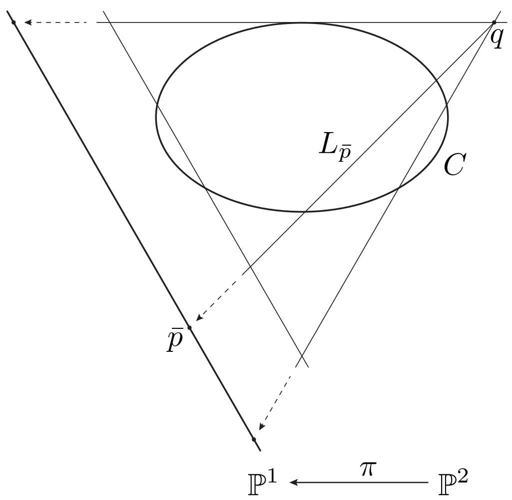
Now $y^2 = (u^2+v^2)+2uv$. The even part of the series y^2 is u^2+v^2 and the odd part is $2uv$. The even series $-2u$ can be written as a (convergent) series in $x = t^2$, say $-2u = b(x)$. Then $by = -2u(u+v) = -2u^2-2uv$, and $y^2 + by = v^2 - u^2$, which is an even series in t. We can write this even serries as a series in $x = t^2$, say $v^2 - u^2 = -c(x)$, so that $y^2 + by + c = 0$. The orders of vanishing of b and c are determined by the equation. $\square$

coverline **(1.8.11) projection to a line**

We denote by π the *projection* $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ that drops the last coordinate, sending a point (x, y, z) to (x, y). It is defined at all points of $\mathbb{P}^2$ except at the *center of projection*, the point $q = (0, 0, 1)$.

The fibre of π over a point $\widetilde{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the line through $p = (x_0, y_0, 0)$ and $q = (0, 0, 1)$, with the point q omitted — the set of points (x_0, y_0, z_0). We denote that line by L_{pq} or by $L_{\widetilde{p}}$

projtoline **1.8.12.**



Projection from the Plane to a Line

The projection π will be defined at all points of a plane curve C, provided that C doesn't contain the center of projection q. Say that C is defined by an irreducible homogeneous polynomial $f(x, y, z)$, and that q isn't a point of C. Let d be the degree of f. We write f as a polynomial in z,

(1.8.13) $$f = c_0 z^d + c_1 z^{d-1} + \cdots + c_d$$ poly-inztwo

with c_i homogeneous, of degree i in x, y. The scalar $c_0 = f(0, 0, 1)$ isn't zero when q isn't in C. We normalize c_0 to 1, so that f becomes a monic polynomial of degree d in z.

The fibre of C over a point $\widetilde{p} = (x_0, y_0)$ of $\mathbb{P}^1$ is the intersection of C with the line L_{pq} described above. It consists of the points (x_0, y_0, α) such that α is a root of the one-variable polynomial

(1.8.14) $$\widetilde{f}(z) = f(x_0, y_0, z)$$ effp

We call C a *branched covering* of $\mathbb{P}^1$ of degree d.

All but finitely many fibres of C over $\mathbb{P}^1$ consist of d points (Lemma 1.7.21). The fibres with fewer than d points are those above the zeros of the discriminant. Those zeros are the *branch points* of the covering. We use the same term for points of C, calling a point of C a *branch point* if its tangent line is $L_{p,q}$, in which case its image in $\mathbb{P}^1$ will also be a branch point.

1.8.15. Proposition. *Let C be a smooth plane curve, let q be a generic point of the plane, and let p be a branch point of C, so that the tangent line L at p contains q. The intersection multiplicity of L and C at p is* 2*, and L and C have $d-2$ other intersections of multiplicity* 1. *flexbitan*

The proof is below, but first, we explain the word *generic*.

(1.8.16) generic and general position genpos

In algebraic geometry, the word *generic* is used for an object such as a point, that has no special 'bad' properties. Typically, the object will be parametrized somehow, and the adjective generic indicates that the parameter representing that particular object avoids a proper closed subset of the parameter space that may be described explicitly or not. The phrase *general position* has a similar meaning. It indicates that an object is not in a 'bad' position. In Proposition 1.8.15, what is required of the generic point q is that it shall not lie on a flex tangent line or on a *bitangent line* — a line that is tangent to C at two or more points. We have seen that a smooth curve C has finitely many flex points (1.4.17). Lemma 1.8.17 below shows that it has finitely many bitangents. So q must avoid a finite set of lines. Most points of the plane will be generic in this sense. □

proof of Proposition 1.8.15. The intersection multipicity of L and C at p is at least 2 because L is a tangent line at p. It will be equal to 2 unlss p is a flex point. The generic point q won't lie on any of the finitely many flex tangents, so the intersection multiplicity at p is 2. Next, the intersection multiplicity at another point p' of $L \cap C$ will be 1 unless L is tangent to C at p' as well as at p, i.e., unless L is a bitangent. The generic point q won't lie on a bitangent. □

1.8.17. Lemma. *A plane curve has finitely many bitangent lines.* *finbitan*

proof. This is an opportunity to use the map $U \xrightarrow{t} C^*$ from the set U of smooth points of C to the dual curve C^*. If L is tangent to C at distinct smooth points p and p', then t is defined at those points, and $t(p) = t(p') = L^*$. Therefore L^* will be a singular point of C^*. Since C^* has finitely many singular points, C has finitely many bitangents. □

(1.8.18) the genus of a plane curve genus

We describe the topological structure of a smooth plane curve in the classical topology.

1.8.19. Theorem. *A smooth projective plane curve of degree d is a compact, orientable and connected manifold of dimension two.* *curveshome-omorphic*

The fact that a smooth curve is a two-dimensional manifold follows from the Implicit Function Theorem. (See the discussion (**1.4.4**)).

orientability: A two-dimensional manifold is orientable if one can choose one of its two sides (as in front and back of a sheet of paper) in a continuous, consistent way. A smooth curve C is orientable because its tangent space at a point, the affine line with the equation (1.4.11), is a one-dimensional complex vector space. Multiplication by i orients the tangent space by defining the counterclockwise rotation. Then the right-hand rule tells us which side of C is "up".

compactness: A plane projective curve is compact because it is a closed subset of the compact space $\mathbb{P}^2$.

connectedness: The fact that a plane curve is connected is subtle, and its proof mixes topology and algebra. Unfortunately, I don't know a proof that fits into our discussion here. It will be proved later (see Theorem 8.2.11).

The topological *Euler characteristic* of a compact, orientable two-dimensional manifold M is the alternating sum $b^0 - b^1 + b^2$ of its Betti numbers. The Euler characteristic, which we denote by e, can be computed using a *topological triangulation*, a subdivision of M into topological triangles, called *faces*, by the formula

vef (1.8.20)
$$e \;=\; |\text{vertices}| - |\text{edges}| + |\text{faces}|$$

For example, a sphere is homeomorphic to a tetrahedron, which has four vertices, six edges, and four faces. Its Euler characteristic is $4-6+4=2$. Any other topological triangulation of a sphere, such as the one given by the icosahedron, yields the same Euler characteristic.

Every compact, connected, orientable two-dimensional manifold is homeomorphic to a sphere with a finite number of "handles". Its *genus* is the number of handles. A torus has one handle. Its genus is one. The projective line $\mathbb{P}^1$, a two-dimensional sphere, has genus zero.

The Euler characteristic and the genus are related by the formula

genuseuler (1.8.21)
$$e = 2 - 2g$$

The Euler characteristic of a torus is zero, and the Euler characteristic of $\mathbb{P}^1$ is two.

To compute the Euler characteristic of a smooth curve C of degree d, we analyze a generic projection, to represent C as a branched covering of the projective line: $C \xrightarrow{\pi} \mathbb{P}^1$ (see **1.8.16**). We choose generic coordinates x, y, z in $\mathbb{P}^2$ and project from the point $q = (0,0,1)$. When the defining equation of C is written as a monic polynomial in z: $f = z^d + c_1 z^{d-1} + \cdots + c_d$ where c_i is a homogeneous polynomial of degree i in the variables x, y, the discriminant $\mathrm{Discr}_z(f)$ with respect to z will be a homogeneous polynomial of degree $d(d-1) = d^2 - d$ in x, y.

Let $\widetilde{p}$ be the image in $\mathbb{P}^1$ of a point p of C. The covering $C \xrightarrow{\pi} \mathbb{P}^1$ will be branched at $\widetilde{p}$ when the tangent line at p is the line L_{pq} through p and q. Proposition 1.8.15 tells us that if L_{pq} is a tangent line, there will be one intersection of multiplicity 2 and $d-1$ simple intersections. The discriminant will have a simple zero at such a point $\widetilde{p}$. This is proved in Proposition 1.9.11 below. Let's assume it for now.

Since the discriminant has degree d^2-d, there will be d^2-d points $\widetilde{p}$ in $\mathbb{P}^1$ at which the discriminant vanishes and the fibre over such a point contains $d-1$ points. They are the *branch points* of the covering. All other fibres consist of d points.

We triangulate the sphere $\mathbb{P}^1$ in such a way that the branch points are among the vertices, and we use the inverse images of the vertices, edges, and faces to triangulate C. Then C will have d faces and d edges lying over each face and each edge of $\mathbb{P}^1$, respectively. There will also be d vertices of C lying over a vertex of $\mathbb{P}^1$, except when the vertex is one of the d^2-d branch points. In that case the the fibre will contain only $d-1$ vertices. So the Euler characteristic of C can be obtained by multiplying the Euler characteristic of $\mathbb{P}^1$ by d and subtracting the number d^2-d of branch points:

eulercover (1.8.22)
$$e(C) \;=\; d\,e(\mathbb{P}^1) - (d^2-d) \;=\; 2d - (d^2-d) \;=\; 3d - d^2$$

This is the Euler characteristic of any smooth curve of degree d, so we denote it by e_d :

equatione (1.8.23)
$$e_d = 3d - d^2$$

Formula (1.8.21) shows that the genus g_d of a smooth curve of degree d is

$$g_d \;=\; \tfrac{1}{2}(d^2-3d+2) = \binom{d-1}{2} \tag{1.8.24}$$ equationg

Thus smooth curves of degrees $1, 2, 3, 4, 5, 6, ...$ have genus $0, 0, 1, 3, 6, 10, ...$, respectively. A smooth plane curve cannot have genus two.

The generic projection to $\mathbb{P}^1$ also computes the degree of the dual C^* of a smooth curve C of degree d. The degree of C^* is the number of its intersections with the generic line q^* in $\mathbb{P}^*$. The intersections of C^* and q^* are the points L^*, where L is a tangent line that contains q. As we saw above, there are d^2-d such lines.

1.8.25. Corollary. *Let C be a plane curve of degree d.* *degdual*
(i) *The degree d^* of the dual curve C^* is the number of tangent lines at smooth points of C that pass through a generic point q of the plane.*
(ii) *If C is smooth, $d^* = d^2 - d$.* □

When C is a singular curve, the degree of its dual curve will be less than $d^2 - d$.

If $d = 2$, C will be a smooth conic, and $d^* = d$. The dual curve is also a conic, as we have seen. But when $d > 2$, $d^* = d^2 - d$ will be greater than d. In this case the dual curve C^* must be singular. If it were smooth, the degree of its dual curve C^{**} would be ${d^*}^2 - d^*$, which would be greater than d. This would contradict the fact that $C^{**} = C$.

1.9 Hensel's Lemma

The resultant matrix (1.7.5) arises in a second context that we explain here. hensel

Suppose given a product $P = FG$ of two polynomials in a variable x, say

$$\big(c_0x^{m+n} + c_1x^{m+n-1} + \cdots + c_{m+n}\big) \;=\; \big(a_0x^m + a_1x^{m-1} + \cdots + a_m\big)\big(b_0x^n + b_1x^{n-1} + \cdots + b_n\big) \tag{1.9.1}$$ multiply-polys

We call the relations among the coefficients implied by this polynomial equation the *product equations*. The product equations are

$$c_i = a_ib_0 + a_{i-1}b_1 + \cdots + a_0b_i$$

for $i = 0, ..., m+n$. For instance, when $m = 3$ and $n = 2$, the poduct equations are

1.9.2. prodeqns

$$\begin{aligned}
c_0 &= a_0b_0 \\
c_1 &= a_1b_0 + a_0b_1 \\
c_2 &= a_2b_0 + a_1b_1 + a_0b_2 \\
c_3 &= a_3b_0 + a_2b_1 + a_1b_2 \\
c_4 &= \qquad\quad\; a_3b_1 + a_2b_2 \\
c_5 &= \qquad\qquad\qquad\;\; a_3b_2
\end{aligned}$$

Let J denote the Jacobian matrix of partial derivatives of $c_1, ..., c_{m+n}$ with respect to the variables $b_1, ..., b_n$ and $a_1, ..., a_m$, treating a_0, b_0 and c_0 as constants. When $m, n = 3, 2$,

$$J = \frac{\partial(c_i)}{\partial(b_j, a_k)} = \begin{pmatrix} a_0 & . & b_0 & . & . \\ a_1 & a_0 & b_1 & b_0 & . \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & . & b_2 & b_1 \\ . & a_3 & . & . & b_2 \end{pmatrix} \tag{1.9.3}$$ prodjacob

1.9.4. Lemma. *The Jacobian matrix J is the transpose of the resultant matrix $\mathcal{R}$ (1.7.5).* □ *jacres*

1.9.5. Corollary. *Let F and G be polynomials with complex coefficients. The Jacobian matrix is singular if and only if F and G have a common root, or else $a_0 = b_0 = 0$.* □ *jacobian-notzero*

This corollary has an application to polynomials with analytic coefficients. Let

poly-forhensel (1.9.6)
$$P(t,x)=c_0(t)x^d+c_1(t)x^{d-1}+\cdots+c_d(t)$$

be a polynomial in x whose coefficients c_i are analytic functions of t, and let $\overline{P}=P(0,x)=\overline{c}_0x^d+\overline{c}_1x^{d-1}+\cdots+\overline{c}_d$ be the evaluation of P at $t=0$, so that $\overline{c}_i=c_i(0)$. Suppose given a factorization $\overline{P}=\overline{F}\,\overline{G}$, where $\overline{F}=x^m+\overline{a}_1x^{m-1}+\cdots+\overline{a}_m$ and $\overline{G}=\overline{b}_0x^n+\overline{b}_1x^{n-1}+\cdots+\overline{b}_n$ are polynomials with complex coefficients, and $\overline{F}$ is monic. Are there polynomials $F(t,x)=x^m+a_1x^{m-1}+\cdots+a_m$ and $G(t,x)=b_0x^n+b_1x^{n-1}+\cdots+b_n$, with F monic, whose coefficients a_i and b_i are analytic functions of t, and such that $F(0,x)=\overline{F},\ G(0,x)=\overline{G}$, and $P=FG$?

hensellemma **1.9.7. Hensel's Lemma.** *With notation as above, suppose that $\overline{F}$ and $\overline{G}$ have no common root. Then P factors: $P=FG$, where F and G are polynomials in x, whose coefficients are analytic functions of t (1.4.18) and F is monic.*

proof. We look at the product equations. Since F is supposed to be monic, we set $a_0(t)=1$. The first product equation tells us that $b_0(t)=c_0(t)$. Corollary 1.9.5 tells us that the Jacobian matrix for the remaining product equations is nonsingular at $t=0$, so according to the Implicit Function Theorem, the product equations have a unique solution in analytic functions $a_i(t),b_j(t)$ for small t. □

Note that P isn't assumed to be monic. If $\overline{c}_0=0$, the degree of $\overline{P}$ will be less than the degree of P. In that case, $\overline{G}$ will have lower degree than G.

henselex **1.9.8. Example.** Let $P=c_0(t)x^2+c_1(t)x+c_2(t)$. The product equations $P=FG$ with $F=x+a_1$ monic and $G=b_0x+b_1$, are

quadfactor (1.9.9)
$$c_0=b_0\,,\quad c_1=a_1b_0+b_1\,,\quad c_2=a_1b_1$$

and the Jacobian matrix is
$$\frac{\partial(c_1,c_2)}{\partial(b_1,a_1)}=\begin{pmatrix}1&b_0\\a_1&b_1\end{pmatrix}$$

Suppose that $\overline{P}=P(0,x)$ factors: $\overline{c}_0x^2+\overline{c}_1x+\overline{c}_2=(x+\overline{a}_1)(\overline{b}_0x+\overline{b}_1)=\overline{F}\,\overline{G}$. The determinant of the Jacobian matrix at $t=0$ is $\overline{b}_1-\overline{a}_1\overline{b}_0$. It is nonzero if and only if the two factors are relatively prime, in which case P factors too.

On the other hand, the one-variable Jacobian criterion allows us to solve the equation $P(t,x)=0$ for x as function of t with $x(0)=-\overline{a}_1$, provided that $\frac{\partial P}{\partial x}=2c_0x+c_1$ isn't zero at the point $(t,x)=(0,-\overline{a}_1)$. If $\overline{P}$ factors as above, then when we substitute into (1.9.9 into $\overline{P}$, we find that $\frac{\partial P}{\partial x}(0,-\overline{a}_1)=-2\overline{c}_0\overline{a}_1+\overline{c}_1=\overline{b}_1-\overline{a}_1\overline{b}_0$. Not surprisingly, $\frac{\partial P}{\partial x}(0,-\overline{a}_1)$ is equal to the determinant of the Jacobian matrix at $t=0$. □

vandisc **(1.9.10) order of vanishing of the discriminant**

Let $f(x,y,z)$ be a homogeneous polynomial with no multiple factors, and let C be the (possibly reducible) plane curve $\{f=0\}$. Suppose that the center of projection $q=(0,0,1)$ is in general position (see **1.8.16**). Let L_{pq} denote the line through a point $p=(x_0,y_0,0)$ and q, the set of points (x_0,y_0,z_0), as before, and let $\widetilde{p}=(x_0,y_0)$.

discrim-vanishing **1.9.11. Proposition. (i)** *If p is a smooth point of C with tangent line L_{pq}, the discriminant* $\mathrm{Discr}_z(f)$ *has a simple zero at $\widetilde{p}$.*

(ii) *If p is a node of C,* $\mathrm{Discr}_z(f)$ *has a double zero at $\widetilde{p}$.*

(iii) *If p is a cusp,* $\mathrm{Discr}_z(f)$ *has a triple zero at $\widetilde{p}$.*

(iv) *If p is a an ordinary flex point of C (**1.4.8**) with tangent line L_{pq},* $\mathrm{Discr}_z(f)$ *has a double zero at $\widetilde{p}$.*

To be precise about what is required of the generic point q in this case, we ask that q not lie on any of these lines:

generic-cond (1.9.12)

flex tangent lines and bitangent lines,
lines that contain more than one singular point,
special lines through singular points,
tangent lines that contain a singular point of C.

1.9.13. Lemma. *This is a list of finitely many lines that q must avoid.* □ *finlines*

##?? $z \leftrightarrow x$?? set up notation## *proof of Proposition 1.9.11.* **(i)–(iii)** We'll use Hensel's Lemma. We set $z = 1$, to work in the standard affine open set $\mathbb{U}$ with coordinates x, y. In affine coordinates, the projection π is the map $(x, y) \to y$. The image $\widetilde{p}$ of p will be the point $y = 0$ of the affine y-line, and the intersection of the line L_{pq} with $\mathbb{U}$ will be the line $\widetilde{L} : \{y = 0\}$. We'll denote the defining polynomial of the curve C, restricted to $\mathbb{U}$, by $f(x, y)$ instead of $f(x, y, 1)$. Let $\widetilde{f}(x) = f(0, x)$.

In each of the cases **(i) – (iii)**, the polynomial $\overline{f}(x) = f(0, x)$ will have a double xero at $x = 0$, so we will have $\overline{f}(x) = x^2\overline{h}(x)$, with $\overline{h}(0) \neq 0$. Then x^2 and $\overline{h}(x)$ have no common root, so we may apply Hensel's Lemma: $f(x, y) = g(x, y)h(x, y)$, where g and h are polynomials in z whose coefficients are analytic functions of y, g is monic, $g(0, x) = z^2$, and $h(0, x) = \overline{h}$. Then $\mathrm{Discr}_x(f) = \pm\,\mathrm{Discr}_z(g)\,\mathrm{Discr}_x(h)\,\mathrm{Res}_x(g, h)^2 d$ (1.7.20). Since q is in general position, $\overline{h}$ will have simple zeros (1.8.15). Then $\mathrm{Discr}_x(h)$ doesn't vanish at $y = 0$, and neither does $\mathrm{Res}_x(g, h)$. So the orders of vanishing of $\mathrm{Discr}_x(f)$ and $\mathrm{Discr}_x(g)$ are equal. We replace f by g.

Havingdonethat, f is a monic quadratic polynomial, of the form

$$f(x, y) = x^2 + b(y)x + c(y)$$

The coefficients b and c are analytic functions of y, and $f(0, x) = x^2$. The discriminant $\mathrm{Discr}_x(f) = b^2 - 4c$ is unchanged when we complete the square by the substitution of $x - \frac{1}{2}b$ for x, and if p is smooth or has a node or a cusp, that property isn't affected by this change of coordinates. So we may assume that f has the form $x^2 + c(y)$. The discriminant is then $D = 4c(y)$.

We write $c(y)$ as a series in y:

$$c(y) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots$$

The constant coefficient c_0 is zero. If $c_1 \neq 0$, p is a smooth point with tangent line $\widetilde{L}$, and D has a simple zero. If p is a node, $c_0 = c_1 = 0$ and $c_2 \neq 0$. Then D has a double zero. If p is a cusp, $c_0 = c_1 = c_2 = 0$, and $c_3 \neq 0$. Then D has a triple zero at p.

(iv) In this case, the polynomial $\widetilde{f}(x) = f(0, x)$ will have a triple zero at $x = 0$. Proceding as above, we may factor: $f = gh$ where g and h are polynomials in x with analyic coefficients in y, and $g(x, y) = x^3 + a(y)x^2 + b(y)x + c(y)$. We eliminate the quadratic coefficient a by substituting $x - \frac{1}{3a}$ for x. With $g = x^3 + bx + c$ in the new coordinates, the discriminant $\mathrm{Discr}_x(g)$ is $4b^3 + 27c^2$ (1.7.15). We write $c(y) = c_0 + c_1 y + \cdots$ and $b(y) = b_0 + b_1 y + \cdots$. Since p is a point of C with tangent line $\{y = 0\}$, $c_0 = 0$ and $c_1 \neq 0$. Since the intersection multiplicity of C with the line $\{y = 0\}$ at p is three, $b_0 = 0$. The discriminant $4b^3 + 27c^2$ has a zero of order two. □

1.9.14. Corollary. *Let $C : \{g = 0\}$ and $D : \{h = 0\}$ be plane curves that intersect transversally at a point $p = (z_0, y_0, z_0)$. With coordinates in general position, $\mathrm{Res}_z(g, h)$ has a simple zero at (x_0, y_0).* *transres*

Two curves are said to intersect *transversally* at a point p if they are smooth at p and their tangent lines there are distinct.

proof. Proposition 1.9.11 **(ii)** applies to the product gh, whose zero locus is the union $C \cup D$. It shows that the discriminant $\mathrm{Discr}_z(gh)$ has a double zero at p. We also have the formula (1.7.20) with $f = gh$. When coordinates are in general position, $\mathrm{Discr}_z(g)$ and $\mathrm{Discr}_z(h)$ will not be zero at p. Since $\mathrm{Discr}_z(gh) = \mathrm{Discr}_z(g)\,\mathrm{Discr}_z(h)\,\mathrm{Res}(g, h)^2$, $\mathrm{Res}_z(g, h)$ has a simple zero at p. □

1.10 Bézout's Theorem

bezoutthm

Bézout's Theorem counts intersections of plane curves. We state it here in a form that is ambiguous because it contains a term "multiplicity" that hasn't yet been defined.

1.10.1. Bézout's Theorem. Let C and D be distinct curves of degrees m and n, respectively. When intersections are counted with an appropriate multiplicity, the number of intersections is equal to mn. Moreover, the multiplicity at a transversal intersection is 1. bezoutone

As before, C and D intersect transversally at p if they are smooth at p and their tangent lines there are distinct.

bezoutline **1.10.2. Corollary.** *Bézout's Theorem is true when one of the curves is a line.*

See Corollary 1.3.10. The multiplicity of intersection of a curve and a line is the one that was defined there. □

The proof in the general case requires some algebra that we would rather defer. The proof will be given later (Theorem 7.8.1), but we will use the theorem in the rest of this chapter.

It is possible to determine the intersections by counting the zeros of the resultant with respect to one of the variables. To do this, one chooses coordinates x, y, z, so that neither C nor D contains the point $(0, 0, 1)$. One writes their defining polynomials f and g as polynomials in z with coefficients in $\mathbb{C}[x, y]$. The resultant R with respect to z will be a homogeneous polynomial in x, y, of degree mn. It will have mn zeros in $\mathbb{P}^1_{x,y}$, counted with multiplicity. If $\widetilde{p} = (x_0, y_0)$ is a zero of R, $f(x_0, y_0, z)$ and $g(x_0, y_0, z)$, which are polynomials in z, have a common root $z = z_0$, and then $p = (x_0, y_0, z_0)$ will be a point of $C \cap D$. It is a fact that the multiplicity of the zero of the resultant R at the image $\widetilde{p}$ is the (as yet undefined) intersection multiplicity of C and D at p. Unfortunately, this won't be obvious when the multiplicity is defined. However, one can prove the next proposition using this approach.

nocommonfactor **1.10.3. Proposition.** *Let C and D be distinct plane curves of degrees m and n, respectively.*

(i) *The curves C and D have at least one point of intersection, and the number of intersections is at most mn.*

(ii) *If all intersections are transversal, the number of intersections is precisely mn.*

It isn't obvious that two curves in the projective plane intersect. If two curves in the affine plane have no intersection, if they are parallel lines, for instance, their closures in the projective plane meet on the line at infinity.

resnotzero **1.10.4. Lemma.** *Let f and g be homogeneous polynomials in x, y, z of degrees m and n, respectively, and suppose that the point $(0, 0, 1)$ isn't a zero of f or g. If the resultant $\mathrm{Res}_z(f, g)$ with respect to z is identically zero, then f and g have a common factor.*

proof. Let the degrees of f and g be m and n, respectively, and let F denote the field of rational functions $\mathbb{C}(x, y)$. If the resultant is zero, f and g have a common factor in $F[z]$ (Corollary 1.7.7). There will be polynomials p and q in $F[z]$, of degrees at most $n-1$ and $m-1$ in z, respectively, such that $pf = qg$ (1.7.3). We may clear denominators, so we may assume that the coefficients of p and q are in $\mathbb{C}[x, y]$. This doesn't change the degree in z. Then $pf = qg$ is an equation in $\mathbb{C}[x, y, z]$. Since p has degree at most $n-1$ in z, it isn't divisible by g, which has degree n in z. Since $\mathbb{C}[x, y, z]$ is a unique factorization domain, f and g have a common factor. □

proof of Proposition 1.10.3. **(i)** Let C and D be distinct curves, defined by irreducible homogeneous polynomials f and g. Proposition 1.3.12 shows that there are finitely many intersections. We project to $\mathbb{P}^1$ from a point q that doesn't lie on any of the finitely many lines through pairs of intersection points. Then a line through q passes through at most one intersection, and the zeros of the resultant $\mathrm{Res}_z(f, g)$ that correspond to the intersection points will be distinct. The resultant has degree mn (1.7.9). It has at least one zero, and at most mn of them. Therefore C and D have at least one and at most mn intersections.

(ii) Every zero of the resultant will be the image of an intersection of C and D. To show that there are mn intersections if all intersections are transversal, it suffices to show that the resultant has simple zeros. This is Corollary 1.9.14. □

smoothirred **1.10.5. Corollary.** *If the curve X defined by a homogeneous polynomial $f(x, y, z)$ is smooth, then f is irreducible, and therefore X is a smooth curve.*

proof. Suppose that $f = gh$, and let p be a point of intersection of the loci $\{g = 0\}$ and $\{h = 0\}$. The previous proposition shows that such a point exists. All partial derivatives of f vanish at p, so p is a singular point of the locus $f = 0$ (1.4.7). □

numberofflexes **1.10.6. Proposition.** **(i)** *Let d be an integer ≥ 3. A smooth plane curve of degree d has at least one flex point, and the number of flex points is at most $3d(d-2)$.*

(ii) *If all flex points are ordinary, the number of flex points is equal to $3d(d-2)$.*

Thus smooth curves of degrees $2, 3, 4, 5, ...$ have at most $0, 9, 24, 45, ...$ flex points, respectively.

proof. **(i)** The flex points are intersections of a smooth curve C with its *Hessian divisor* $D : \{\det H = 0\}$. (If $detH = h_1^{e_1} \cdots h_k^{e_k}$ is the factorization into irreducible polynomials h_i and Z_i is the locus of zeros of h_i, the Hessian divisor is $D = e_1 z_1 + \cdots + e_k Z_k$ (1.3.13).)

We use the definition of divisor that is given in (1.3.13). Let $C : \{f(x_0, x_1, x_2) = 0\}$ be a smooth curve of degree d. The entries of the 3×3 Hessian matrix H are the second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$. They are homogeneous polynomials of degree $d-2$, so the Hessian determinant is homogeneous, of degree $3(d-2)$. Propositions 1.4.17 and 1.10.3 tell us that there are at most $3d(d-2)$ intersections.

(ii) Recall that a flex point is ordinary if the multiplicity of intersection of the curve and its tangent line is 3. Bézout's Theorem asserts that the number of flex points is equal to $3d(d-2)$ if the intersections of C with its Hessian divisor D are transversal, and therefore have multiplicity 1. So the next lemma completes the proof.

1.10.7. Lemma. *A curve $C : \{f = 0\}$ intersects its Hessian divisor D transversally at a point p if and only* *transver-*
p is an ordinary flex point of C. *salH*

proof. We prove this by computation. I don't know a conceptual proof.

Let L be the tangent line to C at the flex point p, and let h denote the restriction of the Hessian determinant to L. The Hessian divisor D will be transversal to C at p if and only if it is transversal to L, and this will be true if and only if the order of vanishing of h at p is 1.

We adjust coordinates x, y, z so that $p = (0, 0, 1)$ and L is the line $\{y = 0\}$, and we write the polynomial f of degree d as

(1.10.8) $$f(x, y, z) = \sum_{i+j+k=d} a_{ij} x^i y^j z^k,$$ fwithco-effs

We set $y = 0$ and $z = 1$, to restrict f to L. The restriction of the polynomial f is

$$f(x, 0, 1) = \sum_{i \le d} a_{i0} x^i$$

When p is a flex point with tangent line L, the coefficients a_{00}, a_{10}, and a_{20} will be zero, and p is an ordinary flex point if and only if the coefficient a_{30} isn't zero.

With this notation, the restiction of $\det H$ to L becomes $h = \det H(x, 0, 1)$. We must show that p is an ordinary flex point if and only if h has a simple zero at $x = 0$.

To evaluate the restriction $f_{xx}(x, 0, 1)$ of the partial derivative to L, the relevant terms in the sum (1.10.8) have $j = 0$. Since $a_{00} = a_{10} = a_{20} = 0$,

$$f_{xx}(x, 0, 1) \;=\; 6a_{30}x + 12a_{40}x^2 + \cdots \;=\; 6a_{30}x + O(2)$$

Similarly,

$$\begin{aligned} f_{xz}(x, 0, 1) &= 0 + O(2) \\ f_{zz}(x, 0, 1) &= 0 + O(2) \end{aligned}$$

For the restriction of f_{yz}, the relevant terms are those with $j = 1$:

$$f_{yz}(x, 0, 1) \;=\; (d-1)a_{01} + (d-2)a_{11}x + O(2)$$

We won't need f_{xy} or f_{yy}.

Let $v = 6a_{30}x$ and $w = (d-1)a_{01} + (d-2)a_{11}x$. The restricted Hessian matrix has the form

(1.10.9) $$H(x, 0, 1) \;=\; \begin{pmatrix} v & * & 0 \\ * & * & w \\ 0 & w & 0 \end{pmatrix} \;+\; O(2)$$

where $*$ are entries that don't affect terms of degree at most one in the determinant. The determinant is

$$h \;=\; -vw^2 + O(2) \;=\; -6(d-1)^2 a_{30} a_{01}^2 x + O(2)$$

It has a zero of order 1 at $x = 0$ if and only if a_{30} and a_{01} aren't zero. Since C is smooth at p and $a_{10} = 0$, the coefficient a_{01} isn't zero. Thus the curve C and its Hessian divisor D intersect transversally, and C and L intersect with multiplicity 3, if and only if a_{30} is nonzero, which is true if and only if p is an ordinary flex point. □

nineflexes **1.10.10. Corollary.** *A smooth cubic curve contains exactly 9 flex points.*

proof. Let f be the irreducible cubic polynomial whose zero locus is a smooth cubic C. The degree of the Hessian divisor D is also 3, so Bézout predicts at most 9 intersections of D with C. To derive the corollary, we show that C intersects D transversally. According to Proposition 1.10.7, a nontransversal intersection would correspond to a point at which the curve and its tangent line intersect with multiplicity greater than 3. This is impossible when the curve is a cubic. □

singdual **(1.10.11) singularities of the dual curve**

Let C be a plane curve. As before, an *ordinary flex point* is a smooth point p such that the intersection multiplicity of the curve and its tangent line L at p is precisely 3. A bitangent, a line L that is tangent to C at distinct points p and p', is an *ordinary bitangent* if neither p nor p' is a flex point. A tangent line L at a smooth point p of C is an *ordinary tangent* if p isn't a flex point and L isn't a bitangent.

The tangent line L at a point p will have other intersections with C. Most often, these other intersections will be transversal. However, it may happen that one of those other intersections is a singular point of C. If L is a bitangent, it may happen that it is a *tritangent*, tangent to C at a third point, or that L contains a singular point of C Let's call such occurences *accidents*.

ordcurve **1.10.12. Definition.** A plane curve C is *ordinary* if it is smooth, all of its bitangents and flex points are ordinary, and if there are no accidents.

genisord **1.10.13. Lemma.** *A generic curve C is ordinary.*

We'll verify this using counting constants. The reasoning is quite convincing, though imprecise. There are three ways in which a curve C might fail to be ordinary:

(a) C may be singular.

(b) C may have a flex point that isn't an ordinary flex.

(c) A bitangent to C may be a flex tangent or a tritangent.

The curve will be ordinary if none of these occurs.

Let the coordinates be x, y, z, and let $f(x, y, z)$ be the defining polynomial of a curve. The homogeneous polynomials of given degree d form a vector space whose dimension is equal to the number of monomials $x^i y^j z^k$ of degree d. That number isn't important here, but it happens to be $\binom{d+1}{2}$. The curves of degree d are parametrized by points of a projective space D of dimension $N = \binom{d+1}{2} - 1$.

(a) We look at the point $p_0 = (0, 0, 1)$, and we set $z = 1$. If p_0 is singular, the coefficients of $1, x, y$ in the polynomial $f(x, y, 1)$ will be zero. This is three conditions. The curves that are singular at p_0 are parametrized by a space of dimension $N - 3$. The points of $\mathbb{P}^2$ depend on only 2 parameters. Therefore, in the space of curves, the singular curves form a subset of dimension at most $N - 1$. (In fact, that dimension is equal to $N - 1$.) Most curves are smooth.

(b) Let's look at curves that have a four-fold tangency with the line $L : \{y = 0\}$ at p_0. Setting $z = 1$ as before, we see that the coefficients of $1, y, y^2, y^3$ in f must be zero. This is four conditions. The lines through p_0 depend on one parameter, and the points of $\mathbb{P}^2$ depend on two parameters, giving us three parameters to vary. We can't get all curves this way. Most curves have no four-fold tangencies.

(c) To be tangent to the line $L : \{y = 0\}$ at the point p_0, the coefficients of 1 and y in f must be zero. This is two conditions. Then to be tangent to L at three given points p_0, p_1, p_2 imposes 6 conditions. A set of three points of L depends on three parameters, and a line depends on two parameters, giving us 5 parameters in all. Most curves don't have a tritangent. Similar reasoning rules out bitangents that are flex tangents on a generic curve. □

1.10.14. Proposition. *Let p be a point of an ordinary curve C, and let L be the tangent line at p.* *dualcusp*
(i) *If L is an ordinary tangent at p, then L^* is a smooth point of C^*.*
(ii) *If L is a bitangent, then L^* is a node of C^*.*
(iii) *If p is a flex point, then L^* is a cusp of C^*.*

proof. We refer to the map $U \xrightarrow{t} C^*$ from the set of smooth points of C to the dual curve (**1.6.3**).

We dehomogenize by setting $z = 1$, and choose affine coordinates so that p is the origin, and the tangent line L at p is the line $\{y = 0\}$. Let $\widetilde{f}(x, y) = f(x, y, 1)$. We solve $\widetilde{f} = 0$ for $y = y(x)$ as an analytic function of x, as before. The tangent line L_1 to C at a nearby point $p_1 = (x, y)$ has the equation (1.6.11), and L_1^* is the point $(u, v, w) = (-y', 1, y'x - y)$ of $\mathbb{P}^*$ (1.6.12). Since there are no accidents, this path traces out all points of C^* near to L^* (Corollary 1.6.17 **(iii)**).

(i) If L is an ordinary tangent line, $y(x)$ will have a zero of order 2 at $x = 0$. Then $u = -y'$ will have a simple zero. So the path $(-y', 1, y'x - y)$ is smooth at $x = 0$, and therefore C^*, is smooth at the origin.

(ii) If L is an ordinary bitangent, tangent to C at two points p and p', the reasoning given for an ordinary tangent shows that the images in C^* of small neighborhoods of p and p' in C will be smooth at L^*. Their tangent lines p^* and ${p'}^*$ will be distinct, so p is a node.

(iii) Suppose that p is an ordinary flex point. Then, in the analytic function $y(x) = c_0 + c_1x + c_2x^2 + \cdots$ that solves $f(x, y) = 0$ the coefficients of x^i are zero when $i < 3$, so $y(x) = c_3x^3 + \cdots$. Since the flex is ordinary, we may assume that $c_3 = 1$. Then, in the local equation $(u, v, w) = (-y', 1, y'x - y)$ for the dual curve, $u = -3x^2 + \cdots$ and $w = 2x^3 + \cdots$. Proposition 1.8.10 tells us that the singularity at the origin is a cusp. $\square$

1.11 The Plücker Formulas

Recall (1.10.12) that a plane curve C is *ordinary* if it is smooth, all of its bitangents and flex points are ordinary (see (**1.10.11**), and there are no accidents. The *Plücker formulas* compute the number of flexes and bitangents of an ordinary plane curve. The formulas are especially interesting because it isn't easy to count the number of bitangents directly. plucker

1.11.1. Theorem: Plücker Formulas. *Let C be an ordinary curve of degree d at least two, and let C^* be its dual curve. Let f and b denote the numbers of flex points and bitangents of C, and let d^*, δ^* and κ^* denote the degree, the numbers of nodes, and the number of cusps of C^*, respectively. Then:* *plform*

(i) *The dual curve C^* has no flexes or bitangents. Its singularities are nodes or cusps.*

(ii) $d^* = d^2 - 2,\quad f = \kappa^* = 3d(d-2),\quad$ *and* $\quad b = \delta^* = \frac{1}{2}d(d-2)(d^2-9).$

proof. **(i)** A bitangent or a flex on C^* would produce a singularity on the bidual C^{**}, which is the smooth curve C.

(ii) The degree d^* was computed in Corollary 1.8.25. Bézout's Theorem counts the flex points (see (1.10.6)). The facts that $\kappa^* = f$ and $\delta^* = b$ are in Proposition 1.10.14. Thus $\kappa^* = f = 3d(d-2)$.

When we project C^* to $\mathbb{P}^1$ from a generic point s of $\mathbb{P}^*$. The number of branch points that correspond to tangent lines through s at smooth points of C^* is the degree d of $C^{**} = C$ (see (1.8.25)).

Next, let $F(u, v, w)$ be the defining polynomial for C^*. The discriminant $\mathrm{Discr}_w(F)$ has degree $d^{*2} - d^*$. Proposition 1.9.11 describes the order of vanishing of the discriminant at the images of the d tangent lines through s, the δ nodes of C^*, and the κ cusps of C^*. It tells us that

$$d^{*2} - d^* = d + 2\delta^* + 3\kappa^*$$

Substituting the known values $d^* = d^2 - d$, and $\kappa^* = 3d(d-2)$ into this formula gives us

$$(d^2-d)^2 - (d^2-d) = d + 2\delta^* + 9d(d-2) \quad \text{or} \quad 2\delta^* = d^4 - 2d^3 - 9d^2 + 18d \qquad \square$$

some-plucker-formulas **1.11.2. Examples.**

(i) All curves of degree 2 and all smooth curves of degree 3 are ordinary.

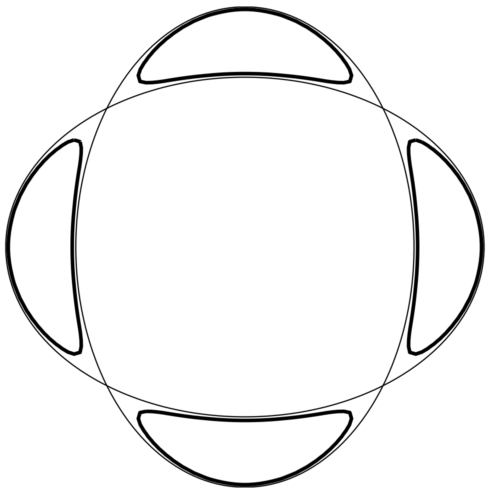
(ii) A curve of degree 2 has no flexes and no bitangents. Its dual curve has degree 2.

(iii) A smooth curve of degree 3 has 9 flexes and no bitangents. Its dual curve has degree 6.

(iv) An ordinary curve C of degree 4 has 24 flexes and 28 bitangents. Its dual curve has degree 12. $\square$

We will make use of the fact that a quartic curve has 28 bitangents in Chapter 4 (see (4.7.18)). The Plücker Formulas are rarely used for curves of degree greater than four.

goober13 **1.11.3.**



A Quartic Curve whose 28 **Bitangents are Real**

To obtain this quartic, we added a small constant ϵ to the product of the quadratic equations of the two ellipses that are shown. The equation of the quartic is $(2x^2 + y^2 - 1)(x^2 + 2y^2 - 1) + \epsilon = 0$.

1.12 Exercises

1.12.1. Prove that a plane curve contains infinitely many points. chaponeex xinfpoints

1.12.2. Prove that the path $x(t) = t,\ y(t) = \sin t$ doesn't lie on any plane algebraic curve in $\mathbb{A}^2$. xsin

1.12.3. Using counting constants, prove that most (nonhomogeneous) polynomials in two or more variables are irreducible. xtwovarirred

1.12.4. Let f a homogeneous polynomial in x, y, z, not a power of z. Prove that f is irreducible if and only if $f(x, y, 1)$ is irreducible. xstaysirreducible

1.12.5. Describe the points that lie in the interior of the coordinate triangle in the real projective space. xcoordtriangle

1.12.6. Prove that all affine conics can be put into one of the forms 1.1.6 by linear change of variable, translation, and scalar multiplication. xaffconictwo

1.12.7. Figure 1.2.11 doesn't give enough information to determine the equation of the conic that is depicted there. What can be deduced about the equation bu looking at this figure? xconicinfig

1.12.8. **(i)** Classify conics in $\mathbb{P}^2$ by writing an irreducible quadratic polynomial in three variables in the form X^tAX where A is symmetric, and diagonalizing the quadratic form. xdiagform

(ii) Quadrics in $\mathbb{P}^3$ are zero sets of irreducible homogeneous quadratic polynomials in four variables. Classify quadrics in $\mathbb{P}^3$.

1.12.9. Let f and g be irreducible homogeneous polynomials in x, y, z. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ are equal, then $g = cf$. xlociequal

1.12.10. Prove that a plane cubic curve can have at most one singular point. Do this without using Bézout's Theorem. xcubicsing

1.12.11. Let C be the plane projective curve defined by the equation $x_0x_1 + x_1x_2 + x_2x_0 = 0$, and let p be the point $(-1, 2, 2)$. What is the equation of the tangent line to C at p? xtanconic

1.12.12. Let C be a smooth cubic curve in $\mathbb{P}^2$, and let p be a flex point of C. Choose coordinates so that p is the point $(0, 1, 0)$ and the tangent line to C at p is the line $\{z = 0\}$. xeqforcubic

(i) Show that the coefficients of x^2y, xy^2, and y^3 in the defining polynomial f of C are zero.

(ii) Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form $f = y^2z + x^3 + axz^2 + bz^3$, and that $x^3 + ax + b$ will be a polynomial with distinct roots.

(iii) Show that one of the coefficients a or b can be eliminated, and therefore that smooth cubic curves in $\mathbb{P}^2$ depend on just one parameter.

1.12.13. Let p be a smooth point of a projective curve X, and suppose that coordinates are chosen so that $p = (0, 0, 1)$ and the tangent line ℓ is the line $\{x_1 = 0\}$. Prove that p is a flex point if and only if the Hessian determinant is zero by computing the Hessian. xflexhess

1.12.14. Using Euler's formula together with row and column operations, show that the Hessian determinant is equal to $a \det H'$, where xhessian

$$H' = \begin{pmatrix} cf & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}, \quad a = \left(\tfrac{d-1}{x_0}\right)^2, \quad \text{and} \quad c = \tfrac{d}{d-1}$$

1.12.15. Prove that a smooth point of a curve is a flex point if and only if the Hessian determinant is zero, in the following way: Given a smooth point p of X, choose coordinates so that $p = (0, 0, 1)$ and the tangent line ℓ is the line $\{x_1 = 0\}$. Then compute the Hessian. To complete the proof, verify that the vanishing of its determinant isn't affected by a change of coordinates. xhessianZero

1.12.16. Prove that the elementary symmetric functions $s_1 = x_1 + \cdots + x_n,\ \ldots,\ s_n = x_1 \cdots x_n$ are algebaically independent. xsymmfnindep

1.12.17. Let K be a field extension of F, and let α be an element of K that is transcendental over F. Then every element of the field $F(\alpha)$ that isn't in F is transcendental over F. xelementtransc xtdadds **1.12.18.** Let $tr(K/F)$ denote the transcendence degree of a field extension K/F. Prove that, if $L \supset K \supset F$ are fields, then $tr(L/F) = \mathrm{tr}(L/K) + tr(L/F)$.

xdualiscurve **1.12.19.** Let $f(x_0, x_1, x_2)$ be a homogeneous polynomial of degree d, and let $f_i = \frac{\partial f}{\partial x_i}$, and let C be the plane curve $\{f = 0\}$. Use the following method to prove that the image in the dual plane of the set of smooth points of C is contained in a curve C^*: Let $N_r(k)$ be the dimension of the space of polynomials of degree $\leq k$ in r variables. Determine $N_r(k)$ for $r = 3$ and 4. Show that $N_4(k) > N_3(kd)$ if k is sufficiently large. Use this fact to prove that there is a nonzero polynomial $G(x_0, x_1, x_2)$ such that $G(f_0, f_1, f_2) = 0$.

xtangentq **1.12.20.** Let C be a smooth cubic curve in the plane $\mathbb{P}^2$, and let q be a generic point of $\mathbb{P}^2$. How many lines through q are tangent lines to C?

xthreenodes **1.12.21.** Let C be a plane curve of degree 4 with three nodes. Use projection to $\mathbb{P}^1$ from a generic point of the plane to determine the degree of the dualcurve C^*.

Cstarcount **1.12.22.** Let C be the curve defined by a homogeneous polynomial f of degree d. To prove that the images in the dual plane of the smooth points of C lie on a curve C^*, we used transcendence degree to conclude that there is a polynomial $G(t, s_0, s_1, s_2)$ such that $G(f, f_0, f_1, f_2)$ is identically zero. Use the following method to give an alternate proof: Determine the dimensions $N_r(k)$ of the spaces of polynomials of degree $\leq k$ in r variables, for $r = 3$ and $r = 4$. Show that $N_4(k) > N_3(kd)$ if k is large enough. Use counting constants to show that there has to be a polynomial G that maps to zero by the substitution.

We note that this method doesn't give a good bound for the degree of C^*. One reason may be that f and its derivatives are related by Euler's Formula. It is tempting try using Euler's Formula to help compute the equation of C^*, but I haven't succeeded in getting anywhere that way.

xprojcurve **1.12.23.** Let X and Y be the surfaces in $\mathbb{A}^3_{x,y,z}$ defined by the equations $z^3 = x^2$ and $yz^2 + z + y = 0$, respectively. The intersection $C = X \cap Y$ is a curve. Determine the equation of the projection of C to the x, y-plane.

xercresformula **1.12.24.** Compute the resultant of the polynomials x^m and $x^n - 1$.

xresfgh **1.12.25.** Let f, g, and h be polynomials. Prove that

(i) $\mathrm{Res}(f, gh) = \mathrm{Res}(f, g)\,\mathrm{Res}(f, h)$.

(ii) If the degree of gh is less than or equal to the degree of f, then $\mathrm{Res}(f, g) = \mathrm{Res}(f + gh, g)$.

xres **1.12.26.** With notation as in 1.7.3, suppose that a_0 and b_0 are not zero, and let α_i and β_j be the roots of $f(x, 1)$ and $g(x, 1)$, respectively. Then $\mathrm{Res}(f, g) = a_0^n b_0^m \prod(\alpha_i - \beta_j)$.

generalline **1.12.27.** Prove that a general line meets a plane projective curve of degree d in d distinct points.

xrestwopol **1.12.28.** Let $f = x^2 + xz + yz$ and $g = x^2 + y^2$. Compute the resultant $\mathrm{Res}_x(f, g)$ with respect to the variable x.

1.12.29. Compute $\prod_{i \neq j}(\zeta^i - \zeta^j)$ when $\zeta = e^{2\pi i/n}$.

xsignindisc xsign **1.12.30.** If $F(x) = \prod(x - \alpha_i)$, then $\mathrm{Discr}(F) = \pm \prod_{i<j}(\alpha_i - \alpha_j)^2$. Determine the sign.

etdegres **1.12.31.** Let $f = a_0x^m + a_1x^{m-1} + \cdots a_m$ and $g = b_0x^n + b_1x^{n-1} + \cdots b_n$, and let $R = \mathrm{Res}(f, g)$ be the resultant of these polynomials. Prove that

(i) R is a polynomial that is homogeneous in each of the sets of variables a and b, and determine its degree.

(ii) If one assigns weighted degree i to the coefficients a_i and b_i, then R is homogeneous, of weighted degree mn.

xamap **1.12.32.** Let coordinates in $\mathbb{A}^4$ be x, y, z, w, let Y be the variety defined by $z^2 = x^2 - y^2$ and $w(z - x) = 1$, and let π denote projection from Y to (x, y)-space. Describe the fibres and the image of π.

xcusptan **1.12.33.** Let p be a cusp of the curve C defined by a homogeneous polynomial f. Prove that there is just one line L through p such that the restriction of f to L has as zero of order > 2 at p, and that the order of zero for that line is precisely 3.

geometryofnode **1.12.34.** Describe the intersection of the node $xy = 0$ at the origin with the unit sphere in $\mathbb{A}^2$.

1.12.35. Prove that the Fermat curve $C : \{x^d + y^d + z^d = 0\}$ is connected by studying its projection to $\mathbb{P}^1$ from the point $(0, 0, 1)$. fermat-conn

1.12.36. Let $p(t, x) = x^3 + x^2 + t$. Then $p(0, x) = x^2(x + 1)$. Since x^2 and $x + 1$ are relatively prime, Hensel's Lemma predicts that p factors: $p = fg$, where g and g are polynomials in x whose coefficients are analytic functions in t, and f is monic, $f(0, x) = x^2$, and $g(0, x) = x + 1$. Determine this factorization up to degree 3 in t. Do the same for the polynomial $tx^4 + x^3 + x^2 + t$. xhensel

1.12.37. Let $f(t, y) = ty^2 - 4y + t$.

(i) Solve $f = 0$ for y by the quadratic formula, and sketch the real locus $f = 0$ in the t, y plane. xxhensellemma

(ii) What does Hensel's Lemma say tell us?

(iii) Factor f, modulo t^4.

1.12.38. Factor $f(t, x) = x^3 + 2tx^2 + t^2x + x + t$, modulo t^2. xxhensellem-matwo

1.12.39. Prove that a plane curve X of degree 4 can have at most three singular points. Begin by showing that there is a conic C that passes through any five points of X. xthreepts

1.12.40. By parametrizing a conic C, show that C meets a plane curve X of degree d and distinct from C in $2d$ points, when counted with multiplicity. xintconic

1.12.41. Determine the degree of the dual of a plane cubic curve C with a cusp using a generic projection to $\mathbb{P}^1$. cus-pcurved-ual

1.12.42. Let C be a cubic curve with a node. Determine the degree of the dual curve C^*, and the numbes of flexes, bitangents, nodes, and cusps of C and of C^*. xdualnode

1.12.43. Prove that any cusp (1.8.6) is analyicially equivalent with the standard cusp. xcuspstan-dard

Chapter 2 AFFINE ALGEBRAIC GEOMETRY

affine

The next chapters are about varieties of arbitrary dimension. We will use some of the basic terminology, such as discriminant and transcendence degree, that was introduced in Chapter 1, but many of the results in Chapter 1 won't be used until we come back to curves in Chapter 8.

To begin, we review some basic facts about rings and modules, omitting proofs. Give this section a quick read, but don't spend too much time on it. You can refer to it as needed, and look up information on the concepts that aren't familiar.

2.1 Rings and Modules

ringreview

By the word 'ring', we mean 'commutative ring': $ab = ba$, unless the contrary is stated explicitly. A *domain* is a ring that has no zero divisors and isn't the zero ring. An *algebra* is a ring that contains the field $\mathbb{C}$ of complex numbers as a subring.

A set of elements $\alpha = \{\alpha_1, ..., \alpha_n\}$ *generates* an algebra A if every element of A can be expressed (usually not uniquely) as a polynomial in $\alpha_1, ..., \alpha_n$, with complex coefficients. Another way to state this is that α generates A if the homomorphism $\mathbb{C}[x_1, ..., x_n] \xrightarrow{\tau} A$ that evaluates a polynomial at $x = \alpha$ is surjective. If α generates A, then A will be isomorphic to the quotient $\mathbb{C}[x]/I$ of the polynomial algebra $\mathbb{C}[x]$, where I is the kernel of τ. a *finite-type algebra* is an algebra that can be generated by a finite set of elements.

If I and J are ideals of a ring R, the *product ideal*, which is denoted by IJ, is the ideal whose elements are finite sums of products $\sum a_i b_i$, with $a_i \in I$ and $b_i \in J$. The *power* I^k of I is the product of k copies of I — the ideal generated by products of k elements of I.

The intersection $I \cap J$ of two ideals is an ideal, and

intersect-product

(2.1.1) $$(I \cap J)^2 \subset IJ \subset I \cap J$$

An ideal M of a ring R is a *maximal ideal* if there is no ideal I with $M < I < R$, and if M isn't the unit ideal R. An ideal M is a maximal ideal if and only if the quotient ring R/M is a field.

An ideal P of a ring R is a *prime ideal* if the quotient R/P is a domain. A maximal ideal is a prime ideal.

invim-prime

2.1.2. Lemma. *Let $A \xrightarrow{\varphi} B$ be a ring homomorphism. The inverse image of a prime ideal of B is a prime ideal of A.*

proof. Let P be the inverse image of a prime ideal Q of B. Then P is the kernel of the composed homomorphism $A \to B \to B/Q$. The quotient A/P maps injectively to a subring of the domain B/Q. Therefore A/P is a domain. $\square$

2.1.3. Lemma. *Let P be an ideal of a ring R, not the unit ideal. The following conditions are equivalent.* *defprime*
(i) *P is a prime ideal.*
(ii) *If a and b are elements of R, and if the product ab is in P, then $a \in P$ or $b \in P$.*
(iii) *If A and B are ideals of R, and if the product ideal AB is contained in P, then $A \subset P$ or $B \subset P$.* $\square$

The following equivalent version of **(iii)** is sometimes convenient:

(iii') *If A and B are ideals that contain P, and if the product ideal AB is contained in P, then $A = P$ or $B = P$.*

2.1.4. Mapping Property of Quotients. *mapprop*
(i) *Let K be an ideal of a ring R, let $R \xrightarrow{\tau} \overline{R}$ denote the canonical map from R to the quotient ring $\overline{R} = R/K$, and let S be another ring. Ring homomorphisms $\overline{R} \xrightarrow{\overline{\varphi}} S$ correspond bijectively to homomorphisms $R \xrightarrow{\varphi} S$ whose kernels contain K, the correspondence being $\varphi = \overline{\varphi} \circ \tau$:*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\tau}\big\downarrow & & \big\| \\ \overline{R} & \xrightarrow{\overline{\varphi}} & S \end{array}$$

If $\ker\varphi = I$, *then* $\ker\overline{\varphi} = I/K$.
(ii) *Let M be a module over a ring R, let K be a submodule of M, and let $M \xrightarrow{\tau} \overline{M}$ denote the canonical map from M to the quotient module $\overline{M} = M/K$. Homomorphisms of modules $\overline{M} \xrightarrow{\overline{\varphi}} N$ to another R-module N correspond bijectively to homomorphisms $M \xrightarrow{\varphi} N$ whose kernels contain K, the correspondence being $\varphi = \overline{\varphi} \circ \tau$. If* $\ker\varphi = L$, *then* $\ker\overline{\varphi} = L/K$. $\square$

The word *canonical* is used to mean a construction that is natural. Exactly what that means is often left unspecified.

(**2.1.5**) **commutative diagrams** commdiag

In the diagram displayed above, the maps $\overline{\varphi}\tau$ and φ from R to S are equal. This is referred to by saying that the diagram is *commutative*. A commutative diagram is one in which every map that can be obtained by composing its arrows depends only on the domain and range of that map. In these notes, almost all diagrams of maps are commutative. We won't mention commutativity most of the time. $\square$

2.1.6. Correspondence Theorem. *corrthm*
(i) *Let $R \xrightarrow{\varphi} S$ be a **surjective** ring homomorphism with kernel K. (For instance, φ might be the canonical map from R to the quotient ring R/K. In any case, S will be isomorphic to R/K.) There is a bijective correspondence*

$$\{\textit{ideals of } R \textit{ that contain } K\} \quad \longleftrightarrow \quad \{\textit{ideals of } S\}$$

This correspondence associates an ideal I of R that contains K with its image $\varphi(I)$ in S and it associates an ideal J of S with its inverse image $\varphi^{-1}(J)$ in R.

If an ideal I of R that contains K corresponds to the ideal J of S, then φ induces an isomorphism of quotient rings $R/I \to S/J$. If one of the ideals, I or J, is prime or maximal, they both are.
(ii) *Let R be a ring, and let $M \xrightarrow{\varphi} N$ be a surjective homomorphism of R-modules with kernel L. There is a bijective correspondence*

$$\{\textit{submodules of } M \textit{ that contain } L\} \quad \longleftrightarrow \quad \{\textit{submodules of } N\}$$

This correspondence associates a submodule S of M that contains L with its image $\varphi(S)$ in N and it associates a submodule T of N with its inverse image $\varphi^{-1}(T)$ in M. $\square$

Ideals $I_1, ..., I_k$ of a ring R are said to be *comaximal* if the sum of any two of them is the unit ideal.

comax **2.1.7. Chinese Remainder Theorem.** *Let $I_1, ..., I_k$ be comaximal ideals of a ring R.*
(i) *The product ideal $I_1 \cdots I_k$ is equal to the intersection $I_1 \cap \cdots \cap I_k$.*
(ii) *The map $R \longrightarrow R/I_1 \times \cdots \times R/I_k$ that sends an element a of R to the vector of its residues in R/I_ν is a surjective homomorphism, and its kernel is $I_1 \cap \cdots \cap I_k$, which is equal to $I_1 \cdots I_k$.*
(iii) *Let M be an R-module. The canonical homomorphism $M \to M/I_1M \times \cdots \times M/I_kM$ is surjective.* □

quotof-prod **2.1.8. Proposition.** *Let R be a product of rings, $R = R_1 \times \cdots \times R_k$, let I be an ideal of R, and let $\overline{R} = R/I$ be the quotient ring. There are ideals I_j of R_j such that $I = I_1 \times \cdots \times I_k$ and $\overline{R} = R_1/I_1 \times \cdots \times R_k/I_k$.* □

noethr (2.1.9) Noetherian rings

Let M be a module over a ring R. We will most often view M as a *left module*. writing the scalar product of an element m of M by an element a of R as am. However, it is sometimes convenient to view M as a right module, writing ma instead of am. This is permissible because our rings are commutative.

Let M and N be modules over a ring R. A homomorphism of R-modules $M \to N$, may also be called an *R-linear map*. When we refer to a map as being linear without mentioning a ring, we mean a $\mathbb{C}$-linear map.

A *finite module* M over a ring R is a module that is spanned, or generated, by a finite set $\{m_1, ..., m_k\}$ of elements. To say that the set *generates* M means that every element of M can be obtained as a combination $r_1m_1 + \cdots + r_km_k$ with coefficients r_i in R, or that the homomorphism from the free R-module R^k to M that sends a vector $(r_1, ..., r_k)$ to the combination $r_1m_1 + \cdots + r_km_k$ is surjective.

An ideal of a ring R is *finitely generated* if, when regarded as an R-module, it is a finite module. A ring R is *noetherian* if all of its ideals are finitely generated. The ring $\mathbb{Z}$ of integers is noetherian. Fields are notherian. If I is an ideal of a noetherian ring R, the quotient ring R/I is noetherian.

basisthm **2.1.10. Hilbert Basis Theorem.** *Let R be a noetherian ring. The ring $R[x_1, ..., x_n]$ of polynomials with coefficients in R is noetherian.* □

Thus $\mathbb{Z}[x_1, ..., x_n]$ and $F[x_1, ..., x_n]$, F a field, are noetherian rings.

qnoeth **2.1.11. Corollary.** *Every finite-type algebra is noetherian.* □

Note. It is important not to confuse the concept of a finite-type algebra with that of a finite module. An R-module M is a finite module if every element of M can be written as a *(linear) combination* $r_1m_1+\cdots+r_km_k$ of some finite set $\{m_1, ..., m_k\}$ of elements of M, with coefficients in R. An algebra A is finite-type algebra if which every element of A can be written as a *polynomial* $f(\alpha_1, ..., \alpha_k)$ in some finite set $\{\alpha_1, ..., \alpha_k\}$ of elements of A, with complex coefficients.

ascchcond (2.1.12) the ascending chain condition

The condition that a ring R be noetherian can be rewritten in several ways that we review here.

Our convention is that, if X' and X are sets, the notation $X' \subset X$ means that X' is a subset of X, while $X' < X$ means that X' is a subset that is distinct from X. A *proper subset* X' of a set X is a nonempty subset distict from X — a set such that $\emptyset < X' < X$.

A sequence $X_1, X_2, ...$, finite or infinite, of subsets of a set Z forms an *increasing chain* if $X_n \subset X_{n+1}$ for all n, equality $X_n = X_{n+1}$ being permitted. If $X_n < X_{n+1}$ for all n, the chain is *strictly increasing*.

Let $\mathcal{S}$ be a set whose elements are subsets of a set Z. A member M of $\mathcal{S}$ is a *maximal member* if there is no member M' of $\mathcal{S}$ such that $M < M'$. For example, the set of proper subsets of a set of five elements contains five maximal members, the subsets of order four. The set of finite subsets of the set of integers contains no maximal member.

A maximal ideal of a ring R is a maximal member of the set of ideals of R different from the unit ideal.

2.1.13. Proposition. *The following conditions on a ring R are equivalent:* *noether-conds*
(i) *Every ideal of R is finitely generated.*
(ii) *The* ***ascending chain condition****: Every strictly increasing chain $I_1 < I_2 < \cdots$ of ideals of R is finite.*
(iii) *Every nonempty set of ideals of R contains a maximal member.* $\square$

The next corollary follows from the ascending chain condition, but the conclusions are true whether or not R is noetherian.

2.1.14. Corollary. *Let R be a noetherian ring.* *idealin-maximal*
(i) *If R isn't the zero ring, every ideal of R except the unit ideal is contained in a maximal ideal.*
(ii) *A nonzero ring R contains at least one maximal ideal.*
(iii) *An element of R that isn't in any maximal ideal is a unit — an invertible element of R.* $\square$

2.1.15. Corollary. *Let $s_1, ..., s_k$ be elements that generate the unit ideal of a ring R. For any positive integer n, the powers $s_1^n, ..., s_k^n$ generate the unit ideal.* *powers-generate*

proof. When $s_1, ..., s_k$ generate the unit ideal, there will be an equation of the form $1 = \sum r_i s_i$, and for any N, $1 = 1^N = (\sum r_i s_i)^N$. If $N \geq nk$, then when the right side is expanded, every term will be divisible by s_i^n for some n.

Or, one could say that if a maximal ideal M contains s_i^n, it contains s_i. But when $s_1, ..., s_k$ generate the unit ideal, there is no maximal ideal that contains all of them. $\square$

2.1.16. Proposition. *Let R be a noetherian ring, and let M be a finite R-module.* *noetheri-anmodule*
(i) *Every submodule of M is a finite module.*
(ii) *The set of submodules of M satisfies the ascending chain condition.*
(iii) *Every nonempty set of submodules of M contains a maximal member.* $\square$

(**2.1.17**) exact sequences exactseq

Let R be a ring. A sequence

$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

of homomorphisms of R-modules is an *exact sequence* if, for all k, the image of d^{k-1} is equal to the kernel of d^k. For instance, a sequence $0 \to V \xrightarrow{d} V'$ is exact if d is injective, and a sequence $V \xrightarrow{d} V' \to 0$ is exact, if d is surjective.

A *short exact sequence* is an exact sequence of the form

$$0 \to V \xrightarrow{a} V' \xrightarrow{b} V'' \to 0.$$

To say that this sequence is exact means that the map a is injective, and that V'' is isomorphic to the quotient module V'/aV.

Let $V' \xrightarrow{d} V$ be a homomorphism of R-modules, and let W be the image of d. The *cokernel* of d is the module $C = V/W$. The homomorphism d embeds into an exact sequence

(2.1.18) $$0 \to K \to V' \xrightarrow{d} V \to C \to 0,$$ kercok-erseq

where K and C are the kernel and cokernel of d, respectively.

The mapping property (2.1.4)**(ii)** tells us that a module homomorphism $V \xrightarrow{f} M$ induces a homomorphism $C \to M$ if and only if the composed homomorphism fd is zero.

A finite-dimensional $\mathbb{C}$-module V (a vector space) has a *dual module* V^*, the module of linear functions $V \to \mathbb{C}$. When $V' \xrightarrow{f} V$ is a homomorphism of $\mathbb{C}$-modules, there is a canonical dual homomorphism $V'^* \xleftarrow{f^*} V^*$. The dual of the sequence (2.1.18) is an exact sequence

$$0 \leftarrow K^* \leftarrow V'^* \xleftarrow{d^*} V^* \leftarrow C^* \leftarrow 0$$

so the dual of K is the cokernel K^* and the dual of C is the kernel C^*. This is the reason for the term "cokernel".

snake **2.1.19. Proposition.** *(functorial property of the kernel and cokernel) Suppose given a diagram of R-modules*

$$\begin{tikzcd} & V \arrow[r, "u"] \arrow[d, "f"'] & V' \arrow[r] \arrow[d, "f'"'] & V'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & W \arrow[r] & W' \arrow[r, "v"'] & W'' & \end{tikzcd}$$

whose rows are exact sequences. Let K, K', K'' and C, C', C'' denote the kernels and cokernels of f, f', and f'', respectively.

(i) (kernel is left exact) *The kernels form an exact sequence $K \to K' \to K''$. If u is injective, the sequence $0 \to K \to K' \to K''$ is exact.*

(ii) (cokernel is right exact) *The cokernels form an exact sequence $C \to C' \to C''$. If v is surjective, the sequence $C \to C' \to C'' \to 0$ is exact.*

(iii) (Snake Lemma) *There is a canonical homomorphism $K'' \xrightarrow{d} C$ that combines with the above sequences to form an exact sequence*

$$K \to K' \to K'' \xrightarrow{d} C \to C' \to C''.$$

If u is injective and/or v is surjective, the sequence remains exact with zeros at the appropriate ends. □

present-module **(2.1.20) presenting a module**

A *presentation* of an A-module M is an exact sequence of the form $A^\ell \to A^k \to M \to 0$. Every finite module over a noetherian ring A has such a presentation. To obtain a presentation, one may choose a finite set of elements $m = (m_1, ..., m_k)$ that generates the finite module M, so that multiplication by m is a surjective map $A^k \to M$. Let N be its kernel. Because A is noetherian, N is a finite module. Choosing generators of N gives us a surjective map $A^\ell \to N$, and composition with the inclusion $N \subset A^\ell$ produces an exact sequence $A^\ell \to A^k \to M \to 0$.

sumprod **(2.1.21) direct sum and direct product**

Let M and N be modules over a ring R. The *product module* $M \times N$ is the set-theoretic product, whose elements are pairs (m, n) with m in M and n in N. The laws of composition are the same as the laws for vectors: $(m_1+n_1) + (m_2+n_2) = (m_1+m_2, n_1+n_2)$ and $r(m, n) = (rm, rn)$. There are homomorphisms $M \xrightarrow{i_1} M \times N$ and $M \times N \xrightarrow{\pi_1} M$, defined by $i_1(m) = (m, 0)$ and $\pi_1(m, n) = m$, and similarly, homomorphisms $N \xrightarrow{i_2} M \times N$ and $M \times N \xrightarrow{\pi_2} N$. So i_1 and i_2 are inclusions, and π_1 and π_2 are projections.

The product module is characterized by this mapping property:

• Let T be an R-module. Homomorphisms $T \xrightarrow{\varphi} M \times N$ correspond bijectively to pairs of homomorphisms $T \xrightarrow{\alpha} M$ and $T \xrightarrow{\beta} N$. The homomorphism φ that corresponds to the pair α, β is $\varphi(m, n) = (\alpha m, \beta n)$, and when φ is given, $\alpha = \pi_1 \varphi$ and $\beta = \pi_2 \varphi$.

There is also a second product, the *tensor product* module $M \otimes_R N$ that is defined below.

The product module $M \times N$ is also the *direct sum* $M \oplus N$. The direct sum is characterized by this mapping property:

• Let S be an R-module. Homomorphisms $M \oplus N \xrightarrow{\psi} S$ correspond bijectively to pairs of homomorphisms $M \xrightarrow{u} S$ and $N \xrightarrow{v} S$. The homomorphism ψ that corresponds to the pair u, v is $\psi(m, n) = um + vn$, and when ψ is given, $\alpha = \psi i_1$ and $\beta = \psi i_2$.

We use the direct product and direct sum notations interchangeably, though the direct sum of an infinite set of modules is not the same as the product set.

(2.1.22) tensor products tensprod

Let U and V be modules over a ring R. The *tensor product* $U \otimes_R V$ is an R-module that is generated by elements $u \otimes v$ called *tensors*, one for each u in U and each v in V. Its elements are combinations of tensors with coefficients in R.

The defining relations among the tensors are the *bilinear relations*:

$$(u_1+u_2) \otimes v = u_1 \otimes v + u_2 \otimes v\,, \quad u \otimes (v_1+v_2) = u \otimes v_1 + u \otimes v_2 \tag{2.1.23}$$ bilinrels

and
$$r(u \otimes v) = (ru) \otimes v = u \otimes (rv)$$

for all u in U, v in V, and r in R. The symbol $\otimes$ is used as a reminder that the tensors are to be manipulated using these relations.

One can absorb a coefficient from R into either one of the factors of a tensor. So every element of $U \otimes_R V$ can be written as a finite sum $\sum u_i \otimes v_i$ with u_i in U and v_i in V.

2.1.24. Examples. (i) If U is the space of m dimensional (complex) column vectors, and V is the space of n-dimensional row vectors. Then $U \otimes_{\mathbb{C}} V$ identifies naturally with the space of $m \times n$-matrices. colxrow
(ii) An R-module U is a free module of *rank* n if it is isomorphic to R^n, or if it has a basis of n elements. If U and V are free R-modules of ranks m and n, with bases $\{u_i\}$, $i = 1, ..., m$ and $\{v_j\}$, $j = 1, ..., n$ respectively, the tensor product $U \otimes_R V$ is a free R-module of rank mn, with basis $\{u_i \otimes v_j\}$. The product module $U \times V$ is a free module of rank $m+n$, with basis $\{u_i\} \cup \{v_j\}$.

There is an obvious map of sets

$$U \times V \xrightarrow{\beta} U \otimes_R V \tag{2.1.25}$$ bilin

from the product *set* to the tensor product, that sends a pair (u, v) to the tensor $u \otimes v$. This map isn't a homomorphism of R-modules. The defining relations (2.1.23) show that it is *R-bilinear*, not linear.

2.1.26. Corollary. *Let U, V, and W be R-modules. Homomorphisms of R-modules $U \otimes_R V \to W$ correspond bijectively to R-bilinear maps $U \times V \to W$.* *tensorbilin*

This follows from the defining relations. $\square$

Thus the map $U \times V \to U \otimes_R V$ is a *universal* bilinear map. Any R-bilinear map $U \times V \xrightarrow{f} W$ to a module W can be obtained from a module homomorphism $U \otimes_R V \xrightarrow{\tilde{f}} W$ by composition with the bilinear map β defined above: $U \times V \xrightarrow{\beta} U \otimes_R V \xrightarrow{\tilde{f}} W$.

2.1.27. Proposition. *There are canonical isomorphisms* *canonisom*

- $U \otimes_R R \approx U$, *defined by* $u \otimes r \leftrightsquigarrow ur$
- $(U \oplus U') \otimes_R V \approx (U \otimes_R V) \oplus (U' \otimes_R V)$, *defined by* $(u_1 + u_2) \otimes v \leftrightsquigarrow u_1 \otimes v + u_2 \otimes v$
- $U \otimes_R V \approx V \otimes_R U$, *defined by* $u \otimes v \leftrightsquigarrow v \otimes u$
- $(U \otimes_R V) \otimes_R W \approx U \otimes_R (V \otimes_R W)$, *defined by* $(u \otimes v) \otimes w \leftrightsquigarrow u \otimes (v \otimes w)$ $\square$

2.1.28. Proposition. Tensor product is right exact *Let $U \xrightarrow{f} U' \xrightarrow{g} U'' \to 0$ be an exact sequence of R-modules. For any R-module V, the sequence* *rexacttensor*

$$U \otimes_R V \xrightarrow{f \otimes 1} U' \otimes_R V \xrightarrow{g \otimes 1} U'' \otimes_R V \to 0$$

in which $[f \otimes 1](u \otimes v) = f(u) \otimes v$, is exact. $\square$

Tensor product isn't left exact. For example, if $R = \mathbb{C}[x]$, then $R/xR \approx \mathbb{C}$. There is an exact sequence $0 \to R \xrightarrow{x} R \to \mathbb{C} \to 0$. When we tensor with $\mathbb{C}$ we get a sequence $0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to 0$, in which the first map $\mathbb{C} \to \mathbb{C}$ is the zero map. That sequence isn't exact on the left.

proof of Proposition 2.1.28. We are given an exact sequence of R-modules $U \xrightarrow{f} U' \xrightarrow{g} U'' \to 0$ and another R-module V. We are to prove that the sequence $U \otimes_R V \xrightarrow{f\otimes 1} U' \otimes_R V \xrightarrow{g\otimes 1} U'' \otimes_R V \to 0$ is exact. It is evident that the composition $(g\otimes 1)(f\otimes 1)$ is zero, and that $g\otimes 1$ is surjective. We must prove that $U'' \otimes_R V$ is isomorphic to the cokernel of $f\otimes 1$.

Let C denote the cokernel of $f \otimes 1$. The mapping property (2.1.4)**(ii)** gives us a canonical map $C \xrightarrow{\varphi} U'' \otimes_R V$ that we want to show is an isomorphism. To show this, we construct the inverse of φ. We choose an element v of V, and form a diagram of R-modules

$$\begin{tikzcd}
U\times v \arrow[r] \arrow[d, "\beta_v"'] & U'\times v \arrow[r] \arrow[d, "\beta'_v"'] & U''\times v \arrow[r] \arrow[d, "\gamma_v"] & 0 \\
U \otimes_R V \arrow[r, "f\otimes 1"] & U' \otimes_R V \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

in which $U\times v$ denotes the module of pairs (u,v) with $u \in U$, which is isomorphic to U.

The rows in the diagram are exact sequences of modules, and the vertical arrows β_v and β'_v are the homomorphisms obtained by restriction from the canonical bilinear maps (2.1.25). Since $U\times v$ maps to zero in $U''\times v$, γ_v is determined by the definition of the cokernel. Putting the maps γ_v together for all v in V gives us a bilinear map $U\times V \to C$, that induces a linear map $U\otimes_R V \to C$ (2.1.26). That map is the inverse of φ. $\square$

####

locistensor **2.1.29. Corollary.** *Let U and V be modules over a domain R and let s be a nonzero element of R. Let R_s, U_s, V_s be the localizations of R, U, V, respectively.*
(i) *There is a canonical isomorphism $U\otimes_R (R_s) \approx U_s$.*
(ii) *Tensor product is compatible with localization: $U_s \otimes_{R_s} V_s \approx (U\otimes_R V)_s$*

proof. **(ii)** The composition of the canonical maps $U\times V \to U_s\times V_s \to U_s \otimes_{R_s} V_s$ is R-bilinear. It defines an R-linear map $U\otimes_R V \to U_s \otimes_{R_s} V_s$. Since s is inverible in $U_s \otimes_{R_s} V_s$, this map extends to an R_s-linear map $(U\otimes_R V)_s \to U_s \otimes_{R_s} V_s$. Next, we define an R_s-bilinear map $U_s\times V_s \to (U\otimes_R V)_s$ by mapping a pair (us^{-m}, vs^{-n}) to $(u\otimes v)s^{-m+n}$. This bilinear map induces the inverse map $U_s\otimes_{R_s} V_s \to (U\otimes_R V)_s$. $\square$

extendscalars **(2.1.30) extension of scalars**

Let $A \xrightarrow{\rho} B$ be a ring homomorphism. Extension of scalars is an operation that constructs an B-module from an A-module.

Let's write scalar multiplication on the right. So M will be a right A-module. Then $M\otimes_A B$ becomes a right B-module, multiplication by $b\in B$ being defined by $(m\otimes b')b = m\otimes(b'b)$. This gives the functor

$$A{-}modules \xrightarrow{\otimes B} B{-}modules$$

called the *extension of scalars* from A to B.

resscal **(2.1.31) restriction of scalars**

If $A \xrightarrow{\rho} B$ is a ring homomorphism, a B-module N can be made into an A-module by *restriction of scalars*. Scalar multiplication by an element a of A is defined by the formula

restrscal (2.1.32)
$$an = \rho(a)n$$

It is customary to denote a module and the one obtained by restriction of scalars by the same symbol. But if it seems necessary in order to avoid confusion, we may denote a B-module N and the A-module obtained from N by restriction of scalars by ${}_BN$ and ${}_AN$, respectively. The additive groups of ${}_BN$ and ${}_AN$ are the same.

xtscaltens **2.1.33. Lemma.** *(extension and restriction of scalars are adjoint operators)*
Let $A \xrightarrow{\rho} B$ be a ring homomorphism, let M be an A-module, and let M' be an B-module. Homomorphisms $M \xrightarrow{\varphi}{}_A N$ of A-modules correspond bijectively to homomorphisms of B-modules $M\otimes_A B \xrightarrow{\psi}{}_B N$. $\square$

This concludes our review of rings and modules.

2.2 The Zariski Topology

zartop

Algebraic geometry studies polynomial equations in terms of their solutions in the affine space $\mathbb{A}^n$ of n-tuples $(a_1, ..., a_n)$ of complex numbers.

Let $f_1, ..., f_k$ be polynomials in $x_1, ..., x_n$. The set of points of $\mathbb{A}^n$ that solve the system of equations

$$f_1 = 0\,,\ \ldots,\ f_k = 0 \tag{2.2.1}$$ fequat

the *locus of zeros* of f, is a *Zariski closed* set. A subset U of $\mathbb{A}^n$ is a *Zariski open* set if its *complement*, the set of points not in U, is Zariski closed.

The locus of solutions of the equations $f = 0$, the *zeros* of the polynomials f, may be denoted by $V(f_1, ..., f_k)$ or by $V(f)$. We use analogous notation for infinite sets. If $\mathcal{F}$ is any set of polynomials, $V(\mathcal{F})$ denotes the set of points of affine space at which all elements of $\mathcal{F}$ are zero. In particular, if I is an ideal of the polynomial ring, $V(I)$ denotes the set of points at which all elements of I vanish.

The ideal I of $\mathbb{C}[x]$ that is *generated* by the polynomials $f_1, ..., f_k$ is the set of combinations $r_1f_1 + \cdots + r_kf_k$ with polynomial coefficients r_i. Some notations for this ideal are $(f_1, ..., f_k)$ and (f). All elements of this ideal vanish on the zero set $V(f)$, so $V(f) = V(I)$. The Zariski closed subsets of $\mathbb{A}^n$ can be described as the sets $V(I)$, where I is an ideal.

We note a few simple relations among ideals and their zero sets here. To begin with, we note that, for any $k > 0$, the power f^k of a polynomial f has the same zeros as f. So an ideal I isn't determined by its zero locus.

2.2.2. Lemma. *Let I and J be ideals of the polynomial ring $\mathbb{C}[x]$.* *IinJ*
(i) *If $I \subset J$, then $V(I) \supset V(J)$.*
(ii) $V(I^k) = V(I)$.
(iii) $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.
(iv) *If I_ν are ideals, then $V(\sum I_\nu)$ is the intersection $\bigcap V(I_\nu)$ or the ideals I_ν.*

proof. **(iii)** $V(I \cap J) = V(IJ)$ because the two ideals have the same radical, and because I and J contain IJ, $V(IJ) \supset V(I) \cup V(J)$. To prove that $V(IJ) \subset V(I) \cup V(J)$, we note that $V(IJ)$ is the locus of common zeros of the products fg with f in I and g in J. Suppose that a point p is a common zero: $f(p)g(p) = 0$ for all f in I and all g in J. If $f(p) \neq 0$ for some f in I, we must have $g(p) = 0$ for every g in J, and then p is a point of $V(J)$. If $f(p) = 0$ for all f in I, then p is a point of $V(I)$. In either case, p is a point of $V(I) \cup V(J)$. $\square$

The *radical* of an ideal I of a ring R is the set of elements α of R such that some power α^r is in I. The radical will be denoted by $\operatorname{rad} I$.

$$\operatorname{rad} I = \{\alpha \in R \,|\, \alpha^r \in I \text{ for some } r > 0\} \tag{\textbf{2.2.3}}$$ raddef

The radical of I is an ideal that contains I.

An ideal that is equal to its radical is a *radical ideal*. A prime ideal is a radical ideal.

2.2.4. Lemma. *If I is an ideal of the polynomial ring $\mathbb{C}[x]$, then $V(I) = V(\operatorname{rad} I)$.* $\square$ *VradI*

Consequently, if I and J are ideals and if $\operatorname{rad} I = \operatorname{rad} J$, then $V(I) = V(J)$. The converse of this statement is also true: If $V(I) = V(J)$, then $\operatorname{rad} I = \operatorname{rad} J$. This is a consequence of the *Strong Nullstellensatz* that is proved below (see (2.4.9)).

Because $(I \cap J)^2 \subset IJ \subset I \cap J$,

$$\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) \tag{2.2.5}$$ radIJ

Also, $\operatorname{rad}(I \cap J) = (\operatorname{rad} I) \cap (\operatorname{rad} J)$.

2.2.6. The Zariski closed sets defined above are the closed sets in the *Zariski topology* on $\mathbb{A}^n$. To verify that the Zariski closed sets are the closed sets of a topology, one must show that ztop

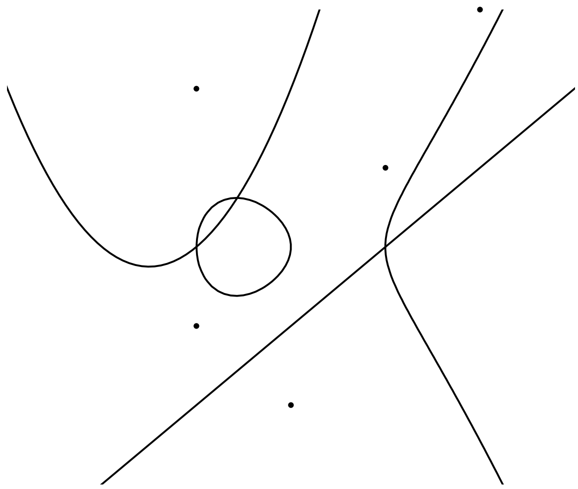
- the empty set and the whole space are Zariski closed,

- the intersection $\bigcap C_\nu$ of an arbitrary family of Zariski closed sets is Zariski closed, and
- the union $C \cup D$ of two Zariski closed sets is Zariski closed.

The empty set and the whole space are the zero sets of the elements 1 and 0, respectively. The other conditions follow from Lemma 2.2.2. $\square$

ztopdimone **2.2.7. Example.** The proper Zariski closed subsets of the affine line, or of a plane affine curve, are the nonempty finite sets. The proper Zariski closed subsets of the affine plane are finite unions of points and curves. Let's omit the proofs of these facts. The corresponding facts for loci in the projective line and the projective plane have been noted before. (See (**1.3.4**) and (1.3.15).) $\square$

2.2.8.



A Zariski closed subset of the affine plane (real locus)

A subset S of a topological space X becomes a topological space with its *induced topology*. The closed (or open) subsets of S in the induced topology are intersections $S \cap Y$, where Y is closed (or open) in X.

The topology induced on a subset S from the Zariski topology on $\mathbb{A}^n$ will be called the *Zariski topology* on S. A subset of S is closed in its Zariski topology if it has the form $S \cap Z$ for some Zariski closed subset Z of $\mathbb{A}^n$. If Y is a Zariski closed subset of $\mathbb{A}^n$, a closed subset of Y can also be described as a closed subset of $\mathbb{A}^n$ that is contained in Y.

where to explain that if cover affine by opens, then cover by localizations. Therefore $s-1, ..., s_k$ generate the unit ideal###

Affine space also has a *classical topology* (**1.3.17**). A subset U of $\mathbb{A}^n$ is open in the classical topology if, whenever a point p is in U, all points sufficently near to p are in U. Since polynomial functions are continuous, their zero sets are closed in the classical topology. Therefore Zariski closed sets are closed in the classical topology too. Though the Zariski topology is very different from the classical topology, it is very useful in algebraic geometry.

When two topologies T and T' on a set X are given, T' is said to be *coarser* than T if every closed set in T' is closed in T, i.e., if T' contains fewer closed sets (or fewer open sets) than T, and T' is *finer* than T if it contains more closed sets (or more open sets) than T. The Zariski topology is coarser than the classical topology, and as the next proposition shows, it is much coarser.

opendense **2.2.9. Proposition.** *Any nonempty Zariski open subset U of $\mathbb{A}^n$ is dense and path connected in the classical topology.*

proof. The (complex) *line* L through distinct points p and q of $\mathbb{A}^n$ is a Zariski closed set whose points can be written as $p + t(q - p)$, with t in $\mathbb{C}$. It corresponds bijectively to the affine t-line $\mathbb{A}^1$, and the Zariski closed subsets of L correspond to Zariski closed subsets of $\mathbb{A}^1$. They are the finite subsets, and L itself.

Let U be a nonempty Zariski open set, and let C be its Zariski closed complement. To show that U is dense in the classical topology, we choose distinct points p and q of $\mathbb{A}^n$, with p in U. If L is the line through p and q, $C \cap L$ will be a Zariski closed subset of L, a finite set that doesn't contain p. The complement of this finite set in L is $U \cap L$. In the classical topology, the closure of $U \cap L$, will be the whole line L. So it contains q. Thus the closure of U contains q, and since q was arbitrary, the closure of U is $\mathbb{A}^n$.

Next, let L be the line through two points p and q of U. As before, $C \cap L$ will be a finite set of points. In the classical topology, L is a complex plane. The points p and q can be joined by a path in this plane that avoids the finite set. $\square$

Though we will use the classical topology from time to time, the Zariski topology will appear more often. Because of this, we will refer to a Zariski closed subset simply as a *closed set*. Similarly, by an *open set* we mean a Zariski open set. We will mention the adjective "Zariski" only for emphasis.

(**2.2.10**) **irreducible closed sets** irrclosed

The fact that the polynomial algebra is a noetherian ring has an important consequence for the Zariski topology that we discuss here.

A topological space X satisfies the *descending chain condition* on closed subsets if X has no infinite, strictly descending chain $C_1 > C_2 > \cdots$ of closed subsets. The descending chain condition on closed subsets is equivalent with the *ascending chain condition* on open sets.

A *noetherian space* is a topological space that satisfies the descending chain condition on closed sets. In a noetherian space, every nonempty family $\mathcal{S}$ of closed subsets has a minimal member, one that doesn't contain any other member of $\mathcal{S}$, and every nonempty family of open sets has a maximal member. (See (**2.1.12**).)

2.2.11. Lemma. *A noetherian topological space is* ***quasicompact****: Every open covering has a finite subcovering.* $\square$ *noethq-comp*

2.2.12. Proposition. *With its Zariski topology, $\mathbb{A}^n$ is a noetherian space.* *deschain*

proof. Suppose that a strictly descending chain $C_1 > C_2 > \cdots$ of closed subsets of $\mathbb{A}^n$ is given. Let I_j be the ideal of all elements of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ that are identically zero on C_j. Then $C_j = V(I_j)$. Since $C_j > C_{j+1}$, $I_j < I_{j+1}$. The ideals I_j form a strictly increasing chain. Since $\mathbb{C}[x_1, ..., x_n]$ is noetherian, this chain is finite. Therefore the chain C_j is finite too. $\square$

2.2.13. Definition. A topological space X is *irreducible* if it isn't the union of two proper closed subsets. de-firrspace

Another way to say that a topological space X is irreducible is this:

2.2.14. *If C and D are closed subsets of an irreducible toplogical space X, and if $X = C \cup D$, then $X = C$ or $X = D$.* ir-redspacetwo

The concept of irreducibility is useful primarily for noetherian spaces. The only irreducible subsets of a Hausdorff space are its points. So, in the classical topology, the only irreducible subsets of affine space are points.

Irreducibility is somewhat analogous to connectedness. A topological space is *connected* if it isn't the union $C \cup D$ of two proper *disjoint* closed subsets. However, the condition that a space be irreducible is much more restrictive because, in Definition 2.2.13, the closed sets C and D aren't required to be disjoint. In the Zariski topology on the affine plane, lines are irreducible closed sets. The union of two intersecting lines is connected, but not irreducible.

2.2.15. Lemma. *The following conditions on topological space X are equivalent.* *ir-redlemma*

- *X is irreducible.*
- *The intersection $U \cap V$ of two nonempty open subsets U and V of X is nonempty.*
- *Every nonempty open subset U of X is dense — its closure is X.* $\square$

The *closure* of a subset U of a topological space X is the smallest closed subset of X that contains U. The closure exists because it is the intersection of all closed subsets that contain S.

closureinters **2.2.16. Lemma.** *Let Y be a subspace of a topological space X, let S be a subset of Y, let C be the closure of S in X. The closure of S in Y is $C \cap Y$.*

proof. Let $D = C \cap Y$. There is a closed subset V of X such that $D = V \cap Y$. Then V contains S, and $V \supset C \supset D$. So $D = V \cap Y \supset C \cap Y \supset D \cap Y = D$. Therefore $D = C \cap Y$. $\square$

quasicompact **2.2.17. Lemma. (i)** *Let $\overline{Z}$ be the closure of a subspace Z of a topological space X. Then $\overline{Z}$ is irreducible if and only if Z is irreducible.*

(ii) *A nonempty open subspace W of an irreducible space X is irreducible.*

(iii) *Let $Y \to X$ be a continuous map of topological spaces. The image in X of an irreducible subset D of Y is irreducible.*

proof. **(i)** Let Z be an irreducible subset of X, and suppose that its closure $\overline{Z}$ is the union $\overline{C} \cup \overline{D}$ of two closed sets $\overline{C}$ and $\overline{D}$. Then Z is the union of the sets $C = \overline{C} \cap Z$ and $D = \overline{D} \cap Z$, and they are closed in Z. Therefore Z is one of those two sets; say $Z = C$. Then $Z \subset \overline{C}$, and since $\overline{C}$ is closed, $\overline{Z} \subset \overline{C}$. Because $\overline{C} \subset \overline{Z}$ as well, $\overline{C} = \overline{Z}$. Conversely, suppose that the closure $\overline{Z}$ of a subset Z of X is irreducible, and that Z is a union $C \cup D$ of closed subsets. Then $\overline{Z} = \overline{C} \cup \overline{D}$, and therefore $\overline{Z} = \overline{C}$ or $\overline{Z} = \overline{D}$. Let's say that $\overline{Z} = \overline{C}$. Then $Z = \overline{C} \cap Z = C$. So C isn't a proper subset of X.

(ii) The closure of W is the irreducible space X.

(iii) Let C be the image of D, and suppose that C is the union $C_1 \cup C_2$ of closed subsets of X. The inverse images D_i of C_i are closed in Y, and $D = D_1 \cup D_2$. Therefore one ofthe two, say D_1, is equal to D. The map $D \to C$ is surjective, and so is the map $D_1 \to C_1$. Therefore $C_1 = C$. $\square$

unionirred **2.2.18. Theorem.** *In a noetherian topological space, every closed subset is the union of finitely many irreducible closed sets.*

proof. If a closed subset C_0 of a topological space X isn't a union of finitely many irreducible closed sets, then it isn't irreducible, so it is a union $C_1 \cup D_1$, where C_1 and D_1 are proper closed subsets of C_0, and therefore closed subsets of X. Since C_0 isn't a finite union of irreducible closed sets, C_1 and D_1 cannot both be finite unions of irreducible closed sets. Say that C_1 isn't such a union. We have the beginning $C_0 > C_1$ of a chain of closed subsets. We repeat the argument, replacing C_0 by C_1, and we continue in this way, to construct an infinite, strictly descending chain $C_0 > C_1 > C_2 > \cdots$. So X isn't a noetherian space. $\square$

defadffvar **2.2.19. Definition.** An *affine variety* is an irreducible closed subset of affine space $\mathbb{A}^n$.

Theorem 2.2.18 tells us that every closed subset of $\mathbb{A}^n$ is a finite union of affine varieties. Since an affine variety is irreducible, it is connected in the Zariski topology. An affine variety is also connected in the classical topology, but this isn't easy to prove. We may not get to the proof.

noethind (**2.2.20**) **noetherian induction**

In a noetherian space Z one can use *noetherian induction* in proofs. Suppose that a statement Σ is to be proved for every closed subvariety X (every irreducible closed subset) of Z. Then it suffices to prove Σ for X under the assumption that Σ is true for every closed subvariety that is a proper subset of X.

Or, to prove a statement Σ for every proper closed subset X, it suffices to permissible prove it for X under the assumption that Σ is true for every proper closed subset of X.

The justification of noetherian induction is similar to the justification of complete induction. Let $\mathcal{S}$ be the family of irreducible closed subsets for which Σ is false. If $\mathcal{S}$ isn't empty, it will contain a minimal member X. Then Σ will be true for every proper closed subset of X, etc.

coordinateagebra (**2.2.21**) **the coordinate algebra of a variety**

2.2.22. Proposition. *The affine varieties in $\mathbb{A}^n$ are the sets $V(P)$, where P is a prime ideal of the polynomial algebra $\mathbb{C}[x] = \mathbb{C}[x_1, ..., x_n]$. If P is a radical ideal of $\mathbb{C}[x]$, then $V(P)$ is an affine variety if and only if P is a prime ideal.* *irredprime*

We will use this proposition in the next section, where we give a few examples of varieties, but we defer the proof to Section 2.5, where the proposition will be proved in a more general form. (See Proposition 2.5.13).)

2.2.23. Definition. Let P be a prime ideal of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, and let V be the affine variety $V(P)$ in $\mathbb{A}^n$. The *coordinate algebra* of V is the quotient algebra $A = \mathbb{C}[x]/P$. defcoordalg

Geometric properties of the variety are reflected in algebraic properties of its coordinate algebra and vice versa. In a primitive sense, one can regard the geometry of an affine variety V as given by closed subsets and incidence relations — the inclusion of one closed set into another, as when a point lies on a line. A finer study of the geometry takes into account other things, tangency, for instance, but it is reasonable to begin by studying incidences $C' \subset C$ among closed subvarieties. Such incidences translate into inclusions $P' \supset P$ in the opposite direction among prime ideals.

2.3 Some affine varieties

This section contains a few simple examples of varieties. somevarieties

2.3.1. A *point* $p = (a_1, \ldots, a_n)$ of affine space $\mathbb{A}^n$ is the set of solutions of the n equations $x_i - a_i = 0$, $i = 1, \ldots, n$. A point is a variety because the polynomials $x_i - a_i$ generate a maximal ideal in the polynomial algebra $\mathbb{C}[x]$, and a maximal ideal is a prime ideal. We denote the maximal ideal that corresponds to the point p by $\mathfrak{m}_p$. It is the kernel of the substitution homomorphism $\pi_p : \mathbb{C}[x] \to \mathbb{C}$ that evaluates a polynomial $g(x_1, ..., x_n)$ at p: $\pi_p(g(x)) = g(a_1, ..., a_n) = g(p)$. As here, we denote the homomorphism that evaluates a polynomial at a point p by π_p. ptvar

The coordinate algebra of the point p is the quotient $\mathbb{C}[x]/\mathfrak{m}_p$. This quotient algebra is also called the *residue field* at p, and it will be denoted by $k(p)$. The residue field $k(p)$ is isomorphic to the image of π_p, which is the field of complex numbers, but it is a particular quotient of the polynomial ring.

2.3.2. The varieties in the affine line $\mathbb{A}^1$ are points and the whole line $\mathbb{A}^1$. The varieties in the affine plane $\mathbb{A}^2$ are points, plane affine curves, and the whole plane. varinline

This is true because the varieties correspond to the prime ideals of the polynomial ring. The prime ideals of $\mathbb{C}[x_1, x_2]$ are the maximal ideals, the principal ideals generated by irreducible polynomials, and the zero ideal. The proof of this is an exercise.

2.3.3. The set X of solutions of a single irreducible polynomial equation $f_1(x_1, ..., x_n) = 0$ in $\mathbb{A}^n$ is a variety called an *affine hypersurface*. hsurf

For instance, the *special linear group SL_2*, the group of complex 2×2 matrices with determinant 1, is a hypersurface in $\mathbb{A}^4$. It is the locus of zeros of the irreducible polynomial $x_{11}x_{22} - x_{12}x_{21} - 1$.

The reason that an affine hypersurface is a variety is that an irreducible element of a unique factorization domain is a prime element, and a prime element generates a prime ideal. The polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is a unique factorization domain.

A hypersurface in the affine plane $\mathbb{A}^2$ is a plane affine curve.

2.3.4. A *line* in the plane, the locus of a linear equation $ax + by - c = 0$, is a plane affine curve. Its coordinate algebra is isomorphic to a polynomial ring in one variable. Every line is isomorphic to the affine line $\mathbb{A}^1$. apc

2.3.5. Let $p = (a_1, \ldots, a_n)$ and $q = (b_1, \ldots, b_n)$ be distinct points of $\mathbb{A}^n$. The *point pair* (p, q) is the closed set defined by the system of n^2 equations $(x_i - a_i)(x_j - b_j) = 0$, $1 \le i, j \le n$. A point pair isn't a variety because the ideal I that is generated by the polynomials $(x_i - a_i)(x_j - b_j)$ isn't a prime ideal. The next corollary, which follows from the Chinese Remainder Theorem 2.1.7, describes that ideal: pointpr

2.3.6. Corollary. *The ideal I of polynomials that vanish on a point pair p, q is the product $\mathfrak{m}_p\mathfrak{m}_q$ of the maximal ideals at the points, and the quotient algebra $\mathbb{C}[x]/I$ is isomorphic to the product algebra $\mathbb{C}\times\mathbb{C}$.* $\square$ *pointpair*

2.4 Hilbert's Nullstellensatz

nullone null **2.4.1. Nullstellensatz (version 1).** *Let $\mathbb{C}[x]$ be the polynomial algebra in the variables $x_1, \dots, x_n$. There are bijective correspondences between the following sets:*

- *points p of the affine space $\mathbb{A}^n$,*
- *algebra homomorphisms $\pi_p : \mathbb{C}[x] \to \mathbb{C}$,*
- *maximal ideals $\mathfrak{m}_p$ of $\mathbb{C}[x]$.*

If $p = (a_1, ..., a_n)$ is a point of $\mathbb{A}^n$, the homomorphism π_p evaluates a polynomial at p: $\pi_p(g) = g(p)g(a_1, ..., a_n)$. The maximal ideal $\mathfrak{m}_p$ is the kernel of π_p. It is the ideal generated by the linear polynomials $x_1 - a_1, \dots, x_n - a_n$.
□

It is obvious that every algebra homomorphism $\mathbb{C}[x] \to \mathbb{C}$ is surjective, and that its kernel is a maximal ideal. It isn't obvious that every maximal ideal of $\mathbb{C}[x]$ is the kernel of such a homomorphism. The proof can be found manywhere.[1]

The Nullstellensatz gives us a way to describe the closed set $V(I)$ of zeros of an ideal I in affine space in terms of maximal ideals. The points of $V(I)$ are those at which all elements of I vanish — the points p such that the ideal I is contained in $\mathfrak{m}_p$:.

maxidealscheme (2.4.2)
$$V(I) = \{p \in \mathbb{A}^n \mid I \subset \mathfrak{m}_p\}$$

Vempty **2.4.3. Proposition.** *Let I be an ideal of the polynomial ring $\mathbb{C}[x]$. If the zero locus $V(I)$ is empty, then I is the unit ideal.*

proof. Every ideal I except the unit ideal is contained in a maximal ideal (Corollary 2.1.14). □

nulltwo **2.4.4. Nullstellensatz (version 2).** *Let A be a finite-type algebra. There are bijective correspondences between the following sets:*

- *algebra homomorphisms $\overline{\pi} : A \to \mathbb{C}$,*
- *maximal ideals $\overline{\mathfrak{m}}$ of A.*

The maximal ideal $\overline{\mathfrak{m}}$ that corresponds to a homomorphism $\overline{\pi}$ is the kernel of $\overline{\pi}$.

If A is presented as a quotient of a polynomial ring, say $A \approx \mathbb{C}[x_1, ..., x_n]/I$, then these sets also correspond bijectively to points of the set $V(I)$ of zeros of I in $\mathbb{A}^n$.

The symbol $\approx$ stands for an isomorphism that is often unspecified.

As before, a finite-type algebra is an algebra that can be generated by a finite set of elements.

A *presentation* of a finite-type algebra A is an isomorphism of A with a quotient $\mathbb{C}[x_1, ..., x_n]]/I$ of a polynomial ring. (This isn't the same as a presentation of a module (**2.1.20**).)

proof. We choose a presentation of A as a quotient of a polynomial ring to identify A with a quotient $\mathbb{C}[x]/I$. The Correspondence Theorem tells us that maximal ideals of A correspond to maximal ideals of $\mathbb{C}[x]$ that contain I. Those maximal ideals correspond to points of $V(I)$.

Let τ denote the canonical homomorphism $\mathbb{C}[x] \to A$. The Mapping Property 2.1.4, applied to τ, tells us that homomorphisms $A \xrightarrow{\overline{\pi}} \mathbb{C}$ correspond to homomorphisms $\mathbb{C}[x] \xrightarrow{\pi} \mathbb{C}$ whose kernels contain I. Those homomorphisms also correspond to points of $V(I)$.

polyringtoA (2.4.5)
$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\pi"] \arrow[d, "\tau"'] & \mathbb{C} \arrow[d, equal] \\ A \arrow[r, "\overline{\pi}"] & \mathbb{C} \end{tikzcd}$$
□

[1] While writing a paper, the mathematician Nagata decided that the English language needed this unusual word. Then he managed to find it in a dictionary.

2.4.6. Strong Nullstellensatz. *Let I be an ideal of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$, and let V be the locus of zeros of I in $\mathbb{A}^n$: $V = V(I)$. If a polynomial $g(x)$ vanishes at every point of V, then I contains a power of g.* *strongnull*

proof. This beautiful proof is due to Rainich. Let $g(x)$ be a polynomial that is identically zero on V. We are to show that I contains a power of g. The zero polynomial is in I, so we may assume that g isn't zero.

The Hilbert Basis Theorem tells us that I is a finitely generated ideal. Let $f = \{f_1, \ldots, f_k\}$ be a set of generators. We introduce a new variable y. In the $n{+}1$–dimensional affine space with coordinates $(x_1, \ldots, x_n, y)$, let W be the locus of solutions of the $k{+}1$ equations

(2.4.7) $$f_1(x) = 0, ..., f_k(x) = 0 \quad \text{and} \quad g(x)y - 1 = 0$$ fgy

Suppose that we have a solution x of the equations $f(x) = 0$, say $(x_1, ..., x_n) = (a_1, ..., a_n)$. Then a is a point of V, and our hypothesis tells us that $g(a) = 0$ too. So there can be no b such that $g(a)b = 1$. There is no point $(a_1, ..., a_n, b)$ that solves the equations (2.4.7): The locus W is empty. Proposition 2.4.3 tells us that the polynomials $f_1, ..., f_k, gy - 1$ generate the unit ideal of $\mathbb{C}[x_1, ..., x_n, y]$. There are polynomials $p_1(x, y), \ldots, p_k(x, y)$ and $q(x, y)$ such that

(2.4.8) $$p_1 f_1 + \cdots + p_k f_k + q(gy - 1) = 1$$ rabinowitz

The ring $R = \mathbb{C}[x, y]/(gy - 1)$ can be described as the one obtained by adjoining an inverse of g to the polynomial ring $\mathbb{C}[x]$. The residue of y becomes the inverse. Since g isn't zero, $\mathbb{C}[x]$ is a subring of R. In R, $gy - 1 = 0$. The equation (2.4.8) becomes $p_1 f_1 + \cdots + p_k f_k = 1$. When we multiply both sides of this equation by a large power g^N of g, we can use the equation $gy = 1$, which is true in R, to cancel all occurences of y in the polynomials $p_i(x, y)$. Let $h_i(x)$ denote the polynomial in x that is obtained by cancelling y in $g^N p_i$. Then
$$h_1(x)f_1(x) + \cdots + h_k(x)f_k(x) \;=\; g^N(x)$$
is a polynomial equation that is true in R and in its subring $\mathbb{C}[x]$. Since $f_1, ..., f_k$ are in I, this equation shows that g^N is in I. □

2.4.9. Corollary. *Let $\mathbb{C}[x]$ denote the polynomial ring $\mathbb{C}[x_1, ..., x_n]$.* *VIsupVJ*

(i) *Let P be a prime ideal of $\mathbb{C}[x]$, and let $V = V(P)$ be the variety of zeros of P in $\mathbb{A}^n$. If a polynomial g vanishes at every point of V, then g is an element of P.*

(ii) *Let f be an irreducible polynomial in $\mathbb{C}[x]$. If a polynomial g vanishes at every point of $V(f)$, then f divides g.*

(iii) *Let I and J be ideals of $\mathbb{C}[x]$. Then $V(I) \supset V(J)$ if and only if $\operatorname{rad} I \subset \operatorname{rad} J$, and $V(I) > V(J)$ if and only if $\operatorname{rad} I > \operatorname{rad} J$ (see (**2.2.3**)).* □

2.4.10. Examples.

(i) Let I be the ideal generated by y^5 and $y^2 - x^3$ in the polynomial algebra $\mathbb{C}[x, y]$ in two variables. In the affine plane, the origin $y = x = 0$, is the only common zero of these polynomials, and the polynomial x also vanishes at the origin. The Strong Nullstellensatz predicts that I contains a power of x. This is verified by the following equation: strongnullex
$$yy^5 - (y^4 + y^2x^3 + x^6)(y^2 - x^3) \;=\; x^9$$

(ii) We may regard pairs A, B of $n \times n$ matrices as points of an affine space $\mathbb{A}^{2n^2}$ with coordinates a_{ij}, b_{ij}, $1 \le i, j \le n$. The pairs of *commuting matrices* ($AB = BA$) form a closed subset of $\mathbb{A}^{2n^2}$, the locus of common zeros of the n^2 polynomials c_{ij} that compute the entries of the matrix $AB - BA$:

(2.4.11) $$c_{ij}(a, b) = \sum_\nu a_{i\nu} b_{\nu j} - b_{i\nu} a_{\nu j}$$ comm-mateq

If I is the ideal of the polynomial algebra $\mathbb{C}[a, b]$ generated by the set of polynomials $\{c_{ij}\}$, then $V(I)$ is the set of pairs of commuting complex matrices. The Strong Nullstellensatz asserts that, if a polynomial $g(a, b)$ vanishes on every pair of commuting matrices, some power of g is in I. Is g itself in I? It is a famous conjecture that I is a prime ideal. If so, g would be in I. Proving the conjecture would establish your reputation as a mathematician, but I don't recommend spending very much time on it right now. □

2.5 The Spectrum

spectrumalg When a finite-type domain A is presented as a quotient of a polynomial ring $\mathbb{C}[x]/P$, where P is a prime ideal, A becomes the coordinate algebra of the variety $V(P)$ in affine space. The points of $V(P)$ correspond to maximal ideals of A and also to homomorphisms $A \to \mathbb{C}$.

The Nullstellensatz allows us to associate a set of points to a finite-type domain A without reference to a presentation. We can do this because the maximal ideals of A and the homomorphisms $A \to \mathbb{C}$ don't depend on a presentation. We replace the variety $V(P)$ by an abstract set of points, the *spectrum* of A, that we denote by $\operatorname{Spec} A$ and call an *affine variety*. We put one point p into the spectrum for every maximal ideal of A, and then we turn around and denote the maximal ideal that corresponds to a point p by $\overline{\mathfrak{m}}_p$. The Nullstellensatz tells us that p also corresponds to a homomorphism $A \to \mathbb{C}$ whose kernel is $\overline{\mathfrak{m}}_p$. We denote that homomorphism by $\overline{\pi}_p$. In analogy with (2.2.23), A is called the *coordinate algebra* of the affine variety $\operatorname{Spec} A$. To work with $\operatorname{Spec} A$, we may interpret its points as maximal ideals or as homomorphisms to $\mathbb{C}$, whichever is convenient.

When defined in this way, the variety $\operatorname{Spec} A$ isn't embedded into any affine space, but because A is a finite-type domain, it can be presented as a quotient $\mathbb{C}[x]/P$, where P is a prime ideal. When this is done, points of $\operatorname{Spec} A$ correspond to points of the subset $V(P)$ in $\mathbb{A}^n$.

Even when the coordinate ring A of an affine variety X is presented as $\mathbb{C}[x]/P$, we will often denote the variety X by $\operatorname{Spec} A$ rather than by $V(P)$.

Let $X = \operatorname{Spec} A$. An element α of A defines a (complex-valued) function on X that we denote by the same letter α. The definition of the function α is as follows: A point p of X corresponds to a homomorphism $A \xrightarrow{\overline{\pi}_p} \mathbb{C}$. By definition The *value* $\alpha(p)$ of the function α at p is $\overline{\pi}_p(\alpha)$:

defalphap (2.5.1)
$$\alpha(p) \overset{def}{=} \overline{\pi}_p(\alpha)$$

Thus the kernel of $\overline{\pi}_p$, which is $\overline{\mathfrak{m}}_p$, is the set of elements α of the coordinate algebra A at which the value of α is 0:

$$\overline{\mathfrak{m}}_p = \{\alpha \in A \mid \alpha(p) = 0\}$$

The functions defined in this way by the elements of A are called the *regular functions* on X. (See Proposition 2.7.2 below.)

When A is a polynomial algebra $\mathbb{C}[x_1, ..., x_n]$, the function defined by a polynomal $g(x)$ is simply the usual polynomial function, because π_p is defined by evaluating a polynomial at p: $\pi_p(g) = g(p)$ (2.3.1).

gpequalsalphap **2.5.2. Lemma.** *Let A be a quotient $\mathbb{C}[x]/P$ of the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, modulo a prime ideal P, so that $\operatorname{Spec} A$ becomes the closed subset $V(P)$ of $\mathbb{A}^n$. Then a point p of $\operatorname{Spec} A$ becomes a point of $\mathbb{A}^n$: $p = (a_1, ..., a_n)$. When an element α of A is represented by a polynomial $g(x)$, the value of α at p can be obtained by evaluating g: $\alpha(p) = g(p) = g(a_1, ..., a_n)$.*

proof. The point p of $\operatorname{Spec} A$ gives us a diagram (2.4.5), with $\pi = \pi_p$ and $\overline{\pi} = \overline{\pi}_p$, and where τ is the canonical map $\mathbb{C}[x] \to A$. Then $\alpha = \tau(g)$, and

redefinefn (2.5.3)
$$g(p) \overset{defn}{=} \pi_p(g) = \overline{\pi}_p\tau(g) = \overline{\pi}_p(\alpha) \overset{defn}{=} \alpha(p).$$
□

Thus the value $\alpha(p)$ at a point p of $\operatorname{Spec} A$ can be obtained by evaluating a suitable polynomial g. However, that polynomial won't be unique unless P is the zero ideal.

regfndetelt **2.5.4. Lemma.** *The regular functions determined by distinct elements α and β of A are distinct. In particular, the only element α of A that is zero at all points of $\operatorname{Spec} A$ is the zero element.*

proof. We replace α by $\alpha - \beta$. Then what is to be shown is that, if the function determined by an element α is the zero function, then α is the zero element.

We present A as $\mathbb{C}[x]/P$, $x = x_1, ..., x_n$, where P is a prime ideal. Then X is the locus of zeros of P in $\mathbb{A}^n$, and Corollary 2.4.9 **(i)** tells us that P is the ideal of all elements that are zero on X. Let $g(x)$ be a polynomial that represents α. If p is a point of X at which α is zero, then $g(p) = 0$ (see (2.4.5)). So if α is the zero function, then g is in P, and therefore $\alpha = 0$. □

Note. In modern terminology, the word "spectrum" is usually used to denote the set of prime ideals of a ring. This becomes important when one studies rings that aren't finite-type algebras. When working with finite-type domains, there are enough maximal ideals. The other prime ideals aren't needed, so we have eliminated them. primespec

(2.5.5) the Zariski topology on an affine variety ztoponvar

Let $X = \operatorname{Spec} A$ be an affine variety with coordinate algebra A. An ideal $\overline{J}$ of A defines a locus in X, a *closed subset*, that we denote by $V(\overline{J})$, using the same notation as for loci in affine space. The points of $V(\overline{J})$ are the points of X at which all elements of $\overline{J}$ vanish. This is analogous to (2.4.2):

$$V(\overline{J}) = \{p \in \operatorname{Spec} A \,|\, \overline{J} \subset \overline{\mathfrak{m}}_p\} \tag{2.5.6}$$

locusin-spec

2.5.7. Lemma. *Let A be a finite-type domain that is presented as $A = \mathbb{C}[x]/P$. An ideal $\overline{J}$ of A corresponds to an ideal J of $\mathbb{C}[x]$ that contains P. Let $V(J)$ denote the zero locus of J in $\mathbb{A}^n$. Wghenwe regard* $\operatorname{Spec} A$ *as a subvariety of $\mathbb{A}^n$, the loci $V(\overline{J})$ and $V(J)$ are equal.* $\square$ *zerolocus-inX*

2.5.8. Proposition. *Let $\overline{J}$ be an ideal of a finite-type domain A, and let $X = \operatorname{Spec} A$. The zero set $V(\overline{J})$ is empty if and only if $\overline{J}$ is the unit ideal of A. If X is empty, then A is the zero ring.* *empty*

proof. The only ideal that isn't contained in a maximal ideal is the unit ideal. $\square$

2.5.9. Note. We have put bars on the symbols $\overline{\mathfrak{m}}$, $\overline{\pi}$, and $\overline{J}$ here, in order to distinguish ideals of A from ideals of $\mathbb{C}[x]$ and homomorphisms $A \to \mathbb{C}$ from homomorphisms $\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$. In the future, we will put bars over the letters only when there is a danger of confusion. Most of the time, we will drop the bars, and write $\mathfrak{m}$, π, and J instead of $\overline{\mathfrak{m}}$, $\overline{\pi}$, and $\overline{J}$. $\square$ nobar

2.5.10. Lemma. *An ideal I of a noetherian ring R contains a power of its radical.* *radpower*

proof. Since R is noetherian, the ideal $\operatorname{rad} I$ is generated by a finite set of elements $\alpha = \{\alpha_1, ..., \alpha_k\}$, and for large r, α_i^r is in I. We can use the same large integer r for every i. A monomial $\beta = \alpha_1^{e_1} \cdots \alpha_k^{e_k}$ of sufficiently large degree n in α will be divisible α_i^r for at least one i, and therefore it will be in I. The monomials of degree n generate $(\operatorname{rad} I)^n$, so $(\operatorname{rad} I)^n \subset I$. $\square$

The properties of closed sets in affine space that are given in Lemmas 2.2.4 and 2.2.2 are true for closed subsets of an affine variety. In particular, $V(\overline{J}) = V(\operatorname{rad} \overline{J})$, and $V(\overline{IJ}) = V(\overline{I} \cap \overline{J}) = V(\overline{I}) \cup V(\overline{J})$.

2.5.11. Corollary. *Let I and J be ideals of a finite-type domain A, and let $X = \operatorname{Spec} A$. Then $V(I) \supset V(J)$ if and only if $\operatorname{rad} I \subset \operatorname{rad} J$.* *subsof-specA*

This follows from the case of a polynomial ring, which is Corollary 2.4.9 **(iii)**, and from Lemma 2.5.7. $\square$

2.5.12. Proposition. *Let I be an ideal of noetherian ring R. The radical* $\operatorname{rad} I$ *is the intersection of the prime ideals of R that contain I.* *inter-sprimes*

proof. Let x be an element of $\operatorname{rad} I$. Some power x^k is in I. If P is a prime ideal and $I \subset P$, then x^k is in P, and since P is a prime ideal, $x \in P$. Therefore $\operatorname{rad} I$ is contained in every such prime ideal. Conversely, let x be an element not in $\operatorname{rad} I$, an element such that no power is in I. We show that there is a prime ideal that contains I but doesn't contain x. Let $\mathcal{S}$ be the set of ideals that contian I, but don't contain a power of x. The ideal I is one such ideal, so $\mathcal{S}$ isn't empty. Since R is noetherian, $\mathcal{S}$ contains a maximal member P (**2.1.12**). We show that P is a prime ideal by showing that, if two ideals A and B are strictly larger than P, their product AB isn't contained in P. Since P is a maximal member of $\mathcal{S}$, A and B aren't in $\mathcal{S}$. They contain I and also contain powers of x, say $x^k \in A$ and $x^\ell \in B$. Then $x^{k+\ell}$ is in AB but not in P. Therefore $AB \not\subset P$. $\square$

The next proposition includes Proposition 2.2.22 as a special case.

2.5.13. Proposition. *Let A be a finite-type domain, and let $X = \operatorname{Spec} A$. The closed subset $V(P)$ defined by a radical ideal P is irreducible if and only if P is a prime ideal.* *proofirred-prime* *proof.* Let P be a radical ideal of A, and let $Y = V(P)$. Let C and D be closed subsets of X such that $Y = C \cup D$. Say $C = V(I)$, $D = V(J)$. We may suppose that I and J are radical ideals. Then the inclusion $C \subset Y$ implies that $I \supset P$ (2.5.11). Similarly, $J \supset P$. Because $Y = C \cup D$, we also have $Y = V(I \cap J) = V(IJ)$. So $IJ \subset P$ (2.5.11). If P is a prime ideal, then $I = P$ or $J = P$, and therefore $C = Y$ or $D = Y$. So Y is irreducible. Conversely, if P isn't a prime ideal, there are ideals I, J strictly larger than the radical ideal P, such that $IJ \subset P$ (2.1.3) **(iii')**. Then Y will be the union of the two proper closed subsets $V(I)$ and $V(J)$ (2.5.11). So Y isn't irreducible. □

thenilradical

(2.5.14) the nilradical

The *nilradical* of a ring is the set of its nilpotent elements. It is the radical of the zero ideal. The nilradical of a domain is the zero ideal. If a ring R is noetherian, its nilradical will be *nilpotent*: some power of it will be the zero ideal (Lemma 2.5.10).

The next corollary follows from Proposition 2.5.12.

intersectprimes

2.5.15. Corollary. *The nilradical of a noetherian ring R is the intersection of its prime ideals.* □

Note. The conclusion of this corollary is true whether or not the ring R is noetherian.

2.5.16. Corollary.

strongnullA

(i) *Let A be a finite-type algebra. An element that is in every maximal ideal of A is nilpotent.*

(ii) *Let A be a finite-type domain. The intersection of the maximal ideals of A is the zero ideal.*

proof. **(i)** Say that A is presented as $\mathbb{C}[x_1, ..., x_n]/I$. Let α be an element of A that is in every maximal ideal, and let $g(x)$ be a polynomial whose residue in A is α. Then α is in every maximal ideal of A if and only if $g = 0$ at all points of the variety $V(I)$ in $\mathbb{A}^n$. If so, the Strong Nullstellensatz asserts that some power g^N is in I. Then $\alpha^N = 0$. □

fndetermineselt

2.5.17. Corollary. *An element α of a finite-type domain A is determined by the function that it defines on $X = \operatorname{Spec} A$.*

proof. It is enough to show that an element α that defines the zero function is the zero element. Such an element is in every maximal ideal (2.5.8), so α is nilpotent, and since A is a domain, $\alpha = 0$. □

2.6 Localization

boldloc

Let s be a nonzero element of a domain A. The ring $A[s^{-1}]$, obtained by adjoining an inverse of s to A is called a *localization* of A. It is isomorphic to the quotient $A[z]/(sz-1)$ of the polynomial ring $A[z]$ in one variable, by the principal ideal generated by $sz - 1$. We will often denote this localization by A_s. If A is a finite-type domain, so is A_s. Then if X denotes the variety $\operatorname{Spec} A$, X_s will denote the variety $\operatorname{Spec} A_s$, and X_s will be called a localization of X too.

topologyonlocalization

2.6.1. Proposition. **(i)** *With terminology as above, points of the localization $X_s = \operatorname{Spec} A_s$ correspond bijectively to the open subset of X of points at which the function defined by s isn't zero.*

(ii) *When we identify a localization X_s with the corresponding subset of X, the Zariski topology on X_s is the induced topology from X. So X_s becomes an open subspace of X.*

proof. **(i)** Let p be a point of X, let $A \xrightarrow{\pi_p} \mathbb{C}$ be the corresponding homomorphism. If s isn't zero at p, say $s(p) = c \neq 0$, then π_p extends uniquely to a homomorphism $A_s \to \mathbb{C}$ that sends s^{-1} to c^{-1}. This gives us a unique point of X_s whose image in X is p. If $c = 0$, then π_p doesn't extend to A_s.

(ii) Let C be a closed subset of X, say the zero set of a set of elements $a_1, ..., a_k$ of A. Then $C \cap X_s$ is the zero set in X_s of those same elements, so it is closed in X_s. Conversely, let D be a closed subset of X_s, say the zero set in X_s of some elements $\beta_1, ..., \beta_k$, where $\beta_i = b_i s^{-n_i}$ with b_i in A. Since s^{-1} doesn't vanish on X_s, the elements b_i and β_i have the same zeros in X_s. If we let C be the zero set of $b_1, ..., b_k$ in X, we will have $C \cap X_s = D$. □

We usually identify a localization X_s with the open subset of X of points at which the value of s isn't zero. Then the effect of adjoining the inverse is to throw out the points of X at which s vanishes. For example, the spectrum of the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ becomes the complement of the origin in the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$.

Most varieties X will contain open sets that aren't localizations. For example, the complement X' of the origin in the affine plane $X = \operatorname{Spec} \mathbb{C}[x_1, x_2]$ isn't a localization of X. Every polynomial that vanishes at the origin vanishes on an affine curve, which has points distinct from the origin. So the inverse of such a polynomial doesn't define a function on X'. This is rather obvious, but in other situations, it is often hard to tell whether or not a given open set is a localization.

Localizations are important for two reasons:

2.6.2. basistop

- The relation between an algebra A and a localization A_s is easy to understand, and
- The localizations X_s of an affine variety X form a basis for the Zariski topology on X.

A *basis* for the topology on a topological space X is a family $\mathcal{B}$ of open sets with the property that every open subset of X is a union of open sets that are members of $\mathcal{B}$.

To verify that the localizations X_s form a basis for the topology on an affine variety X, we must show that every open subset U of $X = \operatorname{Spec} A$ can be covered by sets of the form X_s. Let U be an open subset and let C be its closed complement $X - U$ in X. Since C is closed, it is the set of common zeros of some nonzero elements $s_1, ..., s_k$ of A. The zero set $V(s_i)$ of s_i is the complement of the locus X_{s_i} in X. Then C is the intersection of the zero sets $\bigcap V(s_i)$, and U is the union of the sets X_{s_i}. □

2.6.3. Lemma. *Let X be an affine variety.* *locloc*
(i) *If X_s and X_t are localizations of X, and if $X_s \subset X_t$, then X_s is a localization of X_t.*
(ii) *If u is an element of a localization X_s of X, then $(X_s)_u$ is a localization of X.*

proof. **(i)** Let $X = \operatorname{Spec} A$. Since $A \subset A_t$, $A_s \subset (A_t)_s$. If $X_s \subset X_t$, then $A_t \subset A_s$, and so $(A_t)_s \subset A_s$.

(ii) If $u \in A_s$, say $u = ts^{-k}$ with $t \in A$, then $(A_s)_u = (A_s)_t = A_{st}$. □

(2.6.4) extension and contraction of ideals extcontr

Let $A \subset B$ be the inclusion of a ring A as a subring of a ring B. The *extension* of an ideal I of A is the ideal IB of B generated by I. Its elements are finite sums $\sum_i z_i b_i$ with z_i in I and b_i in B. The *contraction* of an ideal J of B is the intersection $J \cap A$. It is an ideal of A.

If A_s is a localization of A and I is an ideal of A, the elements of the extended ideal IA_s are fractions of the form zs^{-k}, with z in I. We denote this extended ideal by I_s.

2.6.5. Lemma. *xtcontr-prop*
(i) *Let s bea nonzero element of a domain A, let J be an ideal of the localization A_s and let $I = J \cap A$. Then $J = I_s$. Every ideal of A_s is the extension of an ideal of A.*
(ii) *Let P be a prime ideal of A. If s isn't in P, the extended ideal P_s is a prime ideal of A_s. If s is in P, the extended ideal P_s is the unit ideal.* □

(2.6.6) localizing a module locmod

Let A be a domain, let M be an A-module, and let's regard M as a right module here. A *torsion element* of M is an element that is annihilated by some nonzero element s of A: $ms = 0$. A nonzero element m such that $ms = 0$ is an s-*torsion* element.

The set of all torsion elements of M is its *torsion submodule*, and a module whose torsion submodule is zero is *torsion-free*.

Let s be a nonzero element of a domain A. The *localization* M_s of an A-module M is defined in the natural way, as the A_s-module whose elements are equivalence classes of fractions $m/s^r = ms^{-r}$, with m in M and $r \geq 0$. An alternate notation for the localization M_s is $M[s^{-1}]$.

The only complication comes from the fact that M may contain s-torsion elements. If $ms = 0$, then m must map to zero in M_s, because in M_s, we will have $mss^{-1} = m$. To define M_s, it suffices to modify the equivalence relation. Two fractions $m_1 s^{-r_1}$ and $m_2 s^{-r_2}$ are defined to be equivalent if $m_1 s^{r_2+n} = m_2 s^{r_1+n}$ when n is sufficiently large. This takes care of torsion, and M_s becomes an A_s-module. There is a homomorphism $M \to M_s$ that sends an element m to the fraction $m/1$.

This is also how one localizes a ring, that isn't a domain.

multsys **(2.6.7) multiplicative systems**

To work with the inverses of finitely many nonzero elements, one may simply adjoin the inverse of their product. For working with an infinite set of inverses, the concept of a multiplicative system is useful. A *multiplicative system* S in a domain A is a subset of A that consists of nonzero elements, is closed under multiplication, and contains 1. If S is a multiplicative system, the ring of S*-fractions* AS^{-1} is the ring obtained by adjoining inverses of all elements of S. Its elements are equivalence classes of fractions as^{-1} with a in A and s in S, the equivalence relation and the laws of composition being the usual ones for fractions. The ring AS^{-1} will be called a *localization* too. To avoid confusion, the ring obtained by inverting a single nonzero element s may be called a *simple localization*.

inverseexamples **2.6.8. Examples.** **(i)** The set consisting of the powers of a nonzero element s of a domain A is a multiplicative system. Its ring of fractions is the simple localization $A_s = A[s^{-1}]$.

(ii) The set S of all nonzero elements of a domain A is a multiplicative system. Its ring of fractions is the field of fractions of A.

(iii) An ideal P of a domain A is a prime ideal if and only if its complement, the set of elements of A **not** in P, is a multiplicative system. □

extendidealtoloc **2.6.9. Proposition.** *Let S be a multiplicative system in a domain A, and let A' be the localization AS^{-1}.*

(i) *Let I be an ideal of A. The extended ideal IA' (**2.6.4**) is the set IS^{-1} whose elements are classes of fractions xs^{-1}, with x in I and s in S. The extended ideal is the unit ideal if and only if I contains an element of S.*

(ii) *Let J be an ideal of the localization A' and let I denote its contraction $J \cap A$. The extended ideal IA' is equal to J: $J = (J \cap A)A'$.*

(iii) *If Q is a prime ideal of A and if $Q \cap S$ is empty, the extended ideal $Q' = QA'$ is a prime ideal of A', and the contraction $Q' \cap A$ is equal to Q. If $Q \cap S$ isn't empty, the extended ideal is the unit ideal. Thus prime ideals of AS^{-1} correspond bijectively to prime ideals of A that don't meet S.* □

locfintype **2.6.10. Corollary.** *Every localization AS^{-1} of a noetherian domain A is noetherian.* □

locmodtwo **2.6.11.** Let S be a multiplicative system in a domain A. The *localization* MS^{-1} of an A-module M is defined in a way analogous to the one used for simple localizations: MS^{-1} is the AS^{-1}-module whose elements are equivalence classes of fractions ms^{-1} with m in M and s in S. To take care of torsion, two fractions $m_1 s_1^{-1}$ and $m_2 s_2^{-1}$ are defined to be equivalent if there is a nonzero element t in S such that $m_1 s_2 t = m_2 s_1 t$. Then $ms_1^{-1} = 0$ if and only if $mt = 0$ for some nonzero t in S. As with simple localizations, there will be a homomorphism $M \to MS^{-1}$ that sends an element m to the fraction $m/1$.

localexact **2.6.12. Proposition.** *Let S be a multiplicative system in a domain A.*

(i) *Localization is an exact functor: A homomorphism $M \xrightarrow{\varphi} N$ of A-modules induces a homomorphism $MS^{-1} \xrightarrow{\varphi'} NS^{-1}$ of AS^{-1}-modules, and if $M \xrightarrow{\varphi} N \xrightarrow{\psi} P$ is an exact sequence of A-modules, the localized sequence $MS^{-1} \xrightarrow{\varphi'} NS^{-1} \xrightarrow{\psi'} PS^{-1}$ is exact.*

(ii) *Let M be an A-module and let N be an AS^{-1}-module. When N is made into an A-module by restriction of scalars, homomorphisms of A-modules $M \to N$ correspond bijectively to homomorphisms of AS^{-1}-modules $MS^{-1} \to N$.*
(iii) *If multiplication by s is an injective map $M \to M$ for every s in S, then $M \subset MS^{-1}$. If multiplication by every s is a bijective map $M \to M$, then $M \approx MS^{-1}$.* $\square$

(2.6.13) **a general principle** import-princ

An important, elementary principle for working with fractions is that any finite sequence of computations in a localization AS^{-1} will involve only finitely many denominators, and can therefore be done in a simple localization A_s, where s is a common denominator for the fractions that occur.

2.7 Morphisms of Affine Varieties

morphism

Morphisms are the allowed maps between varieties. Morphisms between affine varieties will be defined in this section. They correspond to algebra homomorphisms in the opposite direction between their coordinate algebras. Morphisms of projective varieties will be defined in the next chapter.

(2.7.1) **regular functions** regfnone

The *function field* K of an affine variety $X = \operatorname{Spec} A$ is the field of fractions of A. A *rational function* on X is a nonzero element of the function field. A rational function f is *regular* at a point p of X if it can be written as a fraction $f = a/s$ with $s(p) \neq 0$, and f is regular on a subset U of X if it is regular at every point of U.

In (2.5.1), we have seen that an element a of the coordinate algebra A defines a function on X. The value $a(p)$ of the function a at a point p is $\pi_p(a)$, where π_p is the homomorphism $A \to \mathbb{C}$ that corresponds to p. The value of a rational function $f = a/s$ will be an element of A_s, and it defines a function on the open subset X_s of X: $f(p) = a(p)/s(p)$.

2.7.2. Proposition. *The regular functions on an affine variety $X = \operatorname{Spec} A$ are the elements of the coordinate algebra A.* *deloca*

proof. Let f be a rational function that is regular on X. Then for every point p of X, there is a localization $X_s = \operatorname{Spec} A_s$ that contains p, such that f is an element of A_s. Because X is quasicompact, a finite set of these localizations, say $X_{s_1}, \dots, X_{s_k}$, will cover X. Then $s_1, ..., s_k$ have no common zeros on X, so they generate the unit ideal of A (2.5.8). Since f is in A_{s_i}, we can write $f = s_i^{-n} b_i$, or $s_i^n f = b_i$, with b_i in A, and we can use the same exponent n for each i. Since the elements s_i generate the unit ideal of A, so do the powers s_i^n. Writing $\sum s_i^n c_i = 1$, with c_i in A, $f = \sum s_i^n c_i f = \sum c_i b_i$ is an element of A. $\square$

Note. This reasoning, in which one writes the identity element as a combination, will occur several times.

(2.7.3) **morphisms** morphtwo

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties, and let $A \xrightarrow{\varphi} B$ be an algebra homomorphism. A point q of Y corresponds to an algebra homomorphism $B \xrightarrow{\pi_q} \mathbb{C}$. When we compose π_q with φ, we obtain a homomorphism $A \xrightarrow{\pi_q \varphi} \mathbb{C}$. By definition, points p of $\operatorname{Spec} A$ correspond to homomorphisms $A \xrightarrow{\pi_p} \mathbb{C}$. So there is a unique point p of $X = \operatorname{Spec} A$ such that $\pi_p = \pi_q \varphi$.

2.7.4. Definition. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. A *morphism* $Y \xrightarrow{u} X$ is a map that can be defined, as above, by an algebra homomorphism $A \xrightarrow{\varphi} B$: If q is a point of Y, then uq is the point p of X such that $\pi_p = \pi_q \varphi$: defmor-phaff

piqphi (2.7.5)
$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\pi_p}\downarrow & & \downarrow{\scriptstyle\pi_q} \\ \mathbb{C} & = & \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} X & \xleftarrow{u} & Y \\ \uparrow & & \uparrow \\ p & \longleftarrow & q \end{array}$$

So $p = uq$ means that $\pi_q\varphi = \pi_p$, and if α is an element of A, then

$$[\varphi\alpha](q) = \alpha(uq)$$

because $[\varphi\alpha](q) = \pi_q(\varphi\alpha) = \pi_p(\alpha) = \alpha(p)$.

Or, if we denote $\varphi(\alpha)$ by β and and uq by p, then $\alpha(p) = \beta(q)$.

bisphia (2.7.6)
$$\beta(q) = \pi_q(\beta) = \pi_q(\varphi\alpha) = \pi_p(\alpha) = \alpha(p)$$

A morphism $Y \xrightarrow{u} X$ is an *isomorphism* if and only if there is an inverse morphism. This will be true if and only if $A \xrightarrow{\varphi} B$ is an isomorphism of algebras. □

Thus the homomorphism φ is determined by the morphism u, and vice-versa. But just as a map $A \to B$ needn't be a homomorphism, a map $Y \to X$ needn't be a morphism.

The description of a morphism can be confusing because the direction of the arrow is reversed. It will become clearer as we expand the discussion, though the reversal of arrows will remain a source of confusion.

Morphisms to affine space.

A morphism $Y \xrightarrow{u} \mathbb{A}^1$ from a variety $Y = \operatorname{Spec} B$ to the affine line $\operatorname{Spec}\mathbb{C}[x]$ is defined by an algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} B$, and such a homomorphism substitutes an element β of B for x. The corresponding morphism u sends a point q of Y to the point $x = \beta(q)$ of the x-line.

For example, let Y be the space of 2×2 matrices, $Y = \operatorname{Spec}\mathbb{C}[y_{ij}]$, where y_{ij}, $1 \le i, j \le 2$, are the matrix entries. The determinant defines a morphism $Y \to \mathbb{A}^1$ that sends a matrix to its determinant. The corresponding algebra homomorphism $\mathbb{C}[x] \xrightarrow{\varphi} \mathbb{C}[y_{ij}]$ substitutes $y_{11}y_{22} - y_{12}y_{21}$ for x. It sends a polynomial $f(x)$ to $f(y_{11}y_{22} - y_{12}y_{21})$.

In the other direction, a morphism from the affine line $\mathbb{A}^1$ to a variety X may be called a (complex) polynomial path in X. For example, when Y is the space of matrices, a morphism $\mathbb{A}^1 \to Y$ corresponds to a homomorphism $\mathbb{C}[y_{ij}] \to \mathbb{C}[x]$, which substitutes a polynomial in x for each variable y_{ij}.

A morphism from an affine variety $Y = \operatorname{Spec} B$ to affine space $\mathbb{A}^n$ is defined by a homomorphism $\mathbb{C}[x_1, ..., x_n] \xrightarrow{\Phi} B$. Such a homomorphism substitutes elements β_i of B for x_i: $\Phi(f(x)) = f(\beta)$. (We use an upper case Φ here, keeping φ in reserve.) The corresponding morphism $Y \xrightarrow{u} \mathbb{A}^n$ sends a point q of Y to the point $(\beta_1(q), ..., \beta_n(q))$ of $\mathbb{A}^n$.

Morphisms to affine varieties.

Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ be affine varieties. Say that we have chosen a presentation $A = \mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$ of A, so that X becomes the closed subvariety $V(f)$ of affine space $\mathbb{A}^m$. There is no need to choose a presentation of B. A natural way to define a morphism from a variety Y to X is as a morphism $Y \xrightarrow{u} \mathbb{A}^m$ to affine space, whose image is contained in X. We check that this agrees with Definition 2.7.4.

As above, a morphism $Y \xrightarrow{u} \mathbb{A}^m$ corresponds to a homomorphism $\mathbb{C}[x_1, ..., x_m] \xrightarrow{\Phi} B$. It will be determined by the set $(\beta_1, ..., \beta_m)$ of elements of B with the rule that $\Phi(x_i) = \beta_i$. Since X is the locus of zeros of the polynomials f, the image of Y will be contained in X if and only if $f_i(\beta_1(q), ..., \beta_m(q)) = 0$ for every point q of Y and every i, i.e., if and only if $f_i(\beta)$ is in every maximal ideal of B, in which case $f_i(\beta) = 0$ (2.5.16)**(i)**. A better way to say this is: The image of Y is contained in X if and only if $\beta = (\beta_1, ..., \beta_m)$ solves the equations $f(x) = 0$. And, if β is a solution, the homomorphism Φ defines a homomorphism $A \xrightarrow{\varphi} B$.

$$\begin{array}{ccc} \mathbb{C}[x] & \xrightarrow{\Phi} & B \\ \downarrow & & \| \\ A & \xrightarrow{\varphi} & B \end{array}$$

There is an elementary, but important, principle at work here:

- *Homomorphisms from the algebra $A = \mathbb{C}[x]/(f)$ to an algebra B correspond to solutions of the equations $f = 0$ in B.*

2.7.7. Example. Let $B = \mathbb{C}[x]$ be the polynomial ring in one variable, and let A be the coordinate algebra $\mathbb{C}[u, v]/(v^2 - u^3)$ of a cusp curve. A homomorphism $A \to B$ is determined by a solution of the equation $v^2 = u^3$ in $\mathbb{C}[x]$. The solutions have the form $u = g^3$, $v = g^2$ with g in $\mathbb{C}[x]$. For example, $u = x^3$ and $v = x^2$ is a solution. $\square$ maptocusp

2.7.8. Corollary. *Let $X = \operatorname{Spec} A$ and let $Y = \operatorname{Spec} B$ be affine varieties. Suppose that A is presented as the quotient $\mathbb{C}[x_1, ..., x_m]/(f_1, ..., f_k)$ of a polynomial ring. There are bijective correspondences between the following sets:* *morphandhomom*

- *algebra homomorphisms $A \to B$, or morphisms $Y \to X$,*
- *morphisms $Y \to \mathbb{A}^m$ whose images are contained in X,*
- *solutions of the equations $f_i(x) = 0$ in B, $i = 1, ..., k$.* $\square$

The second and third sets refer to an embedding of the variety X into affine space, but the first one does not. It shows that a morphism depends only on the varieties X and Y, not on their embeddings.

We note a few more facts about morphisms here. Their geometry will be analyzed further in Chapters 4 and 5.

2.7.9. Proposition. *Let $X \xleftarrow{u} Y$ be the morphism of affine varieties that corresponds to a homomorphism of coordinate algebras $A \xrightarrow{\varphi} B$.* *phisurj*

(i) *Let $Y \xleftarrow{v} Z$ be another morphism, that corresponds to a homomorphism $B \xrightarrow{\psi} R$ of finite-type domains. The composed map $Z \xrightarrow{uv} X$ is the morphism that corresponds to the composed homomorphism $A \xrightarrow{\psi\varphi} R$.*

(ii) *Suppose that $B = A/P$, where P is a prime ideal of A, and that φ is the canonical homomorphism $A \to A/P$. Let $Y = V(P)$ be the closed subvariety of X zeros of P. Then u is the inclusion of Y into X.*

(iii) *The map φ is surjective if and only if u maps Y isomorphically to a closed subvariety of X.* $\square$

It can be useful to rephrase the definition of the morphism $Y \xrightarrow{u} X$ that corresponds to a homomorphism $A \xrightarrow{\varphi} B$ in terms of maximal ideals. Let $\mathfrak{m}_q$ be the maximal ideal of B at a point q of Y. The inverse image of $\mathfrak{m}_q$ in A is the kernel of the composed homomorphism $A \xrightarrow{\varphi} B \xrightarrow{\pi_q} \mathbb{C}$, so it is a maximal ideal of A: $\varphi^{-1}\mathfrak{m}_q = \mathfrak{m}_p$, for some p in X. That point p is the image of q.

We describe the fibre of the morphism $Y \xrightarrow{u} X$ defined by a homomorphism $A \xrightarrow{\varphi} B$. let $\mathfrak{m}_p$ be the maximal ideal at a point p of X, and let J be the extended ideal $\mathfrak{m}_p B$ — the ideal generated by the image of $\mathfrak{m}_p$ in B (**2.6.4**). Its elements are finite sums $\sum \varphi(z_i)b_i$ with z_i in $\mathfrak{m}_p$ and b_i in B. If q is is a point of Y, then $uq = p$ if and only if $\mathfrak{m}_p = \varphi^{-1}\mathfrak{m}_q$. This will be true if and only if te extended ideal J is contained in $\mathfrak{m}_q$.

2.7.10. Corollary. *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $Y \xrightarrow{u} X$ be the morphism corresponding to a homomorphism $A \xrightarrow{\varphi} B$. Let $\mathfrak{m}_p$ be the maximal ideal at a point p of X, and let $J = \mathfrak{m}_p B$ be the extended ideal. The fibre of Y over p is the set of points q such that $J \subset \mathfrak{m}_q$ — the set $V(J)$. The fibre is empty if and only if J is the unit ideal of B.* $\square$ *xtmax*

2.7.11. Example. *(blowing up the plane)* cuspnormx

Let W and X be planes with coordinates (x, w) and (x, y), respectively. The blowup map $W \xrightarrow{\pi} X$ was described before (**1.8.5**). It is defined by the substitution $\pi(x, w) = (x, xw)$, which corresponds to the algebra homomorphism $\mathbb{C}[x,y] \xrightarrow{\varphi} \mathbb{C}[x,w]$ that is defined by $\varphi(x) = x,\ \varphi(y) = xw$. To be specific, the image of the point $q : (x,w) = (a,c)$ of W is the point $p : (x,y) = (a,ac)$ of X.

As was explained in (**1.8.5**), the morphism π is bijective at points (x,y) at which $x \neq 0$. The fibre of Z over a point of Y of the form $(0,y)$ is empty unless $y = 0$, and the fibre over the origin $(0,0)$ in Y is the w-axis $\{(0,z)\}$ in the plane W. $\square$

mcont **2.7.12. Proposition.** *A morphism $Y \xrightarrow{u} X$ of affine varieties is a continuous map in the Zariski topology and also in the classical topology.*

proof. the Zariski topology: Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, so that u corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. A closed subset C of X will be the zero locus of a set $\alpha = \{\alpha_1, ..., \alpha_k\}$ of elements of A. Let $\beta_i = \varphi\alpha_i$. The inverse image $u^{-1}C$ is the set of points q such that $p = uq$ is in C, i.e., such that $\alpha_i(uq) = \beta_i(q) = 0$ (2.7.6). So $u^{-1}C$ is the zero locus in Y of the elements $\beta_i = \varphi(\alpha_i)$. It is a closed set.

the classical topology: We use the fact that polynomials are continuous functions. First, a morphism of affine spaces $\mathbb{A}^n_y \xrightarrow{U} \mathbb{A}^m_x$ is defined by an algebra homomorphism $\mathbb{C}[x_1, ..., x_m] \xrightarrow{\Phi} \mathbb{C}[y_1, ..., y_n]$, and this homomorphism is determined by the polynomials $h_1(y), ..., h_m(y)$ that are the images of the variables $x_1, ..., x_m$. The morphism U sends the point $(y_1, ..., y_n)$ of $\mathbb{A}^n$ to the point $(h_1(y), ..., h_m(y))$ of $\mathbb{A}^m$. It is continuous.

Next, a morphism $Y \xrightarrow{u} X$ is defined by a homomorphism $A \xrightarrow{\varphi} B$. We choose presentations $A = \mathbb{C}[x]/I$ and $B = \mathbb{C}[y]/J$, and we form a diagram of homomorphisms and the associated diagram of morphisms:

$$\begin{tikzcd} \mathbb{C}[x] \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & \mathbb{C}[y] \arrow[d, "\beta"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^n_x & \mathbb{A}^m_y \arrow[l, "U"'] \\ X \arrow[u] & Y \arrow[l, "u"'] \arrow[u] \end{tikzcd}$$

Here α and β are the canonical maps of a ring to a quotient ring. The map α sends $x_1, ..., x_n$ to $\alpha_1, ..., \alpha_n$. Then Φ is obtained by choosing elements h_i of $\mathbb{C}[y]$, such that $\beta(h_i) = \varphi(\alpha_i)$.

In the diagram of morphisms on the right, U is continuous, and the vertical arrows are the embeddings of X and Y into their affine spaces. Since the topologies on X and Y are induced from their embeddings, u is continuous. $\square$

As we see here, every morphism of affine varieties can be obtained by restriction from a morphism of affine spaces. However, in the diagram above, the morphism U isn't unique. It depends on the choice of the polynomials h_i as well as on the presentations of A and B.

2.8 Finite Group Actions

grpi

Let G be a finite group of automorphisms of a finite-type domain B. An *invariant* element of B is an element that is sent to itself by every element σ of G. For example, the product and the sum

invarelts (2.8.1)
$$\prod_{\sigma \in G} \sigma b \quad , \quad \sum_{\sigma \in G} \sigma b$$

are invariant elements. The invariant elements form a subalgebra of B that is often denoted by B^G. Theorem 2.8.5 below asserts that B^G is a finite-type domain, and that points of the variety $\operatorname{Spec} B^G$ correspond bijectively to G-orbits in the variety $\operatorname{Spec} B$.

actonpla-neex **2.8.2. Examples.**

(i) The symmetric group $G = S_n$ operates on the polynomial algebra $R = \mathbb{C}[x_1, ..., x_n]$ by permuting the variables, and the Symmetric Functions Theorem asserts that the elementary symmetric functions

$$s_1(x) = \sum_i x_i\ , \quad s_2(x) = \sum_{i<j} x_i x_j\ ,\ \ldots,\ s_n(x) = x_1 x_2 \cdots x_n$$

generate the algebra R^G of invariant polynomials: $R^G = \mathbb{C}[s_1, ..., s_n]$. Moreover, $s_1, ..., s_n$ are algebraically independent, so R^G is a polynomial algebra. The inclusion of R^G into R gives us a morphism from affine

x-space $\mathbb{A}^n_x$ to affine s-space $\mathbb{A}^n_s = \operatorname{Spec} R^G$. If $c = (c_1, ..., c_n)$ is a point of $\mathbb{A}^n_s$, the points $a = (a_1, ..., a_n)$ of $\mathbb{A}^n_x$ that map to c are those such that $s_i(a) = c_i$. The components a_i are the roots of the polynomial $x^n - c_1x^{n-1} + \cdots \pm c_n$. Since the roots form a G-orbit, the set of G-orbits in $\mathbb{A}^n_x$ maps bijectively to $\mathbb{A}^n_s$.

(ii) Let $\zeta = e^{2\pi i/n}$, and let σ be the automorphism of the polynomial ring $B = \mathbb{C}[y_1, y_2]$ that is defined by $\sigma y_1 = \zeta y_1$ and $\sigma y_2 = \zeta^{-1} y_2$. Let G be the cyclic group of order n generated by σ, and let A denote the algebra B^G of invariant elements. A monomial $m = y_1^i y_2^j$ is invariant if and only if n divides $i-j$, and an invariant polynomial is a linear combination of invariant monomials. You will be able to show that the three monomials

(2.8.3) $$u_1 = y_1^n\ ,\ u_2 = y_2^n\ ,\ \text{and } w = y_1y_2$$ zetainvar

generate the algebra A of invariants. Let's use the same symbols u_1, u_2, w to denote variables in a polynomial ring $\mathbb{C}[u_1, u_2, w]$. Let J be the kernel of the canonical homomorphism $\mathbb{C}[u_1, u_2, w] \xrightarrow{\tau} A$ that sends u_1, u_2, and w to y_1^n, y_2^n, and y_1y_2, respectively.

2.8.4. Lemma. *With notation as above, the kernel J of τ is a principal ideal of $\mathbb{C}[u_1, u_2, w]$. It is generated by the polynomial $f = w^n - u_1u_2$. Thus $A \approx \mathbb{C}[u_1, u_2, w]/(w^n = u_1u_2)$.* *slaction*

proof. First, f is obviously in J. Let $g(u_1, u_2, w)$ be any element of J. So $g(y_1^n, y_2^n, y_1y_2) = 0$. We divide g by f, considered as a monic polynomial in w, say $g = fq + r$, where the remainder $r(u_1, u_2, w)$ has degree $< n$ in w. The remainder will be in J too: $r(y_1^n, y_2^n, y_1y_2) = 0$. We write r as a polynomial in w: $r = r_0(u_1, u_2) + r_1(u_1, u_2)w + \cdots + r_{n-1}(u_1, u_2)w^{n-1}$. When we substitute y_1^n, y_2^n, y_1y_2, the term $r_i(u_1, u_2)w^i$ becomes $r_i(y_1^n, y_2^n)(y_1y_2)^i$. The degree in y_1 of every monomial that appears there will be congruent to i modulo n, and the same is true for the degree of y_2. Since $r(y_1^n, y_2^n, y_1y_2) = 0$, and since the indices i are distinct, $r_i(y_1^n, y_2^n)$ will be zero for every i. This implies that $r_i(u_1, u_2) = 0$ for every i. So $r = 0$, which means that f divides g. □

We go back to the operation of the cyclic group G on $B = \mathbb{C}[y_1, y_2]$ and the algebra of invariants A. Let Y denote the affine plane $\operatorname{Spec} B$, and let $X = \operatorname{Spec} A$. The group G operates on Y, and except for the origin, which is a fixed point, the orbit of a point (y_1, y_2) consists of the n points $(\zeta^i y_1, \zeta^{-i} y_2)$, $i = 0, \dots, n-1$. To show that G-orbits in Y correspond bijectively to points of X, we fix complex numbers u_1, u_2, w with $w^n = u_1u_2$, and we look for solutions of the equations (2.8.3). When $u_1 \neq 0$, the equation $u_1 = y_1^n$ has n solutions for y_1, and given a soluion, y_2 is determined by the equation $y_1y_2 = w$. So the fibre has order n. Similarly, there are n points in the fibre if $u_2 \neq 0$. If $u_1 = u_2 = 0$, then $y_1 = y_2 = w = 0$. In all cases, the fibres are the G-orbits. □

2.8.5. Theorem. *Let G be a finite group of automorphisms of a finite-type domain B, and let A denote the algebra B^G of invariant elements. Let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.* *groupoper-one*

(i) *A is a finite-type domain and B is a finite A-module.*

(ii) *G operates by automorphisms on Y.*

(iii) *The morphism $Y \to X$ defined by the inclusion $A \subset B$ is surjective, and its fibres are the G-orbits of points of Y.*

When a group G operates on a set Y, one often denotes the set of G-orbits of Y by Y/G, which is read as 'Y mod G'. With that notation, part **(iii)** of the theorem asserts that there is a bijective map

$$Y/G \to X$$

proof of 2.8.5 **(i)**: *The invariant algebra $A = B^G$ is a finite-type algebra, and B is a finite A-module.*

This is an interesting indirect proof. To show that A is a finite-type algebra, one constructs a finite-type subalgebra R of A such that B is a finite R-module.

Let $\{z_1, \dots, z_k\}$ be the G-orbit of an element z_1 of B. The orbit is the set of roots of the polynomial

$$f(t) \;=\; (t - z_1) \cdots (t - z_k) \;=\; t^k - s_1 t^{k-1} + \cdots \pm s_k$$ spoly

whose coefficients s_i are the elementary symmetric functions in $\{z_1, ..., z_k\}$. Let R_1 denote the algebra generated by those symmetric functions. Because the symmetric functions are invariant, $R_1 \subset A$. Using the equation $f(z_1) = 0$, we can write any power of z_1 as a polynomial in z_1 of degree less than k, with coefficients in R_1.

We choose a finite set of generators $\{y_1, \dots, y_r\}$ for the algebra B. If the order of the orbit of y_j is k_j, then y_j will be the root of a monic polynomial f_j of degree k_j with coefficients in A. Let R denote the finite-type algebra generated by all of the coefficients of all of the polynomials $f_1, ..., f_r$. We can write any power of y_j as a polynomial in y_j with coefficients in R, and of degree less than k_j, for every $j = 1, ..., r$. Using such expressions, we can write every monomial in $y_1, ..., y_r$ as a polynomial with coefficients in R, whose degree in the variable y_j is less than k_j. Since $y_1, ..., y_r$ generate B, we can write every element of B as such a polynomial. Then the finite set of monomials $y_1^{e_1} \cdots y_r^{e_r}$ with $e_j < k_j$ spans B as an R-module. Therefore B is a finite R-module.

The algebra A of invariants is a subalgebra of B that contains R. Since R is a finite-type algebra, it is noetherian. When regarded as an R-module, A is a submodule of the finite R-module B. Therefore A is also a finite R-module. When we put a finite set of algebra generators for R together with a finite set of R-module generators for A, we obtain a finite set of algebra generators for A, so A is a finite-type algebra. And, since B is a finite R-module, it is also a finite module over the larger ring A.

proof of 2.8.5 **(ii)**: *The group G operates on Y.*

A group element σ is a homomorphism $B \xrightarrow{\sigma} B$. It defines a morphism $Y \xleftarrow{u_\sigma} Y$, as in Definition 2.7.4. Since σ is an invertible homomorphism, i.e., an automorphism of B, u_σ is an automorphism of Y. Thus G operates on Y. However, there is a point that should be mentioned.

Let's write the operation of G on B on the left as usual, so that a group element σ maps an element β of B to σb. Then if σ and τ are two group elements, the product $\sigma\tau$ acts as *first do τ, then σ*: $(\sigma\tau)\beta = \sigma(\tau\beta)$.

BBB (2.8.6)
$$B \xrightarrow{\tau} B \xrightarrow{\sigma} B$$

We substitute $u = u_\sigma$ into Definition 2.7.4: If q is a point of Y, the morphism $Y \xleftarrow{u_\sigma} Y$ sends q to the point p such that $\pi_p = \pi_q\sigma$. It seems permissible to drop the symbol u, and to write the morphism simply as $Y \xleftarrow{\sigma} Y$. But since arrows are reversed when going from homomorphisms of algebras to morphisms of their spectra (2.7.5), the maps displayed in (2.8.6) above, give us morphisms

YYY (2.8.7)
$$Y \xleftarrow{\tau} Y \xleftarrow{\sigma} Y$$

On $Y = \operatorname{Spec} B$, the product $\sigma\tau$ acts as *first do σ, then τ.*

We can get around this problem by putting the symbol σ on the right when it operates on Y, so that σ sends a point q to $q\sigma$. Then if q is a point of Y, we will have $q(\sigma\tau) = (q\sigma)\tau$, as required of the operation.

- *If G operates on the left on B, then it operates on the right on* $\operatorname{Spec} B$.

This is important only when one wants to compose morphisms. In Definition 2.7.4, we followed custom and wrote the morphism u that corresponds to an algebra homomorphism φ on the left. We will continue to write morphisms on the left when possible, but not here.

Let β be an element of B and let q be a point of Y. The value of the function $\sigma\beta$ at a point q is the same as the value of β at the point $q\sigma$ (2.7.6):

bsigmay (2.8.8)
$$[\sigma\beta](q) = \beta(q\sigma)$$
$\square$

proof of 2.8.5 **(iii)**: *The fibres of the morphism $Y \to X$ are the G-orbits in Y.*

We go back to the subalgebra $A = B^G$. For σ in G, we have a diagram of algebra homomorphisms and the corresponding diagram of morphisms

actonB (2.8.9)
$$\begin{array}{ccc} B & \xrightarrow{\sigma} & B \\ \uparrow & & \uparrow \\ A & = & A \end{array} \qquad\qquad \begin{array}{ccc} Y & \xleftarrow{\sigma} & Y \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

The diagram of morphisms shows that all points of Y that are in a G-orbit have the same image in X, and therefore that the set of G-orbits in Y, which we may denote by Y/G, maps to X. We show that the map $Y/G \to X$ is bijective.

2.8.10. Lemma. **(i)** *Let $p_1, \dots, p_k$ be distinct points of affine space $\mathbb{A}^n$, and let $c_1, \dots, c_k$ be complex numbers. There is a polynomial $f(x_1, \dots, x_n)$ such that $f(p_i) = c_i$ for $i = 1, \dots, n$.* arbvals

(ii) *Let B be a finite-type algebra, let $q_1, \dots, q_k$ be points of* $\operatorname{Spec} B$*, and let $c_1, \dots, c_k$ be complex numbers. There is an element β in B such that $\beta(q_i) = c_i$ for $i = 1, \dots, k$.* $\square$

injectivity of the map $Y/G \to X$: Let O_1 and O_2 be distinct G-orbits. Lemma 2.8.10 tells us that there is an element β in B whose value is 0 at every point of O_1, and is 1 at every point of O_2. Since G permutes the orbits, $\sigma\beta$ will also be 0 at points of O_1 and 1 at points of O_2. Then the product $\gamma = \prod_\sigma \sigma\beta$ will be 0 at points of O_1 and 1 at points of O_2, and the product γ is invariant. If p_i denotes the image in X of the orbit O_i, the maximal ideal $\mathfrak{m}_{p_i}$ of A is the intersection $A \cap \mathfrak{m}_q$, where q is any point in O_i. Therefore γ is in the maximal ideal $\mathfrak{m}_{p_1}$, but not in $\mathfrak{m}_{p_2}$. The images of the two orbits are distinct.

surjectivity of the map $Y/G \to X$: It suffices to show that the map $Y \to X$ is surjective.

2.8.11. Lemma. *If I is an ideal of the invariant algebra A, and if the extended ideal IB is the unit ideal of B, then I is the unit ideal of A.* extideal

As before, the extended ideal IB is the ideal of B generated by I.

Let's assume the lemma for the moment, and use it to prove surjectivity of the map $Y \to X$. Let p be a point of X. The lemma tells us that the extended ideal $\mathfrak{m}_p B$ isn't the unit ideal. So it is contained in a maximal ideal $\mathfrak{m}_q$ of B, where q is a point of Y. Then $\mathfrak{m}_p \subset (\mathfrak{m}_p B) \cap A \ \subset \mathfrak{m}_q \cap A$.

The contraction $\mathfrak{m}_q \cap A$ is an ideal of A, and it isn't the unit ideal. It doesn't contain 1, which isn't in $\mathfrak{m}_q$. Since $\mathfrak{m}_p \subset \mathfrak{m}_q \cap A$ and $\mathfrak{m}_p$ is a maximal ideal, $\mathfrak{m}_p = \mathfrak{m}_q \cap A$. This means that q maps to p in X. $\square$

proof of the lemma. If $IB = B$, there will be an equation $\sum_i z_i b_i = 1$, with z_i in I and b_i in B. The sums $\alpha_i = \sum_\sigma \sigma b_i$ are invariant, so they are elements of A, and the elements z_i are invariant. Therefore $\sum_\sigma \sigma(z_i b_i) = z_i \sum_\sigma \sigma b_i = z_i \alpha_i$ is in I. Then

$$\sum_\sigma 1 = \sum_\sigma \sigma(1) = \sum_{\sigma, i} \sigma(z_i b_i) = \sum_i z_i \alpha_i$$

The right side is in I, and the left side is the order of the group which, because A contains the complex numbers, is an invertible element of A. So I is the unit ideal. $\square$

2.9 Exercises

2.9.1. Prove that if A, B are finite-type domains, then $A \otimes B$ is a finite-type domain.

2.9.2. Describe all prime ideals of the two-variable polynomial ring $\mathbb{C}[x, y]$.

2.9.3. Prove that if a noetherian ring contains just one prime ideal, then that ideal is nilpotent.

2.9.4. Prove that, if an algebra A is a complex vector space of dimension d, it contains at most d maximal ideals.

2.9.5. Let T denote the ring $\mathbb{C}[\epsilon]$, with $\epsilon^2 = 0$. If A is the coordinate ring of an affine variety X, an (infinitesimal) tangent vector to X is, by definition, given by an algebra homomorphism $\varphi : A \to T$.

(i) Show that such a homomorphism can be written in the form $\varphi(a) = f(a) + d(a)\epsilon$, where f and d are functions $A \to \mathbb{C}$. Show that f is an algebra homomorphism, and that d is an *f-derivation*, a linear map that satisfies the identity $d(ab) = f(a)d(b) + d(a)f(b)$.

(ii) Show that, when $A = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_r)$ the tangent vectors are defined by the equations $\nabla f_i(p)x = 0$.

2.9.6. Prove that, in the ring $\mathbb{C}[[x_1, ..., x_n]]$ of formal power series, an element whose constant term is nonzero is invertible.

2.9.7. Prove that that the varieties in the affine plane $\mathbb{A}^2$ are points, curves, and the affine plane $\mathbb{A}^2$ itself.

2.9.8. Classify algebras that are complex vector spaces of dimensions two and three.

2.9.9. Prove that if $f(x_0, x_1, x_2)$ is an irreducible homogeneous polynomial, not x_0, then its dehomogneniza-tion $f(1, x_1, x_2)$ is also irreducible.

2.9.10. Derive version 1 of the Nullstellensatz from the Strong Nulletellensatz.

2.9.11. Let K be a field extension of an infinite field k, and suppose that K is a finite-type k-algebra. prove that K is a finite field extension of k.

2.9.12. Find generators for the ideal of $\mathbb{C}[x, y]$ of polynomials that vanish on the three points $(0, 0), (0, 1), (1, 0)$.

2.9.13. Let A be a noetherian ring. Prove that a radical ideal I of A is the intersection of finitely many prime ideals.

2.9.14. Let C and D be closed subsets of an affine variety $X = \operatorname{Spec} A$. Suppose that no component of D is contained in C. Prove that there is a regular function f that vanishes on C and isn't identically zero on any component of D.

2.9.15. A minimal prime ideal is an ideal that doesn't properly contain any other prime ideal. Prove that a nonzero, finite-type algebra A (not necessarily a domain) contains at least one and only finitely many minimal prime ideals. Try to find a proof that doesn't require much work.

2.9.16. Let K be a field and let R be the polynomial ring $K[x_1, ..., x_n]$, with $n > 0$. Prove that the field of fractions of R is not a finitely generated K-algebra.

2.9.17. Prove that the algebra $A = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is isomorphic to the Laurent Polynomial Ring $\mathbb{C}[t, t^{-1}]$, but that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not isomorphic to $\mathbb{R}[t, t^{-1}]$.

2.9.18. Let B be a finite type domain, and let p and q be points of the affine variety $Y = \operatorname{Spec} B$. Let A be the set of elements $f \in B$ such that $f(p) = f(q)$. Prove

(i) A is a finite type domain.

(ii) B is a finite A-module.

(iii) Let $\varphi :\ \operatorname{Spec} B \to\ \operatorname{Spec} A$ be the morphism obtained from the inclusion $A \subset B$. Show that $\varphi(p) = \varphi(q)$, and that φ is bijective everywhere else.

2.9.19. The equation $y^2 = x^3$ defines a plane curve X with a cusp at the origin, the spectrum of the algebra $A = \mathbb{C}[x, y]/(y^2 - x^3)$. There is a homomorphism $A \xrightarrow{\varphi} \mathbb{C}[t]$, with $\varphi(x) = t^2$ and $\varphi(y) = t^3$, and the associated morphism $\mathbb{A}^1_t \xrightarrow{u} X$ sends a point t of $\mathbb{A}^1$ to the point $(x, y) = (t^2, t^3)$ of X. Prove that u is a homeomorphism the Zariski topology and also in the classical topology.

2.9.20. Explain what a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ means in terms of polynomials, when $A = \mathbb{C}[x_1, \dots, x_m]/(f_1, \dots, f_r)$ and $B = \mathbb{C}[y_1, \dots, y_n]/(g_1, \dots, g_k)$. xspellout-morph

2.9.21. Let $A = \mathbb{C}[x_1, ..., x_2]$, and let $B = A[\alpha]$, where α is an element of the fraction field $\mathbb{C}(x)$ of A. Describe the fibres of the morphism $Y = \operatorname{Spec} B \to \operatorname{Spec} A = X$. xadjoin-frac

2.9.22. Let X be the plane curve $y^2 = x(x-1)^2$, let $A = \mathbb{C}[x, y]/(y^2 - x(x-1)^2)$ be its coordinate algebra, and let x, y denote the residues of those elements in A too. xparam-curve

(i) Points of the curve can be parametrized by a variable t. Use the lines $y = t(x-1)$ to determine such a parametrization.

(ii) Let $B = \mathbb{C}[t]$ and let T be the affine line $\operatorname{Spec} \mathbb{C}[t]$. The parametrization gives us an injective homomorphism $A \to B$. Describe the corresponding morphism $T \to X$.

2.9.23. Let $A = \mathbb{C}[u, v]/(v^2 - u(1-u))$ and $B = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$, and let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Show that the substitution $u = x^2$, $v = xy$ defines a morphism $Y \to X$. xamorph

2.9.24. Let X be the affine line $\operatorname{Spec} \mathbb{C}[x]$. When we view $\operatorname{Spec} \mathbb{C}[x_1, x_2]$ as the product $X \times X$, a homomorphism $\mathbb{C}[x] \to \mathbb{C}[x_1, x_2]$ defines a law of composition on X — a morphism $X \times X \to X$. Determine the homomorphisms such that the law makes X into a group with the point $x = 0$ as the identity. grplaw

2.9.25. The cyclic group $G = \langle \sigma \rangle$ of order n operates on the polynomial algebra $A = \mathbb{C}[x, y]$ by $\sigma(x) = \zeta x$ and $\sigma(y) = \zeta y$, where $\zeta = e^{2\pi i/n}$. xzetaxy

(i) Describe the invariant ring A^G by exhibiting generators and defining relations.

(ii) Prove that the there is a $2 \times n$ matrix whose 2×2-minors are defining relations for A^G.

(iii) Prove directly that the morphism $\operatorname{Spec} A = \mathbb{A}^2 \to \operatorname{Spec} B$ defined by the inclusion $B \subset A$ is surjective, and that its fibres are the G-orbits.

2.9.26. Let A be a finite-type algebra, and let f be an irreducible element of $\mathbb{C}[x_1, ..., x_n]$ of positive degree. Prove that f is an irreducible element of $A[x_1, ..., x_n]$. remainirrd

Chapter 3 PROJECTIVE ALGEBRAIC GEOMETRY

projgeom

3.1 Projective Varieties

pvariety

The projective space $\mathbb{P}^n$ was described in Chapter 1. Its points are equivalence classes of nonzero vectors $(x_0, ..., x_n)$, the equivalence relation being that

xlambdax (3.1.1)
$$(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n).$$

for any nonzero complex number λ.

A subset of $\mathbb{P}^n$ is *Zariski closed* if it is the set of common zeros of a family of *homogeneous* polynomials $f_1, ..., f_k$ in the coordinate variables $x_0, ..., x_n$, or if it is the set of zeros of the ideal $\mathcal{I}$ generated by such a family. As was explained in (1.3.1), $f(\lambda x) = 0$ for all λ if and only if f is homogeneous.

The Zariski closed sets are the closed sets in the *Zariski topology* on $\mathbb{P}^n$. We usually refer to the Zariski closed sets simply as *closed sets*.

Because the polynomial ring $\mathbb{C}[x_0, ..., x_n]$ is noetherian, the projective space $\mathbb{P}^n$ is a noetherian space: Every strictly increasing family of ideals of $\mathbb{C}[x]$ is finite, and every strictly decreasing family of closed subsets of $\mathbb{P}^n$ is finite. Therefore every closed subset of $\mathbb{P}^n$ is a finite union of irreducible closed sets (2.2.18).

defprojvar **3.1.2. Definition.** A *projective variety* is an irreducible closed subset of a projective space $\mathbb{P}^n$.

We will want to know when two projective varieties are isomorphic. This will be explained in Section 3.5, where morphisms are defined.

The Zariski topology on a projective variety X is induced from the topology on the projective space that contains it (see (2.2.6). Since a projective variety X is closed in $\mathbb{P}^n$, a subset of X is closed in X if it is closed in $\mathbb{P}^n$.

point-closed **3.1.3. Lemma.** *The one-point sets in projective space are closed.*

proof. This simple proof illustrates a general method. Let p be the point $(a_0, ..., a_n)$. The first guess might be that the one-point set $\{p\}$ is defined by the equations $x_i = a_i$, but the polynomials $x_i - a_i$ aren't homogeneous in x. This is reflected in the fact that, for any $\lambda \neq 0$, the vector $(\lambda a_0, ..., \lambda a_n)$ represents the same point, though it won't satisfy those equations. The equations that define the set $\{p\}$ are

define-point (3.1.4)
$$a_i x_j = a_j x_i,$$

for $i, j = 0, ..., n$, which imply that the ratios a_i/a_j and x_i/x_j are equal. □

3.1.5. Lemma. *The proper closed subsets of the projective line are its nonempty finite subsets, and the proper closed subsets of the projective plane are finite unions of points and curves.* □ *closedin-line*

Though affine varieties are important, most of algebraic geometry concerns projective varieties. It isn't completely clear why this is so, but one property of projective space gives a hint of its importance: With its classical topology, projective space is *compact*.

A topological space is compact if

I=it has the *Hausdorff* property: Distinct points p, q of X have disjoint open neighborhoods, and
it is *quasicompact*: If X is covered by a family $\{U^i\}$ of open sets, then a finite subfamily covers X.

By the way, when we say that the sets $\{U^i\}$ *cover* a topological space X, we mean that X is the union $\bigcup U^i$. We don't allow U^i to contain elements that aren't in X, though that would be a customary usage in English.

In the classical topology, affine space $\mathbb{A}^n$ isn't quasicompact, and therefore it isn't compact. The *Heine-Borel Theorem* asserts that a subset of $\mathbb{A}^n$ is compact in the classical topology if and only if it is closed and bounded.

We'll show that $\mathbb{P}^n$ is compact, assuming that the Hausdorff property has been verified. The $2n+1$-dimensional sphere $\mathbb{S}$ of unit length vectors in $\mathbb{A}^{n+1}$ is a bounded set, and because it is the zero locus of the equation $\overline{x}_0 x_0 + \cdots + \overline{x}_n x_n = 1$, it is closed. The Heine-Borel Theorem tells us that $\mathbb{S}$ is compact. The map $\mathbb{S} \to \mathbb{P}^n$ that sends a vector $(x_0, ..., x_n)$ to the point of projective space with that coordinate vector is continuous and surjective, so the next lemma of topology shows that $\mathbb{P}^n$ is compact.

3.1.6. Lemma. *Let $Y \xrightarrow{f} X$ be a continuous map. Suppose that Y is compact and that X is a Hausdorff space. Then the image $Z = f(Y)$ is a closed, compact subset of X.* □ *image-compact*

The rest of this section contains a few examples of projective varieties.

(3.1.7) linear subspaces

linsubsp

If W is a subspace of dimension $r+1$ of the vector space $\mathbb{C}^{n+1}$, the points of $\mathbb{P}^n$ that are represented by the nonzero vectors in W form a *linear subspace* L of $\mathbb{P}^n$, of dimension r. If $(w_0, ..., w_r)$ is a basis of W, the linear subspace L corresponds bijectively to a projective space of dimension r, by

$$c_0 w_0 + \cdots + c_r w_r \quad \longleftrightarrow \quad (c_0, ..., c_r)$$

For example, the set of points $(x_0, ..., x_r, 0, ..., 0)$ is a linear subspace of dimension r. □

(3.1.8) a quadric surface

quadric-surface

A *quadric* in projective three-space $\mathbb{P}^3$ is the locus of zeros of an irreducible homogeneous quadratic equation in four variables.

We describe a bijective map from the product $\mathbb{P}^1 \times \mathbb{P}^1$ of projective lines to a quadric. Let coordinates in the two copies of $\mathbb{P}^1$ be (x_0, x_1) and (y_0, y_1), respectively, and let the four coordinates in $\mathbb{P}^3$ be w_{ij}, with $0 \le i, j \le 1$. The map is defined by $w_{ij} = x_i y_j$. Its image is the quadric Q whose equation is

$$w_{00} w_{11} = w_{01} w_{10} \tag{3.1.9}$$

ponepone

Let's check that the map $\mathbb{P}^1 \times \mathbb{P}^1 \to Q$ is bijective: If w is a point of Q, one of its coordinates, say w_{00}, will be nonzero. Then if (x, y) is a point of $\mathbb{P}^1 \times \mathbb{P}^1$ whose image is w, so that $w_{ij} = x_i y_j$, the coordinates x_0 and y_0 must be nonzero. When we normalize w_{00}, x_0, and y_0 to 1, the equation becomes $w_{11} = w_{10} w_{01}$. This equation has a unique solution for x_1 and y_1 such that $w_{ij} = x_i y_j$, namely $x_1 = w_{10}$ and $y_1 = w_{01}$.

The quadric with the equation (3.1.9) contains two families of *lines* — one dimensional linear subspaces, the images of the subsets $x \times \mathbb{P}^1$ and $\mathbb{P}^1 \times y$ of $\mathbb{P} \times \mathbb{P}$.

The equation (3.1.9) can be diagonalized by the substitution $w_{00} = s{+}t$, $w_{11} = s{-}t$, $w_{01} = u{+}v$, $w_{10} = u{-}v$, which changes the equation (3.1.9) to $s^2 - t^2 = u^2 - v^2$. When we look at the affine open set $\{u = 1\}$, the equation becomes $s^2 + v^2 - t^2 = 1$. The real locus of this equation is a one-sheeted hyerboloid in $\mathbb{R}^3$, and the two families of complex lines in the quadric correspond to the familiar rulings of that hyperboloid by real lines.

hsurftwo

(**3.1.10**) hypersurfaces

A *hypersurface* in projective space $\mathbb{P}^n$ is the locus of zeros of an irreducible homogeneous polynomial $f(x_0, ..., x_n)$. Its *degree* is the degree of the polynomial f.

Plane projective curves and quadric surfaces are hypersurfaces.

segreemb

(**3.1.11**) the Segre embedding of a product

The product $\mathbb{P}^m_x \times \mathbb{P}^n_y$ of projective spaces can be embedded by its *Segre embedding* into a projective space $\mathbb{P}^N_w$ that has coordinates w_{ij}, with $i = 0, ..., m$ and $j = 0, ..., n$. So $N = (m{+}1)(n{+}1) - 1$. The Segre embedding is defined by

segreco-ords

$$w_{ij} = x_i y_j \tag{3.1.12}$$

We call the coordinates w_{ij} the *Segre variables*.

The map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ that was described in (**3.1.8**) is the simplest case of a Segre embedding.

segreeq

3.1.13. Proposition. *The Segre embedding maps the product $\mathbb{P}^m \times \mathbb{P}^n$ bijectively to the locus S of the* ***Segre equations***

segree-quations

$$w_{ij} w_{k\ell} - w_{i\ell} w_{kj} = 0 \tag{3.1.14}$$

proof. The proof is analogous to the one given in (**3.1.8**). When one substitutes (3.1.12) into the Segre equations, one obtains equations in $\{x_i, y_j\}$ that are true. So the locus S contains the image of the Segre embedding.

Say that a point p of S is the image of a point (x, y) of $\mathbb{P}^m \times \mathbb{P}^n$. Some coordinate of p, say w_{00}, will be nonzero, and then x_0 and y_0 are also nonzero. We normalize w_{00}, x_0, and y_0 to 1. Then $w_{ij} = w_{i0} w_{0j}$ for all i, j. The unique solution of these equations is $x_i = w_{i0}$ and $y_j = w_{0j}$. $\square$

The Segre embedding is important because it makes the product of projective spaces into a projective variety, the closed subvariety of $\mathbb{P}^N$ defined by the Segre equations. However, to show that the product is a variety, we need to show that the locus S of the Segre equations is irreducible, and this isn't obvious. We defer the proof to Section 3.3 (see Proposition 3.3.4).

verone-seemb

(**3.1.15**) the Veronese embedding of projective space

Let the coordinates in $\mathbb{P}^n$ be x_i, and let those in $\mathbb{P}^N$ be v_{ij}, with $0 \le i \le j \le n$. So $N = \binom{n+2}{2} - 1$. The *Veronese embedding* is the map $\mathbb{P}^n \xrightarrow{f} \mathbb{P}^N$ defined by $v_{ij} = x_i x_j$. The Veronese embedding resembles the Segre embedding, but in the Segre embedding, there are distinct sets of coordinates x and y, and $i \le j$ isn't required.

The proof of the next proposition is similar to the proof of (3.1.13), once one has untangled the inequalities.

veroneq

3.1.16. Proposition. *The Veronese embedding f maps $\mathbb{P}^n$ bijectively to the locus X in $\mathbb{P}^N$ of the equations*

$$v_{ij} v_{k\ell} = v_{i\ell} v_{kj} \quad \textit{for} \quad 0 \le i \le k \le j \le \ell \le n$$

$\square$

For example, the Veronese embedding maps $\mathbb{P}^1$ bijectively to the conic $v_{00}v_{11} = v_{01}^2$ in $\mathbb{P}^2$.

(3.1.17) the twisted cubic twistcubic

There are higher order Veronese embeddings, defined by evaluating the monomials of some degree $d > 2$. The first example is the embedding of $\mathbb{P}^1$ by the cubic monomials in two variables, which maps $\mathbb{P}^1_x$ to $\mathbb{P}^3_v$. Let the coordinates in $\mathbb{P}^3$ be $v_0, ..., v_3$. The cubic Veronese embedding is defined by

$$v_0 = x_0^3, \quad v_1 = x_0^2 x_1, \quad v_2 = x_0 x_1^2, \quad v_3 = x_1^3$$

Its image, the locus $(v_0, v_1, v_2, v_3) = (x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$, is a *twisted cubic* in $\mathbb{P}^3$, the set of common zeros of the three polynomials

(3.1.18) $$v_0v_2 - v_1^2\ , \ \ v_1v_2 - v_0v_3\ , \ \ v_1v_3 - v_2^2$$ twcubic

which are the 2×2 minors of the 2×3 matrix

(3.1.19) $$\begin{pmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \end{pmatrix}$$ twothree-matrix

A 2×3 matrix has rank ≤ 1 if and only if its 2×2 minors are zero. So a point (v_0, v_1, v_2, v_3) lies on the twisted cubic if (3.1.19) has rank one. This means that the vectors (v_0, v_1, v_2) and (v_1, v_2, v_3) represent the same point of $\mathbb{P}^2$, provided that they are both nonzero.

Setting $x_0 = 1$ and $x_1 = t$, the twisted cubic becomes the locus of points $(1, t, t^2, t^3)$. There is one point on the twisted cubic at which $x_0 = 0$, the point $(0, 0, 0, 1)$. □

3.2 Homogeneous Ideals

We denote the polynomial algebra $\mathbb{C}[x_0, ..., x_n]$ by R here. homogen

3.2.1. Lemma. *Let $\mathcal{I}$ be an ideal of R. The following conditions are equivalent.* *homogideal*

(i) *$\mathcal{I}$ can be generated by homogeneous polynomials.*

(ii) *A polynomial is in $\mathcal{I}$ if and only if its homogeneous parts are in $\mathcal{I}$.* □

An ideal $\mathcal{I}$ of R that satisfies these conditions is a *homogeneous ideal.*

3.2.2. Corollary. *Let S be a subset of projective space $\mathbb{P}^n$. The set of elements of R that vanish at all points of S is a homogeneous ideal.* *idealishom*

This follows from Lemma 1.3.2. □

3.2.3. Lemma. *The radical of a homogeneous ideal is homogeneous.* *radicalhomogeneous*

proof. Let $\mathcal{I}$ be a homogeneous ideal, and let f be an element of its radical $\operatorname{rad}\mathcal{I}$. So f^r is in $\mathcal{I}$ for some r. When f is written as the sum $f_0 + \cdots + f_d$ of its homogeneous parts, the highest degree part of f^r is $(f_d)^r$. Since $\mathcal{I}$ is homogeneous, $(f_d)^r$ is in $\mathcal{I}$ and f_d is in $\operatorname{rad}\mathcal{I}$. Then $f_0 + \cdots + f_{d-1}$ is also in $\operatorname{rad}\mathcal{I}$. By induction on d, all of the homogeneous parts $f_0, ..., f_d$ are in $\operatorname{rad}\mathcal{I}$. □

3.2.4. If f is a set of homogeneous polynomials, its set of zeros in $\mathbb{P}^n$ may be denoted by $V(f)$, and the set of zeros of a homogeneous ideal $\mathcal{I}$ may be denoted by $V(\mathcal{I})$. This is also the notation that we use for closed subsets of affine space. defVXI

The complement of the origin in the affine space $\mathbb{A}^{n+1}$ is mapped to the projective space $\mathbb{P}^n$ by sending a vector $(x_0, ..., x_n)$ to the point of $\mathbb{P}^n$ defined by that vector. This map can be useful when one studies projective space. A homogeneous ideal $\mathcal{I}$ has a zero locus in projective space $\mathbb{P}^n$ and also a zero locus in the affine space $\mathbb{A}^{n+1}$. We can't use the $V(\mathcal{I})$ notation for both of them here, so let's denote these two loci by V and W, respectively. Unless $\mathcal{I}$ is the unit ideal, the origin $x = 0$ will be a point of W, and the complement of the origin will map surjectively to V. If a point x other than the origin is in W, then every point of the line spanned by x, the one-dimensional subspace of $\mathbb{A}^{n+1}$, is in W, because a homogeneous polynomial f vanishes at x if and only if it vanishes at λx. An affine variety that is the union of such lines through the origin is called an *affine cone*. If the locus W contains a point x other than the origin, it is an affine cone.

The loci $x_0^2 + x_1^2 - x_2^2 = 0$ and $x_0^3 + x_1^3 - x_2^3 = 0$ are cones in $\mathbb{A}^3$.

Note. The real locus $x_0^2 + x_1^2 - x_2^2 = 0$ in $\mathbb{R}^3$ decomposes into two parts when the origin is removed. Because of this, it is sometimes called a "double cone". However, the complex locus doesn't decompose.

irrel **(3.2.5) the irrelevant ideal**

In the polynomial algebra $R = \mathbb{C}[x_0, ..., x_n]$, the maximal ideal $\mathcal{M} = (x_0, ..., x_n)$ generated by the variables is called the *irrelevant ideal* because its zero locus in projective space is empty.

nozeros **3.2.6. Proposition.** *The zero locus $V(\mathcal{I})$ in $\mathbb{P}^n$ of a homogeneous ideal $\mathcal{I}$ of R is empty if and only if $\mathcal{I}$ contains a power of the irrelevant ideal $\mathcal{M}$.*

Another way to say this is that the zero locus of a homogeneous ideal $\mathcal{I}$ is empty if and only if either $\mathcal{I}$ is the unit ideal R, or its radical is the irrelevant ideal.

proof of Proposition 3.2.6. Let Z be the zero locus of $\mathcal{I}$ in $\mathbb{P}^n$. If $\mathcal{I}$ contains a power of $\mathcal{M}$, it contains a power of each variable. Powers of the variables have no common zeros in projective space, so Z is empty.

Suppose that Z is empty, and let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$. Since the complement of the origin in W maps to the empty locus Z, it is empty. The origin is the only point that might be in W. If W is the one point space consisting of the origin, then $\operatorname{rad} \mathcal{I} = \mathcal{M}$. If W is empty, $\mathcal{I}$ is the unit ideal. $\square$

homprime **3.2.7. Lemma.** *Let $\mathcal{P}$ be a homogeneous ideal in the polynomial algebra R, not the unit ideal. The following conditions are equivalent:*

(i) *$\mathcal{P}$ is a prime ideal.*

(ii) *If f and g are homogeneous polynomials, and if $fg \in \mathcal{P}$, then $f \in \mathcal{P}$ or $g \in \mathcal{P}$.*

(iii) *If $\mathcal{A}$ and $\mathcal{B}$ are homogeneous ideals, and if $\mathcal{AB} \subset \mathcal{P}$, then $\mathcal{A} \subset \mathcal{P}$ or $\mathcal{B} \subset \mathcal{P}$.*

In other words, a homogeneous ideal is a prime ideal if the usual conditions for a prime ideal are satisfied when the polynomials or ideals are homogeneous.

proof of the lemma. When the word homogeneous is omitted, **(ii)** and **(iii)** become the definition of a prime ideal. So **(i)** implies **(ii)** and **(iii)**. The fact that **(iii)** $\Rightarrow$ **(ii)** is proved by considering the principal ideals generated by f and g.

(ii) $\Rightarrow$ **(i)** Suppose that a homogeneous ideal $\mathcal{P}$ satisfies condition **(ii)**, and that the product fg of two polynomials, not necessarily homogeneous, is in $\mathcal{P}$. If f has degree d and g has degree e, the highest degree part of fg is the product $f_d g_e$ of the homogeneous parts of f and g of maximal degree. Since $\mathcal{P}$ is a homogeneous ideal that contains fg, it contains $f_d g_e$. Therefore one of the factors, say f_d, is in $\mathcal{P}$. Let $h = f - f_d$. Then $hg = fg - f_d g$ is in $\mathcal{P}$, and it has lower degree than fg. By induction on the degree of fg, h or g is in $\mathcal{P}$, and if h is in $\mathcal{P}$, so is f. $\square$

3.2.8. Proposition. *Let V be the zero locus in $\mathbb{P}^n$ of a homogeneous radical ideal $\mathcal{I}$ (**2.2.3**) that isn't the*
closedirred *irrelevant ideal. Then V is a projective variety, an irreducible closed subset of $\mathbb{P}^n$, if and only if $\mathcal{I}$ is a prime ideal. Thus a subset V of $\mathbb{P}^n$ is a projective variety if and only if it is the zero locus of a homogeneous prime ideal other than the irrelevant ideal.*

proof. The locus W of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ is irreducible if and only if V is irreducible (2.2.17 **(iii)**). Proposition 2.2.22 tells us that W is irreducible if and only if the radical ideal $\mathcal{I}$ is a prime ideal. $\square$

homstrnull **3.2.9. Strong Nullstellensatz, projective version.**

(i) *Let g be a nonconstant homogeneous polynomial in $x_0, ..., x_n$, and let $\mathcal{I}$ be a homogeneous ideal of $\mathbb{C}[x]$. If g vanishes at every point of the zero locus $V(\mathcal{I})$ in $\mathbb{P}^n$, then $\mathcal{I}$ contains a power of g.*

(ii) *Let f and g be homogeneous polynomials. If f is irreducible and if $V(f) \subset V(g)$, then f divides g.*

(iii) *Let $\mathcal{I}$ and $\mathcal{J}$ be homogeneous ideals, and suppose that $\operatorname{rad}\mathcal{I}$ isn't the irrelevant ideal or the unit ideal. Then $V(\mathcal{I}) = V(\mathcal{J})$ if and only if $\operatorname{rad}\mathcal{I} = \operatorname{rad}\mathcal{J}$.*

proof. **(i)** Let W be the locus of zeros of $\mathcal{I}$ in the affine space $\mathbb{A}^{n+1}$ with coordinates $x_0, ..., x_n$. The homogeneous polynomial g vanishes at every point of W different from the origin, and since g isn't a constant, it vanishes at the origin too. So the affine Strong Nullstellensatz applies: Since g vanishes the locus of zeros W, $\mathcal{I}$ contains a power of g. a □

(3.2.10) quasiprojective varieties var-quasiproj

In addition to projective varieties, we will want to study nonempty open subsets of projective varieties. We call such a subset a *variety* too. A variety will be an irreducible topological space (Lemma 2.2.17).

For example, the complement of a point in a projective variety is a variety. An affine variety $X = \operatorname{Spec} A$ may be embedded as a closed subvariety into the standard affine space $\mathbb{U}^0 : \{x_0 \neq 0\}$. It becomes an open subset of its closure in $\mathbb{P}^n$, which is a projective variety (Lemma 2.2.17). So it is a variety. And of course, a projective variety is a variety. The topology on a variety is induced from the topology on projective space.

We will use this terminology, though most of the the varieties we encounter will be affine or projective. Elsewhere, what we call a variety is usually called a *quasiprojective variety*. We drop the adjective 'quasiprojective'. There are abstract varieties that aren't quasiprojective — abstract varieties that cannot be embedded into any projective space. But such varieties aren't very important. We won't study them. In fact, it is hard enough to find convincing examples that we won't try to give one here. So for us, the adjective 'quasiprojective' is superfluous as well as ugly.

3.2.11. Lemma. *The topology on the affine open subset $\mathbb{U}^0 : x_0 \neq 0$ of $\mathbb{P}^n$ that is induced from the Zariski topology on $\mathbb{P}^n$ is same as the Zariski topology that is obtained by viewing $\mathbb{U}^0$ as the affine space $\operatorname{Spec} \mathbb{C}[u_1, ..., u_n]$, $u_i = x_i/x_0$.* □ *topon-standaff*

3.3 Product Varieties

The properties of products of varieties are intuitively plausible, but because the Zariski topology on a product of varieties isn't the product topology, one must be careful. prodvar

(3.3.1) the Zariski topology on $\mathbb{P}^m \times \mathbb{P}^n$ zartopprod

The *product topology* on the product $X \times Y$ of topological spaces is the coarsest topology such that the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are continuous. If C and D are closed subsets of X and Y, then $C \times D$ is a closed subset of $X \times Y$ in the product topology, and every closed set in the product topology is a finite union of such subsets.

The product topology on $\mathbb{P}^m \times \mathbb{P}^n$ is much coarser than the Zariski topology. For example, the proper Zariski closed subsets of $\mathbb{P}^1$ are the nonempty finite subsets. In the product topology, the proper closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ are finite unions of sets of the form $\mathbb{P}^1 \times q$, $p \times \mathbb{P}^1$, and $p \times q$ ('horizontal' lines, 'vertical' lines, and points). Most Zariski closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ aren't of this form. The diagonal $\Delta = \{(p,p) \,|\, p \in \mathbb{P}^1\}$ is one example.

As has been mentioned, the product of projective spaces $\mathbb{P}^m \times \mathbb{P}^n$ can be embedded into a projective space $\mathbb{P}^N$ by the Segre map, which identifies it as a closed subset of $\mathbb{P}^N$, the locus of the Segre equations $w_{ij}w_{k\ell} = w_{i\ell}w_{kj}$. The integer N is unimportant. Since $\mathbb{P}^m \times \mathbb{P}^n$, with its Segre embedding, is a closed subset of $\mathbb{P}^N$, we don't really need a separate definition of its Zariski topology. Its closed subsets are the zero sets of families of homogeneous polynomials in the Segre variables w_{ij}, families that include the Segre equations.

However, it is important to show that the Segre embedding maps the product $\mathbb{P}^m \times \mathbb{P}^n$ to an *irreducible* closed subset of $\mathbb{P}^N$, so that the product becomes a projective variety. This will be done below, in Corollary 3.3.5.

One can also describe the closed subsets of $\mathbb{P}^m \times \mathbb{P}^n$ directly, in terms of bihomogeneous polynomials. A polynomial $f(x,y)$ in $x = (x_0, ..., x_m)$ and $y = (y_0, ..., y_n)$ is *bihomogeneous* if it is homogeneous in the variables x and also in the variables y. For example, $x_0^2y_0 + x_0x_1y_1$ is a bihomogeneous polynomial, of degree 2 in x and degree 1 in y.

Because (x, y) and $(\lambda x, \mu y)$ represent the same point of $\mathbb{P}^m \times \mathbb{P}^n$ for all nonzero scalars λ and μ, we want to know that $f(x, y) = 0$ if and only if $f(\lambda x, \mu y) = 0$ for all nonzero λ and μ, and this is true if and only if all of bihomogeneous parts of f are zero. (The bihomogeneous part of f of bidegree i, j is the sum of terms whose degrees in x and y are i and j, respectively.)

closedin-pxp **3.3.2. Proposition.** **(i)** *Let Z be a subset of $\mathbb{P}^m \times \mathbb{P}^n$. The Segre image of Z is closed if and only if Z is the locus of zeros of a family of bihomogeneous polynomials.*

(ii) *If X and Y are closed subsets of $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively, then $X \times Y$ is a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$.*

(iii) *The projection maps $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\pi_1} \mathbb{P}^m$ and $\mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\pi_2} \mathbb{P}^n$ are continuous.*

(iv) *For all x in $\mathbb{P}^m$ the fibre $x \times \mathbb{P}^n$ is homeomorphic to $\mathbb{P}^n$ and for all y in $\mathbb{P}^n$, the fibre $\mathbb{P}^m \times y$ is homeomorphic to $\mathbb{P}^m$.*

proof. **(i)** For this proof, we denote the Segre image of $\mathbb{P}^m \times \mathbb{P}^n$ by V. Let $f(w)$ be a homogeneous polynomial in the Segre variables w_{ij}. When we substitute $w_{ij} = x_iy_j$ into f, we obtain a polynomial $\widetilde{f}(x, y)$ that is bihomogeneous and whose degree in x and in y is the same as the degree of f. The inverse image of the zero set of f in V is the zero set of $\widetilde{f}$ in $\mathbb{P}^m \times \mathbb{P}^n$. Therefore the inverse image of a closed subset of V is the zero set of a family of bihomogeneous polynomials in $\mathbb{P}^m \times \mathbb{P}^n$.

Conversely, let $\widetilde{g}(x, y)$ be a bihomogeneous polynomial, say of degrees r in x and degree s in y. If $r = s$, we may collect variables that appear in $\widetilde{g}$ in pairs x_iy_j and replace each pair x_iy_j by w_{ij}. We will obtain a homogeneous polynomial g in w such that $g(w) = \widetilde{g}(x, y)$ when $w_{ij} = x_iy_j$. The zero set of g in V is the image of the zero set of $\widetilde{g}$ in $\mathbb{P}^m \times \mathbb{P}^n$.

Suppose that $r \geq s$, and let $k = r - s$. Because the variables y cannot all be zero at any point of $\mathbb{P}^n$, the equation $g = 0$ on $\mathbb{P}^m \times \mathbb{P}^n$ is equivalent with the system of equations $gy_0^k = gy_1^k = \cdots = gy_n^k = 0$. The polynomials gy_i^k are bihomogeneous, of same degree in x as in y. This puts us back in the first case.

(ii) A homogeneous polynomial $f(x)$ can be viewed as a bihomogeneous polynomial of degree zero in y, and a homogeneous polynomial $g(y)$ as a bihomogeneous polynomial of degree zero in x. So $X \times Y$, which is a locus of the form $f(x) = g(y) = 0$ in $\mathbb{P}^m \times \mathbb{P}^n$, is closed in $\mathbb{P}^m \times \mathbb{P}^n$.

(iii) For the projection π_1, we must show that if X is a closed subset of $\mathbb{P}^m$, its inverse image is closed. This is the case $Y = \mathbb{P}^n$ of **(ii)**.

(iv) It will be best to denote the chosen point of $\mathbb{P}^m$ by a symbol other than x here. We'll denote it by $\overline{x}$. Part **(i)** tells us that the bijective map $\overline{x} \times \mathbb{P}^n \to \mathbb{P}^n$ is continuous. To show that the inverse map is continuous, we must show that a closed subset Z of $\overline{x} \times \mathbb{P}^n$ is the inverse image of a closed subset of $\mathbb{P}^n$. Say that Z is the zero locus of a set of bihomogeneous polynomials $f(x, y)$. The polynomials $\overline{f}(y) = f(\overline{x}, y)$ are homogeneous in y, and the inverse image of their zero locus is Z. $\square$

projcont **3.3.3. Corollary.** *Let X and Y be projective varieties, and let Π denote the product $X \times Y$, regarded as a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$.*

- *The projections $\Pi \to X$ and $\Pi \to Y$ are continuous.*
- *For all x in X and all y in Y, the fibres $x \times Y$ and $X \times y$ are homeomorphic to Y and X, respectively.* $\square$

Pirred **3.3.4. Proposition.** *Suppose that a topology is given on the product $\Pi = X \times Y$ of irreducible topological spaces X and Y, and that it has these properties:*

- *The projections $\Pi \xrightarrow{\pi_1} X$ and $\Pi \xrightarrow{\pi_2} Y$ are continuous.*
- *For all x in X and all y in Y, the fibres $x \times Y$ and $X \times y$ are homeomorphic to Y and X, respectively.*

Then Π is an irreducible topological space.

The first condition tells us that the topology on $X \times Y$ is at least as fine as the product topology, and the second one tells us that the topology isn't too fine. (We don't want the discrete topology on Π.)

We introduce some notation for use in the proof of the proposition. Let x be a point of X. If W is a subset of $X \times Y$, we denote the intersection $W \cap (x \times Y)$ by ${}_xW$. Similarly, if y is a point of Y, we denote $W \cap (X \times y)$ by W_y. By analogy with the x, y-plane, we call ${}_xW$ a *vertical slice* and W_y a *horizontal slice*, of W.

proof of Proposition 3.3.4. We prove irreducibility by showing that the intersection of two nonempty open subsets W and W' of $X \times Y$ isn't empty (2.2.15).

We show first that the image $U = \pi_2 W$ of an open subset W of $X \times Y$ via the projection to Y is open in Y. We are given that, for every x, the fibre $x \times Y$ is homeomorphic to Y. Since W is open in $X \times Y$, the vertical slice ${}_xW$ is open in $x \times Y$, and its image $\pi_2({}_xW)$ is open in Y. Since W is the union of the sets ${}_xW$, U is the union of the open sets $\pi_2({}_xW)$. So U is open.

Now let W and W' be nonempty open subsets of $X \times Y$, and let U and U' be their images via projection to Y. So U and U' are nonempty open subsets of Y. Since Y is irreducible, $U \cap U'$ isn't empty. Let y be a point of $U \cap U'$.

Since U is the image of W and y is a point of U, the horizontal slice W_y, which is an open subset of the fibre $X \times y$, isn't empty. Similarly, W'_y isn't empty. Since $X \times y$ is homeomorphic to the irreducible space X, it is irreducible. So $W_y \cap W'_y$ isn't empty. Therefore $W \cap W'$ isn't empty, as was to be shown. □

3.3.5. Corollary. *The product $X \times Y$ of two projective varieties X and Y is a projective variety.* □ *prodirred*

(3.3.6) products of affine varieties

We inspect the product $X \times Y$ of the affine varieties $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Say that X is embedded as a closed subvariety of $\mathbb{A}^m$, so that $A = \mathbb{C}[x_1, ..., x_m]/P$ for some prime ideal P, and that Y is embedded similarly into $\mathbb{A}^n$, $B = \mathbb{C}[y_1, ..., y_n]/Q$ for some prime ideal Q. Then in affine x, y-space $\mathbb{A}^{m+n}$, $X \times Y$ is the locus of the equations $f(x) = 0$ and $g(y) = 0$, with f in P and g in Q. Proposition 3.3.4 shows that $X \times Y$ is irreducible, so it is a variety. Let P' and Q' be the ideals of $\mathbb{C}[x, y]$ generated by the elements of P and Q, respectively. Sp P' consists of sums of products of elements of P with polynomials in x, y, and Q' is described analogously.

3.3.7. Proposition. *The ideal $I = P' + Q'$ of $\mathbb{C}[x, y]$ consists of all elements of $\mathbb{C}[x, y]$ that vanish on the variety $X \times Y$. Therefore I is a prime ideal.* *fgprime*

The fact that $X \times Y$ is a variety tells us only that the radical of I is a prime ideal.

proof of Proposition 3.3.7. Let $A = \mathbb{C}[x]/P$, $B = \mathbb{C}[y]/Q$, and $R = \mathbb{C}[x, y]/I$. The map $X \times Y \to X$ is surjective. Therefore the map $A \to R$ is injective. Similarly, $B \to R$ is injective. We identify A and B with their images in R. Any polynomial in x, y can the written, in many ways, as a sum, each of whose terms is a product of a polynomial in x with a polynomial in y: $F(x, y) = \sum a_i(x)b_i(y)$. Therefore any element ρ of R can be written as a finite sum of products

$$\rho = \sum_{i=1}^{k} a_i b_i \tag{3.3.8}$$ pab

with a_i in A and b_i in B. We show that if ρ vanishes identically on $X \times Y$, then $\rho = 0$. To do this, we show that ρ can also be written as a sum of $k-1$ products.

If $a_k = 0$, then $\rho = \sum_{i=1}^{k-1} a_i b_i$, so ρ is a sum of $k-1$ products. If $a_k \neq 0$, the function defined by a_k isn't identically zero on X. We choose a point $\overline{x}$ of X such that $a_k(\overline{x}) \neq 0$. Let $\overline{a}_i = a_i(\overline{x})$ and $\overline{\rho}(y) = \rho(\overline{x}, y)$. So $\overline{\rho}(y) = \sum_{i=1}^{k} \overline{a}_i b_i$ is an element of B. Since ρ vanishes on $X \times Y$, $\overline{\rho}$ vanishes on Y. Therefore $\overline{\rho} = 0$. Then $b_k = \sum_{i=1}^{k-1} c_i b_i$, where $c_i = -\overline{a}_i/\overline{a}_k$. Substituting into ρ and collecting coefficients of $b_1, ..., b_{k-1}$ gives us an expression for ρ as a sum of $k-1$ terms. Finally, when $k = 1$, $\rho = a_1 b_1$, and $\overline{a}_1 b_1 = 0$. Then $b_1 = 0$, and therefore $\rho = 0$. □

3.4 Rational Functions

ratfnsec

(3.4.1) the function field fnfld

Let X be a projective variety, and let $U = \operatorname{Spec} A$ be an affine open subset of X. The function field of X is the field of fractions the coordinate algebra A. The general definition of an affine open set is still to come

(Section 3.6). However, we do know certain affine open sets. If $\mathbb{U}^i$ is one of the standard affine open subsets of the ambient projective space, the intersection $X^i = X \cap \mathbb{U}^i$, if it isn't empty, will be an affine variety — a closed subvariety of $\mathbb{U}^i$. Its localizations will also be affine varieties. The next lemma tells us that there are enough of these affine open sets to work with.

enufaff **3.4.2. Lemma.** *The open subsets of a variety X that are localizations of the nonempty sets $X^i = X \cap \mathbb{U}^i$ form a basis for the topology on X.*

This follows from (2.6.2). □

We'll call the subsets $X^i = X \cap \mathbb{U}^i$ that are nonempty the *standard open subsets* of X.

Up to Section 3.6, when we refer to an affine open subset of a variety X, we mean one of the standard open subsets.

Say that X is a closed subvariety of $\mathbb{P}^n$, and let $x_0, ..., x_n$ be coordinates in $\mathbb{P}^n$. For each $i = 0, ..., n$, let $X^i = X \cap \mathbb{U}^i$. We omit the indices for which X^i is empty. Then X^i will be affine. The intersection $X^{ij} = X^i \cap X^j$ will be a localization of X^i and also a localization of X^j. If $X^i = \operatorname{Spec} A_i$, where A_i is the algebra generated by the images of the elements $u_{ij} = x_j/x_i$ in A_i, and if we denote those images by u_{ij} too. Then $X^{ij} = \operatorname{Spec} A_{ij}$, where $A_{ij} = A_i[u_{ij}^{-1}] = A_j[u_{ji}^{-1}]$.

deffnfld **3.4.3. Definition.** The *function field* K of a projective variety X is the function field of any one of the standard open subsets X^i, and the function field of an open subvariety X' of a projective variety X is the function field of X. All open subvarieties have the same function field. A *rational function* on a variety X is an element of its function field.

For example, let x_0, x_1, x_2 be coordinates in $\mathbb{P}^2$. To write the function field of $\mathbb{P}^2$, we can use the standard open set $\mathbb{U}^0$, which is an affine plane $\operatorname{Spec} \mathbb{C}[u_1, u_2]$ with $u_i = x_i/x_0$. The function field of $\mathbb{P}^2$ is the field of rational functions: $K = \mathbb{C}(u_1, u_2)$. We must use u_1, u_2 as coordinates here. It wouldn't be good to normalize x_0 to 1 and use coordinates x_1, x_2, because we may want to change to another open set such as $\mathbb{U}^1$. The coordinates in $\mathbb{U}^1$ are $v_0 = x_0/x_1$ and $v_2 = x_2/x_1$, so the function field K is also the field of rational functions $\mathbb{C}(v_0, v_2)$. The two fields $\mathbb{C}(u_1, u_2)$ and $\mathbb{C}(v_0, v_2)$ are the same, because $v_0 = u_1^{-1}$ and $v_2 = u_2/u_1$

Let p be a point of X that lies in the standard open set $X^i = \operatorname{Spec} A_i$. A rational function α on X is *regular* at p if it can be written as a fraction a/s of elements of A_i with $s(p) \neq 0$. The *value* of a regular function α at p is $\alpha(p) = a(p)/s(p)$.

Thus a rational function α on a projective variety X can be evaluated at some points of X, usually not at all of them. It will define a function on a nonempty open subset of X.

If X' is an open subvariety of a projective variety S, a rational function on X' is regular at a point p of X' if it is a regular rational function on X at p.

When we regard an affine variety $X = \operatorname{Spec} A$ as a closed subvariety of $\mathbb{U}^0$, its function field will be the field of fractions of A. Proposition 2.7.2 shows that the regular functions on an affine variety $\operatorname{Spec} A$ are the elements of A.

regindep **3.4.4. Lemma.** *Let p be a point of a projective variety X. The regularity of a rational function at p doesn't depend on the choice of a standard open set X^i that contains p.* □

ratfndet **3.4.5. Lemma.** *Let X be a projective variety. A rational function that is regular on a nonempty open set X' is determined by the function it defines on X'.*

This follows from Corollary 2.5.17. □

ptvlfld (3.4.6) points with values in a field

Let K be a field that contains the complex numbers, and let $\mathbb{P}^n$ be the projective space with coordinates $x_0, ..., x_n$. A *point* of $\mathbb{P}^n$ *with values in* K is an equivalence class of nonzero vectors $(\alpha_0, ..., \alpha_n)$ with α_i in K, the equivalence relation being analogous to the one for ordinary points: $\alpha \sim \alpha'$ if $\alpha' = \lambda\alpha$ for some λ in

K. If X is the subvariety of $\mathbb{P}^n$ defined by a homogeneous prime ideal $\mathcal{P}$ of $\mathbb{C}[x]$, a *point α of X with values in K* is a point of $\mathbb{P}^n$ with values in K such that $f(\alpha) = 0$ for all f in $\mathcal{P}$.

Let X be a subvariety of projective space $\mathbb{P}^n$, and let K be the function field of X. The projective embedding defines a point $(\alpha_0, ..., \alpha_n)$ of X with values in K. To get this point, we choose a standard affine open set $\mathbb{U}^i$ of $\mathbb{P}^m$ such that $X^0 = X \cap \mathbb{U}^i$ isn't empty. Say $i = 0$. Then X^0 is affine, say $X^0 = \operatorname{Spec} A_0$. The embedding of X^0 into the affine space $\mathbb{U}^0$ is defined by a homomorphism $\mathbb{C}[u_1, ..., u_n] \to A_0$, with $u_i = x_i/x_0$. If α_i denotes the image of u_i in A_0, for $i = 1, ..., n$, and $\alpha_0 = 1$, then $(\alpha_0, ..., \alpha_n)$ is the point of $\mathbb{P}^n$ with values in the function field K of X.

(3.4.7) the function field of a product ffofprod

To define the function field of the product $X \times Y$ of projective varieties, we use the Segre embedding $\mathbb{P}^m_x \times \mathbb{P}^n_y \to \mathbb{P}^N_w$. Let Π denote the Segre image of $X \times Y$ in $\mathbb{P}^N$. The coordinates in the three projective spaces $\mathbb{P}^m$, $\mathbb{P}^n$, and $\mathbb{P}^N$ are x_i, y_j, and w_{ij}, respectively, and the Segre map is defined by $w_{ij} = x_i y_j$. Let $\mathbb{U}^i$, $\mathbb{V}^j$, and $\mathbb{W}^{ij}$ be the standard affine open sets in the three projective spaces, and let $X^i = X \cap \mathbb{U}^i$, $Y^j = Y \cap \mathbb{V}^j$, and $\Pi^{ij} = \Pi \cap \mathbb{W}^{ij}$, with indices i, j chosen so that X^i and Y^j are nonempty. The product $X^i \times Y^j$ maps bijectively to Π^{ij}, and the function field of Π will be the field of fractions of Π^{ij}.

Since $\Pi^{ij} = X^i \times Y^j$, all that remains to do is to describe the field of fractions of a product Π of affine varieties $X \times Y$, when $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. If $A = \mathbb{C}[x]/P$ and $B = \mathbb{C}[y]/Q$, and if P' and Q' are the ideals of $\mathbb{C}[x, y]$ generated by P and Q, respectively, then the coordinate algebra of Π is the algebra $\mathbb{C}[x, y]/(P' + Q')$ (see Proposition 3.3.7). This is the tensor product algebra $A \otimes B$. We don't need to know much about the tensor product yet, but let's use the tensor notation.

The function field K_X of X is the field of fractions of A. Similarly, K_Y is the field of fractions of B and $K_{X \times Y}$ is the field of fractions of $A \otimes B$. The one fact to note is that $K_{X \times Y}$ isn't generated by K_X and K_Y. For example, if $A = \mathbb{C}[x]$ and $B = \mathbb{C}[y]$ (one x and one y), then $K_{X \times Y}$ is the field of rational functions in two variables $\mathbb{C}(x, y)$. The algebra generated by the fraction fields $\mathbb{C}(x)$ and $\mathbb{C}(y)$ consists of the rational functions $p(x, y)/q(x, y)$ in which $q(x, y)$ is a product of a polynomial in x and a polynomial in y. Most rational functions, $1/(x + y)$ for example, aren't of this type.

The function field $K_{X \times Y}$ of$X \times Y$ is the fraction field of $A \otimes B$. The denominator in a fraction can be any nonzero element of $A \otimes B$.

(3.4.8) interlude: rational functions on projective space ratfn-spspace

Let R denote the polynomial ring $\mathbb{C}[x_0, ..., x_n]$. A *homogeneous fraction* f is a fraction of homogeneous polynomials in $x_0, ..., x_n$. The *degree* of a homogeneous fraction $f = g/h$ is the difference of degrees: $\deg f = \deg g - \deg h$.

If f is a homogeneous fraction of degree d, then $f(\lambda x) = \lambda^d f(x)$. So when d isn't zero, f won't define a function anywhere on projective space. In particular, a homogeneous polynomial g of nonzero degree won't define a function, though it makes sense to say that a homogeneous polynomial g vanishes at a point of $\mathbb{P}^n$.

On the other hand, let $f = g/h$ be homogeneous fraction of degree zero, so that g and h are homogeneous polynomials of the same degree r. Then f does define a function wherever h isn't zero, because $g(\lambda x)/h(\lambda x) = \lambda^r g(x)/\lambda^r h(x) = g(x)/h(x)$.

A homogeneous fraction f is *regular at a point* p of $\mathbb{P}^n$ if, when it is written as a fraction g/h of relatively prime homogeneous polynomials, the denominator h isn't zero at p, and f is *regular on a subset* U if it is regular at every point of U.

3.4.9. Lemma. (i) *Let h be a homogeneous polynomial of positive degree d, and let V be the open subset of $\mathbb{P}^n$ of points at which h isn't zero. The rational functions that are regular on V are those of the form g/h^k, where $k \geq 0$ and g is a homogeneous polynomial of degree dk.* *ho-mogfractsfn-fld*
(ii) *The only rational functions that are regular at every point of $\mathbb{P}^n$ are the constant functions.*

For example, the homogeneous polynomials that don't vanish at any point of the standard affine open set $\mathbb{U}^0$ are the scalar multiples of powers of x_0. So the rational functions that are regular on $\mathbb{U}^0$ are those of the form

g/x_0^k, with g homogeneous of degree k. This agrees with the fact that the coordinate algebra of $\mathbb{U}^0$ is the polynomial ring $\mathbb{C}[u_1,...,u_n]$, $\;u_i = x_i/x_0$, because $g(x_0,...,x_m)/x_0^k = g(u_0,...,u_n)$.

proof of Lemma 3.4.9. **(i)** Let α be a regular function on the open set V, say $\alpha = g_1/h_1$, where g_1 and h_1 are relatively prime homogeneous polynomials. Then h_1 doesn't vanish on V, so its zero locus in $\mathbb{P}^n$ is contained in the zero locus of h. According to the Strong Nullstellensatz, h_1 divides a power of h. Say that $h^k = fh_1$. Then $g_1/h_1 = fg_1/fh_1 = fg_1/h^k$.

(ii) If a rational function f is regular at every point of $\mathbb{P}^n$, then it is regular on $\mathbb{U}^0$. It will have the form g/x_0^k, where g is a homogeneous polynomial of degree k not divisible by x_0. And since f is regular on $\mathbb{U}^1$, it will have the form h/x_1^ℓ, where h is homogeneous and not divisible by x_1. Then $gx_1^\ell = hx_0^k$. Since x_0 doesn't divide g, $\;k = 0$, $\;g$ is a constant, and $f = g$. □

It is also true that the only rational functions that are regular at every point of a projective variety are the constants. The proof of this will be given later (see Corollary 8.2.9). When studying projective varieties, the constant functions are useless, so one has to look at at regular functions on open subsets. One way that affine varieties appear in projective algebraic geometry is as open subsets on which there are enough regular functions.

3.5 Morphisms and Isomorphisms

sectmorph Some morphisms, such as the projection from a product $X \times Y$ to X, are sufficiently obvious that they don't really require discussion. But there are many morphisms that aren't obvious.

Let X and Y be subvarieties of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, respectively. A morphism $Y \to X$, as will be defined below, is determined by a morphism $Y \xrightarrow{f} \mathbb{P}^m$ whose image is contained in X. However, such a morphism neen't be the restriction of a morphism from $\mathbb{P}^n$ to $\mathbb{P}^m$. Most often, there will be no way to extend f to $\mathbb{P}^n$. Put another way, it is usually impossible to define f using polynomials in the coordinate variables of $\mathbb{P}^n$.

veronex **3.5.1. Example.** The Veronese map from the projective line $\mathbb{P}^1$ to $\mathbb{P}^2$, defined by $(x_0, x_1) \rightsquigarrow (x_0^2, x_0x_1, x_1^2)$, is an obvious morphism. Let's denote the coordinates in the projective plane $\mathbb{P}^2$ by y_0, y_1, y_2 here, instead of v_{ij}. The image of the Veronese map is the conic $C : \{y_0y_2 - y_1^2 = 0\}$ in $\mathbb{P}^2$. The Veronese defines a bijective morphism $\mathbb{P}^1 \xrightarrow{f} C$ whose inverse function π sends a point (y_0, y_1, y_2) of C with $y_0 \neq 0$ to the point $(x_0, x_1) = (y_1, y_2)$, and it send the remaining point, which is $(0, 0, 1)$, to $(0, 1)$. Though π is a morphism, there is no way to extend it to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$. In fact, the only morphisms from $\mathbb{P}^2$ to $\mathbb{P}^1$ are the constant morphisms whose images are points.

We define morphisms using points with values in a field.

morphtoP **(3.5.2) morphisms to projective space**

A morphism from a variety X to projective space $\mathbb{P}^n$ will be defined by a point of $\mathbb{P}^n$ with values in the function field K of X. We must keep in mind that points of projective space are equivalence classes of vectors, not the vectors themselves. As we will see, this complication turns out to be useful.

For the rest of this section, it will be helpful to have a separate notation for the point with values in a field K that is determined by a nonzero vector $\alpha = (\alpha_0, ..., \alpha_n)$, with entries in K. We'll denote that point by $\underline{\alpha}$. So if α and α' are points with values in K, then $\underline{\alpha} = \underline{\alpha}'$ if $\alpha' = \lambda\alpha$ for some nonzero λ in K. We'll drop this notation later.

Let $\alpha = (\alpha_0, ..., \alpha_n)$ be a nonzero vector with entries in K be the function field of a variety Y. We try to use the point $\underline{\alpha}$ with values in K to define a morphism from Y to projective space $\mathbb{P}^n$. To define the image $\underline{\alpha}(q)$ of a point q of Y (an ordinary point), we look for a vector $\alpha' = (\alpha'_0, ..., \alpha'_n)$, with $\underline{\alpha}' = \underline{\alpha}$, i.e., $\alpha' = \lambda\alpha$ with λ in K, such that the rational functions α'_i are regular at q and not all zero there. This vector may exist or not. If there exists such a vector, we define

(3.5.3) $$\underline{\alpha}(q) = (\alpha'_0(q), ..., \alpha'_n(q)) \quad \big(= \underline{\alpha}'(q)\big)$$ defmor-phP

If such a vector α' exists for every point q of Y, we call $\underline{\alpha}$ a *good point*.

3.5.4. Lemma. *A point $\underline{\alpha}$ of $\mathbb{P}^n$ with values in the function field K_Y of Y is a good point if either one of the two following conditions holds for every point q of Y:* *twoconds*

- *There is an element λ of K_Y such that the rational functions $\alpha'_i = \lambda\alpha_i$, $i = 0, ..., n$, are regular and not all zero at q.*
- *There is an index j such that the rational functions α_i/α_j are regular at q, for $i = 0, ..., n$.*

proof. The first condition simply restates the definition. We show that it is equivalent with the second one. Suppose that α_i/α_j are regular at q for all i. Let $\lambda = \alpha_j^{-1}$, and let $\alpha'_i = \lambda\alpha_i = \alpha_i/\alpha_j$. The rational functions α'_i are regular at q, and they aren't all zero there because $\alpha'_j = 1$. Conversely, suppose that for some nonzero λ in K_Y, $\alpha'_i = \lambda\alpha_i$ are all regular at q and that α'_j isn't zero there. Then ${\alpha'_j}^{-1}$ is a regular function at q, so the rational functions α'_i/α'_j, which are equal to α_i/α_j, are regular at q for all i. □

3.5.5. Lemma. *With notation as in (3.5.3), the image $\underline{\alpha}(q)$ in $\mathbb{P}^n$ of a point q is independent of the choice of the vector α'.* *doesntde-pend*

proof. This follows from Lemma 3.5.4 because the second assertion of that lemma doesn't involve λ. □

3.5.6. Definition. Let Y be a variety with function field K. A *morphism* from Y to projective space $\mathbb{P}^n$ is a map that is defined, as in (3.5.3), by a good point $\underline{\alpha}$ with values in K. defmor-phtoP

We will often denote the morphism defined by a good point $\underline{\alpha}$ by the same symbol $\underline{\alpha}$.

3.5.7. Examples. identmap

(i) *The identity map* $\mathbb{P}^1 \to \mathbb{P}^1$. Let $X = \mathbb{P}^1$, and let (x_0, x_1) be coordinates in X. The function field of X is the field $K = \mathbb{C}(t)$ of rational functions in the variable $t = x_1/x_0$. The identity map $X \to X$ is the map $\underline{\alpha}$ defined by the point $\alpha = (1, t)$ with values in K. For every point p of X except the point $(0, 1)$, $\underline{\alpha}(p) = \alpha(p) = (1, t(p))$. For the point $q = (0, 1)$, we let $\alpha' = t^{-1}\alpha = (t^{-1}, 1)$. Then $\underline{\alpha}(q) = \alpha'(q) = (x_0(q)/x_1(q), 1) = (0, 1)$.

(ii) We go back to Example 3.5.1, in which C is the conic $y_0y_2 = y_1^2$ and f is the morphism $\mathbb{P}^1 \to C$ defined by $f(x_0, x_1) = (x_0^2, x_0x_1, x_1^2)$. The inverse morphism π can be described as the projection from C to the line $L_0 : \{y_0 = 0\}$, $\pi(y_0, y_1, y_2) = (y_1, y_2)$. This formula is undefined at the point $q = (1, 0, 0)$, though the map extends to the whole conic C. Let's write this projection using a point with values in the function field K of C. The affine open set $\{y_0 \neq 0\}$ of $\mathbb{P}^2$ is the polynomial algebra $\mathbb{C}[u_1, u_2]$, with $u_1 = y_1/y_0$ and $u_2 = y_2/y_0$. We denote the restriction of the function u_i to $C^0 = C \cap \mathbb{U}^0$ by u_i too. The restricted functions are related by the equation that is obtained by dehomogenizing f: $u_2 - u_1^2 = 0$, or $u_2 = u_1^2$, The function field K is $\mathbb{C}(u_1)$.

The projection π is defined by the point $\alpha = (u_1, u_1^2)$ with values in K: $\pi(y_0, y_1, y_2) = \pi(1, u_1, u_2) = (u_1, u_1^2)$. Multiplying by $\lambda = u_1^{-1}$, we see that $\underline{\alpha} = \underline{\alpha}'$, where $\alpha' = (1, u_1)$. This formula defines the projection at all points of C at which $u_1 = y_1/y_0$ is regular — at all points such that $y_0 \neq 0$. When $y_0 = 0$, the equation of C shows that $y_1 = 0$ as well. The only point at which u_1 fails to be regular is the point $p = (0, 0, 1)$.

Here $\alpha' = (1, u_1) = (\alpha'_0, \alpha'_1)$. If α is a good point, Lemma 3.5.4 tells us that $\alpha'_0/\alpha'_1 = (u_1^{-1}, 1) = (y_0/y_1, 1)$ must be regular at p. Since $y_2 = 1$ at p, we may set $y_2 = 1$ into the equation $y_0y_2 = y_1^2$ for C, obtaining $y_0 = y_1^2$. Then $y_0/y_1 = y_1$ is regular at p, as required. □

(3.5.8) morphisms to projective varieties morphtoV

3.5.9. Definition. Let Y be a variety, and let X be a subvariety of a projective space $\mathbb{P}^m$. A morphism of varieties $Y \xrightarrow{\alpha} X$ is the restriction of a morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ whose image is contained in X. defmor-phtoX

3.5.10. Corollary. *LetX be a projective variety X that is the locus of zeros of a family f of homogeneous polynomials. A morphism $Y \xrightarrow{\alpha} \mathbb{P}^m$ defines a morphism $Y \to X$ if and only if $f(\alpha) = 0$.* □ *ptfvalK-fzero*

Note. A morphism $Y \xrightarrow{\alpha} X$ doesn't define a map of function fields $K_X \to K_Y$ unless the image of Y is dense in X. $\square$

morphcont **3.5.11. Proposition.** *A morphism of varieties $Y \xrightarrow{\alpha} X$ is a continuous map in the Zariski topology, and also a continuous map in the classical topology.*

proof. Since the topologies on X are induced from those on $\mathbb{P}^m$, we may suppose that $X = \mathbb{P}^m$. Let $\mathbb{U}^i$ be a standard affine open subset of $\mathbb{P}^m$ whose inverse image in Y isn't empty, and let Y' be an affine open subset of that inverse image. The restriction $Y' \to \mathbb{U}^i$ of the morphism $\underline{\alpha}$ is continuous in either topology because it is a morphism of affine varieties, as was defined in Section 2.7. Since Y is covered the affine open sets such as Y', $\underline{\alpha}$ is continuous. $\square$

comp-morph **3.5.12. Proposition.** *Let X, Y, and Z be varieties and let $Z \xrightarrow{\beta} Y$ and $Y \xrightarrow{\alpha} X$ be morphisms. The composed map $Z \xrightarrow{\alpha\beta} X$ is a morphism.*

proof. We spell this simple proof out, perhaps in too much detail. Say that X is a subvariety of $\mathbb{P}^m$. Let K_Y and K_Z bee the functions fields of Y and Z, respectively. The morphism $\underline{\alpha}$ is the restriction of a morphism $Y \to \mathbb{P}^m$ whose image is contained in X, and that is defined by a good point $\alpha = (\alpha_0, ..., \alpha_m)$ of $\mathbb{P}^m$ with values K_Y. Similarly, if Y is a subvariety of $\mathbb{P}^n$, the morphism $\underline{\beta}$ is the restriction of a morphism $Z \xrightarrow{\beta} \mathbb{P}^n$ whose image is contained in Y, that is defined by a good point $\beta = (\beta_0, ..., \beta_n)$ of $\mathbb{P}^n$, with values K_Z.

Let z be an ordinary point of Z. Since $\underline{\beta}$ is a good point, we may adjust β by a factor in K_Z so that the rational functions β_i are regular and not all zero at z. Then $\underline{\beta}(z)$ is the point $(\beta_0(z), ..., \beta_n(z))$. Let's denote that point by $q = (q_0, ..., q_n)$, where $q_i = \beta_i(z)$. The elements α_j are rational functions on Y. Since α is a good point, we may adjust by a factor in K_Y, so that α_i are all regular and not all zero at q. Then $[\underline{\alpha\beta}](z) = \underline{\alpha}(q) = (\alpha_0(q), ..., \alpha_m(q))$, and $\alpha_j(q) = \alpha_j(\beta_0(z), ..., \beta_n(z)) = \alpha_j(\beta(z))$ are not all zero. When these adjustments have been made, the point of $\mathbb{P}^m$ with values in K_Z that defines $\underline{\alpha\beta}$ is $(\alpha_0(\beta(z)), ..., \alpha_m(\beta(z)))$. $\square$

closedinaff **3.5.13. Lemma.** *Let $\{X^i\}$ be a covering of a topological space X by open sets. A subset V of X is open if and only if $V \cap X^i$ is open in X^i for every i, and a subset V of X is closed if and only if $V \cap X^i$ is closed in X^i for every i.*

In particular, if $\{\mathbb{U}^i\}$ is the standard affine cover of $\mathbb{P}^n$, a subset V of $\mathbb{P}^n$ is open (or closed) if and only if $V \cap \mathbb{U}^i$ is open (or closed) in $\mathbb{U}^i$ for every i. $\square$

firstpropmorph **3.5.14. Lemma.**

(i) *The inclusion of an open or a closed subvariety Y into a variety X is a morphism.*

(ii) *Let $Y \xrightarrow{f} X$ be a map whose image lies in an open or a closed subvariety Z of X. Then f is a morphism if and only if its restriction $Y \to Z$ is a morphism.*

(iii) *Let $\{Y^i\}$ be an open covering of a variety Y, and let $Y^i \xrightarrow{f^i} X$ be morphisms. If the restrictions of f^i and f^j to the intersections $Y^i \cap Y^j$ are equal for all i, j, there is a unique morphism f whose restriction to Y^i is f^i.*

We omit the proofs of **(i)** and **(ii)**. Part **(iii)** is true because the points with values in K that define the morphisms f^i are equal. $\square$

mappropprod **(3.5.15) the mapping property of a product**

The product $X \times Y$ of sets is characterized by this mapping property: Maps from a set T to the product $X \times Y$, correspond bijectively to pairs of maps $T \xrightarrow{f} X$ and $T \xrightarrow{g} Y$. The map $T \xrightarrow{(f,g)} X \times Y$ that corresponds to a pair of maps f, g sends a point t to the point pair $(f(t), g(t))$. If $T \xrightarrow{h} X \times Y$ is a map to the product, the corresponding maps to X and Y are the compositions with the projections $X \times Y \xrightarrow{\pi_1} X$ and $X \times Y \xrightarrow{\pi_2} Y$: $T \xrightarrow{\pi_1 h} X$ and $T \xrightarrow{\pi_2 h} Y$.

Parts **(i)** and **(ii)** of the next proposition assert that the analogous statements are true for morphisms of varieties.

3.5.16. Proposition. *Let X and Y be varieties, and let $X\times Y$ be the product variety.* *mapprop-var*

(i) *The projections $X\times Y \xrightarrow{\pi_1} X$ and $X\times Y \xrightarrow{\pi_2} Y$ are morphisms.*

(ii) *Morphisms from a variety T to the product variety $X\times Y$ correspond bijectively to pairs of morphisms $T\to X$ and $T\to Y$, the correspondence being the same as for maps of sets.*

(iii) *If $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} W$ are morphisms of varieties, the product map $X\times Y \xrightarrow{f\times g} Z\times W$, which is defined by $[f\times g](x,y) = (f(x), g(y))$, is a morphism.* □

(3.5.17) isomorphisms

isomorphisms

An *isomorphism* of varieties is a bijective morphism $Y \xrightarrow{u} X$ whose inverse function is also a morphism. Isomorphisms are important because they allow us to identify different incarnations of what might be called the "same" variety, i.e., to describe an isomorphism class of varieties. For example, the projective line $\mathbb{P}^1$, a conic in $\mathbb{P}^2$, and a twisted cubic in $\mathbb{P}^3$ are isomorphic.

3.5.18. Example. Let Y denote the projective line with coordinates y_0, y_1. As before, the function field K of Y is the field of rational functions in $t = y_1/y_0$. The degree 3 Veronese map $Y \longrightarrow \mathbb{P}^3$ (**3.1.17**) defines an isomorphism of Y to its image X, a twisted cubic. It is defined by the vector $\alpha = (1, t, t^2, t^3)$ of $\mathbb{P}^3$ with values in K, and $\alpha' = (t^{-3}, t^{-2}, t^{-1}, 1)$ defines the same point. The rational functions t^k are regular and not all zero on the open set $\{y_0 \neq 0\}$ of Y, and the functions t^{-k} are regular on the open set $\{y_1 \neq 0\}$. So $\underline{\alpha}$ is a good point. It defines the degree 3 Veronese map. twistcubicinverse

The twisted cubic X is the locus of zeros of the equations $v_0v_2 = v_1^2$, $v_2v_1 = v_0v_3$, $v_1v_3 = v_2^2$. To identify the function field of X, we put $v_0 = 1$, obtaining relations $v_2 = v_1^2, v_3 = v_1^3$. The function field is the field $F = \mathbb{C}(v_1)$. The point of $Y = \mathbb{P}^1$ with values in F that defines the inverse of the morphism $\underline{\alpha}$ is $\beta = (1, v_1)$. □

3.5.19. Lemma. *Let $Y \xrightarrow{f} X$ be a morphism of varieties, let $\{X^i\}$ and $\{Y^i\}$ be open coverings of X and Y, respectively, such that the image of Y^i in X is contained in X^i. If the restrictions $Y^i \xrightarrow{f^i} X^i$ of f are isomorphisms, then f is an isomorphism.* *propisom*

proof. Let g^i denote the inverse of the morphism f^i. Then $g^i = g^j$ on $X^i \cap X^j$ because $f^i = f^j$ on $Y^i \cap Y^j$. By (3.5.14) **(iii)**, there is a unique morphism $X \xrightarrow{g} Y$ whose restriction to Y^i is g^i. That morphism is the inverse of f. □

(3.5.20) the diagonal

diagonal

Let X be a variety. In $X\times X$, the *diagonal* X_Δ is the set of points (p,p). It is an example of a subset of $X\times X$ that is closed in the Zariski topology, but not closed in the product topology.

3.5.21. Proposition. *Let X be a variety. The diagonal X_Δ is a closed subvariety of the product variety $X\times X$, and it is isomorphic to X.* *diagclosed*

proof. Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$ that contains X, and let $x_0, ..., x_n$ and $y_0, ..., y_n$ be coordinates in the two factors of $\mathbb{P}\times\mathbb{P}$. The diagonal $\mathbb{P}_\Delta$ in $\mathbb{P}\times\mathbb{P}$ is the closed subvariety defined by the bilinear equations $x_iy_j = x_jy_i$, or in the Segre variables, by the equations $w_{ij} = w_{ji}$, which show that the ratios x_i/x_j and y_i/y_j are equal.

Next, let X be the closed subvariety of $\mathbb{P}$ defined by a system of homogeneous equations $f(x) = 0$. The diagonal X_Δ can be identified as the intersection of the product $X\times X$ with the diagonal $\mathbb{P}_\Delta$ in $\mathbb{P}\times\mathbb{P}$, so it is a closed subvariety of $X\times X$. As a closed subvariety of $\mathbb{P}\times\mathbb{P}$, the diagonal X_Δ is defined by the equations

$$x_iy_j = x_jy_i \quad \text{and} \quad f(x) = 0 \tag{3.5.22}$$

Xdelta

The equations $f(y) = 0$ hold too. They are redundant. The morphisms $X \xrightarrow{(id,id)} X_\Delta \xrightarrow{\pi_1} X$ show that X_Δ is isomorphic to X. □

It is interesting to compare Proposition 3.5.21 with the Hausdorff condition for a topological space. The proof of the next lemma is often assigned as an exercise in topology.

hausdorff-diagonal **3.5.23. Lemma.** *A topological space X is a Hausdorff space if and only if, when $X\times X$ is given the product topology, the diagonal X_Δ is a closed subset of $X\times X$.* $\square$

Though a variety X with its Zariski topology isn't a Hausdorff space unless it is a point, Lemma 3.5.23 doesn't contradict Proposition 3.5.21 because the Zariski topology on $X\times X$ is finer than the product topology.

ggraph **(3.5.24) the graph of a morphism**

Let $Y \xrightarrow{f} X$ be a morphism of varieties. The *graph* Γ of f is the subset of $Y\times X$ of pairs (q,p) such that $p=f(q)$.

graph **3.5.25. Proposition.** *The graph Γ_f of a morphism $Y \xrightarrow{f} X$ is a closed subvariety of $Y\times X$, and it is isomorphic to Y.*

proof. We form a diagram of morphisms

graphdiagram (3.5.26)
$$\begin{array}{ccc} \Gamma_f & \longrightarrow & Y\times X \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle f\times id} \\ X_\Delta & \longrightarrow & X\times X \end{array}$$

where v sends a point (q,p) of Γ_f with $f(q)=p$ to the point (p,p) of X_Δ. The graph is the inverse image in $Y\times X$ of the diagonal. Since the diagonal is closed in $X\times X$, Γ_f is closed in $Y\times X$.

Let π_1 denote the projection from $Y\times X$ to Y. The composition of the morphisms $Y \xrightarrow{(id,f)} Y\times X \xrightarrow{\pi_1} Y$ is the identity map on Y, and the image of the map (id,f) is the graph Γ_f. Therefore Y maps bijectively to Γ_f. The two maps $Y\to\Gamma_f$ and $\Gamma_f\to Y$ are inverses, so Γ_f is isomorphic to Y. $\square$

defprojection **(3.5.27) projection**

The map

projectiontwo (3.5.28)
$$\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^{n-1}$$

that drops the last coordinate of a point: $\pi(x_0,...,x_n)=(x_0,...,x_{n-1})$ is called a *projection*. (The projection from $\mathbb{P}^2$ to $\mathbb{P}^1$ was defined in Chapter 1.) The projection is defined at all points of $\mathbb{P}^n$ except at the point $q=(0,...,0,1)$, the *center of projection*. So π is a morphism from the complement $U=\mathbb{P}^n-\{q\}$ to $\mathbb{P}^{n-1}$:

$$U \xrightarrow{\pi} \mathbb{P}^n$$

The points of U are those of the form $(x_0,...,x_{n-1},1)$

Let the coordinates in $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ be $x=x_0,...,x_n$ and $y=y_0,...,y_{n-1}$, respectively. The fibre $\pi^{-1}(y)$ over a point $(y_0,...,y_{n-1})$ is the set of points $(x_0,...,x_n)$ such that $(x_0,...,x_{n-1})=\lambda(y_0,...,y_{n-1})$, while x_n is arbitrary. It is the line in $\mathbb{P}^n$ through the points $(y_1,...,y_{n-1},0)$ and $q=(0,...,0,1)$, with the center of projection q omitted.

In Segre coordinates, the graph Γ of π in $U\times\mathbb{P}^{n-1}_y$ is the locus of solutions of the equations $w_{ij}=w_{ji}$ for $0\le i,j\le n-1$, which imply that the vectors $(x_0,...,x_{n-1})$ and $(y_0,...,y_{n-1})$ are proportional.

projgrah **3.5.29. Proposition.** *In $\mathbb{P}^n_x\times\mathbb{P}^{n-1}_y$, the locus of the equations $x_iy_j=x_jy_i$, or $w_{ij}=w_{ji}$, with $0\le i,j\le n-1$ is the closure $\overline{\Gamma}$ of the graph Γ of π.*

proof. At points $x\neq q$, the solutions of the equations are the points of Γ, and the equations hold at all remaining points of $\mathbb{P}^n\times\mathbb{P}^{n-1}$, the points (q,y). So the locus $\overline{\Gamma}$, a closed set, is contained in the union $\Gamma\cup(q\times\mathbb{P}^{n-1})$. To show that $\overline{\Gamma}$ is equal to that union, we must show that, if a homogeneous polynomial $g(w)$ vanishes on Γ, then it vanishes at all points of $q\times\mathbb{P}^{n-1}$. Given $(y_0,...,y_{n-1})$ in $\mathbb{P}^{n-1}$, let $x=(ty_0,...,ty_{n-1},1)$. For all $t\neq 0$, the point (x,y) is in Γ and therefore $g(x,y)=0$. Since g is a continuous function, $g(x,y)$ approaches $g(q,y)$ as $t\to 0$. So $g(q,y)=0$. $\square$

The projection $\overline{\Gamma}\to\mathbb{P}^n_x$ that sends a point (x,y) to x is bijective except when $x=q$, and the fibre over q, which is $q\times\mathbb{P}^{n-1}$, is a projective space of dimension $n-1$. Because the point q of $\mathbb{P}^n$ is replaced by a projective space in $\overline{\Gamma}$, the map $\overline{\Gamma}\to\mathbb{P}^n_x$ is called a *blowup* of the point q.

3.5.30. Proposition. *Let $Y\xrightarrow{\alpha}X$ and $Z\xrightarrow{\beta}W$ be morphisms of varieties. The map $Y\times Z\xrightarrow{\alpha\times\beta}X\times W$ that sends (y,z) to $(\underline{\alpha}(y),\underline{\beta}(z))$ is a morphism.* *pro-ductmap*

proof. Let p and q be points of X and Y, respectively. We may assume that α_i are regular and not all zero at p and that β_j are regular and not all zero at q. Then, in the Segre coordinates w_{ij}, $[\alpha\times\beta](p,q)$ is the point $w_{ij}=\alpha_i(p)\beta_j(q)$. We must show that $\alpha_i\beta_j$ are all regular at (p,q) and are not all zero there. This follows from the analogous properties of α_i and β_j. $\square$

3.6 Affine Varieties

We have used the term 'affine variety' in several contexts: An irreducible closed subset of affine space $\mathbb{A}^n_x$ is an affine variety, the set of zeros of a prime ideal P of $\mathbb{C}[x]$. The spectrum $\operatorname{Spec}A$ of a finite type domain A is an affine variety. A closed subvariety in $\mathbb{A}^n$ becomes a variety in $\mathbb{P}^n$ when the ambient affine space $\mathbb{A}^n$ is identified with the standard open subset $\mathbb{U}^0$. affineV

We combine these definitions now: An *affine variety* X is a variety that is isomorphic to a variety of the form $\operatorname{Spec}A$.

If X is an affine variety with coordinate algebra A and function field K, then A will be the subalgebra of K whose elements are the regular functions on X. So A and $\operatorname{Spec}A$ are determined uniquely by X, and the isomorphism $\operatorname{Spec}A\to X$ is determined uniquely too. When A is the coordinate algebra of an affine variety X, it seems permissible to identify it with $\operatorname{Spec}A$.

(3.6.1) regular functions on affine varieties

regfnon-affinetwo

Let $X=\operatorname{Spec}A$ be an affine variety. Its function field K is the field of fractions of the coordinate algebra A. As Proposition 2.7.2 shows, the regular functions on X are the elements of A.

3.6.2. Lemma. *Xmaps*

(i) *Let R be the algebra of regular functions on a variety Y, and let $X=\operatorname{Spec}A$ be an affine variety. A homomorphism $A\to R$ defines a morphism $Y\xrightarrow{f}X$.*

(ii) *When X and Y are affine varieties, say $X=\operatorname{Spec}A$ and $Y=\operatorname{Spec}B$, morphisms $Y\to X$, as defined in (3.5.9), correspond bijectively to algebra homomorphisms $A\to B$, as in Definition 2.7.4.*

proof of Lemma 3.6.2. **(i)** Since Y isn't affine, we don't know much about the algebra R, but it is a subring of the function field of Y, whose elements are the rational functions that are regular at every point of Y.

Let $\{Y^i\}$ be an affine open covering of Y, and let R_i be the coordinate algebra of Y^i. A rational function that is regular on Y is regular on Y^i, so $R\subset R_i$. The homomorphisms $A\to R\subset R_i$ define morphisms $Y^i=\operatorname{Spec}R_i\xrightarrow{f^i}\operatorname{Spec}A$ for each i. It is true that $f^i=f^j$ on the affine variety $Y^i\cap Y^j$. So Lemma 3.5.14 **(iii)** shows that there is a unique morphism $Y\xrightarrow{f}\operatorname{Spec}A$ that restricts to f^i on Y^i.

(ii) We choose a presentation of A, to embed X as a closed subvariety of affine space, and we identify that affine space with the standard affine open set $\mathbb{U}^0$ of $\mathbb{P}^n$. Let $x_0,...,x_n$ be coordinates in $\mathbb{P}^n$, and let K be the function field of Y — the field of fractions of B. A morphism $Y\xrightarrow{u}X$ is determined by a point α with values in K, and since the image of u is contained in $\mathbb{U}^0$, $\alpha_0\neq 0$. We may suppose that $\alpha=(1,\alpha_1....,\alpha_n)$. Then the rational functions α_i are regular at every point of Y. (See the second bullet of Lemma 3.5.4.) So α_i are elements of B. The coordinate algebra A of X is generated by the residues of the coordinate variables x, with $x_0=1$, and sending $x_i\to\alpha_i$ defines a homomorphism $A\xrightarrow{\varphi}B$. Conversely, if φ is such a homomorphism, the good point that defines the morphism $Y\xrightarrow{u}X$ is $(1,\varphi(x_1),...,\varphi(x_n))$. $\square$

pullback-reg **3.6.3. Corollary.** *Let $Y \xrightarrow{f} X$ be a morphism of varieties, let q be a point of X, and let $p = f(q)$. If g is a rational function on X that is regular at p, its pullback $g \circ f$ is a regular function on Y at q.*

proof. We choose an affine open neighborhood U of p in X, such that g is a regular function on U, and we choose an affine neighborhood V of q in Y contained in the inverse image $f^{-1}U$. The morphism f restricts to a morphism $V \to U$ that we denote by the same letter f. Let A and B be the coordinate algebras of U and V, respectively. The morphism $V \xrightarrow{f} U$ corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. On U, the function g is an element of A, and $g \circ f = \varphi(g)$. $\square$

affopens **(3.6.4)** **affine open sets**

Now that we have a definition of an affine variety, we can make the next definition. Though fairly obvious, it is important: An *affine open subset* of a variety X is an open subvariety that is an affine variety. This will be the definition from now on.

A nonempty open subset V of X is an affine open subset if and only if the algebra R of rational functions that are regular on V is a finite-type domain, so that $\operatorname{Spec} R$ is defined, and V is isomorphic to $\operatorname{Spec} R$.

affinesbasis **3.6.5. Lemma.** *The affine open subsets of a variety X form a basis for the topology on X.* $\square$

locloctwo **3.6.6. Lemma.** *Let U and V be open subsets of an affine variety X.*
(i) *If U is a localization of X and V is a localization of U, then V is a localization of X.*
(ii) *If $V \subset U$ nd V is a localization of X, then V is a localization of U.*
(iii) *Let p be a point of $U \cap V$. There is an open set Z containing p that is a localization of U and also a localization of V.*

proof. **(i)** Let $X = \operatorname{Spec} A$, let $U = X_s = \operatorname{Spec} A_s$, and let $V = U_t = \operatorname{Spec}(A_s)_t$, where s is a nonzero element of A and t is a nonzero element of A_s. Say that $t = rs^{-k}$ with r in A. The localizations $(A_s)_t$ and $(A_s)_r$ are equal, and $(A_s)_r = A_{sr}$. So $V = X_{sr}$.

(ii) Say that $X = \operatorname{Spec} A$, $U = \operatorname{Spec} B$, and $V = \operatorname{Spec} A_s$, where s is a nonzero element of A. A regular function on X restricts to a regular function on U, and a regular function on U restricts to a regular function on V. So $A \subset B \subset A_s$. Since $A \subset B$, $A_s \subset B_s$ and since $B \subset A_s$, $B_s \subset A_s$. Therefore $A_s = B_s$.

(iii) The localizations form a basis for the topology on X. So $U \cap V$ contains a localization X_s of X that contains p. By **(ii)**, X_s is a localization of U and a localization of V. $\square$

comphyper **3.6.7. Proposition.** *The complement of a hypersurface is an affine open subvariety of $\mathbb{P}^n$.*

proof. Let H be the hypersurface defined by an irreducible homogeneous polynomial f of degree d, and let Y be the complement of H in $\mathbb{P}^n$. Let R and K be the algebra of regular functions and the field of rational functions on Y, respectively.

The elements of R are the homogeneous fractions of degree zero of the form g/f^k (**3.4.8**), and the fractions m/f, where m is a monomial of degree d, generate R. Since there are finitely many monomials of degree d, R is a finite-type domain. Lemma 3.6.2 gives us a morphism $Y \xrightarrow{u} X = \operatorname{Spec} R$. We show that u is an isomorphism.

Let A be the algebra of regular functions on the standard affine open set $\mathbb{U}^0$ of $\mathbb{P}^n$. The intersection $Y^0 = Y \cap \mathbb{U}^0$ is a localization of $\mathbb{U}^0$. It is the spectrum of $A[s^{-1}]$, where s is the element f/x_0^d. Let $t = x_0^d/f$. This is an element of R.

Yzero **3.6.8. Lemma.** *The algebras $A[s^{-1}]$ and $R[t^{-1}]$ are equal.*

proof. The generators m/f of R can be written as products $s^{-1}(m/x_0^d)$. Since m/x_0^d is in A, the generators are in the localization A_s. So $R \subset A_s$, and since $t^{-1} = s$ is in A, $R_t \subset A_s$. Next, the fractions x_i/x_0 generate A, and x_i/x_0 can be written as $t^{-1}(m/f)$, with $m = x_i x_0^{d-1}$, so they are in R_t. Then $A \subset R_t$ and since $s^{-1} = t$ is in R, $A_s \subset R_t$. $\square$

We go back to the proof of the proposition. According Lemma 3.6.8, the morphism $Y \xrightarrow{u} X$ restricts to an isomorphism $Y^0 \to X^0 = \operatorname{Spec} A[s^{-1}]$. Since the index 0 can be replaced by any $i = 0, ..., n$, Lemma 3.5.19 shows that u is an isomorphism $\square$

3.6.9. Theorem. *Let U and V be affine open subvarieties of a variety X, say $U \approx \operatorname{Spec} A$ and $V \approx \operatorname{Spec} B$. The intersection $U \cap V$ is an affine open subvariety. Its coordinate algebra is generated by the two algebras A and B.* *inter-sectaffine*

proof. We denote by $[A, B]$ the subalgebra generated by two subalgebras A and B of the function field K of X. The elements of $[A, B]$ are finite sums of products $\sum a_i b_i$ with a_i in A and b_i in B. If $A = \mathbb{C}[a_1, ..., a_r]$, and $B = \mathbb{C}[b_1, ..., b_s]$, then $[A, B]$ is generated by the set $\{a_i\} \cup \{b_j\}$.

Let $R = [A, B]$ and let $W = \operatorname{Spec} R$. We are to show that W is isomorphic to $U \cap V$. The varieties U, V, W, and X have the same function field K, and the inclusions of coordinate algebras $A \to R$ and $B \to R$ give us morphisms $W \to U$ and $W \to V$. We also have inclusions $U \subset X$ and $V \subset X$, and X is a subvariety of a projective space $\mathbb{P}^n$. Restricting the projective embedding of X gives us the projective embeddings of U and V and it gives us a morphism from W to $\mathbb{P}^n$. All of these morphisms to $\mathbb{P}^n$ will be defined by the same good point α with values in K, the one that defines the projective embedding of X. Let's denote these morphisms to $\mathbb{P}^n$ by $\underline{\alpha}_X, \underline{\alpha}_U, \underline{\alpha}_V$ and $\underline{\alpha}_W$. The morphism $\underline{\alpha}_W$ can be obtained as the composition of the morphisms $W \to U \subset X \xrightarrow{\underline{\alpha}_X} \mathbb{P}^n$, and also as the analogous composition, in which V replaces U. Therefore the image of W in $\mathbb{P}^n$ is contained in $U \cap V$. Thus $\underline{\alpha}_W$ restricts to a morphism $W \xrightarrow{\epsilon} U \cap V$. We show that ϵ is an isomorphism.

Let p be a point of $U \cap V$. We choose an affine open subset Z of $U \cap V$ that is a localization of U and that contains p (3.6.6). Let S be the coordinate algebra of Z. So $S = A_s$ for some nonzero s in A, and $B \subset S$. Then

$$R_s = [A, B]_s = [A_s, B] = [S, B] = S$$

So ϵ maps the localization $W_s = \operatorname{Spec} R_s$ of W isomorphically to the open subset $Z = \operatorname{Spec} S$ of $U \cap V$, and since we can cover $U \cap V$ by open sets such as Z, Lemma 3.5.14 **(ii)** shows that ϵ is an isomorphism. $\square$

3.7 Lines in Projective Three-Space

linesinp

The *Grassmanian* $\mathbf{G}(m, n)$ is a variety whose points correspond to subspaces of dimension m of the vector space $\mathbb{C}^n$, and to linear subspaces of dimension $m-1$ of $\mathbb{P}^{n-1}$. One says that $\mathbf{G}(m, n)$ *parametrizes* those subspaces. Our first example is the Grassmanian $\mathbf{G}(1, n+1)$, which is the projective space $\mathbb{P}^n$. Points of $\mathbb{P}^n$ parametrize the one-dimensional subspaces of $\mathbb{C}^{n+1}$.

The Grassmanian $\mathbf{G}(2, 4)$ parametrizes two-dimensional subspaces of $\mathbb{C}^4$, or lines in $\mathbb{P}^3$. We denote that Grassmanian by $\mathbb{G}$, and we describe $\mathbb{G}$ in this section. The point of $\mathbb{G}$ that corresponds to a line ℓ in $\mathbb{P}^3$ will be denoted by $[\ell]$.

One can get some insight into the structure of $\mathbb{G}$ using row reduction. Let $V = \mathbb{C}^4$, let u_1, u_2 be a basis of a two-dimensional subspace U of V and let M be the 2×4 matrix whose rows are u_1, u_2. The rows of the matrix M' obtained from M by row reduction span the same space U, and the row-reduced matrix M' is uniquely determined by U. Provided that the left hand 2×2 submatrix of M is invertible, M' will have the form

(3.7.1) $$M' = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$ rowre-duced

The Grassmanian $\mathbb{G}$ contains, as an open subset, a four-dimensional affine space whose coordinates are the variable entries of M'.

In any 2×4 matrix M with independent rows, some pair of columns will be independent, and the corresponding 2×2 submatrix will be invertible.. That pair of columns can be used in place of the first two in a row reduction. So $\mathbb{G}$ is covered by six four-dimensional affine spaces that we denote by $\mathbb{W}^{ij}$, $1 \le i < j \le 4$, $\mathbb{W}^{ij}$ being the space of 2×4 matrices such that column i is $(1, 0)^t$ and column j is $(0, 1)^t$.

Since both $\mathbb{P}^4$ and $\mathbb{G}$ are covered by affine spaces of dimension four, they may seem similar, but they aren't the same.

(3.7.2) the exterior algebra extalgone

Let V be a complex vector space. The *exterior algebra* $\bigwedge V$ (read 'wedge V') is a noncommutative algebra — an algebra whose multiplication law isn't commutative. The exterior algebra is generated by the elements of V, with the relations

(3.7.3) $$vw = -wv \quad \text{for all } v, w \text{ in } V.$$ extalg vvzero **3.7.4. Lemma.** *The condition (3.7.3) is equivalent with: $vv = 0$ for all v in V.*

proof. To get $vv = 0$ from (3.7.3), one sets $w = v$. Suppose that $vv = 0$ for all v in V. Then $(v+w)(v+w)$, vv, and ww are all zero. Since $(v+w)(v+w) = vv + vw + wv + ww$, it follows that $vw + wv = 0$. □

To familiarize yourself with computation in $\bigwedge V$, verify that $v_3v_2v_1 = -v_1v_2v_3$ and that $v_4v_3v_2v_1 = v_1v_2v_3v_4$.

Let $\bigwedge^r V$ denote the subspace of $\bigwedge V$ spanned by products of length r of elements of V. The exterior algebra $\bigwedge V$ is the direct sum of the subspaces $\bigwedge^r V$. An algebra A that is a direct sum of subspaces A^i, and such that multiplication maps $A^i \times A^j$ to A^{i+j} is called a *graded algebra*. The exterior algebra is a *noncommutative graded algebra*.

dependentproduct **3.7.5. Proposition.** *If $(v_1, ..., v_n)$ is a basis for V, the products $v_{i_1} \cdots v_{i_r}$ of length r with increasing indices $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$.*

The proof is at the end of the section.

wedgezero **3.7.6. Corollary.** *Let $v_1, ..., v_r$ be elements of V. The product $v_1 \cdots v_r$ is zero in $\bigwedge^r V$ if and only if the set $(v_1, ..., v_r)$ is dependent.* □

For the rest of the section, we let V be a vector space of dimension four with basis $(v_1, ..., v_4)$. Proposition 3.7.5 tells us that

extbasistwo (3.7.7)

$\bigwedge^0 V = \mathbb{C}$ is a space of dimension 1, with basis $\{1\}$
$\bigwedge^1 V = V$ is a space of dimension 4, with basis $\{v_1, v_2, v_3, v_4\}$
$\bigwedge^2 V$ is a space of dimension 6, with basis $\{v_iv_j \,|\, i<j\} = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$
$\bigwedge^3 V$ is a space of dimension 4, with basis $\{v_iv_jv_k \,|\, i<j<k\} = \{v_1v_2v_3, v_1v_2v_4, v_1v_3v_4, v_2v_3v_4\}$
$\bigwedge^4 V$ is a space of dimension 1, with basis $\{v_1v_2v_3v_4\}$
$\bigwedge^q V = 0$ when $q > 4$.

The elements of $\bigwedge^2 V$ are combinations

wedgetwo (3.7.8)
$$w = \sum_{i<j} a_{ij} v_i v_j$$

We regard $\bigwedge^2 V$ as an affine space of dimension 6, identifying the vector $(a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$ with the combination w, and we use the same symbol w to denote the corresponding element of the projective space $\mathbb{P}^5$.

defdecomp **3.7.9. Definition.** An element w of $\bigwedge^2 V$ is *decomposable* if it is a product of two elements of V.

describedecomp **3.7.10. Proposition.** *The decomposable elements w of $\bigwedge^2 V$ are those such that $ww = 0$, and the relation $ww = 0$ is equivalent to the following equation in the coefficients a_{ij}:*

eqgrass (3.7.11)
$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$$

proof. If w is decomposable, say $w = u_1u_2$, then $w^2 = u_1u_2u_1u_2 = -u_1^2u_2^2$ is zero because $u_1^2 = 0$. For the converse, we compute w^2 when $w = \sum_{i<j} a_{ij}v_iv_j$. The answer is

$$ww = 2\big(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\big)v_1v_2v_3v_4$$

To show that w is decomposable if $w^2 = 0$, it seems simplest to factor w explictly. Since the assertion is trivial when $w = 0$, we may suppose that some coefficient of w is nonzero. Say that $a_{12} \neq 0$. Then if $w^2 = 0$,

factorw (3.7.12)
$$w = \frac{1}{a_{12}}\big(a_{12}v_2 + a_{13}v_3 + a_{14}v_4\big)\big(-a_{12}v_1 + a_{23}v_3 + a_{24}v_4\big)$$

The computation for another pair of indices is similar. □

3.7.13. Corollary. **(i)** *Let w be a nonzero decomposable element of $\bigwedge^2 V$, say $w = u_1u_2$, with u_i in V. Then (u_1, u_2) is a basis for a two-dimensional subspace of V.* de-comptwo

(ii) *Let (u_1, u_2) and (u_1', u_2') be bases for the subspace U and U' of V, and let $w = u_1u_2$ and $w' = u_1'u_2'$. Then $U = U'$, if and only if w and w' differ by a scalar factor — if and only if they represent the same point of $\mathbb{P}^5$.*

(iii) *Let u_1, u_2 be a basis for a two-dimensional subspace U of V, and let $w = u_1u_2$. The rule $\epsilon(U) = w$ defines a bijection ϵ from $\mathbb{G}$ to the quadric Q in $\mathbb{P}^5$ whose equation is (3.7.11).*

Thus the Grassmanian $\mathbb{G}$ can be represented as the quadric (3.7.11) in $\mathbb{P}^5$

proof. **(i)** If an element w of $\bigwedge^2 V$ is decomposable, say $w = u_1u_2$, and if w isn't zero, then u_1 and u_2 must be independent (3.7.6). They span a two-dimensional subspace.

(ii) Suppose that $U' = U$. Then, when we write the second basis in terms of the first one, say $(u_1', u_2') = (au_1+bu_2, cu_2+du_2)$, the product w' becomes a scalar multiplie $(ad-bc)w$ of w, and $ad-bc \neq 0$.

If $U' \neq U$, then at least three of the vectors u_1, u_2, u_1', u_2' will be independent. Say that u_1, u_2, u_1' are independent. Then, according to Corollary 3.7.6, $u_1u_2u_1' \neq 0$. Since $u_1'u_2'u_1' = 0$, $u_1'u_2'$ cannot be a scalar multiple of u_1u_2.

(iii) This follows from **(i)** and **(ii)**. □

For the rest of this section, we will use the concept of dimension. The algebraic dimension of a variety X can be defined as the length d of the longest chain $C_0 > C_1 > \cdots > C_d$ of closed subvarieties of X. We refer to the algebraic dimension simply as the *dimension*. We use some of its properties informally here, deferring proofs to the next chapter.

The topological dimension of X, its dimension in the classical topology, is always twice the algebraic dimension. Because the Grassmanian $\mathbb{G}$ is covered by affine spaces of dimension 4, its algebraic dimension is 4 and its topological dimension is 8.

3.7.14. Proposition. *Let $\mathbb{P}^3$ be the projective space associated to a four dimensional vector space V. In the product $\mathbb{P}^3 \times \mathbb{G}$, the locus Γ of pairs $p,[\ell]$ such that the point p of $\mathbb{P}^3$ lies on the line ℓ is a closed subset of dimension 5.* pinlclosed

proof. Let ℓ be the line in $\mathbb{P}^3$ that corresponds to the subspace U with basis (u_1, u_2), and say that p is represented by a vector x in V. Let $w = u_1u_2$. Then $p \in \ell$ means $x \in U$, which is true if and only if (x, u_1, u_2) is a dependent set, and this happens if and only if $xw = 0$ (3.7.5). So Γ is the closed subset of points (x, w) of $\mathbb{P}^3 \times \mathbb{P}^5$ defined by the bihomogeneous equations $w^2 = 0$ and $xw = 0$. □

When we project Γ to $\mathbb{G}$, the fibre over a point $[\ell]$ of $\mathbb{G}$ is the set of pairs p, $[\ell]$ such that p is a point of the line ℓ. The projection to $\mathbb{P}^3$ maps the fibre bijectively to the line ℓ. Thus Γ can be viewed as a family of lines, parametrized by $\mathbb{G}$. Its dimension is $\dim \ell + \dim \mathbb{G} = 1 + 4 = 5$.

(3.7.15) lines on a surface linesina-surface

When one is given a surface S in bbp^3, one may ask: Does S contain a line? One surface that contains lines is the quadric Q in $\mathbb{P}^3$ with equation $w_{01}w_{10} = w_{00}w_{11}$, the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3_w$ (**3.1.8**). It contains two families of lines, the ones that correspond to the two "rulings" $p\times\mathbb{P}^1$ and $\mathbb{P}^1\times q$ of $\mathbb{P}^1 \times \mathbb{P}^1$. There are surfaces of arbitrary degree that contain lines, but a generic surface of degree four or more won't contain any line.

We use coordinates x_i with $i = 1, 2, 3, 4$ for $\mathbb{P}^3$ here. There are $N = \binom{d+3}{3}$ monomials of degree d in four variables, so homogeneous polynomials of degree d are parametrized by an affine space of dimension N, and surfaces of degree d in $\mathbb{P}^3$ by a projective space of dimension $N-1$. Let $\mathbb{S}$ denote that projective space, and let $[S]$ denote the point of $\mathbb{S}$ that corresponds to a surface S, and let f be the polynomial whose zero locus is S. The coordinates of $[S]$ are the coefficients of f. Speaking infomally, we say that a point of $\mathbb{S}$ "is" a surface of degree d in $\mathbb{P}^3$. (When f is reducible, its zero locus isn't a variety, but let's not worry about this.)

Consider the line ℓ_0 defined by $x_3 = x_4 = 0$. Its points are those of the form $(x_1, x_2, 0, 0)$, and a surface $S : \{f = 0\}$ will contain ℓ_0 if and only if $f(x_1, x_2, 0, 0) = 0$ for all x_1, x_2. Substituting $x_3 = x_4 = 0$ into f leaves us with a polynomial in two variables:

(3.7.16) $$f(x_1, x_2, 0, 0) = c_0x_1^d + c_1x_1^{d-1}x_2 + \cdots + c_dx_2^d$$ scon-tainslzero where c_i are some of the coefficients of the polynomial f. If $f(x_1, x_2, 0, 0)$ is identically zero, all of those coefficients must be zero. So the surfaces that contain ℓ_0 correspond to the points of the linear subspace $\mathbb{L}_0$ of $\mathbb{S}$ defined by the equations $c_0 = \cdots = c_d = 0$. Its dimension is $(N-1)-(d+1) = N-d-2$. This is a satisfactory answer to the question of which surfaces contain ℓ_0, and we can use it to make a guess about lines in a generic surface of degree d.

scon-tainslclosed **3.7.17. Lemma.** *In the product variety $\mathbb{G}\times\mathbb{S}$, the set Σ of pairs $[\ell],[S]$ such that ℓ is a line, S is a surface of degree d, and $\ell \subset S$, is closed.*

proof. Let $\mathbb{W}^{ij}$, $1 \le i < j \le 4$ denote the six affine spaces that cover the Grassmanian, as at the beginning of this section. It suffices to show that the intersection $\Sigma^{ij} = \Sigma \cap (\mathbb{W}^{ij} \times \mathbb{S})$ is closed in $\mathbb{W}^{ij} \times \mathbb{S}$ for all i, j (3.5.13). We inspect the case $i, j = 1, 2$.

A line ℓ such that $[\ell]$ is in $\mathbb{W}^{12}$ corresponds to a subspace of $\mathbb{C}^2$ with basis of the form $u_1 = (1, 0, a_2, a_3)$, $u_2 = (0, 1, b_2, b_3)$, and ℓ is the line whose points are combinations $ru_1 + su_2$ of u_1, u_2. Let $f(x_1, x_2, x_3, x_4)$ be the polynomial that defines a surface S of degree d. The line ℓ is contained in S if and only if $f(r, s, ra_2 + sb_2, ra_3 + sb_3)$ is zero for all r and s. This is a homogeneous polynomial of degree d in r, s. Let's call it $\widetilde{f}(r, s)$. If we write $\widetilde{f}(r, s) = z_0 r^d + z_1 r^{d-1} s + \cdots + z_d s^d$, the coefficients z_ν will be polynomials in a_i, b_i and in the coefficients of f. The locus $z_0 = \cdots = z_d = 0$ is the closed subset Σ^{12} of $\mathbb{W}^{12} \times \mathbb{S}$. $\square$

The set of surfaces that contain our special line ℓ_0 corresponds to the linear space $\mathbb{L}_0$ of $\mathbb{S}$ of dimension $N-d-2$, and ℓ_0 can be carried to any other line ℓ by a linear map $\mathbb{P}^3 \to \mathbb{P}^3$. So the sufaces that contain another line ℓ also form a linear subspace of $\mathbb{S}$ of dimension $N-d-2$. Those subspaces are the fibres of Σ over $\mathbb{G}$. The dimension of the Grassmanian $\mathbb{G}$ is 4. Therefore the dimension of Σ is

$$\dim \Sigma \ = \ \dim \mathbb{L}_0 + \dim \mathbb{G} \ = \ (N-d-2) + 4$$

Since $\mathbb{S}$ has dimension $N-1$,

dimspace-lines (3.7.18) $$\dim \Sigma \ = \ \dim \mathbb{S} - d + 3$$

We project the product $\mathbb{G}\times\mathbb{S}$ and its subvariety Σ to $\mathbb{S}$. The fibre of Σ over a point $[S]$ is the set of pairs $[\ell],[S]$ such that ℓ is contained in S — the set of lines in S.

lines-lowdeg **3.7.19.** When the degree d of the surfaces we are studying is 1, $\dim \Sigma = \dim \mathbb{S}+2$. Every fibre of Σ over $\mathbb{S}$ will have dimension at least 2. In fact, every fibre has dimension equal to 2. Surfaces of degree 1 are planes, and the lines in a plane form a two-dimensional family.

When $d = 2$, $\dim \Sigma = \dim \mathbb{S}+1$. We can expect that most fibres of Σ over $\mathbb{S}$ will have dimension 1. This is true: A smooth quadric contains two one-dimensional families of lines. (All smooth quadrics are equivalent with the quadric (3.1.9).) But if a quadratic polynomial $f(x_1, x_2, x_3, x_4)$ is the product of linear polynomials, its locus of zeros will be a union of planes. It will contain two-dimensional families of lines. Some fibres have dimension 2.

When $d \ge 4$, $\dim \Sigma < \dim \mathbb{S}$. The projection $\Sigma \to \mathbb{S}$ cannot be surjective. Most surfaces of degree 4 or more contain no lines.

The most interesting case is that $d = 3$. In this case, $\dim \Sigma = \dim \mathbb{S}$. Most fibres will have dimension zero. They will be finite sets. In fact, a generic cubic surface contains 27 lines. We will have to wait to see why the number is precisely 27 (see Theorem 4.7.14).

Our conclusions are intuitively plausible, but to be sure about them, we need to study dimension carefully. We do this in the next chapters.

proof of Proposition 3.7.5. Let $v = (v_1, ..., v_n)$ be a basis of a vector space V. The proposition asserts that the products $v_{i_1} \cdots v_{i_r}$ of length r with increasing indices $i_1 < i_2 < \cdots < i_r$ form a basis for $\bigwedge^r V$.

To prove this, we need to be more precise about the definition of the exterior algebra $\bigwedge V$. We start with the algebra $T(V)$ of noncommutative polynomials in the basis v, which is also called the *tensor algebra* on V. The part $T^r(V)$ of $T(V)$ of degree r has as basis the n^r noncommutative monomials of degree r, the products $v_{i_1}\cdots v_{i_r}$ of ssulength r of elements of the basis. Its dimension is n^r. For example, when $n=2$ $(x_1^3,\, x_1^2x_2,\, x_1x_2x_1,\, x_2x_1^2,\, x_1x_2^2,\, x_2x_1x_2,\, x_2^2x_1,\, x_2^3)$ is a basis for the eight-dimensional space $T^3(V)$

The exterior algebra $\bigwedge V$ is the quotient of $T(V)$ obtained by forcing the relations $vw+wv=0$ (3.7.3). Using the distributive law, one sees that the relations $v_iv_j+v_jv_i=0$, $1\le i,j\le n$, are sufficient to define this quotient. The relations $v_iv_i=0$ are included as the cases that $i=j$.

We can multiply the relations $v_iv_j+v_jv_i$ on left and right by noncommutative monomials $p(v)$ and $q(v)$ in $v_1,...,v_n$. When we do this with all pairs p,q of monomials whose degrees sum to $r-2$, the noncommutative polynomials

(3.7.20) $$p(v)(v_iv_j+v_jv_i)q(v)$$ vivj

span the kernel of the linear map $T^r(V)\to\bigwedge^r V$. So in $\bigwedge^r V$, $p(v)(v_iv_j)q(v)=-p(v)(v_jv_i)q(v)$. Using these relations, any product $v_{i_1}\cdots v_{i_r}$ in $\bigwedge^r V$ is, up to sign, equal to a product in which the indices i_ν are in increasing order. Thus the products with indices in increasing order span $\bigwedge^r V$, and because $v_iv_i=0$, such a product will be zero unless the indices are strictly increasing.

We go to the proof now. Let $v=(v_1,...,v_n)$ be a basis for V. We show first that the product $w=v_1\cdots v_n$ of the basis elements in increasing order is a basis of the space $\bigwedge^n V$. We have shown that w spans $\bigwedge^n V$, and it remains to show that $w\neq 0$, or that $\bigwedge^n V\neq 0$.

Let's use multi-index notation, writing $(i)=(i_1,...,i_r)$, and $v_{(i)}=v_{i_1}\cdots v_{i_r}$. We define a surjective linear map $T^n(V)\xrightarrow{\varphi}\mathbb{C}$. The products $v_{(i)}=(v_{i_1}\cdots v_{i_n})$ of length n form a basis of $T^n(V)$. If there is no repetition among the indices $i_1,...,i_n$, then (i) will be a permutation of the indices $1,...,n$. In that case, we set $\varphi(v_{(i)})=\varphi(v_{i_1}\cdots v_{i_n})=sign(i)$. If there is a repetition, we set $\varphi(v_{(i)})=0$.

Let p and q be noncommutative monomials whose degrees sum to $n-2$. If the product $p(v_iv_j)q$ has no repeated index, the indices in $p(v_iv_j)q$ and $p(v_jv_i)q$ will be permutations of $1,...,n$, and those permutations will have opposite signs. Then $p(v_iv_j+v_jv_i)q$ will be in the kernel of φ. Since these elements span the space of relations that define $\bigwedge^n V$ as a quotient of $T^n(V)$, the surjective map $T^n(V)\xrightarrow{\varphi}\mathbb{C}$ defines a surjective map $\bigwedge^n V\to\mathbb{C}$. Therefore $\bigwedge^n V\neq 0$.

To prove (3.7.5), we must show that for $r\le n$, the products $v_{i_1}\cdots v_{i_r}$ with $i_1<i_2<\cdots<i_r$ form a basis for $\bigwedge^r V$, and we have seen that those products span $\bigwedge^r V$. We must show that they are independent. Suppose that a combination $z=\sum c_{(i)}v_{(i)}$ is zero, the sum being over the sets $\{i_1,...,i_r\}$ of strictly increasing indices. We choose a particular set $(j_1,...,j_r)$ of n strictly increasing indices, and we let $(k)=(k_1,...,k_{n-r})$ be the set of indices that don't occur in (j), listed in arbitrary order. Then all terms in the sum $zv_{(k)}=\sum c_{(i)}v_{(i)}v_{(k)}$ will be zero except the term with $(i)=(j)$. On the other hand, since $z=0$, $zv_{(k)}=0$. Therefore $c_{(j)}v_{(j)}v_{(k)}=0$, and since $v_{(j)}v_{(k)}$ differs by sign from $v_1\cdots v_n$, it isn't zero. It follows that $c_{(j)}=0$. This is true for all (j), so $z=0$. $\square$

3.8 Exercises

chap threeex closureinP **3.8.1.** Let X be the affine surface in $\mathbb{A}^3$ defined by the equation $x_1^3 + x_1x_2x_3 + x_1x_3 + x_2^2 + x_3 = 0$, and let $\overline{X}$ be its closure in $\mathbb{P}^3$. Describe the intersection of $\overline{X}$ with the plane at infinity in $\mathbb{P}^3$.

xcubiceq **3.8.2.** Let C be a cubic curve, the locus of a homogeneous cubic polynomial $f(x, y, z)$ in $\mathbb{P}^2$. Suppose that (001) and (010) are flex points of C, that the tangent line to C at (001) is the line $\{y = 0\}$, and the tangent line at (010) is the line $\{z = 0\}$. What are the possible polynomials f? Disregard the question of whether f is irreducible.

xtendtoinf **3.8.3.** Let Y and Z be the zero sets in $\mathbb{P}$ of relatively prime homogeneous polynomials g and h of the same degree r. Prove that the rational function $\alpha = g/h$ will tend to infinity as one approaches a point of Z that isn't also a point of Y and that, at intersections of Y and Z, α is indeterminate in the sense that the limit depends on the path.

exsamelocus **3.8.4.** Let f and g be irreducible homogeneous polynomials in x, y, z. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ in $\mathbb{P}^2$ are equal, then $g = cf$.

xfzoneirred **3.8.5.** Let f be a homogeneous polynomial in x, y, z, not divisible by z. Prove that f is irreducible if and only if $f(x, y, 1)$ is irreducible.

xfirredAx **3.8.6.** Let f be an irreducible polynomial in $\mathbb{C}[x_1, ..., x_n]$, and let A finite-type domain. Prove that f an irreducible element of $A[x_1, ..., x_n]$.

xsamelocus **3.8.7.** Let f and g be irreducible homogeneous polynomials in x, y, z. Prove that if the loci $\{f = 0\}$ and $\{g = 0\}$ are equal, then g is a constant multiple of f.

homogprimeconverse **3.8.8.** Let $\mathcal{P}$ be a homogeneous ideal in $\mathbb{C}[x_0, ..., x_n]$, and suppose that its dehomogenization P is a prime ideal. Is $\mathcal{P}$ a prime ideal?

xcondpt **3.8.9.** Let X be the open complement of a closed subset Y in a projective variety $\overline{X}$ in $\mathbb{P}^n$. Say that $\overline{X}$ is the set of solutions of the homogeneous polynomial equations $f = 0$ and that Y is the set of solutions of the equations $g = 0$. What conditions must a point p of $\mathbb{P}^n$ satisfy in order to be a point of X?

xidealinAxP **3.8.10.** Describe the ideals that define closed subsets of $\mathbb{A}^m \times \mathbb{P}^n$.

xlocusinPxP **3.8.11.** With coordinates x_0, x_1, x_2 in the plane $\mathbb{P}$ and s_0, s_1, s_2 in the dual plane $\mathbb{P}^*$, let C be a smooth projective plane curve $f = 0$ in $\mathbb{P}$, where f is an irreducible homogeneous polynomial in x. Let Γ be the locus of pairs (x, s) of $\mathbb{P} \times \mathbb{P}^*$ such that the line $s_0x_0 + s_1x_1 + s_2x_2 = 0$ is the tangent line to C at x. Prove that Γ is a Zariski closed subset of the product $\mathbb{P} \times \mathbb{P}^*$.

xfnbounded **3.8.12.** Let U be a nonempty open subset of $\mathbb{P}^n$. Prove that if a rational function is bounded on U, it is a constant.

xxfinmanyzeros **3.8.13.** Prove that relatively prime polynomials in F, G two variables x, y, not necessarily homogeneous, have finitely many common zeros in $\mathbb{A}^2$.

xmapcusp **3.8.14.** Let Y be the cusp curve $\operatorname{Spec} B$, where $B = \mathbb{C}[x, y]/(y^2 - x^3)$. This algebra embeds as subring into $\mathbb{C}[t]$, by $x = t^2$. $y = t^3$. Show that the two vectors $v_0 = (x-1, y-1)$ and $v_1 = (t+1, t^2+t+1)$ define the same point of $\mathbb{P}^1$ with values in the fraction field K of B, and that they define morphisms from Y to $\mathbb{P}^1$ wherever the entries are regular functions on Y. Prove that the two morphisms they define piece together to give a morphism $Y \to \mathbb{P}^1$.

xdoesntextend **3.8.15.** Let C be a conic in $\mathbb{P}^2$, and let π be the projection to $\mathbb{P}^1$ from a point q of C. Prove that there is no way to extend this map to a morphism from $\mathbb{P}^2$ to $\mathbb{P}^2$.

xverifymorph **3.8.16.** Verify that the following maps are morphisms of projective varieties:

(i) the projection from a product variety $X \times Y$ to X,

(ii) the inclusion of X into the product $X \times Y$ as the set $X \times y$ for a point y of Y,

(iii) the morphism of products $X \times Y \to X' \times Y$ when a morphism $X \to X'$ is given.

3.8.17. A pair f_0, f_1 of homogeneous polynomials in x_0, x_1 of the same degree d can be used to define a morphism $\mathbb{P}^1 \to a\mathbb{P}^1$. At a point q of $\mathbb{P}^1$, the morphism evaluates $(1, f_1/f_0)$ or $(f_0/f_1, 1)$ at q. zautPone

(i) The *degree* of such a morphism is the number of points in a generic fibre. Determine the degree.

(ii) Describe the group of automorphisms of $\mathbb{P}^1$.

3.8.18. **(i)** What are the conditions that a triple of $f = (f_0, f_1, f_2)$ homogeneous polynomials in x_0, x_1, x_2 of the same degree d must satisfy in order to define a morphism $\mathbb{P}^2 \to \mathbb{P}^2$? xmorph-Ptwo

(ii) If f does define a morphism, what is its degree?

3.8.19. Let C be the projective plane curve $x^3 - y^2 z = 0$. cuspto-pone

(i) Show that the function field K of C is the field $\mathbb{C}(t)$ of rational functions in $t = y/x$.

(ii) Show that the point $(t^2 - 1, t^3 - 1)$ of $\mathbb{P}^1$ with values in K defines a morphism $C \to \mathbb{P}^1$.

3.8.20. Describe all morphisms $\mathbb{P}^2 \to \mathbb{P}^1$. xPtwo-Pone

3.8.21. *blowing up a point in* $\mathbb{P}^2$. Consider the Veronese embedding of $\mathbb{P}^2_{xyz} \to \mathbb{P}^5_u$ by monomials of degree 2 defined by $(u_0, u_1, u_2, u_3, u_4, u_5) = (z^2, y^2, x^2, yz, xz, xy)$. If we drop the coordinate u_0, we obtain a map $\mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^4$: $\varphi(x, y, z) = (y^2, x^2, yz, xz, xy)$ that is defined at all points except the point $q = (0, 0, 1)$. Find defining equations for the closure of the image X. Prove that the inverse map $X \xrightarrow{\varphi^{-1}} \mathbb{P}^2$ is everywhere defined, and that the fibre of φ^{-1} over q is a projective line. xblowup

3.8.22. Show that the conic C in $\mathbb{P}^2$ defined by the polynomial $y_0^2 + y_1^2 + y_2^2 = 0$ and the twisted cubic V in $\mathbb{P}^3$, the zero locus of the polynomials $v_0 v_2 - v_1^2$, $v_0 v_3 - v_1 v_2$, $v_1 v_3 - v_2^2$ are isomorphic by exhibiting inverse morphisms between them. xconic-cubic

3.8.23. Let X be the affine plane with coordinates (x, y). Given a pair of polynomials $u(x, y), v(x, y)$ in x, y, one may try to define a morphism $f : X \to \mathbb{P}^1$ by $f(x, y) = (u, v)$. Under what circumstances is f a morphism? xmapplan-etoPone

3.8.24. Let x_0, x_1, x_2 be the coordinate variables in the projective plane X. The function field K of X is the field of rational functions in the variables u_1, u_2, $u_i = x_i/x_0$. Let $f(u_1, u_2)$ and $g(u_1, u_2)$ be polynomials. Under what circumstances does the point $(1, f, g)$ with values in K define a morphism $X \to \mathbb{P}^2$? xmap-toPthree

3.8.25. Prove that every finite subset S of a projective variety X is contained in an affine open subset. easyk-leiman

3.8.26. Describe the affine open subsets of the projective plane $\mathbb{P}^2$. xaffinopen-plane

3.8.27. What is the dimension of the Grassmanian $\mathbf{G}(m, n)$? xdimgrass

3.8.28. According to (3.7.19), a generic quartic surface in $\mathbb{P}^3$ won't contain any lines. Will such a surface contain a plane conic? xconicin-quartic

3.8.29. Let V be a vector space of dimension 5, let $\mathbb{G}$ denote the Grassmanian $\mathbf{G}(2, 5)$ of lines in $\mathbb{P}^4$, let $W = \bigwedge^2 V$, and let D denote the subset of decomposable vectors in the projective space $\mathbb{P}(W)$ of one-dimensional subspaces of W. Prove that there is a bijective correspondence between two-dimensional subspaces U of V and the points of D, and that a vector w in $\bigwedge^2 V$ is decomposable if and only if $ww = 0$. Exhibit defining equations for $\mathbb{G}$ in the space $\mathbb{P}(W)$. xlinesp-five

3.8.30. *a flag variety*. Let $\mathbb{P} = \mathbb{P}^3$. The space of planes in $\mathbb{P}$ is the *dual projective space* $\mathbb{P}^*$. The variety F that parametrizes triples (p, ℓ, H) consisting of a point p, a line ℓ, and a plane H in $\mathbb{P}$, with $p \in \ell \subset H$, is called a *flag variety*. Exhibit defining equations for F in $\mathbb{P}^3 \times \mathbb{P}^5 \times \mathbb{P}^{3*}$. The equations should be homogeneous in each of 3 sets of variables. xflag

Chapter 4 INTEGRAL MORPHISMS

The concept of an algebraic integer was one of the important ideal essential to the development of algebraic number theory in the 19th century. Then, largely through the work of Noether and Zariski, an analog was seen to be essential in algebraic geometry. We study that analog in this chapter.

Section 4.1 The Nakayama Lemma

nakayama

eigen **(4.1.1) eigenvectors**

It won't be a surprise that eigenvectors are important, but the way that they are used to study modules may be unfamiliar.

Let P be an $n \times n$ matrix with entries in a ring A. The concept of an eigenvector for P makes sense when the entries of a vector are in an A-module. A column vector $v = (v_1, ..., v_n)^t$ with entries in an A-module M is an *eigenvector* of P with *eigenvalue* λ in A if $Pv = \lambda v$.

When the entries of a vector are in a module, it becomes hard to adapt the usual requirement that an eigenvector must be nonzero. So we drop it, though the zero vector tells us nothing.

eigenval **4.1.2. Lemma.** *Let P be a square matrix with entries in a ring A and let $p(t)$ be the characteristic polynomial* $\det(tI - P)$ *of P. If v is an eigenvector of P with eigenvalue λ, then $p(\lambda)v = 0$.*

The usual proof, in which one multiplies the equation $(\lambda I - P)v = 0$ by the cofactor matrix of $(\lambda I - P)$, carries over. □

The next lemma is a cornerstone of the theory of modules. In it, JM denotes the set of (finite) sums $\sum_i a_i m_i$ with a_i in J and m_i in M.

nakaya-malem **4.1.3. Nakayama Lemma.** Let M be a finite module over a ring A, and let J be an ideal of A such that $M = JM$. There is an element z in J such that $m = zm$ for all m in M, i.e., such that $(1-z)M = 0$.

Because $M \supset JM$ is always true, the hypothesis $M = JM$ could be replaced by $M \subset JM$.

proof of the Nakayama Lemma. Let $v_1, ..., v_n$ be generators for the finite A-module M. The equation $M = JM$ tells us that there are elements p_{ij} in J such that $v_i = \sum p_{ij} v_j$. We write this equation in matrix notation, as $v = Pv$, where v is the column vector $(v_1, ..., v_n)^t$ and P is the matrix $P = (p_{ij})$. Then v is an eigenvector of P with eigenvalue 1, and if $p(t)$ is the characteristic polynomial of P, then $p(1)v = 0$. Since the entries of P are in the ideal J, inspection of the determinant of $I - P$ shows that $p(1)$ has the form $1 - z$, with z in J. Then $(1-z)v_i = 0$ for all i. Since $v_1, ..., v_n$ generate M, $(1-z)M = 0$. $\square$

4.1.4. Corollary. *With notation as in the Nakayama Lemma, let $s = 1-z$, so that $sM = 0$. The localized module M_s is the zero module.* $\square$ *loczero*

4.1.5. Corollary. *Let I and J be ideals of a noetherian domain A.* *idealzero*
(i) *If $I = JI$, then either I is the zero ideal or J is the unit ideal.*
(ii) *Let B be a domain that contains A and that is a finite A-module. If the extended ideal JB is the unit ideal of B, then J is the unit ideal of A.*

proof. **(i)** Since A is noetherian, I is a finite A-module. If $I = JI$, the Nakayama Lemma tells us that there is an element z of J such that $zx = x$ for all x in I. Suppose that I isn't the zero ideal. We choose a nonzero element x of I. Because A is a domain, we can cancel x from the equation $zx = x$, obtaining $z = 1$. Then 1 is in J, and J is the unit ideal.

(ii) The elements of the extended ideal JB are sums $\sum u_i b_i$ with u_i in J and b_i in B (**2.6.4**). Suppose that $B = JB$. The Nakayama Lemma tells us that there is an element z in J such that $zb = b$ for all b in B. Setting $b = 1$ shows that $z = 1$. So J is the unit ideal. $\square$

4.1.6. Corollary. *Let x be an element of a noetherian domain A, not a unit, and let J be the principal ideal xA.* *xkdividesy*
(i) *The intersection $\bigcap J^n$ is the zero ideal.*
(ii) *If y is a nonzero element of A, the integers k such that x^k divides y in A are bounded.*
(iii) *For every $k > 0$, $J^k > J^{k+1}$.*

proof. Let $I = \bigcap J^n$. The elements of I are the ones that are divisible by x^n for every n. Let y be such an element. So for every n, there is an element a_n in A such that $y = a_n x^n$. Then $y/x = a_n x^{n-1}$, which is an element of J^{n-1}. Since this is true for every n, y/x is in I, and y is in JI. Since y can be any element of I, $I = JI$. But since x isn't a unit, J isn't the unit ideal. Corollary 4.1.5(**i**) tells us that $I = 0$. This proves (**i**), and (**ii**) follows. For (**iii**), we note that if $J^k = J^{k+1}$, then, multiplying by J^{n-k}, we see that $J^n = J^{n+1}$ for every $n \geq k$, and therefore that $J^k = \bigcap J^n = 0$. But since A is a domain and $x \neq 0$, $J^k = x^k A \neq 0$. Therefore $J^k < J^{k+1}$ for all k. $\square$

Section 4.2 Integral Extensions

int

An *extension* of a domain A is a domain B that contains A as a subring.

Let B be an extension of A. An element β of B is *integral over* A if it is a root of a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{4.2.1}$$

eqn

with coefficients a_i in A, and B is an *integral extension* of A if all of its elements are integral over A.

4.2.2. Lemma. *Let $A \subset B$ be an extension of noetherian domains.* *aboutinte-*
(i) *An element b of B is integral over A if and only if the subring $A[b]$ of B generated by b is a finite A-module.* *gral*
(ii) *The set of elements of B that are integral over A is a subring of B.*
(iii) *If B is generated as A-algebra by finitely many integral elements, it is a finite A-module.*
(iv) *Let $R \subset A \subset B$ be domains, and suppose that A is an integral extension of R. An element of B is integral over A if and only if it is integral over R.* $\square$

For example, if $f(x)$ is a monic irreducible polynomial in $A[x]$, and if $B = A[x]/(f)$, then every element of B will be integral over A.

4.2.3. Corollary. *An extension $A \subset B$ of finite-type domains is an integral extension if and only if B is a finite A-module.* $\square$ *inte-graliffinite* betaintegral

4.2.4. Lemma. *Let I be a nonzero ideal of a noetherian domain A, let B be an extension of A, and let β be an element of B. If $\beta I \subset I$, then β is integral over A.*

proof. Because A is noetherian, I is finitely generated. Let $v = (v_1, ..., v_n)^t$ be a vector whose entries generate I. The hypothesis $\beta I \subset I$ allows us to write $\beta v_i = \sum p_{ij} v_j$ with p_{ij} in A, or in matrix notation, $\beta v = Pv$. So v is an eigenvector of P with eigenvalue β, and if $p(t)$ is the characteristic polynomial of P, then $p(\beta)v = 0$. Since at least one v_i is nonzero and since A is a domain, $p(\beta) = 0$. The characteristic polynomial $p(t)$ is a monic polynomial with coefficients in A, so β is integral over A. $\square$

deffinmorphaff

4.2.5. Definition. Let $Y \xrightarrow{u} X$ be a morphism of affine varieties $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, and let $A \xrightarrow{\varphi} B$ be the corresponding homomorphism of coordinate algebras. If φ makes B into a finite A-module, we call u a *finite morphism* of affine varieties. If A is a subring of B, and B is an integral extension of A, we call u an *integral morphism.* of affine varieties.

An integral morphism of affine varieties is a finite morphism whose associated algebra homomorphism $A \xrightarrow{\varphi} B$ is injective. For example, if G is a finite group of automorphisms of a finite-type domain B, then B is an integral extension of its subring B^G of invariants. (See Theorem 2.8.5.) The inclusion of a closed subvariety Y into an affine variety X is a finite morphism, but not an integral morphism.

integralsurj

4.2.6. Proposition. *An integral morphism $Y \xrightarrow{u} X$ of affine varieties is surjective.*

proof. Let $\mathfrak{m}_x$ be the maximal ideal at point x of X. Corollary 4.1.5 **(ii)** shows that the extended ideal $\mathfrak{m}_x B$ isn't the unit ideal of B, so $\mathfrak{m}_x B$ is contained in a maximal ideal of B, say $\mathfrak{m}_y$, where y is a point of Y. Then x is the image of y, and therefore u is surjective. $\square$

The next example is helpful for an intuitive understanding of the geometric meaning of integrality.

integralovercx

4.2.7. Example. Let f be an irreducible polynomial in $\mathbb{C}[x, y]$ (one x and one y), let $A = \mathbb{C}[x]$, and let $B = \mathbb{C}[x, y]/(f)$. So $X = \operatorname{Spec} A$ is an affine line and $Y = \operatorname{Spec} B$ is a plane affine curve. The canonical map $A \to B$ defines a morphism $Y \xrightarrow{u} X$ that is the restriction of the projection $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_x$ to Y.

We write f as a polynomial in y, whose coefficients are polynomials in x:

fxy (4.2.8)
$$f(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$$

Let x_0 be a point of X. The fibre of Y over x_0 consists of the points (x_0, y_0) such that y_0 is a root of the one-variable polynomial $f(x_0, y)$.

The discriminant $\delta(x)$ of $f(x, y)$, viewed as a polynomial in y, isn't identically zero because f is irreducible (1.7.21). For all but finitely many values x_0 of x, both a_0 and δ will be nonzero, $f(x_0, y)$ will have n distinct roots, and the fibre will have order n.

When $f(x, y)$ is a monic polynomial in y, u will be an integral morphism. If so, the leading term y^n of f will be the dominant term, when y is large. Near to any point x_0 of X, there will be a positive real number N such that, when $|y| > N$,

$$|y^n| > |a_1(x)y^{n-1} + \cdots + a_n(x)|$$

and therefore $f(x, y) \neq 0$. So the roots y of $f(x_1, y)$ are bounded by N for all x_1 near to x_0.

On the other hand, when the leading coefficient $a_0(x)$ isn't a constant, B won't be integral over A. Whenn x_0 is a root of $a_0(x)$, $f(x_0, y)$ will have degree less than n. What happens then is that, for points x_1 near to x_0, the roots of $f(x_1, y)$ are unbounded. In calculus, one says that the locus $f(x, y) = 0$ has a vertical asymptote at x_0.

To see this, we divide f by its leading coefficient. Let $g(x, y) = f(x, y)/a_0 = y^n + c_1 y^{n-1} + \cdots + c_n$ with $c_i(x) = a_i(x)/a_0(x)$. For any x at which $a_0(x)$ isn't zero, the roots of g are the same as those of f. However, let x_0 be a root of a_0. Because f is irreducible. At least one coefficient $a_j(x)$ doesn't have x_0 as a root. Then $c_j(x)$ is unbounded near x_0, and because the coefficient c_j is an elementary symmetric function in the roots, the roots aren't all bounded.

This is the general picture: The roots of a polynomial remain bounded where the leading coefficient isn't zero, but some roots are unbounded near to a point at which the leading coefficient vanishes. $\square$

4.2.9. Noether Normalization Theorem. Let A be a finite-type algebra over an infinite field k. There exist elements $y_1, \ldots, y_n$ in A that are algebraically independent over k, such that A is a finite module over the polynomial subalgebra $R = k[y_1, \ldots, y_n]$, i.e., such that A is an integral extension of R. noether-normal

When $k = \mathbb{C}$, the theorem can be stated by saying that every affine variety X admits an integral morphism to an affine space. It is trivial that an affine variety admits a finite morphism to affine space, because its embedding into affine space is a finite morphism.

The Noether Normalization Theorem remains true when A is a finite-type algebra over a finite field, though the proof given below needs to be modified.

4.2.10. Lemma. *Let k be an infinite field, and let $f(x)$ be a nonzero polynomial of degree d in $x_1, \ldots, x_n$, with coefficients in k. After a suitable linear change of variable and scaling, f will be a monic polynomial in x_n.* *nonzero-coeff*

proof. Let f_d be the homogeneous part of f of maximal degree d, which we regard as a function $k^n \to k$. Since k is infinite, that function isn't identically zero. We choose coordinates $x_1, ..., x_n$ so that the point $q = (0, ..., 0, 1)$ isn't a zero of f_d. Then $f_d(0, ..., 0, x_n) = cx_n^d$, and the coefficient c, which is $f_d(0, ..., 0, 1)$, will be nonzero. Multiplication by a suitable scalar makes $c = 1$. $\square$

proof of the Noether Normalization Theorem. Say that the finite-type algebra A is generated by elements $x_1, \ldots, x_n$. If those elements are algebraically independent over k, A will be isomorphic to the polynomial algebra $\mathbb{C}[x]$. In this case we let $R = A$. If $x_1, ..., x_n$ aren't algebraically independent, they satisfy a polynomial relation $f(x) = 0$ of some degree d, with coefficients in k. The lemma tells us that, after a suitable change of variable and scaling, the coefficient of x_n^d in f will be 1. Then f will be a monic polynomial in x_n with coefficients in the subalgebra R of A generated by $x_1, \ldots, x_{n-1}$, x_n will be integral over R, and A will be a finite R-module. By induction on n, we may assume that R is a finite module over a polynomial subalgebra R. Then A will be a finite module over R too. $\square$

The next corollary is an example of the general principle, that, as has been noted before, that in a localization, a construction involving finitely many operations can be done in a simple localization.

4.2.11. Corollary. *Let $A \subset B$ be finite-type domains. There is a nonzero element s in A such that B_s is a finite module over a polynomial subring $A_s[y_1, ..., y_r]$.* *localntheorem*

proof. Let S be the multiplicative system of nonzero elements of A, so that $K = AS^{-1}$ is the fraction field of A, and let $B_K = BS^{-1}$ be the ring obtained from B by inverting all elements of S. Also, let $\beta = (\beta_1, ..., \beta_k)$ be a set of algebra generators for the finite-type algebra B. Then B_K is generated as K-algebra by β. It is a finite-type K-algebra. The Noether Normalization Theorem tells us that B_K is a finite module over a polynomial subring $R_K = K[y_1, ..., y_r]$. So B_K is an integral extension of R_K. An element of B will be in B_K, and therefore it will be the root of a monic polynomial, say

$$f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0$$

where the coefficients $c_j(y)$ are elements of R_K. Each coefficient c_j is a combination of finitely many monomials in y, with coefficients in K. If $d \in A$ is a common denominator for those coefficients, $c_j(x)$ will have coefficients in $A_d[y]$. Since the generators β of B are integral over R_K, we may choose a denominator s so that all of the generators $\beta_1, ..., \beta_k$ are integral over $A_s[y]$. The algebra B_s is generated over A_s by β, so B_s will be an integral extension of $A_s[y]$. $\square$

Section 4.3 Normalization

finint

Let A be a domain with fraction field K. The *normalization* $A^\#$ of A is the set of elements of K that are integral over A. It is a domain, and it contains A (4.2.2) **(ii)**.

A domain A is *normal* if it is equal to its normalization, and a *normal variety* X is a variety that has an affine open covering $\{X^i = \operatorname{Spec} A_i\}$ in which the algebras A_i are normal domains.

To justify the definition of normal variety, we need to show that if an affine variety $X = \operatorname{Spec} A$ has an affine covering $X^i = \operatorname{Spec} A_i$, in which A_i are normal domains, then A is normal. This follows from Lemma 4.3.4 **(iii)** below.

Our goal here is the next theorem, whose proof is at the end of the section.

normalfinite **4.3.1. Theorem.** *Let A be a finite-type domain with fraction field K of characteristic zero. The normalization $A^{\#}$ of A is a finite A-module and a finite-type domain.*

Thus there will be an integral morphism $\operatorname{Spec} A^{\#} \to \operatorname{Spec} A$.

Asharp-modA **4.3.2. Corollary.** *With notation as above, there is a nonzero element s in A such that $sA^{\#} \subset A$.*

proof. We assume that the theorem has been proved. Since A and $A^{\#}$ have the same fraction field, every element α of $A^{\#}$ can be written as a fraction $\alpha = a/s$ with a, s in A, and then $s\alpha$ is in A. Since $A^{\#}$ is a finite A-module, one can find a nonzero element $s \in A$ such that $a\alpha$ is in A for all α in $A^{\#}$. Then $sA^{\#} \subset A$. □

nodecurve **4.3.3. Example.** *(normalization of a nodal cubic curve)* The algebra $A = \mathbb{C}[u,v]/(v^2 - u^3 - u^2)$ can be embedded into the one-variable polynomial algebra $B = \mathbb{C}[x]$, by $u = x^2 - 1$ and $v = x^3 - x$. The fraction fields of A and B are equal because $x = v/u$, and the equation $x^2 - (u+1) = 0$ shows that x is integral over A. The algebra B is normal (Lemma 4.3.4 **(i)**), so it is the normalization of A.

The plane curve $C = \operatorname{Spec} A$ has a node at the origin $p = (0,0)$, and $\operatorname{Spec} B$ is the affine line $\mathbb{A}^1$. The inclusion $A \subset B$ defines an integral morphism $\mathbb{A}^1 \to C$ whose fibre over p is the point pair $x = \pm 1$. The morphism is bijective at all other points. I think of C as the variety obtained by gluing the points $x = \pm 1$ of the affine line together.

In this example, the effect of normalization can be visualized geometrically. This is fairly unusual. Normalization is an algebraic process, whose effect on geometry may be subtle. □

ufdnormal **4.3.4. Lemma.** **(i)** *A unique factorization domain is normal. In particular, a polynomial algebra over a field is normal.*

(ii) *If s is a nonzero element of a normal domain A, the localization A_s is normal.*

(iii) *Let $s_1, ..., s_k$ be nonzero elements of a domain A that generate the unit ideal. If the localizations A_{s_i} are normal for all i, then A is normal.*

proof. **(i)** Let A be a unique factorization domain, and let β be an element of its fraction field that is integral over A. Say that

eqntwo (4.3.5)
$$\beta^n + a_1\beta^{n-1} + \cdots + a_{n-1}\beta + a_n = 0$$

with a_i in A. We write $\beta = r/s$, where r and s are relatively prime elements of A. Multiplying by s^n gives us the equation

$$r^n = -s\,(a_1r^{n-1} + \cdots + a_ns^{n-1})$$

This equation shows that if a prime element of A divides s, it also divides r. Since r and s are relatively prime, there is no such prime element. So s is a unit, and β is in A.

(ii) Let β be an element of the fraction field of A that is integral over A_s. There will be a polynomial relation of the form (4.3.5), except that the coefficients a_i will be elements of A_s. The element $\gamma = s^k\beta$ satisfies the polynomial equation

$$\gamma^n + (s^ka_1)\gamma^{n-1} + (s^{2k}a_2)\gamma^{n-2} + \cdots + +(s^{nk}a_n) = 0$$

Since a_i are in A_s, all coefficients in this polynomial equation will be in A when k is sufficiently large, and then γ will be integral over A. Since A is normal, γ will be in A, and $\beta = s^{-k}\gamma$ will be in A_s.

(iii) This proof follows a common pattern. Suppose that A_{s_i} is normal for every i. If an element β of K is integral over A, it will be in A_{s_i} for all i, and $s_i^n\beta$ will be an element of A when n is large. We can use the same exponent n for all i. Since $s_1, ..., s_k$ generate the unit ideal, so do their powers $s_i^n, ..., s_k^n$. Say that $\sum r_is_i^n = 1$, with r_i in A. Then $\beta = \sum r_is_i^n\beta$ is in A. □

We prove Theorem 4.3.1 in a slightly more general form. Let A be a finite type domain with fraction field K, and let L be a finite field extension of K. The *integal closure* of A in L is the set of all elements of L that are integral over A.

intclo **4.3.6. Theorem.** *Let A be a finite type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. The integal closure B of A in L is a finite A-module.*

The proof that we give here makes use of the characteristic zero hypothesis, though the theorem is true for a finite-type algebra over any field k.

4.3.7. Lemma. *Let A be a normal noetherian domain with fraction field K of characteristic zero, and let L be an algebraic field extension of K. An element β of L is integral over A if and only if the monic irreducible polynomial f for β over K has coefficients in A.* *about-tracetwo*

proof. If the monic polynomial f has coefficients in A, then β is integral over A. Suppose that β is integral over A. Since we may replace L by any field extension that contains β, we may replace L by $K[\beta]$. Then L becomes a finite extension, which embeds into a Galois extension of K. So we may replace L by a Galois extension. Let G be its Galois group, and let $\{\beta_1, \dots, \beta_r\}$ be the G-orbit of β, with $\beta = \beta_1$. Then the irreducible polynomial for β over K is

$$f(x) = (x - \beta_1) \cdots (x - \beta_r) \tag{4.3.8}$$ orbitpoly

If β is integral over A, then all elements of the orbit are integral over A. Therefore the coefficients of f, which are symmetric functions in the orbit, are integral over A, and since A is normal, they are in A. So f has coefficients in A. $\square$

4.3.9. Example. A polynomial $f(x, y)$ in $A = \mathbb{C}[x, y]$ is *square-free* if it has no nonconstant square factors and isn't a constant. Let f be a square-free polynomial, and let B denote the integral extension $\mathbb{C}[x, y, w]/(w^2 - f)$ of A. Let K and L be the fraction fields of A and B, respectively. Then L is a Galois extension of K. Its Galois group is generated by the automorphism σ of order 2 defined by $\sigma(w) = -w$. The elements of L have the form $\beta = a + bw$ with $a, b \in K$, and $\sigma(\beta) = \beta' = a - bw$. dplanenormal

We show that B is the integral closure of A in L. Suppose that $\beta = a + bw$ is integral over A. If $b = 0$, then $\beta = a$. This is an element of A and therefore it is in B. If $b \neq 0$, the irreducible polynomial for β will be

$$(x - \beta)(x - \beta') = x^2 - 2ax + (a^2 - b^2 f)$$

Because β is integral over A, $2a$ and $a^2 - b^2 f$ are in A. This is true if and only if a and $b^2 f$ are in A, because the characteristic isn't 2. We write $b = u/v$, with u, v relatively prime elements of A, so $b^2 f = u^2 f / v^2$. If v weren't a constant, then since f is square-free, we couldn't cancel v^2 from $u^2 f$, so $b^2 f$ wouldn't be in A. From $b^2 f$ in A we can conclude that v is a constant and that b is in A. Summing up, β is integral if and only if a and b are in A, which means that β is in B. $\square$

(4.3.10) trace

deftrace

Let L be a finite field extension of a field K and let β be an element of K. When L is viewed as a K-vector space, multiplication by β becomes a K-linear operator $L \to L$. The *trace* of this operator will be denoted by $\mathrm{tr}(\beta)$. The trace is a K-linear map $L \to K$.

4.3.11. Lemma. *Let L/K be a field extension of degree n, let $K[\beta]$ be the extension of K generated by an element β of L, and let $f(x) = x^r + a_1 x^{r-1} + \cdots + a_r$ be the irreducible polynomial of β over K. Say that $[L : K[\beta]] = d$, so that $n = rd$. Then $\mathrm{tr}(\beta) = -da_1$. If β is an element of K, then $\mathrm{tr}(\beta) = n\beta$.* *about-traceone*

proof. The set $(1, \beta, \dots, \beta^{r-1})$ is a K-basis for $K[\beta]$. On this basis, the matrix of multiplication by β has the form illustrated below for the case $r = 3$. Its trace is $-a_1$.

$$M = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix}.$$

Next, let $(u_1, \dots, u_d)$ be a basis for L over $K[\beta]$. Then the set $\{\beta^i u_j\}$, with $i = 0, \dots, r-1$ and $j = 1, \dots, d$, will be a basis for L over K. When this basis is listed in the order

$$(u_1, u_1\beta, ..., u_1\beta^{n-1}; u_2, u_2\beta, \dots u_2\beta^{n-1}; \; \dots \; ; u_d, u_d\beta, ..., u_d\beta^{n-1}),$$

the matrix of multiplication by β will be made up of d blocks of the matrix M. $\square$

traceinA **4.3.12. Corollary.** *Let A be a normal domain with fraction field K and let L be a finite field extension of K. If an element β of K is integral over A, its trace is in A.*

This follows from Lemmas 4.3.7 and 4.3.11. □

formnon-deg **4.3.13. Lemma.** *Let A be a normal noetherian domain with fraction field K* of characteristic zero*, and let L be a finite field extension of K. The form $L \times L \to K$ defined by $\langle \alpha, \beta \rangle = \mathrm{tr}(\alpha\beta)$ is K-bilinear, symmetric, and nondegenerate. If α and β are integral over A, then $\langle \alpha, \beta \rangle$ is an element of A.*

proof. The form is obviously symmetric, and it is K-bilinear because multiplication is K-bilinear and trace is K-linear. A form is nondegenerate if its nullspace is zero, which means that when α is a nonzero element, there is an element β such that $\langle \alpha, \beta \rangle \neq 0$. Let $\beta = \alpha^{-1}$. Then $\langle \alpha, \beta \rangle = \mathrm{tr}(1)$, which, according to (4.3.11), is the degree $[L\!:\!K]$ of the field extension. It is here that the hypothesis on the characteristic of K enters: The degree is a nonzero element of K.

If α and β are integral over A, so is their product $\alpha\beta$ (4.2.2) **(ii)**. Corollary 4.3.12 shows that $\mathrm{tr}(\alpha\beta) = \langle \alpha, \beta \rangle$ is an element of A. □

clearde-nom **4.3.14. Lemma.** *Let A be a domain with fraction field K, let L be a field extension of K, and let β be an element of L that is algebraic over K. If β is a root of a polynomial $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with a_i in A, then $\gamma = a_n\beta$ is integral over A.*

proof. One finds a monic polynomial with root γ by substituting $x = y/a_n$ into f and multiplying by a_n^{n-1}. □

proof of Theorem 4.3.1. Let A be a finite-type domain with fraction field K of characteristic zero, and let L be a finite field extension of K. We are to show that the integral closure of A in L is a finite A-module.

Step 1. We may assume that A is normal.

We use the Noether Normalization Theorem to write A as a finite module over a polynomial subalgebra $R = \mathbb{C}[y_1, \dots, y_d]$. Let F be the fraction field of R. Then K and L are finite extensions of F. An element of L will be integral over A if and only if it is integral over R (4.2.2) **(iv)**. So the integral closure of A in L is the same as the integral closure of R in L. We replace A by the normal algebra R and K by F.

Step 2. Bounding the integral extension.

We assume that A is normal. Let $(v_1, \dots, v_n)$ be a K-basis for L whose elements are integral over A. Such a basis exists because we can multiply any element of L by a nonzero element of K to make it integral (Lemma 4.3.14). Let

mapvector (4.3.15)
$$T : L \to K^n$$

be the map defined by $T(\beta) = \big(\langle v_1, \beta \rangle, \dots, \langle v_n, \beta \rangle\big)$, where $\langle\ ,\ \rangle$ is the bilinear form defined in Lemma 4.3.13. The map T is K-linear. If $\langle v_i, \beta \rangle = 0$ for all i, then because $(v_1, \dots, v_n)$ is a basis for L, $\langle \gamma, \beta \rangle = 0$ for all γ in L, and since the form is nondegenerate, $\beta = 0$. Therefore T is injective.

Let B be the integral closure of A in L. The basis elements v_i are in B, and if β is in B, $v_i\beta$ will be in B too. Then $\langle v_i, \beta \rangle = \mathrm{tr}(v_i b)$ will be in A, and $T(\beta)$ will be in A^n (4.3.13). When we restrict T to B, we obtain an injective map $B \to A^n$ that we denote by T_0. Since T is K-linear, T_0 is A-linear. It is an injective homomorphism of A-modules that maps B isomorphically to its image, a submodule of A^n. Since A is noetherian, every submodule of the finite A-module A^n is finitely generated. Therefore the image of T_0 is a finite A-module, and so is the isomorphic module B. □

Section 4.4 Geometry of Integral Morphisms

prmint The main geometric properties of an integral morphism of affine varieties are summarized in the theorems in this section, which show that the geometry is as nice as could be expected.

When an integral morphism $Y \xrightarrow{u} X$ of affine varieties is given. we say that a closed subvariety D of Y *lies over* a closed subvariety C of X if C is the image of D.

Similarly, given an integral extension $A \to B$ of finite-type domains, we say that a prime ideal Q of B *lies over* a prime ideal P of A if P is the contraction $Q \cap A$ (**2.6.4**). For example, if $Y \to X$ is the corresponding morphism of affine varieties, and if the point x of X is the image of a point y of Y, the maximal ideal $\mathfrak{m}_y$ lies over the maximal ideal $\mathfrak{m}_x$.

4.4.1. Lemma. *Let $A \subset B$ be an integral extension of finite-type domains, and let J be an ideal of B. If J isn't the zero ideal of B, then the contraction $J \cap A$ isn't the zero ideal of A.* *contrzero*

proof. An element β of J is the root of a monic polynomial with coefficients in A, say $\beta^k + a_{k-1}\beta^{k-1} + \cdots + a_0 = 0$. If $a_0 = 0$, then since B is a domain, we can cancel β from this equation. So we may assume that $a_0 \neq 0$. Then the equation shows that a_0 is in J as well as in A. $\square$

4.4.2. Proposition. *Let $A \to B$ be an integral extension of finite-type domains, and let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$.* *QoverP*

(i) *Let P and Q be prime ideals of A and B, respectively, let C be the locus of zeros of P in X, and let D be the locus of zeros of Q in Y. Then Q lies over P if and only if D lies over C.*

(ii) *Let Q and Q' be prime ideals of B that lie over the same prime ideal P of A. If $Q \subset Q'$, then $Q = Q'$. Therefore, if D' and D are closed subvarieties of Y that lie over the same subvariety C of X and if $D \subset D'$, then $D' = D$.*

proof. **(i)** Suppose that Q lies over P, i.e., that $P = Q \cap A$. Let $\overline{A} = A/P$ and $\overline{B} = B/Q$. We have a canonical injective map $\overline{A} \to \overline{B}$, and $\overline{B}$ will be generated as $\overline{A}$-module by the residues of a set of generators of the finite A-module B. So $\overline{B}$ is a finite $\overline{A}$-module, an integral extension of $\overline{A}$, and the map from $\operatorname{Spec} \overline{B} = D$ to $\operatorname{Spec} \overline{A} = C$ is surjective (Proposition 4.2.6). This means that D lies over C. Conversely, if D lies over C, the morphism $D \to C$ is surjective, and therefore the canonical map $\overline{A} \to \overline{B}$ is injective. This implies that $P = Q \cap A$.

(ii) Suppose that Q and Q' lie over P and that $Q \subset Q'$. Let $\overline{A} = A/P$, $\overline{B} = B/Q$, and $\overline{Q}' = Q'/Q$. Then because B is an integral extension of A, $\overline{B}$ is an integral extension of its subring $\overline{A}$, and $\overline{Q}'$ is an ideal of $\overline{B}$. Since Q nd Q' lie over P, $Q' \cap A = P = Q \cap A$. So $\overline{Q}' \cap \overline{A} = 0$. Lemma 4.4.1 shows that $\overline{Q}' = 0$. Therefore $Q' = Q$. $\square$

4.4.3. Theorem. *Let $Y \xrightarrow{u} X$ be an integral morphism of affine varieties.* *closedimage*

(i) *The fibres of the morphism u have bounded cardinality.*

(ii) *The image of a closed subset of Y is a closed subset of X, and the image of a closed subvariety of Y is a closed subvariety of X.*

(iii) *Let C be a closed subvariety of X. The set of closed subvarieties of Y that lie over C is finite and nonempty.*

proof. Let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, and let $A \subset B$ be the inclusion that corresponds to the integral, and therefore surjective, morphism $Y \to X$.

(i) *(bounding the fibres)* Let $y_1, ..., y_r$ be points of Y in the fibre over a point x of X. For each i, the maximal ideal $\mathfrak{m}_x$ of A at x is the contraction of the maximal ideal $\mathfrak{m}_{y_i}$ of B at y_i. To bound the number r, we use the Chinese Remainder Theorem to show that B cannot be spanned as A-module by fewer than r elements.

Let k_i and k denote the residue fields $B/\mathfrak{m}_{y_i}$, and $A/\mathfrak{m}_x$, respectively, all of these fields being isomorphic to $\mathbb{C}$. Let $\overline{B} = k_1 \times \cdots \times k_r$. We form a diagram of algebra homomorphisms

$$\begin{tikzcd} B \arrow[r, "\varphi"] & \overline{B} \approx \mathbb{C}^r \\ A \arrow[u] \arrow[r] & k \approx \mathbb{C} \arrow[u] \end{tikzcd}$$

which we interpret as a diagram of A-modules. The minimal number of generators of the A-module $\overline{B}$ is equal to its dimension as k-module, which is r. The Chinese Remainder Theorem asserts that φ is surjective, so B cannot be spanned by fewer than r elements.

(ii) (*the image of a closed set is closed*) The image of an irreducible set via a continuous map is irreducible (2.2.17**iii**), so it suffices to show that the image of a closed subvariety is closed. Let D be the closed subvariety of Y that corresponds to a prime ideal Q of B, and let $P = Q \cap A$ be its contraction, which is a prime ideal of A. Let C be the variety of zeros of P in X. The coordinate algebras of the affine varieties D and C are $\overline{B} = B/Q$ and $\overline{A} = A/P$, respectively, and because B is an integral extension of A, $\overline{B}$ is an integral extension of $\overline{A}$. By (4.2.6), the map $D \to C$ is surjective. Therefore C is the image of D.

(iii) (*subvarieties that lie over a closed subvariety*) Let C be a closed subvariety of X. Its inverse image $Z = u^{-1}C$ is closed in Y. It is the union of finitely many irreducible closed subsets, say $Z = D'_1 \cup \cdots \cup D'_k$. Part **(i)** tells us that the image C'_i of D'_i is a closed subvariety of X. Since u is surjective, $C = \bigcup C'_i$, and since C is irreducible, $C'_i = C$ for at least one i. Then for that i, D'_i lies over C. Next, any subvariety D that lies over C will be contained in the inverse image Z, and therefore contained in D'_i for some i. Proposition 4.4.2 **(ii)** shows that $D = D'_i$. Therefore the varieties that lie over C are among the varieties D'_i. $\square$

grpii **4.4.4. Example.** Let G be a finite group of automorphisms of a normal, finite-type domain B, let A be the algebra B^G of invariant elements of B, and let $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. According to Theorem 2.8.5, A is a finite-type domain, B is a finite A-module, and points of X correspond to G-orbits of points of Y.

invariant-denom **4.4.5. Lemma.** With the above notation, let L and K be the fraction fields of B and A, respectively.

(i) The algebra A is normal, and B is an integral extension of A.

(ii) Every element of L can be written as a fraction b/s, with b in B, and s in A.

(iii) L is a Galois extension of K, with Galois group G. The ring L^G of invariant elements of L is K. $\square$

Section 4.5 Dimension

dim Every variety has a dimension, and, as is true for the dimension of a vector space, the dimension is important, though it is a very coarse measure. We give two definitions of dimension of a variety X, though the proof that they are equivalent requires work.

The first definition is that the dimension of a variety X is the transcendence degree of its function field. For now, we'll refer to this as the *t-dimension* of X.

intdim **4.5.1. Corollary.** *Let $Y \to X$ be an integral morphism of affine varieties. The t-dimensions of X and of Y are equal.* $\square$

The second definition of dimension is the combinatorial dimension, which is defined as follows: A *chain* of closed subvarieties of a variety X is a strictly decreasing sequence of closed subvarieties.

chntwo (4.5.2)
$$C_0 > C_1 > C_2 > \cdots > C_k$$

The *length* of this chain is defined to be k. The chain is *maximal* if it cannot be lengthened by inserting another closed subvariety, which means that $C_0 = X$, that there is no closed subvariety $\widetilde{C}$ with $C_i > \widetilde{C} > C_{i+1}$ for $i < k$, and that C_k is a point.

For example, $\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$, where $\mathbb{P}^i$ is the linear subspace of points $(x_0, ..., x_i, 0, ..., 0)$, is a maximal chain, of length n, in projective space $X = \mathbb{P}^n$.

Theorem 4.5.7 below shows that all maximal chains of closed subvarieties have the same length. The *combinatorial dimension* of X is the length of a maximal chain. We'll refer to is as the *c-dimenson*. Theorem 4.5.7 also shows that the t-dimension and the c-dimension of a variety are equal. When we have proved that theorem, we will refer to the t-dimension and to the c-dimension simply as the *dimension*, and we will use the two definitions interchangeably.

Recall that the transcendence degree of a domain A is equal to the transcendence degree of its fraction field (1.5.1). So the t-dimension of an affine variety $X = \operatorname{Spec} A$ is also equal to the transcendence degree of the coordinate algebra A.

In an affine variety $\operatorname{Spec} A$, the decreasing chain (4.5.2) corresponds to a strictly increasing chain

chn **(4.5.3)**
$$P_0 < P_1 < P_2 < \cdots < P_k,$$

of prime ideals of A of length k, a *prime chain*. This prime chain is *maximal* if it cannot be lengthened by inserting another prime ideal, which means that P_0 is the zero ideal, that there is no prime ideal $\widetilde{P}$ with $P_i < \widetilde{P} < P_{i+1}$ for $i < k$, and that P_k is a maximal ideal. The *c-dimension* of a finite-type domain A is the length k of a maximal chain (**4.5.3**) of prime ideals. If $X = \operatorname{Spec} A$, then the c-dimensions of X and of A are equal.

The next theorem is the basic tool for studying dimension. Though the statement is intuitively plausible, the proof isn't easy. It is a subtle theorem.

4.5.4. Krull's Principal Ideal Theorem. Let X be an affine variety of t-dimension d, let α be a nonzero element of its coordinate algebra A, and let V be the zero locus of α in X. Every irreducible component of V has t-dimension $d-1$. krullthm

4.5.5. Corollary. *Let X be an affine variety of t-dimension d, and let C be a component of the zero locus of a nonzero element α of its coordinate algebra A. There is no closed subvariety D wuch that $C < D < X$. So among proper closed subvarieties, C is maximal.* compmax

proof, assuming Krull's Theorem. We show that if X has t-dimension d, and if $C < D < X$ are closed subvarieties of X, then the t-dimension of C is at most $d-2$.

Some nonzero element β of A will vanish on D. Then D will be a subvariety of the zero locus of β, so by Krull's Theorem, its t-dimension will be at most $d-1$. Similarly, if $D = \operatorname{Spec} B$, some nonzero element of B will vanish on C, so the t-dimension of C will be at most $d-2$. □

proof of Krull's Theorem.
Step 1. The case of an affine space: $A = \mathbb{C}[x_1, ..., x_d]$.

The irreducible components of the zero locus V of the nonzero polynomial α are the zero sets of its irreducible factors. We replace α by the irreducible factor that vanishes on the particular component C. Then C is the zero locus of α, and the coordinate algebra of C is $\overline{A} = A/A\alpha$. Next, we choose a transcendence basis $\alpha_1, ..., \alpha_d$ of A, with $\alpha_d = \alpha$ (see (1.5.1)). Let $\overline{\alpha}_i$ be the residue of α_i in $\overline{A}$, for $i = 1, ..., d-1$. To show that the t-dimension of C is $d-1$, we show that $\overline{\alpha}_1, ..., \overline{\alpha}_{d-1}$ is a transcendence basis of $\overline{A}$.

Let $R = \mathbb{C}[\alpha_1, ..., \alpha_d]$ and $\overline{R} = \mathbb{C}[\overline{\alpha}_1, ..., \overline{\alpha}_{d-1}]$. We first show that every element of $\overline{A}$ is algebraic over $\overline{R}$. Let $\overline{\beta}$ be such an element, and say that $\overline{\beta}$ is represented by an element β of A. This element β is algebraic over the fraction field $L = \mathbb{C}(\alpha_1, ..., \alpha_d)$ of R. Let $f(t) = \sum c_i t^i$ be a polynomial with coefficients c_i in L that has β as root. We may clear denominators, so we may assume that the coefficients are in R, and that they aren't all divisible by α_d. Then the residues $\overline{c}_i$ of c_i in $\overline{R}$ aren't all zero, so the residue $\overline{f}$ of f isn't the zero polynomial. But $\overline{\beta} = \overline{f}(\overline{\alpha}_1, ..., \overline{\alpha}_{d-1}) = 0$. So $\overline{\beta}$ is algebraic over $\overline{R}$.

Next, let $g(z_1, ..., z_{d-1})$ be a nonzero polynomial such that $g(\overline{\alpha}_1, ..., \overline{\alpha}_{d-1}) = 0$ in $\overline{R}$. Then $g(\alpha_1, ..., \alpha_{d-1}) \equiv 0$, modulo α_d in R. Therefore there is a polynomial $h(z_1, ..., z_d)$ such that $g(\alpha_1, ..., \alpha_{d-1}) = \alpha_d h(\alpha_1, ..., \alpha_d)$. The polynomial $g(z_1, ..., z_{d-1}) - z_d h(z_1, ..., z_d)$ isn't zero, but it evaluates to zero when α is substituted for z. This contradicts the fact that $\alpha_1, ..., \alpha_d$ are algebraically independent.

Step 2. Reduction to the case that A is normal.

We go back to Krull's Theorem. We are given an affine variety $X = \operatorname{Spec} A$ of t-dimension d, a nonzero element α of A, and an irreducible component C of the zero locus of α. We are to show that the t-dimension of C is $d-1$.

Let $A^\#$ be the normalization of A and let $X^\# = \operatorname{Spec} A^\#$. The t-dimension of $X^\#$ is d. The integral morphism $X^\# \to X$ is surjective, and it sends closed sets to closed sets (4.4.3). Let V' and V be the zero loci of α in $X^\#$ and in X, respectively. Then V' is the inverse image of V, and the map $V' \to V$ is surjective.

Let $D_1, \cdots, D_k$ be the irreducible components of V', and let C_i be the image of D_i in X. The closed sets C_i are irreducible (4.4.3) **(ii)**, and their union is V. So at least one C_i is equal to C. Let D be a component of V' whose image is C. The map $D \to C$ is also an integral morphism, so the t-dimensions of D and of C are equal. We may therefore replace X and C by $X^\#$ and D, respectively. Hence we may assume that A is normal.

Step 3. Reduction to the case that the zero locus of α is irreducible.

We do this by localizing. Suppose that the zero locus of α is $C \cup \Delta$, where C is our irreducible component, and Δ is the union of the other irreducible components. We choose an element s of A that is identically zero on Δ but not identically zero on C. Inverting s eliminates all points of Δ, but $X_s \cap C = C_s$ won't be empty. If X is normal, so is X_s (4.3.4) **(ii)**. Since localization doesn't change t-dimension, we may replace X and C by X_s and C_s, respectively.

Step 4. Completion of the proof.

This is the main step. We assume that X is normal, and that the irreducible closed set C is the zero locus of α in X. We apply the Noether Normalization Theorem. Let $X \to S$ be an integral morphism to an affine space $S = \operatorname{Spec} R$ of dimension d, where R is a polynomial ring $\mathbb{C}[u_1, ..., u_d]$. Let K and F be the function fields of X and S, respectively, and let L be a Galois extension of F that contains K. Let B be the integral closure of A in L, and let $Y = \operatorname{Spec} B$. Then Y is an integral extension of S and of X. The Galois group G of L/F operates on B and on Y. and the algebra B^G of invariants is R. We have morphisms

$$Y \xrightarrow{u} X \xrightarrow{v} S$$

Let $w = vu$ denote the composed morphism $Y \to S$.

Let $\alpha_1, ..., \alpha_r$ be the orbit of α, with $\alpha = \alpha_1$. The coefficients of the polynomial $f(t) = (t-\alpha_1)\cdots(t-\alpha_r)$ are invariant. They are elements of R, and the constant term is the product $\alpha_1 \cdots \alpha_r$. Let's denote that product by β. In A, $t-\alpha$ divides $f(t)$, and therefore α divides β.

The element β defines functions on S, X, and Y. The functions on X and Y are obtained from the one on S by composition with the maps v and w, respectively. We denote all of those functions by β. If y is a point of Y, $x = uy$ and $s = wy$, then $\beta(y) = \beta(x) = \beta(s)$. Similarly, α defines functions on X and on Y that we denote by α.

imageofC **4.5.6. Lemma.** *With notation as above, let Z be the zero locus of β in S, and let C be the zero locus of α in X, as above. Then Z is the image of C via the map $X \xrightarrow{v} S$.*

proof. Let x be a point of C. So $\alpha(x) = 0$. Since α divides β, $\beta(x) = 0$. If s is the image of x in S, then $\beta(s) = \beta(x)$, so $\beta(s) = 0$. This shows that s is a point of Z. Therefore Z contains the image of C.

For the other inclusion, let z be a point of Z. So $\beta(z) = 0$. Let y be a point of Y such that $wy = z$. The fibre of Y over z is the G-orbit of y (2.8.5), and β vanishes at every point of that orbit. Since $\beta = \prod \alpha_i$, $\alpha_i(y) = 0$ for some i. Let σ be an element of G such that $\alpha_i = \sigma\alpha$. Remembering that $[\sigma\alpha](y) = \alpha(y\sigma)$ (2.8.8), we see that $\alpha(y\sigma) = 0$. We replace y by $y\sigma$. Then $\alpha(y) = 0$, and it is still true that $wy = z$.

Let $x = uy$. Because $\alpha(y) = 0$, it is also true that $\alpha(x) = 0$. So x is a point of C. The image of x in S is $vx = vuy = wy = z$. Since z is an arbitrary point of Z, the map $C \to Z$ is surjective. □

Going back to the proof of Step 4 of Krull's Theorem, the image Z of the surjective map $C \to Z$ is irreducible because C is irreducible (2.2.17**(iii)**), so it is a variety. Since $X \to S$ is an integral morphism, so is the map $C \to Z$. Therefore C and Z have the same t-dimension. Moreover, Z is the zero set of β in S. Step 1 tells us that the t-dimenion of Z is $d-1$. So C also has t-dimension $d-1$. This completes the proof of Krull's Theorem. □

dimtheorem **4.5.7. Theorem.** *Let X be a variety of t-dimension d. All chains of closed subvarieties of X have length at most d, and all maximal chains have length d. Therefore the c-dimension and the t-dimension of X are equal.*

proof. We prove the theorem for an affine variety first. Induction allows us to assume that it is true for an affine variety whose t-dimension is less than d. Let $X = \operatorname{Spec} A$ be an affine variety of t-dimension d, and let $C_0 > C_1 > \cdots > C_k$ be a chain of closed subvarieties of X. We must show that $k \le d$ and that $k = d$ if the chain is maximal. We may insert closed subvarieties into the chain where possible, so we may assume that $C_0 = X$. Next, C_1, being a proper closed subset of X, is contained in the zero locus Z of a nonzero element α of A, and it will be contained in some irreducible component $\widetilde{C}$ of Z. If $\widetilde{C} > C_1$, we insert $\widetilde{C}$ into the chain, to reduce ourselves to the case that C_1 is a component of the zero locus of α. By Krull's Theorem, C_1 has t-dimension $d-1$. By Corollary 4.5.5 it is a maximal proper closed subvariety, and induction applies to the chain $C_1 > \cdots > C_k$ of closed subvarieties of C_1. By induction, the length of that chain, which is $k-1$ is less than $d-1$, and it is equal to $d-1$ if the chain is maximal. Therefore the chain $\{P_i\}$ has length at most n, and it has length n if it is a maximal chain.

Theorem 4.5.7 for a variety that isn't affine follows from the next lemma. □

restrictchain **4.5.8. Lemma.** *Let X' be an open subvariety of a variety X. There is a bijective correspondence between chains $C_0 > \cdots > C_k$ of closed subvarieties of X such that $C_k \cap X' \neq \emptyset$ and chains $C'_0 > \cdots > C'_k$ of closed subvarieties of X'. Given the chain $\{C_i\}$ in X, the chain $\{C'_i\}$ in X' is defined by $C'_i = C_i \cap X'$. Given a chain C'_i in X', the corresponding chain in X consists of the closures C_i in X of the varieties C'_i.*

proof. Suppose given a chain C_i and that $C_k \cap X' \neq \emptyset$. Then for every i, the intersection $C_i' = C_i \cap X'$ is a dense open subset of the irreducible closed set C_i (2.2.15). So the closure of C_i' is C_i, and since $C_i > C_{i+1}$, it is also true that $C_i' > C_{i+1}'$. Therefore $C_0' > \cdots > C_k'$ is a chain of closed subsets of X'. Conversely, if $C_0' > \cdots > C_k'$ is a chain in X', the closures in X form a chain in X. $\square$

From now on, we use the word *dimension* to denote either of the two concepts, t-dimension or c-dimension, and we denote the dimension of a variety by $\dim X$.

4.5.9. Examples. (i) The polynomial algebra $\mathbb{C}[x_0, \dots, x_n]$ in $n+1$ variables has dimension $n+1$. The chain of prime ideals dim-polyalg

(**4.5.10**) $$0 < (x_0) < (x_0, x_1) < \cdots < (x_0, \dots, x_n)$$ primechain

is a maximal prime chain. When the irrelevant ideal $(x_0, ..., x_n)$ is removed from this chain, it corresponds to a maximal chain

$$\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$$

of closed subvarieties of projective space $\mathbb{P}^n$, which has c-dimension n.

(ii) The maximal chains of closed subvarieties of $\mathbb{P}^2$ have the form $\mathbb{P}^2 > C > p$, where C is a plane curve and p is a point. $\square$

If (4.5.2) is a maximal chain in X, then $C_0 = X$, and

(**4.5.11**) $$C_1 > C_2 > \cdots > C_k$$ chaini

will be a maximal chain in the variety C_1. So when X has dimension k, the dimension of C_1 is k–1. Similarly, let (**4.5.3**) be a maximal chain of prime ideals in a finite-type domain A, let $\overline{A} = A/P_1$ and let $\overline{P}_j$ denote the image P_j/P_1 of P_j in $\overline{A}$, for $j \geq 1$. Then

$$\overline{0} = \overline{P}_1 < \overline{P}_2 < \cdots < \overline{P}_k$$

will be a maximal prime chain in $\overline{A}$, and therefore the dimension of the domain $\overline{A}$ is $k-1$. There is a bijective correspondence between maximal prime chains in $\overline{A}$ and maximal prime chains in A whose first coefficient is P_0.

4.5.12. Corollary. *Let X be a variety.* *integraldimequal*

(i) *If X' is an open subvariety of a X, then* $\dim X' = \dim X$.

(ii) *If Y is a proper closed subvariety of X, then* $\dim Y < \dim X$.

(iii) *If $Y \to X$ is an integral morphism of varieties, then* $\dim Y = \dim X$. $\square$

One more term: A closed subvariety C of a variety X has *codimension* 1 if $C < X$ and if there is no closed set $\widetilde{C}$ with $C < \widetilde{C} < X$, or if $\dim C = \dim X - 1$. A prime ideal P of a noetherian domain has *codimension* 1 if it isn't the zero ideal, and if there is no prime ideal $\widetilde{P}$ with $(0) < \widetilde{P} < P$. In the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, the prime ideals of codimension 1 are the principal ideals generated by irreducible polynomials.

Section 4.6 Chevalley's Finiteness Theorem

(4.6.1) finite morphisms

finmorph prodagain

The concepts of finite morphisms and integral morphisms of affine varieties were defined in Section 4.2. A morphism $Y \xrightarrow{u} X$ of affine varieties $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ is a finite morphism if the homomorphism $A \xrightarrow{\varphi} B$ that corresponds to u makes B into a finite A-module. As was noted before, the difference between a finite morphism and an integral morphism of affine varieties is that for a finite morphism, the homomorphism φ needn't be injective. If u is a finite morphism and φ is injective, B will be an integral extension of A, and u will be an integral morphism. We extend these definitions to varieties that aren't necessarily affine here.

By the *restriction* of a morphism $Y \xrightarrow{u} X$ to an open subset X' of X, we mean the induced morphism $Y' \to X'$, where Y' is the inverse image of X'.

deffin-morph

4.6.2. Definition. A morphism of varieties $Y \xrightarrow{u} X$ is a *finite morphism* if X can be covered by affine open subsets X^i such that the restriction of u to each X^i is a finite morphism of affine varieties, as defined in (4.2.5). Similarly, a morphism u is an *integral morphism* if X can be overed by affine open sets X^i to which the restriction of u is an integral morphism of affine varieties.

finiteex-amp

4.6.3. Corollary. *An integral morphism is a finite morphism. The composition of finite morphisms is a finite morphism. The inclusion of a closed subvariety into a variety is a finite morphism.* □

When X is affine, Definitions 4.2.5 and 4.6.2 both apply. The next proposition shows that these two definitions are equivalent. Unfortunately, the proof is rather long. Verifications such as this are costs of doing business with affine open subsets of projective varieties.

onecov-erfinite

4.6.4. Proposition. *Let $Y \xrightarrow{u} X$ be a finite or an integral morphism, as defined in (4.6.2), and let X' be an affine open subset of X. The restriction of u to X' is a finite or an integral morphism of affine varieties, as defined in (4.2.5).*

restfin-morph

4.6.5. Lemma. **(i)** *Let $A \xrightarrow{\varphi} B$ be a homomorphism of finite-type domains that makes B into a finite A-module, and let s be a nonzero element of A. Then B_s is a finite A_s-module.*
(ii) *Using Definition 4.6.2, the restriction of a finite (or an integral) morphism $Y \xrightarrow{u} X$ to an open subset of a variety X is a finite (or an integral) morphism.*

proof. **(i)** Here B_s denotes the localization of B as an A-module. This localization can also be obtained by localizing the algebra B with respect to the image $s' = \varphi(s)$, provided that it isn't zero. If s' is zero, then s annihilates B, so $B_s = 0$. In either case, a set of elements that spans B as A-module will span B_s as A_s-module, so B_s is a finite A_s-module.

(ii) Say that X is covered by affine open sets to which the restriction of u is a finite morphism. The localizations of these open sets form a basis for the Zariski topology on X, so X' can be covered by such localizations. Part **(i)** shows that the restriction of u to X' is a finite morphism. □

proof of Proposition 4.6.4. We'll do the case of a finite morphism. The proof isn't difficult, but there are several things to check, and this makes the proof longer than one would like.

Step 1. Preliminaries.

We are given a morphism $Y \xrightarrow{u} X$, and an affine open covering $\{X^i\}$ of X, such that the restriction u^i of u to X^i is a finite morphism of affine varieties for every i. We are to show that the restriction to any affine open set X_1 is a finite morphism of affine varieties.

The affine open set X_1 is covered by the affine open sets $X_1^i = X_1 \cap X^i$. For every i, the restriction u_1^i of u to X_1^i can also be obtained by restricting u^i. So u_1^i is a finite morphism (4.6.5) **(ii)**. We may replace X by X_1. Since the localizations of an affine variety form a basis for its Zariski topology, we see that what is to be proved is this:

A morphism $Y \xrightarrow{u} X$ is given in which $X = \operatorname{Spec} A$ is affine. There are elements $s_1, ..., s_k$ that generate the unit ideal of A, such that for every i, the inverse image Y^i of $X^i = X_{s_i}$, if nonempty, is affine, and its coordinate algebra B_i is a finite module over the localized algebra $A_i = A_{s_i}$. We must show that Y is affine, and that its coordinate algebra B is a finite A-module.

Step 2. The algebra of regular functions on Y.

We assume that X is affine, $X = \operatorname{Spec} A$. Let B be the algebra of regular functions on Y. If Y is affine, B will be its coordinate algebra, and Y will be its spectrum. Since Y isn't assumed to be affine, we don't know very much about B other than that it is a subalgebra of the function field of Y. By hypothesis, the inverse image Y^i of X^i, if nonempty, is the spectrum of a finite A_i-algebra B_i. We throw out the indices i such that Y^i is empty. Then B and B_i are subalgebras of the function field of Y. Since the localizations X^i cover X, the affine varieties Y^i cover Y. A function is regular on Y if and only if it is regular on each Y^i, and therefore

$$B = \bigcap B_i$$

Step 3. The algebras B_j are localizations of B.

Denoting the images in B of the elements s_i by the same symbols s_i, we show that B_j is the localization $B[s_j^{-1}]$. The localization $X^i = X_{s_i}$ it is the set of points of X at which $s_i \neq 0$. The inverse image Y^i of X^i in Y is the set of points of Y at which $s_i \neq 0$, and the affine variety $Y^j \cap Y^i$ is the set of points of Y^j at which $s_i \neq 0$. So the coordinate algebra of $Y^j \cap Y^i$ is the localization $B_j[s_i^{-1}]$. Then

$$B[s_j^{-1}] \overset{(1)}{=} \bigcap_i \big(B_i[s_j^{-1}]\big) \overset{(2)}{=} \bigcap_i B_j[s_i^{-1}] \overset{(3)}{=} B_j[s_j^{-1}] \overset{(4)}{=} B_j$$

The explanation of the numbered equalities is as follows:

(1) A rational function β is in $B_i[s_j^{-1}]$ if $s_j^n\beta$ is in B_i for large n, and we can use the same exponent n for all $i = 1, ..., r$. Then β is in $\bigcap_i \big(B_i[s_j^{-1}]\big)$ if and only if $s_j^n\beta$ is in $\bigcap_i B_i = B$. So β is in $\bigcap_i \big(B_i[s_j^{-1}]\big)$ if and only if it is in $B[s_j^{-1}]$.

(2) $B_i[s_j^{-1}] = B_j[s_i^{-1}]$ because $Y^j \cap Y^i = Y^i \cap Y^j$.

(3),(4) For all i, $B_j \subset B_j[s_i^{-1}]$. Since s_j is among the elements s_i, $\bigcap_i B_j[s_i^{-1}] \subset B_j[s_j^{-1}]$. Moreover, s_j doesn't vanish on Y^j. It is a unit in B_j, and therefore $B_j[s_j^{-1}] = B_j$. Then $B_j \subset \bigcap_i B_j[s_i^{-1}] \subset B_j[s_j^{-1}] = B_j$.

Step 4. B is a finite A-module.

With $A_i = A_{s_i}$ as before, we choose a finite set $(b_1, ..., b_n)$ of elements of B that generates the A_i-module B_i for every i. We can do this because we can span the finite A_i-module $B_i = B[s_i^{-1}]$ by finitely many elements of B, and there are finitely many algebras B_i. We show that the set $(b_1, ..., b_n)$ generates the A-module B.

Let x be an element of B. Then x is in B_i, so it is a combination of $(b_1, ..., b_n)$ with coefficients in A_i. For large k, $s_i^k x$ will be a combination of those elements with coefficients in A, say

$$s_i^k x = \sum_\nu a_{i,\nu} b_\nu$$

with $a_{i,\nu}$ in A. We can use the same exponent k for all i. Then with $\sum r_i s_i^k = 1$,

$$x = \sum_i r_i s_i^k x = \sum_i r_i \sum_\nu a_{i,\nu} b_\nu$$

The right side is a combination of $b_1, ..., b_n$ with coefficients in A.

Step 5. Y is affine.

The algebra B of regular functions on Y is a finite-type domain because it is a finite module over the finite-type domain A. Let $\widetilde{Y} = \operatorname{Spec} B$. The fact that B is the algebra of regular functions on Y gives us a morphism $Y \xrightarrow{\epsilon} \widetilde{Y}$ (Corollary 3.6.2). Restricting to the open subset X^j of X gives us a morphism $Y^j \xrightarrow{\epsilon^j} \widetilde{Y}^j$ in which Y^j and $\widetilde{Y}^j$ are both equal to $\operatorname{Spec} B_j$, $B_j = B[s_j^{-1}]$. Therefore ϵ^j is an isomorphism. Corollary 3.5.14 **(ii)** shows that ϵ is an isomorphism. So Y is affine and by *Step 4*, its coordinate algebra B is a finite A-module. □

We come to Chevalley's theorem now. Let $\mathbb{P}$ denote the projective space $\mathbb{P}^n$ with coordinates $y_0, ..., y_n$.

4.6.6. Chevalley's Finiteness Theorem. *Let X be a variety, let Y be a closed subvariety of the product $\mathbb{P}\times X$, let π denote the projection $Y \to X$, and let i denote the inclusion of Y into $\mathbb{P}\times X$. If all fibres of π are finite sets, then π is a finite morphism.* chevfin

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "\pi"'] & \mathbb{P}\times X \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

4.6.7. Corollary. **(i)** *Let Y be a projective variety and let $Y \xrightarrow{\pi} X$ be a morphism whose fibres are finite sets. Then π is a finite morphism.* pro-jchevfin

(ii) *If Y is a projective curve, every nonconstant morphism $Y \xrightarrow{u} X$ is a finite morphism.*

proof. **(i)** This follows from the theorem when one replaces Y by the graph of π in $Y\times X$, which is isomorphic to Y. If Y is a closed subvariety of $\mathbb{P}$, the graph will be a closed subvariety of $\mathbb{P}\times X$ (Proposition 3.5.25).

(ii) When Y is a curve, the fibres of a nonconstant morphism are finite sets. □

In the next lemma, A denotes a finite-type domain, B denotes a quotient of the algebra $A[u]$ of polynomials in n variables $u_1,...,u_n$ with coefficients in A, and $A \xrightarrow{\varphi} B$ denotes the canonical homomorphism. We'll use capital letters for nonhomogeneous polynomials here, and if $G(u)$ is a polynomial in $A[u]$, we denote its image in B by $G(u)$ too.

powerso-fyenuf **4.6.8. Lemma.** *Let k be a positive integer. Suppose that, for each $i = 1,...,n$, there is a polynomial $G_i(u_1,...,u_n)$ of degree at most $k-1$ with coefficients in A, such that $u_i^k = G_i(u)$ in B. Then B is a finite A-module.*

proof. Any monomial in $u_1,...,u_n$ of degree $d \ge nk$ will be divisible by u_i^k for at least one i. If m is such a monomial, the relations $u_i^k = G_i(u)$ show that, in B, m is equal to a polynomial in $u_1,...,u_n$ of degree less than d, with coefficients in A. It follows by induction that the monomials in $u_1,...,u_n$ of degree at most $nk-1$ span B as an A-module. □

Let $y_0,...,y_n$ be coordinates in $\mathbb{P}^n$, and let $A[y_0,...,y_n]$ be the algebra of polynomials in y with coefficients in A. In analogy with the terminology for complex polynomials, we say that a a polynomial with coefficients in A is *homogeneous* if it is homogeneous as a polynomial in y. An ideal of $A[y]$ that can be generated by homogeneous polynomials is a *homogeneous ideal*.

hompoly-coeffA **4.6.9. Lemma.** *Let $X = \operatorname{Spec} A$ be an affine variety, and let Y be a subset of $\mathbb{P}\times X$.*
(i) *The ideal $\mathcal{I}$ of elements of $A[y]$ that vanish at every point of Y is a homogeneous ideal of $A[y]$, that is equal to its radical. If Y is a closed subvariety of $\mathbb{P}\times X$, then $\mathcal{I}$ is a prime ideal.*
(ii) *If the zero locus of a homogeneous ideal $\mathcal{I}$ of $A[y]$ is empty, then $\mathcal{I}$ contains a power of the irrelevant ideal $\mathcal{M} = (y_0,...,y_n)$ of $A[y]$.*

proof. **(i)** We write a point of $\mathbb{P}\times Y$ as $q = (y_0,...,y_n,p)$, where p is a point of Y. So $(y,p) = (\lambda y,p)$. Then the proof for the case $A = \mathbb{C}$ that is given in (1.3.2) carries over to show that $\mathcal{I}$ is a homogeneous ideal, and it is a radical ideal. The proof of Proposition 2.5.13 applies without change, to show that $\mathcal{I}$ is a prime ideal if Y is a closed subvariety of $\mathbb{P}\times X$.

(ii) Let W be the complement of the origin in the affine $n+1$-space with coordinates y. Then $W\times Y$ maps to $\mathbb{P}\times Y$ (see 3.2.6). If the locus of zeros of $\mathcal{I}$ in $\mathbb{P}\times Y$ is empty, its locus of zeros in $W\times Y$ will be contained in $o\times Y$, o being the origin in $\mathbb{A}^{n+1}$. Then the radical of $\mathcal{I}$ will contain the ideal of $o\times Y$ in $\mathbb{A}^{n+1}\times Y$, which is the irrelevant ideal $\mathcal{M}$. □

familyof-vars **4.6.10. Example.** Let $X = \operatorname{Spec} A$, where A is the polynomial algebra $\mathbb{C}[t]$. The ideal in the algebra $A[y]$ generated by the irreducible polynomial $y_0^3 + y_1^3 + y_2^3 + ty_0y_1y_2 = 0$ is a prime ideal because $A[y]$ is a unique factorization domain. Its zero locus in $\mathbb{P}^2\times Y$, can be regarded as a family of plane cubic curves, parametrized by t. □

proof of Chevelley's Finiteness Theorem. This is Schelter's proof.
By induction on n, we may assume that the theorem is true when $\mathbb{P}$ is a projective space of dimension $n-1$.

We abbreviate the notation for a product of a variety Z with X, denoting $Z\times X$ by $\widetilde{Z}$. We are given a closed subvariety Y of $\widetilde{\mathbb{P}} = \mathbb{P}\times X$, whose fibres over X are finite sets. We are to prove that the projection $Y \to X$ is a finite morphism. We may suppose that X is affine, say $X = \operatorname{Spec} A$ (see Definition 4.6.2).

Case 1. There is a hyperplane H in $\mathbb{P}$ such that Y is disjoint from $\widetilde{H} = H\times X$ in $\widetilde{\mathbb{P}}$.

This is the main case. We adjust coordinates $y_0,...,y_n$ in $\mathbb{P}$ so that H is the hyperplane at infinity $\{y_0 = 0\}$. Because Y is disjoint from $\widetilde{H}$, it is a subset of the affine variety $\widetilde{\mathbb{U}}^0 = \mathbb{U}^0\times X$, $\mathbb{U}^0$ being the standard affine $\{y_0 \neq 0\}$ in $\mathbb{P}$. Since Y is irreducible and closed in $\widetilde{\mathbb{P}}$, it is a closed subvariety of $\widetilde{\mathbb{U}}^0$. So Y is affine.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the homogeneous prime ideals in $A[y]$ whose zero sets in $\widetilde{\mathbb{P}}$ are Y and $\widetilde{H}$, respectively, and let $\mathcal{I} = \mathcal{P}+\mathcal{Q}$. The ideal $\widetilde{q}$ is the principal ideal of $A[y]$ generated by y_0. A homogeneous polynomial of degree k in $\mathcal{I}$ has the form $f(y) + y_0g(y)$, where f is a homogeneous element of $\mathcal{P}$ of degree k, and g is a homogeneous polynomial in $A[y]$ of degree $k-1$.

The closed subsets Y and $\widetilde{H}$ are disjoint: $Y\cap\widetilde{H}$ is empty. Therefore the sum $\mathcal{I}=\mathcal{P}+\mathcal{Q}$ contains a power of the irrelevant ideal $\mathcal{M}=(y_0,...,y_n)$. Say that $\mathcal{M}^k\subset\mathcal{I}$. Then y_i^k is in $\mathcal{I}$, for $i=0,...,n$. So we may write

(4.6.11) $$y_i^k = f_i(y)+y_0g_i(y)$$ fplusg

with f_i in $\mathcal{P}$ homogeneous, of degree k and g_i in $A[y]$ homogeneous, of degree $k-1$. For the index $i=0$, we can set $f_0=0$ and $g_0=y_0^{d-1}$. We can omit this trivial case.

We dehomogenize these equations, multiplying by y_0^d and substituting $u_i=y_i/y_0$ for y_i, with $i=0,...,n$, and $u_0=1$. Writing the dehomogenizations with capital letters, the equations that correspond to (4.6.11) have the form

(4.6.12) $$u_i^k = F_i(u)+G_i(u)$$ FplusG

The important point is that the degree of G_i is at most $k-1$.

Recall that Y is a closed subset of $\mathbb{U}^0$. Its (nonhomogenous) ideal P in $A[u]$ contains the polynomials $F_1(u),...,F_n(u)$, and its coordinate algebra is $B=A[u]/P$. In the quotient algebra B, the terms $F_i(u)$ in (4.6.12) drop out, leaving us with equations $u_i^k=G_i(u)$, which are true in B. Since G_i has degree at most $k-1$, Lemma 4.6.8 tells us that B is a finite A-algebra, as was to be shown.

This completes the proof of Case 1.

Case 2. the general case.

We have taken care of the case in which there exists a hyperplane H such that Y is disjoint from $\widetilde{H}$. The next lemma shows that we can cover the given variety X by open subsets to which this special case applies. Then Lemma 4.6.4 and Proposition 4.6.4 apply to complete the proof.

4.6.13. Lemma. *Let Y be a closed subvariety of $\widetilde{\mathbb{P}}=\mathbb{P}^n\times X$, and suppose that the projection $Y\xrightarrow{\pi}X$ has finite fibres. Suppose also that Chevalley's Theorem has been proved for closed subvarieties of $\mathbb{P}^{n-1}\times X$. For every point p of X, there is an open neighborhood X' of p in X, and there is a hyperplane H in $\mathbb{P}$, such that the inverse image $Y'=\pi^{-1}X'$ is disjoint from $\widetilde{H}$.* *avoidh-plane*

proof. Let p be a point of X, and let $\widetilde{q}=(\widetilde{q}_1,...,\widetilde{q}_r)$ be the finite set of points of Y making up the fibre over p. We project $\widetilde{q}$ from $\mathbb{P}\times X$ to $\mathbb{P}$, obtaining a finite set $q=(q_1,...,q_r)$ of points of $\mathbb{P}$, and we choose a hyperplane H in $\mathbb{P}$ that avoids this finite set. Then $\widetilde{H}$ avoids the fibre $\widetilde{q}$. Let Z denote the closed set $Y\cap\widetilde{H}$. Because the fibres of Y over X are finite, so are the fibres of Z over X. By hypothesis, Chevalley's Theorem is true for subvarieties of $\mathbb{P}^{n-1}\times X$, and $\widetilde{H}$ is isomorphic to $\mathbb{P}^{n-1}\times X$. It follows that, for every component Z' of Z, the morphism $Z'\to X$ is a finite morphism, and therefore its image is closed in X (Theorem 4.4.3). Thus the image C of Z is a closed subset of X, and it doesn't contain p. Then $X'=X-C$ is the required neighborhood of p. $\square$

Section 4.7 Double Planes

dplane

(4.7.1) affine double planes affdplanes

Let A be the polynomial algebra $\mathbb{C}[x,y]$, and let X be the affine plane $\operatorname{Spec}A$. An *affine double plane* is a locus of the form $w^2=f(x,y)$ in affine 3-space with coordinates w,x,y, where f is a square-free polynomial in x,y. (see Example 4.3.9). The affine double plane is $Y=\operatorname{Spec}B$, where $B=\mathbb{C}[w,x,y]/(w^2-f)$, and the inclusion $A\subset B$ gives us an integral morphism $Y\to X$.

We'll denote by w,x,y both the variables and their residues in B. As in Example 4.3.9, B is a normal domain of dimension two, and a free A-module with basis $(1,w)$. It has an automorphism σ of order 2, defined by $\sigma(a+bw)=a-bw$.

The fibres of Y over X are the σ-orbits in Y. If $f(x_0,y_0)\neq 0$, the fibre over the point x_0 of X consists of two points, and if $f(x_0,y_0)=0$, it consists of one point. The reason that Y is called a *double plane* is that most points of the plane X are covered by two points of Y. The *branch locus* of the covering, which will be denoted by Δ, is the (possibly reducible) curve $\{f=0\}$ in X. The fibres over the *branch points*, the points of Δ, are single points.

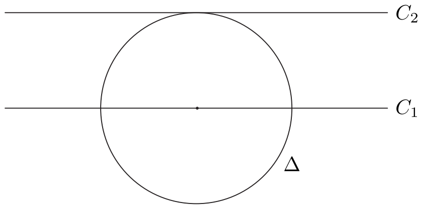
The closed subvarieties D of Y that lie over a curve C in X will have dimension one, and we call them curves too. The map $D \to C$ is surjective, and if D lies over C, so does $D' = D\sigma$. The curves D and D' may be equal or not. Let g be the defining polynomial of C. The components of the zero locus of g in Y have dimension one (Krull's Theorem). If a point q of Y lies over a point p of C, then q and $q\sigma$ are the only points of Y lying over p. One of them will be in D, the other in $D\sigma$. (Recall that since we are writing the operation of σ on B on the left, it operates on the right on Y (2.8.7).) So the inverse image of C is $D \cup D'$. There are no isolated points in the inverse image, and there is no room for another curve.

Thus if $D = D'$, then D is the only curve lying over C. Otherwise, there will be two curves that lie over C, namely D and D'. In that case we say that C *splits* in Y.

A curve C in X will be the zero set of a principal prime ideal P of the polynomial algebra A, and if D lies over C, it will be the zero set of a prime ideal Q of B that lies over P (4.4.2 **(i)**). However, the prime ideal Q needn't be a principal ideal of B.

circleex-ample **4.7.2. Example.** Let $f(x,y) = x^2 + y^2 - 1$. The double plane $Y = \{w^2 = x^2 + y^2 - 1\}$ is an *affine quadric* in $\mathbb{A}^3$. In the affine plane, its branch locus Δ is the curve $\{x^2 + y^2 = 1\}$.

The line $C_1 : \{y = 0\}$ in X meets the branch locus Δ transversally at the points $(x,y) = (\pm 1, 0)$, and when we set $y = 0$ in the equation for Y, we obtain the irreducible polynomial $w^2 - x^2 + 1$. So y generates a prime ideal of B. On the other hand, the line $C_2 : \{y = 1\}$ is tangent to Δ at the point $(0,1)$, and it splits. When we set $y = 1$ in the equation for Y, we obtain $w^2 = x^2$. The locus $\{w^2 = x^2\}$ is the union of the two lines $\{w = x\}$ and $\{w = -x\}$ that lie over C_1. The prime ideals of B that correspond to these lines aren't principal ideals.



$\square$

This example is an illustration of a general fact: A curve which intersects the branch locus transversally doesn't split. We explain this now.

localanal **(4.7.3) local analysis**

Suppose that a plane curve $C : \{g = 0\}$ and the branch locus $\Delta : \{f = 0\}$ of a double plane $w^2 = f$ meet at a point p. We adjust coordinates so that p becomes the origin $(0,0)$, and we write

$$f(x,y) = \sum a_{ij} x^i y^j = a_{10}x + a_{01}y + a_{20}x^2 + \cdots$$

Since p is a point of Δ, the constant coefficient of f is zero. If the two linear coefficients aren't both zero, p will be a smooth point of Δ, and the tangent line to Δ at p will be the line $\{a_{10}x + a_{01}y = 0\}$. Similarly, writing $g(x,y) = \sum b_{ij} x^i y^j$, the tangent line to C, if defined, is the line $\{b_{10}x + b_{01}y = 0\}$.

Let's suppose that the two tangent lines are defined and distinct, i.e., that Δ and C intersect transversally at p. We change coordinates once more, to make the tangent lines the coordinate axes. After adjusting by scalar factors, the polynomials f and g will have the form

$$f(x,y) = x + u(x,y) \quad \text{and} \quad g(x,y) = y + v(x,y),$$

where u and v are polynomials all of whose terms have degree at least 2.

Let $X_1 = \operatorname{Spec} \mathbb{C}[x_1, y_1]$ be another affine plane. The map $X_1 \to X$ defined by the substitution $x_1 = x + u(x,y)$, $y_1 = y + v(x,y)$ is invertible analytically near the origin, because the Jacobian matrix

jacob (4.7.4)
$$\left(\frac{\partial(x_1, y_1)}{\partial(x,y)} \right)_{(0,0)}$$

at the origin p is the identity matrix. When we make the substitution, Δ becomes the locus $\{x_1 = 0\}$ and C becomes the locus $\{y_1 = 0\}$. In this local analytic coordinate system, the equation $w^2 = f$ that defines the double plane becomes $w^2 = x_1$. When we restrict it to C by setting $y_1 = 0$, x_1 becomes a local coordinate function on C. The restriction of the equation remains $w^2 = x_1$. So the inverse image Z of C can't be split analytically. Therefore it doesn't split algebraically either.

4.7.5. Corollary. *A curve that intersects the branch locus transversally at some point doesn't split.* □ *splitnot-transversal*

This isn't a complete analysis. When C and Δ are tangent at every point of intersection, C may split or not, and which possibility occurs cannot be decided locally in most cases. However, one case in which a local analysis suffices to decide splitting is that C is a line. Let t be a coordinate in a line C, so that $C \approx \operatorname{Spec} \mathbb{C}[t]$. The restriction of the polynomial f to C will give us a polynomial $\overline{f}(t)$ in t. A root of $\overline{f}$ corresponds to an intersection of C with Δ, and a multiple root corresponds to an intersection at which C and Δ are tangent, or at which Δ is singular. The line C will split if and only if the polynomial $w^2 - \overline{f}$ factors, i.e., if and only if $\overline{f}$ is a square in $\mathbb{C}[t]$. This will be true if and only if every root of $\overline{f}$ has even multiplicity — if and only if the intersection multiplicity of C and Δ at every intersection point is even.

A *rational curve* is a curve whose function field is a rational function field $\mathbb{C}(t)$ in one variable. One can make a similar analysis for any rational plane curve, a conic for example, but one needs to inspect its points at infinity and its singular points as well as its smooth points at finite distance.

(4.7.6) projective double planes projdplane

Let X be the projective plane $\mathbb{P}^2$, with coordinates x_0, x_1, x_2. A *projective double plane* is a locus of the form

(4.7.7)
$$y^2 = f(x_0, x_1, x_2)$$
wtdplane

where f is a square-free, homogeneous polynomial *of even degree* $2d$. To regard (4.7.7) as a homogeneous equation, we must assign weight d to the variable y (see **1.7.8**). Then, since we have weighted variables, we must work in a *weighted projective space* $\mathbb{WP}$ with coordinates x_0, x_1, x_2, y, where x_i have weight 1 and y has weight d. A point of this weighted space is represented by a nonzero vector (x_0, x_1, x_2, y), with the equivalence relation that, for all $\lambda \neq 0$, $(x_0, x_1, x_2, y) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda^d y)$. The points of the projective double plane Y are the points of $\mathbb{WP}$ that solve the equation (4.7.7).

The projection $\mathbb{WP} \to X$ that sends (x, y) to x is defined at all points except at $(0, 0, 0, 1)$. If (x, y) solves (4.7.7) and if $x = 0$, then $y = 0$ too. So $(0, 0, 0, 1)$ isn't a point of Y. The projection is defined at all points of Y. The fibre of the morphism $Y \to X$ over a point x consists of points (x, y) and $(x, -y)$, which will be equal if and only if x lies on the *branch locus* of the double plane, the (possibly reducible) plane curve $\Delta : \{f = 0\}$ in X. The map $\sigma : (x, y) \rightsquigarrow (x, -y)$ is an automorphism of Y, and points of X correspond bijectively to σ-orbits in Y.

Since the double plane Y is embedded into a weighted projective space, it isn't presented to us as a projective variety in the usual sense. However, it can be embedded into a projective space in the following way: The projective plane X can be embedded by a Veronese embedding of higher order, using as coordinates the monomials $m = (m_1, m_2, \ldots)$ of degree d in the variables x. This embeds X into a projective space $\mathbb{P}^N$ where $N = \binom{d+2}{2} - 1$. When we add a coordinate y of weight d, we obtain an embedding of the weighted projective space $\mathbb{WP}$ into $\mathbb{P}^{N+1}$ that sends the point (x, y) to (m, y). The double plane can be realized as a projective variety by this embedding.

When $Y \to X$ is a projective double plane, then, as with affine double planes, a curve C in X may split in Y or not. If C has a transversal intersection with the branch locus Δ, it will not split. On the other hand, if C is a line all of whose intersections with the branch locus Δ have even multiplicity, it will split.

4.7.8. Corollary. *Let Y be a double plane whose branch locus Δ is a generic quartic curve. The lines that split in Y are the bitangent lines to Δ.* □ *bitansplit*

(4.7.9) homogenizing an affine double plane homogdplane

To construct a projective double plane from an affine double plane, we write the affine double plane as

rela-belaffd-plane (4.7.10)
$$w^2 = F(u_1, u_2)$$

for some nonhomogeneous polynomial F. We suppose that F has even degree $2d$, and we homogenize F, setting $u_i = x_i/x_0$. We multiply both sides of this equation by x_0^{2d} and set $y = x_0^d w$. This produces an equation of the form (4.7.7), where f is the homogenization of F.

If F has odd degree $2d-1$, one needs to multiply F by x_0 in order to make the substitution $y = x_0^d w$ permissible. When we do this, the line at infinity $\{x_0 = 0\}$ becomes a part of the branch locus.

cubicisd-plane

(4.7.11) cubic surfaces and quartic double planes

Let $\mathbb{P}^3$ be the (unweighted) projective 3-space with coordinates x_0, x_1, x_2, z, and let X be be the projective plane $\mathbb{P}^2$ with coordinates x_0, x_1, x_2. We consider the projection $\mathbb{P}^3 \xrightarrow{\pi} X$ that sends (x, z) to x. It is defined at all points except at the center of projection $q = (0, 0, 0, 1)$, and its fibres are the lines through q, with q omitted.

Let S be a cubic surface in $\mathbb{P}^3$, the locus of zeros of an irreducible homogeneous cubic polynomial $g(x, z)$, and suppose that q **is** a point of S. Then the coefficient of z^3 in g will be zero, so g will be quadratic in z: $g(x, z) = az^2 + bz + c$, where a, b, c are homogeneous polynomials in x, of degrees $1, 2, 3$, respectively. The defining equation for S becomes

projequa-tion (4.7.12)
$$az^2 + bz + c = 0$$

The discriminant $f(x) = b^2 - 4ac$ of g with respect to z is a homogeneous polynomial of degree 4 in x. Let Y be the projective double plane

quarticd-plane (4.7.13)
$$y^2 = b^2 - 4ac$$

in which the variable y has weight 2.

The quadratic formula solves for z in terms of the chosen square root y of the disriminant, wherever $a \neq 0$:

quadrfor-mula (4.7.14)
$$z = \frac{-b + y}{2a} \qquad \text{or} \qquad y = 2az + b$$

The formula $y = 2az + b$ remains correct when $a = 0$. It defines a map $S \to Y$. The inverse map $Y \to Z$ given by the quadratic formula (4.7.14) is defined wherever $a \neq 0$. So the cubic surface and the double plane are isomorphic except above the line $\{a = 0\}$ in X.

quarticis-generic **4.7.15. Lemma.** *The discriminants of the cubic polynomial $az^2 + bz + c$ include every homogeneous quartic polynomial $f(x)$ such that the divisor $\Delta : \{f = 0\}$ has at least one bitangent line. Therefore the discriminants form a dense subset of the space of quartic polynomials.*

proof. Let f be a quartic polynomial whose zero locus has a bitangent line ℓ_0. Then ℓ_0 splits in the double plane $y^2 = f$. If ℓ_0 is the zero set of a homogeneous linear polynomial $a(x)$, then f is congruent to a square, modulo a. There is a homogeneous quadratic polynomial $b(x)$ such that $f \equiv b^2$, modulo a. Then $f = b^2 - 4ac$ for some homogeneous cubic polynomial $c(x)$. The cubic polynomial $g(x, z) = az^2 + bz + c$ has discriminant f.

Conversely, if $g(x, z) = az^2 + bz + c$ is given, the line $\{a = 0\}$ will be a bitangent to the discriminant divisor Δ provided that the locus $b = 0$ meets that line in two distinct points, which will be true when g is generic. $\square$

From now on, we suppose that S is a generic cubic surface. With a suitable change of coordinates any point of a generic surface can become the point q, so we may suppose that both S and q are generic. Then S contains only finitely many lines, and those lines won't contain q (see(3.7.19)).

Let ℓ be a line in the plane X, say the locus of zeros of a linear equation $r_0x_0 + r_1x_1 + r_2x_2 = 0$. The same equation defines a plane H in $\mathbb{P}^3_{x,y}$ that contains q. The inverse image of ℓ in S is the cubic curve $C = S \cap H$.

4.7.16. Lemma. *Let S be a generic cubic surface. The lines L contained in S correspond bijectively to lines ℓ in X whose inverse images C are reducible cubic curves. If C is reducible, it will be the union $L \cup Q$ of a line and a conic.* *cubicsplits*

proof. A line L in S won't contain q. Its image in X will be a line ℓ in X, and L will be a component of the inverse image C of ℓ. Therefore C will be reducible.

Let ℓ be a line in X. At least one irreducible component of its inverse image C will contain q, and there are no lines through q. So if the cubic C is reducible, it will be the union of a conic and a line L, q will be a point of the conic, and L will be one of the lines in S. $\square$

Let ℓ_0 be the particular line $\{a = 0\}$. The points of Y that lie above ℓ_0 are the points (x, y) such that $a = 0$ and $y = \pm b$. Also, let H_0 denote the plane $\{a = 0\}$ in $\mathbb{P}^3$, the inverse image of ℓ_0, and let C_0 be the cubic curve $S \cap H_0$. The points of C_0 are the solutions in $\mathbb{P}^3$ of the equations $a = 0$ and $bz + c = 0$.

4.7.17. Lemma. *The curve C_0 is irreducible.* *ae-qualzero*

proof. We may adjust coordinates so that a becomes the linear polynomial x_0. When we restrict to H_0 by setting $x_0 = 0$ in the polynomial $bz+c$, we obtain a polynomial $\overline{b}z+\overline{c}$, where $\overline{b}$ and $\overline{c}$ are generic homogeneous polynomials in x_1, x_2 of degrees 2 and 3, respectively. Such a polynomial is irreducible. $\square$

4.7.18. Theorem. *A generic cubic surface S in $\mathbb{P}^3$ contains precisely 27 lines.* *twenty-seven*

This theorem follows from next lemma, which relates the 27 lines in the generic cubic surface S to the 28 bitangents of its generic quartic discriminant curve Δ (see Example 1.11.2 **(iv)**).

4.7.19. Lemma. *Let S be a generic cubic surface $az^2 + bz + c = 0$, and suppose that coordinates are chosen so that $q = (0, 0, 0, 1)$ is a generic point of S. Let $\Delta : \{b^2 - 4ac = 0\}$ be the quartic discriminant curve, and let Y be the double plane $y^2 = b^2 - 4ac$.* *linesplits*

(i) *If a line L is contained in S, its image in X is a bitangent to the quartic curve Δ. Distinct lines in S have distinct images in X.*

(ii) *The line $\ell_0 : \{a = 0\}$ is a bitangent, and it isn't the image of a line in S.*

(iii) *Every bitangent ℓ except ℓ_0 is the image of a line in S.*

proof. Let L be a line in S, let ℓ be its image in X, and let C be the inverse image of ℓ in S. Lemma 4.7.16 tells us that C is the union of the line L and a conic. So L is the only line in S that has ℓ as its image. The quadratic formula (4.7.14) shows that, because the inverse image C of ℓ is reducible, ℓ splits in the double plane Y too, and therefore ℓ is a bitangent to δ. This proves **(i)**. Moreover, Lemma 4.7.17 shows that ℓ cannot be the line ℓ_0. This proves **(ii)**. If a bitangent ℓ is distinct from ell_0, the map $Y \to Z$ given by the quadratic formula is defined except at the finite set $\ell \cap \ell_0$. Since ℓ splits in Y, its inverse image C in S will be reducible. One component of C is a line in S. This proves **(iii)**. $\square$

Section 4.8 Exercises

chap-xdcc fourex **4.8.1.** A ring A is said to have the *descending chain condition* (dcc) if every strictly decreasing chain of ideals $I_1 > I_2 > \cdots$ is finite. Let A be a finite type $\mathbb{C}$-algebra. Prove

(i) A has dcc if and only if it is a finite dimensional complex vector space.

(ii) If A has dcc, then it has finitely many maximal ideals, and every prime ideal is maximal.

(iii) If a finite-type algebra A has finitely many maximal ideals, then A has dcc.

(iv) Suppose that A has dcc, let M be an arbitrary A-module, and let I denote the intersection of the maximal ideals of A. If $IM = M$, then $M = 0$. (This might be called the srong Nakayama lemma. The usual Nakayama lemma requires that M be finitely generated.)

xfaith **4.8.2.** A module M over a ring B is *faithful* if, for every nonzero element b of B, scalar multiplication by b isn't the zero operation on M. Let A be a domain, let z be an element of its field of fractions, and let B be the ring generated by z over A. Suppose there is a faithful B-module M that is finitely generated as an A-module. Prove that z is integral over A.

xfinmod-fld **4.8.3.** Let $A \subset B$ be noetherian domains and suppose that B is a finite A-module. Prove that A is a field if and only if B is a field.

xaltnull **4.8.4.** Use Noether Normalization to prove this alternate form of the Nullstellensatz: Let k be a field, and let B be a domain that is a finitely generated k-algebra. If B is a field, then $[B : k] < \infty$.

xinver-seintegral **4.8.5.** Let α be an element of a domain A, and let $\beta = \alpha^{-1}$. Prove that if β is integral over A, then it is an element of A.

xopeni-som **4.8.6.** Let X and Y be varieties with the same functions field K. Show that there are nonempty open subsets Y' and X' of Y and X, respectively, that are isomorphic.

xAsint-closed **4.8.7.** Let $A \subset B$ be finite type domains with fraction fields $K \subset L$, and let $Y \to X$ be the corresponding morphism of affine varieties. Prove the following:

(i) There is a nonzero element $s \in A$ such that A_s is integrally closed.

(ii) There is a nonzero element $s \in A$ such that B_s is a finite module over a polynomial ring $A_s[y_1, ..., y_d]$.

(ii) Suppose that L is a finite extension of K of degree d. There is a nonzero element $s \in A$ such that all fibres of the morphism $Y \to X$ consist of d points.

incomp **4.8.8.** Let $Y \to X$ be an integral morphism of affine varieties. Let $D \supset D'$ be closed subvarieties of Y that lie over subvarieties $C \supset C'$ of X. Prove that $C = C'$ if and only if $D = D'$.

chainmax-imalx **4.8.9.** Verify directly that the prime chain **4.5.10** is maximal.

xpchain-max **4.8.10.** Prove that $\mathbb{P}^n > \mathbb{P}^{n-1} > \cdots > \mathbb{P}^0$ is a maximal chain of closed subsets of $\mathbb{P}^n$.

incomp **4.8.11.** Let $Y \to X$ be an integral morphism of affine varieties. With reference to Diagram **??**, prove that $C' = C$ if and only if $D' = D$.

xorbits **4.8.12.** Let G be a finite group of automorphisms of a normal, finite-type domain B, let A be the algebra of invariant elements of B, and let $Y \xrightarrow{u} X$ be the integral morphism of varieties corresponding to the inclusion $A \subset B$. Prove that there is a bijective correspondence between G-orbits of closed subvarieties of Y and closed subvarieties of X.

xatmost-deg **4.8.13.** Let $A \subset B$ be an extension of finite-type algebras such that B is a finite A-module, and let P be a prime ideal of A. Prove that the number of prime ideals of B that lie over P is at most equal to the degree $[L : K]$ of the field extension.

xnotidzero **4.8.14.** Let $Y = \operatorname{Spec} B$ be an affine variety, let $D_1, \ldots, D_n$ be distinct closed subvarieties of Y and let V be a closed subset of Y. Assume that V doesn't contain any of the sets D_j. Prove that there is an element β of B that vanishes on V, but isn't identically zero on any D_j.

dimdim **4.8.15.** Let $Y \xrightarrow{u} X$ be a surjective morphism, and let K and L be the function fields of X and Y, respectively. Show that if $\dim Y = \dim X$, there is a nonempty open subset X' of X such that all fibres over points of X' have the same order n, and that $n = [L : K]$.

4.8.16. Work out the proof of Chevalley's Theorem in the case that Y is a closed subset of $\widetilde{X} = X\times\mathbb{P}^1$ that doesn't meet the locus at infinity $\widetilde{H}$. (In $\mathbb{P}^1$, H will be the point at infinity, and $\widetilde{H} = X\times H$.) Do this in the following way: Say that $X = \operatorname{Spec} A$. Let $B_0 = A[u]$, $B_1 = A[v]$, and $B_{01} = A[u,v]$, where $u = y_1/y_0$ and $v = u^{-1} = y_0/y_1$. Then $X\times U^0 = \widetilde{U}^0 = \operatorname{Spec} B_0$, $\widetilde{U}^1 = \operatorname{Spec} B_1$, and $tU^{01} = \operatorname{Spec} B_{01}$. Let P_1 be the ideal of B_1 that defines $Y\cap\widetilde{U}^1$, and let P_0 be the analogous ideal of B_0. In B_1, the ideal of $\widetilde{H}$ is the principal ideal vB_1. Since $Y\cap\widetilde{H} = \emptyset$, $P_1 + vB_1$ is the unit ideal of B_1. Write out what this means. Then go over to the open set $\widetilde{U}^0$, and show that the residue of u in the coordinate algebra B_0/P_0 of Y is the root of a monic polynomial. xchevth-mdimone

4.8.17. Prove that a nonconstant morphism from a curve Y to $\mathbb{P}^1$ is a finite morphism without appealing to Chevalley's Theorem. xmapcurvefin

4.8.18. Let A be a finite type domain, $R = \mathbb{C}[t]$, $X = \operatorname{Spec} A$, and $Y = \operatorname{Spec} R$. Let $\varphi : A \to R$ be a homomorphism whose image is not $\mathbb{C}$, and let $\pi : Y \to X$ be the corresponding morphism. ximclosed

(i) Show that R is a finite A-module.

(ii) Show that the image of π is a closed subset of X.

4.8.19. Let Y be a closed subvariety of projective space $\mathbb{P}^n$ with coordinates $y_0,...,y_n$, let d be a positive integer, and let $w_0,...,w_k$ be homogeneous polynomials in y of degree d that have no common zeros on Y. Prove that sending a point q of Y to $(w_0(q),...,w_k(q))$ defines a finite morphism $Y \xrightarrow{u} \mathbb{P}^k$. Consider the case that $w = u_1,...,u_k$ first. xmapYtoP

4.8.20. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism. xPtwoPt-wofinite

4.8.21. Let $Y \xrightarrow{u} X$ be a finite morphism of curves, and let K and L be the function fields of X and Y, respectively, and suppose $[L:K] = n$. Prove that all fibres have order at most n, and all but finitely many fibres of Y over X have order equal to n. countfib

4.8.22. Prove that a variety of any dimension contains no isolated point. xnoisolpt

4.8.23. Let X be the subset obtained by deleting the origin from $\mathbb{A}^2$. Prove that there is no injective morphism from an affine variety Y to $\mathbb{A}^2$ whose image is X. xmiss-point

4.8.24. With reference to Example 4.7.2, show that the prime ideal that corresponds to the line $w = x$ is not a principal ideal. xnotprinc

4.8.25. Identify the double plane $y^2 = f(x)$ defined as in (**4.7.6**) by a quadratic polynomial f. xquadrd-plane

4.8.26. A *double line* is a locus $y^2 = f(x_0, x_1)$ analogous to a double plane (4.7.7), where f is a homogeneous polynomial of even degree $2d$ with distinct roots. Determine the genus of a double line. xdblcurve

4.8.27. Let $Y \to X$ be an affine double plane, and let D be a curve in Y whose image in X is a plane curve C. Say that C has degree d. Define $\deg D$ to be d if C splits and $2d$ if C remains prime or ramifies. Most curves C in X will intersect the branch locus transversally. Therefore they won't split. On the other hand, most curves D in Y will not be symmetric with respect to the automorphism σ of Y over X. Then there will be two curves $D, D\sigma$ lying over C, so C will split. Try to explain this curious point, with reference to the degrees of C and D. xstrange

4.8.28. Let Y be a closed subvariety of projective space $\mathbb{P}^n$ with coordinates $y = (y_0,...,y_n)$, let d be a positive integer, and let $w = (w_0,...,w_k)$ be homogeneous polynomials in y of degree d with no common zeros on Y. Prove that sending a point q of Y to $(w_0(q),...,w_k(q))$ defines a finite morphism $Y \xrightarrow{u} \mathbb{P}^k$. Consider the case that w_i are linear polynomials first. xfmorph

Chapter 5 STRUCTURE OF VARIETIES IN THE ZARISKI TOPOLOGY

A major goal of this chapter is to show how algebraic curves control the geometry of higher dimensional varieties. We do this, beginning in Section 5.4.

Section 5.1 Local Rings

localrings

A *local ring* is a noetherian ring that contains just one maximal ideal. We make a few general comments about local rings here though we will be interested mainly in some special ones, the discrete valuation rings that are discussed below.

Let R be a local ring with maximal ideal M. The quotient $R/M = k$ is a field called the *residue field* of R. For us, the residue field will most often be the field of complex numbers. An element of R that isn't M isn't in any maximal ideal, so it is a unit.

The Nakayama Lemma 4.1.3 has a useful version for local rings:

local-nakayama

5.1.1. Local Nakayama Lemma. *Let R be a local ring with maximal ideal M and residue field k. Let V be a finite R-module, and let $\overline{V} = V/MV$. If $\overline{V} = 0$, then $V = 0$.*

proof. If $\overline{V} = 0$, then $V = MV$. The usual Nakayama Lemma tells us that M contains an element z such that $1-z$ annihilates V. Then $1-z$ isn't in M, so it is a unit. A unit annihilates V, and therefore $V = 0$. $\square$

gen-modmsq

5.1.2. Corollary *Let R be a local ring. A set $z_1, ..., z_k$ of elements generates M if its residues generate M/M^2.* $\square$

A local domain R with maximal ideal M has *dimension one* if it contains only two prime ideals, (0) and M, and if they are distinct. We describe the *normal* local domains of dimension one in this section. They are the discrete valuation rings that are defined below.

local

5.1.3. A note about the overused word local.

A property is true *locally* on a topological space X if every point p of X has an open neighborhood U such that the property is true on U.

In these notes, the words *localize* and *localization* usually refer to the process of adjoining inverses. The (simple) localizations of an affine variety $X = \operatorname{Spec} A$ form a basis for the topology on X. So if some property is true locally on X, one can cover X by localizations on which the property is true. There will be elements $s_1, ..., s_k$ of A that generate the unit ideal, such that the property is true on each of the localizations X_{s_i}.

An A-module M is *locally free* if there are elements $s_1, ..., s_k$ that generate the unit ideal of A, such that M_{s_i} is a free A_{s_i}-module for each i. The free modules M_{s_i} will have the same rank. That rank is the *rank* of the locally free A-module M.

An ideal I of A is *locally principal* if there are elements s_i that generate the unit ideal, such that I_{s_i} is a principal ideal of A_{s_i} for every i. A locally principal ideal is a locally free module of rank one. $\square$

(5.1.4) Valuations dvr

Let K be a field. A *discrete valuation* v on K is a surjective homomorphism

$$K^\times \xrightarrow{\mathrm{v}} \mathbb{Z}^+ \quad , \quad \mathrm{v}(ab) = \mathrm{v}(a) + \mathrm{v}(b) \tag{5.1.5}$$ dval

from the multiplicative group of nonzero elements of K to the additive group of integers, such that, if a, b are elements of K, and if a, b and $a+b$ aren't zero, then

- $\mathrm{v}(a+b) \geq \min\{\mathrm{v}(a), \mathrm{v}(b)\}$.

The word "discrete" refers to the fact that $\mathbb{Z}^+$ has the discrete topology. Other valuations exist. They are interesting, but less important. We won't use them. To simplify terminology, we refer to a discrete valuation simply as a *valuation*.

5.1.6. Lemma. *Let* v *be a valuation on a field K that contains the complex numbers. Every nonzero complex number has value zero.* *valczero*

proof. This is true because $\mathbb{C}$ contains n th roots. If γ is an n th root of a nonzero complex number c, $c = \gamma^n$, then because v is a homomorphism, $\mathrm{v}(c) = n\,\mathrm{v}(\gamma)$, so $\mathrm{v}(c)$ is divisible by n. The only integer that is divisible by every positive integer n is zero. $\square$

The *valuation ring R* associated to a valuation v on a field K is the subring of elements with non-negative values, together with zero:

$$R = \{a \in K^\times \,|\, \mathrm{v}(a) \geq 0\} \,\cup\, \{0\} \tag{5.1.7}$$ valnring

Valuation rings are usually called "discrete valuation rings", but since we drop the adjective discrete from the valuation, we drop it from the valuation ring too.

5.1.8. Proposition. *Valuations of the field $\mathbb{C}(t)$ of rational functions in one variable correspond bijectively to points of the projective line $\mathbb{P}^1_t$. The valuation ring that corresponds to a point $p \neq \infty$ is the ring of rational functions of the form $f = g/h$ where g and h are polynomials in t, and $h(p) \neq 0$.* *valsinCt*

beginning of the proof. Let a be a complex number. To define the valuation v that corresponds to the point $p: \; t = a$ of $\mathbb{P}^1$, we write a nonzero polynomial f as $(t-a)^k h$, where $t-a$ doesn't divide h, and we define, $\mathrm{v}(f) = k$. Then we define $\mathrm{v}(f/g) = \mathrm{v}(f) - \mathrm{v}(g)$. You will be able to check that, with this definition, v becomes a valuation whose valuation ring is the algebra of regular functions at p (**2.7.1**). That valuation ring is called the *local ring at* of $\mathbb{P}^1$ at p (see (**5.1.11**) below). Its elements are rational functions in t whose denominators aren't divisible by $t-a$. The valuation that corresponds to the point of $\mathbb{P}^1$ at infinity is obtained by working with t^{-1} in place of t.

The proof that these are all of the valuations of $\mathbb{C}(t)$ will be given at the end of the section.

5.1.9. Proposition. *Let* v *be a valuation on a field K, and let x be a nonzero element of K with value* $\mathrm{v}(x) = 1$. *idealsin-valring*

(i) *The valuation ring R of* v *is a normal local domain of dimension one. Its maximal ideal M is the principal ideal xR. The elements of M are the elements of K with positive value, together with zero:*

$$M = \{a \in K^\times \mid \mathrm{v}(a) > 0\} \cup \{0\}$$

(ii) *The units of R are the elements of $K^\times$ with value zero. Every nonzero element z of K has the form $z = x^k u$, where u is a unit and $k = \mathrm{v}(z)$ is an integer.*

(iii) *The proper R-submodules of K are the sets x^kR, where k is an integer. The set x^kR consists of zero and the elements of $K^\times$ with value $\geq k$. The sets x^kR with $k \geq 0$ are the nonzero ideals of R. They are the powers of the maximal ideal, and are principal ideals.*

(iv) *There is no ring properly between R and K: If R' is a ring and if $R \subset R' \subset K$, then either $R = R'$ or $R' = K$.*

proof. We prove **(i)** last.

(ii) Since v is a homomorphism, $\mathrm{v}(u^{-1}) = -\mathrm{v}(u)$ for any nonzero u in K. So u and u^{-1} are both in R, i.e., u is a unit, if and only if $\mathrm{v}(u)$ is zero. If z is a nonzero element of K with $\mathrm{v}(z) = k$, then $u = x^{-k}z$ has value zero, so it is a unit, and $z = x^k u$.

(iii) It follows from **(ii)** that x^kR consists of the elements of K of value at least k. Suppose that a nonzero R-submodule N of K contains an element z with value k. Then N contains x^k and therefore N contains x^kR. If $x^kR < N$, then N contains an element with value $< k$. So if k is the smallest integer such that $x^k \subset N$, then $N = x^kR$. If there is no minimum value among the elements of N, then N contains x^kR for every k, and $N = K$.

(iv) This follows from **(iii)**. The ring R' will be a nonzero R-submodule of K. If $R' < K$, then $R' = x^kR$ for some k, and if $R \subset R'$, then $k \leq 0$. But x^kR isn't closed under multiplication when $k < 0$. So if $R \subset R' < K$, then $k = 0$ and $R = R'$.

(i) First, R is noetherian because **(iii)** tells us that it is a principal ideal domain. Its maximal ideal is $M = xR$. It also follows from **(iii)** that M and $\{0\}$ are the only prime ideals of R. So R is a local ring of dimension 1. If the normalization of R were larger than R, then according to **(iv)**, it would be equal to K, and x^{-1} would be integral over R. There would be a polynomial relation $x^{-r} + a_1x^{-(r-1)} + \cdots + a_r = 0$ with a_i in R. When one multiplies this relation by x^r, one sees that 1 would be a multiple of x. Then x would be a unit, which it is not, because $\mathrm{v}(x^{-1}) = -1$. □

characterizedvr

5.1.10. Theorem.

(i) *A local domain whose maximal ideal is a nonzero principal ideal is a valuation ring.*

(ii) *Every normal local domain of dimension 1 is a valuation ring.*

proof. **(i)** Let R be a local domain whose maximal ideal M is a nonzero principal ideal, say $M = xR$, with $x \neq 0$, and let y be a nonzero element of R. The integers k such that x^k divides y are bounded (4.1.6). Let x^k be the largest power that divides y. Then $y = ux^k$, where $k \geq 0$ and u is in R but not in M. So u is a unit. A nonzero element z of the fraction field K of R will have the form $z = ux^r$ where u is a unit and r is an integer, possibly negative. This is shown by writing the numerator and denominator of a fraction in such a form.

The valuation whose valuation ring is R is defined by $\mathrm{v}(z) = r$ when $z = ux^r$ with u a unit, as above. If $z_i = u_i x^{r_i}$ for $i = 1, 2$, where u_i are units and $r_1 \leq r_2$, then $z_1 + z_2 = \alpha x^{r_1}$, where $\alpha = u_1 + u_2 x^{r_2 - r_1}$ is an element of R. Therefore $\mathrm{v}(z_1 + z_2) \geq r_1 = \min\{\mathrm{v}(z_1), \mathrm{v}(z_2)\}$. We also have $\mathrm{v}(z_1z_2) = \mathrm{v}(z_1) + \mathrm{v}(z_2)$. Thus v is a surjective homomorphism. The requirements for a valuation are satisfied.

(ii) The fact that a valuation ring is a normal, one-dimensional local ring is Proposition 5.1.9 **(i)**. We show that a normal local domain R of dimension 1 is a valuation ring by showing that its maximal ideal is a principal ideal. This proof is tricky.

Let z be a nonzero element of M. Because R is a local ring of dimension 1, M is the only prime ideal that contains z, so M is the radical of the principal ideal zR, and $M^r \subset zR$ if r is large. (See Proposition 2.5.12.) Let r be the smallest integer such that $M^r \subset zR$. Then there is an element y in M^{r-1} that isn't in zR, such that $yM \subset zR$. We restate this by saying that $w = y/z \notin R$, but $wM \subset R$. Since M is an ideal, multiplication by an element of R carries wM to wM. So wM is also an ideal of R. Since M is the maximal ideal of the local ring R, either $wM \subset M$, or $wM = R$. If $wM \subset M$, Corollary 4.1.5 **(iii)** shows that w is integral over R. This can't happen because R is normal and w isn't in R. Therefore $wM = R$ and $M = w^{-1}R$. This implies that w^{-1} is in R and that M is a principal ideal. □

(5.1.11) the local ring at a point local-ringatp

Let $\mathfrak{m}$ be the maximal ideal at a point p of an affine variety $X = \operatorname{Spec} A$, and let S be the complement of $\mathfrak{m}$ in A. This is a multiplicative system (**2.6.7**), and the prime ideals P of the localization AS^{-1} are extensions of the prime ideals Q of A that are contained in $\mathfrak{m}$: $P = QS^{-1}$ (2.6.9). Thus $\mathfrak{m}S^{-1}$ is the only maximal ideal of AS^{-1}, and AS^{-1} is a local ring. This ring is called the *local ring of A at p*. It is often denoted by A_p. Lemma 4.3.4 shows that, if A is a normal domain, then A_p is a normal domain.

For example, let $X = \operatorname{Spec} A$ be the affine line, $A = \mathbb{C}[t]$, and let p be the point $t = 0$. The elements of the local ring A_p are fractions of polynomials $f(t)/g(t)$ with $g(0) \neq 0$.

The local ring at a point p of any variety, not necessarily affine, is the the local ring at p of an affine open neighborhood of p.

5.1.12. Corollary. *Let $X = \operatorname{Spec} A$ be an affine variety.* *intloc*

(i) *The coordinate algebra A is the intersection of the local rings A_p at the points of X.*

$$A = \bigcap_{p \in X} A_p$$ *intersectcodimoneagain*

(ii) *The coordinate algebra A is normal if and only if all of its local rings A_p are normal.* □

5.1.13. Proposition. *Let M be a finite module over a finite-type domain A. If, for some point p of $\operatorname{Spec} A$, the localized module M_p (2.6.11) is a free A_p-module, then there is an element s, not in $\mathfrak{m}_p$, such that M_s is a free A_s-module.* *generateMinloc*

proof. This is an example of the general principle (**2.6.13**). □

Completion of the proof of Proposition 5.1.8. We show that every valuation v of the function field $\mathbb{C}(t)$ of $\mathbb{P}^1$ corresponds to a point of $\mathbb{P}^1$.

Let R be the valuation ring of v. If $\mathrm{v}(t) < 0$, we replace t by t^{-1}. So we may assume that $\mathrm{v}(t) \geq 0$. Then t is an element of R, and therefore $\mathbb{C}[t] \subset R$. The maximal ideal M of R isn't zero. It contains a nonzero fraction g/h of polynomials in t. The denominator h is in R, so M also contains the numerator g. Since M is a prime ideal, it contains a monic irreducible factor of g of the form $t - a$ for some complex number a. When $c \neq a$, the scalar $c-a$ isn't in M, so $t-c$ won't be in M. Since R is a local ring, $t-c$ must be a unit of R for all $c \neq a$. The localization R_0 of $\mathbb{C}[t]$ at the point $t = a$ is a valuation ring, and it is contained in R (5.1.8). There is no ring properly containing R_0 except K, so $R_0 = R$. □

Section 5.2 Smooth Curves

A *curve* is a variety of dimension 1. The proper closed subsets of a curve are finite sets. smaffcurve

A point p of a curve X is a *smooth point* if the local ring at p is a valuation ring. Otherwise, it is a *singular point*. A curve X is *smooth* if all of its points are smooth.

Note. Suppose that an affine curve X is the spectrum of an algebra $A = \mathbb{C}[x_1, ..., x_n]/P$, and that $f_1, ..., f_k$ generate the prime ideal P. A better definition of a smooth point p is that the rank of the Jacobian matrix $J = \frac{\partial f_i}{\partial x_j}$ at p is $n-1$. However, we will use the Jacobian matrix just once, at the end of this section. For us, the definition given above is more convenient. □ jacobian

Let r be a positive integer. If v is a valuation and if $\mathrm{v}(a) = r$, then r is the *order of zero of* a, and if $\mathrm{v}(a) = -r$, then r is the *order of pole of* a, with respect to the valuation. If v_p is the valuation associated to a smooth point of a curve X, a rational function α on X has a *zero of order* $r > 0$ at p if $\mathrm{v}_p(\alpha) = r$, and it has a *pole of order* r at p if $\mathrm{v}_p(\alpha) = -r$.

5.2.1. Lemma. **(i)** *An affine curve X is smooth if and only if its coordinate algebra is a normal domain.* *smptsopen*

(ii) *A curve has finitely many singular points.*

(iii) *The normalization $X^\#$ of a curve X is a smooth curve, and the finite morphism $X^\# \to X$ becomes an isomorphism when singular points of X and their inverse images are deleted.*

proof. **(i)** This follows from Theorem 5.1.10 and Proposition 4.3.4.

(ii) The statement that a morphism is an isomorphism can be verified locally, so we may replace X by an affine open subset, say $\operatorname{Spec} A$. Let $A^{\#}$ be the normalization of A. There is a nonzero element s in A such that $sA^{\#} \subset A$ (Corollary 4.3.2). Then $A_s = A^{\#}_s$. So $\operatorname{Spec} A_s$, which is the complement of a finite set in $\operatorname{Spec} A$, is smooth.

(iii) This is rather obvious. □

node-curvetwo **5.2.2. Example.** We go back to Example 4.3.3 of a nodal cubic curve $C = \operatorname{Spec} A$, $A = \mathbb{C}[u,v]/(v^2–u^3–u^2)$ and its normalization $B = \mathbb{C}[x]$, the map $A \xrightarrow{\varphi} B$ being defined by $\varphi(u) = x^2-1$ and $\varphi(v) = x^3-x$. So the normalization $C^{\#} = \operatorname{Spec} B$ is the affine line. The curve C has a node at the origin $p = (0,0)$, and the fibre of $C^{\#}$ over p is the point pair $x = \pm 1$. Let's denote the points $x = 1$ and $x = -1$ by q and q', respectively, and denote the polynomial $x^2 - 1$ by w. The complement U of p in C can be identified as the spectrum of the localization $A[u^{-1}]$ of A. Its inverse image in $C^{\#}$ is the complement W of the point pair q, q', which is the spectrum of the localization $B[w^{-1}]$. Since $\varphi(u) = w$, φ extends to a map $A_u \to B_w$, and its inverse maps x to v/u. So W and U are isomorphic, as stated in Lemma 5.2.1. □

pointsofcurve **5.2.3. Proposition.** *Let X be a smooth curve with function field K. Every point of $\mathbb{P}^n$ with values in K defines a morphism $X \to \mathbb{P}^n$.*

proof. A point $(\alpha_0, ..., \alpha_n)$ of $\mathbb{P}^n$ with values in K determines a morphism $X \to \mathbb{P}^n$ if it is a good point, which means that, for every point p of X, there is an index j such that the functions α_i/α_j are regular at p for all $i = 0, ..., n$ (3.5.6). This will be true when j is chosen so that the order of zero of α_j at p is the minimal integer among the orders of zero of α_i for indices i such that $\alpha_i \neq 0$. □

The next example shows that this proposition doesn't extend to varieties X of dimension greater than one.

nomaptop **5.2.4. Example.** Let Y be the complement of the origin in the affine plane $X = \operatorname{Spec} \mathbb{C}[x,y]$, and let $K = \mathbb{C}(x,y)$ be the function field of X. The vector (x,y) defines a good point of Y with values in K, andthereforea morphism $Y \to \mathbb{P}^1$. If (x,y) were a good point of X then, according to Proposition 3.5.4, at least one of the two rational functions x/y or y/x would be regular at the origin $q = (0,0)$. This isn't the case, so (x,y) isn't a good point of X. The morphism $Y \to \mathbb{P}^1$ doesn't extend to X. □

ptsvals **5.2.5. Proposition.** *Let $X = \operatorname{Spec} A$ be a smooth affine curve with function field K. The local rings of X are the valuation rings of K that contain A. The maximal ideals of A are locally principal.*

In fact, it follows from Proposition 5.2.8 below that every ideal of A is locally principal.

proof. Since A is a normal domain of dimension one, its local rings are valuation rings that contain A (see Theorem 5.1.10 and Corollary 5.1.12). Let R be a valuation ring of K that contains A, let v be the associated valuation, and let M be the maximal ideal of R. The intersection $M \cap A$ is a prime ideal of A (2.1.2). Since A has dimension 1, the zero ideal is the only prime ideal of A that isn't a maximal ideal. We can clear the denominator of an element of M, multiplying by an element of R, to obtain an element of A while staying in M. So $M \cap A$ isn't the zero ideal. It is the maximal ideal $\mathfrak{m}_p$ of A at a point p of X. The elements of A that aren't in $\mathfrak{m}_p$ aren't in M either. They are invertible in R. So the local ring A_p at p, which is a valuation ring, is contained in R, and therefore it is equal to R (5.1.9) **(iii)**. Since M is a principal ideal, $\mathfrak{m}_p$ is locally principal. □

5.2.6. Proposition. *Let X' and X be smooth curves with the same function field K.*

pointsvalns

(i) *A morphism $X' \xrightarrow{f} X$ that is the identity on the function field K maps X' isomorphically to an open subvariety of X.*

(ii) *If X is projective, X' is isomorphic to an open subvariety of X.*

(iii) *If X' and X are both projective, they are isomorphic.*

(iv) *If X is projective, every valuation ring of K is the local ring at a point of X.*

proof. **(i)** Let p be the image in X of a point p' of X', let U be an affine open neighborhood of p in X, and let V be an affine open neighborhood of p' in X' that is contained in the inverse image of U. Say $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$. The morphism f gives us a homomorphism $A \to B$, and since p' maps to p, this homomorphism extends to an inclusion of local rings $A_p \subset B_{p'}$. They are valuation rings with the same field of fractions, so they are equal. Since B is a finite-type algebra, there is an element s in A, with $s(p') \neq 0$, such that $A_s = B_s$. Then the open subsets $\operatorname{Spec} A_s$ of X and $\operatorname{Spec} B_s$ of X' are equal. Since the point p' is arbitrary, X' is covered by open subvarieties of X. So X' is an open subvariety of X.

(ii) The projective embedding $X \subset \mathbb{P}^n$ is defined by a point $(\alpha_0, ..., \alpha_n)$ with values in K. That point also defines a morphism $X' \to \mathbb{P}^n$. If $f(x_0, ..., x_n) = 0$ is a set of defining equations of X in $\mathbb{P}^n$, then $f(\alpha) = 0$ in K, and therefore f vanishes on X' too. So the image of X' is contained in the zero locus of f, which is X. Then **(i)** shows that X' is an open subvariety of X.

(iii) This follows from **(ii)**.

(iv) The local rings of X are normal and of dimension one, so they are valuation rings of K. Let R be a valuation ring of K, let v be the corresponding valuation, and let $\beta = (\beta_0, ..., \beta_n)$ be the point with values in K that defines the projective embedding of X. When we order the coordinates so that $\mathrm{v}(\beta_0)$ is minimal, the ratios $\gamma_j = \beta_j/\beta_0$ will be in R. The coordinate algebra A_0 of the affine variety $X^0 = X \cap \mathbb{U}^0$ is generated by the coordinate functions γ_j, so $A_0 \subset R$. Prposition 5.2.5 tells us that R is the local ring of X^0 at some point. $\square$

5.2.7. Proposition. *Let p be a smooth point of an affine curve $X = \operatorname{Spec} A$, and let $\mathfrak{m}$ and v be the maximal ideal and valuation, respectively, at p. The valuation ring R of v is the local ring of A at p.* *truncate-curve*

(i) *The power $\mathfrak{m}^k$ consists of the elements of A whose values are at least k. If I is an ideal of A whose radical is $\mathfrak{m}$, then $I = \mathfrak{m}^k$ for some $k > 0$.*

(ii) *For every $n \geq 0$, the algebras $A/\mathfrak{m}^n$ and R/M^n are isomorphic to the truncated polynomial ring $\mathbb{C}[t]/(t^n)$.*

proof. **(i)** Proposition 5.1.9 tells us that the nonzero ideals of R are powers of its maximal ideal M, and M^k is the set of elements of R with value $\geq k$.

Let I be an ideal of A whose radical is $\mathfrak{m}$, and let k be the minimal value $\mathrm{v}(x)$ of the nonzero elements x of I. We will show that I is the set of all elements of A with value $\geq k$, i.e., that $I = M^k \cap A$. Since we can apply the same reasoning to $\mathfrak{m}^k$, it will follow that $I = \mathfrak{m}^k$.

We must show that if an element y of A has value $\mathrm{v}(y) \geq k$, then it is in I. We choose an element x of I with value k. Then x divides y in R, say $y/x = w$, with w in R. The element w will be a fraction a/s with s and a in A, and s not in $\mathfrak{m}$. then $sy = ax$, and s will vanish at a finite set of points $q_1, ..., q_r$, but not at p. We choose an element z of A that vanishes at p but not at any of the points $q_1, ..., q_r$. Then z is in $\mathfrak{m}$, and since the radical of I is $\mathfrak{m}$, some power of z is in I. We replace z by that power, so that z is in I. By our choice, z and s have no common zeros in X. They generate the unit ideal of A, say $1 = cs + dz$ with c and d in A. Then $y = csy + dzy = cax + dzy$. Since x and z are in I, so is y.

(ii) Since p is a smooth point, the local ring of A at p is the valuation ring R. We may localize A by inverting an element s of A that isn't in $\mathfrak{m}$, because $A/\mathfrak{m}^k$ will be isomorphic to the corresponding quotient $A_s/\mathfrak{m}_s^k$. Doing so suitably, we may suppose that $\mathfrak{m}$ is a principal ideal, say tA. Then $\mathfrak{m}^k = t^kA$. Let P be the subring $\mathbb{C}[t]$ of A, and let $\overline{P}_k = P/t^kP$, $\overline{A}_k = A/\mathfrak{m}^k = A/t^kA$, and $\overline{R}_k = R/M^k = R/t^kR$. The quotients $t^{k-1}P/t^kP$, $\mathfrak{m}^{k-1}/\mathfrak{m}^k$, and M^{k-1}/M^k are one-dimensional vector spaces. So the map labelled g_{k-1} in the diagram below is bijective.

$$\begin{tikzcd}
0 \arrow[r] & t^{k-1}P/t^kP \arrow[r] \arrow[d, "g_{k-1}"'] & \overline{P}_k \arrow[r] \arrow[d, "f_k"'] & \overline{P}_{k-1} \arrow[r] \arrow[d, "f_{k-1}"'] & 0 \\
0 \arrow[r] & \mathfrak{m}^{k-1}/\mathfrak{m}^k \arrow[r] & \overline{A}_k \arrow[r] & \overline{A}_{k-1} \arrow[r] & 0
\end{tikzcd}$$

By induction on k, we may assume that the map f_{k-1} is bijective, and then f_k is bijective too. So $\overline{P}_k$ and $\overline{A}_k$ are isomorphic. The analogous reasoning shows that $\overline{P}_k$ and $\overline{R}_k$ are isomorphic.

5.2.8. Proposition. *Let X be a smooth affine curve. Every nonzero ideal I of the coordinate algebra A of X is a product $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$ of powers of maximal ideals.* *idealsin-curve*

proof. Let I be a nonzero ideal of A. Because X has dimension one, the locus of zeros of I is a finite set $\{p_1, ..., p_k\}$. Therefore the radical of I is the intersection $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$ of the maximal ideals $\mathfrak{m}_j$ at p_j, which, by the Chinese Remainder Theorem, is the product ideal $\mathfrak{m}_1 \cdots \mathfrak{m}_k$, Moreover, I contains a power of that product, say $I \supset \mathfrak{m}_1^N \cdots \mathfrak{m}_k^N$. Let $J = \mathfrak{m}_1^N \cdots \mathfrak{m}_k^N$. The quotient algebra A/J is the product $B_1 \times \cdots \times B_k$, with $B_j = A/\mathfrak{m}_j^N$, and A/I is a quotient of A/J. Proposition 2.1.8 tells us that A/I is a product $\overline{A}_1 \times \cdots \times \overline{A}_k$, where $\overline{A}_j$ is a quotient of B_j. By Proposition 5.2.7(**ii**), each B_j is a truncated polynomial ring, so the quotient $\overline{A}_j$ is also a truncated polynomial ring, and the kernel of the maps $A \to A_j$ is a power of $\mathfrak{m}_j$. The kernel I of the map $A \to \overline{A}_1 \times \cdots \times \overline{A}_k$ is a product of powers of the maximal ideals $\mathfrak{m}_j$. $\square$

jacobtwo **(5.2.9) isolated points, again**

isolptofy **5.2.10. Lemma.**
(i) *Let Y' be an open subvariety of a variety Y. A point q of Y' is an isolated point of Y if and only if it is an isolated point of Y'.*

(ii) *Let $Y' \xrightarrow{u'} Y$ be a nonconstant morphism of curves, let q' be a point of Y', and let q be its image in Y. If q is an isolated point of Y, then q' is an isolated point of Y'.*

proof. **(i)** A point q of Y is isolated if the set $\{q\}$ is open in Y. If $\{q\}$ is open in Y' and Y' is open in Y, then $\{q\}$ is open in Y, and if $\{q\}$ is open in Y, it is open in Y'.

(ii) Because Y' has dimension one, the fibre over q will be a finite set, say $\{q'\} \cup F$, where F is the finite set of points of the fibre distinct form q. Let Y'' denote the (open) complement $Y' - F$ of F in Y', and let u'' be the restriction of u' to Y''. The fibre of Y'' over q is the point q'. If $\{q\}$ is open in Y, then because u'' is continuous, $\{q'\}$ will be open in Y''. By **(i)**, $\{q'\}$ is open in Y'. $\square$

jacfor-curve **5.2.11. Lemma.** *Let q be a smooth point of an affine curve $Y = \operatorname{Spec} B$. Say that $B = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$, and that q is the origin $(0, ..., 0)$ in $\mathbb{A}^n_x$. Suppose that the maximal ideal $\mathfrak{m}_q$ of B at q is a principal ideal, generated by the residue of a polynomial f_0.*
(i) *The polynomials $f_0, f_1, ..., f_k$ generate the maximal ideal $M = (x_1, ..., x_n)$ of $\mathbb{C}[x]$ at the origin.*
(ii) *Let J denote the $(n+1)\times k$ Jacobian matrix $\frac{\partial f_i}{\partial x_j}$. The evaluation J_0 of J at the origin q is a matrix of rank n.*

proof. **(i)** This is true because $B = \mathbb{C}[x]/(f_1, ..., f_k)$, and because the residue of f_0 generates the maximal ideal $\mathfrak{m}_q$ of B.

(ii) Writing $x = (x_1, ..., x_n)$ and $f = (f_0, ..., f_k)$ as row vectors, $f = J_0 x^t + O(2)$, where $O(2)$ denotes undetermined polynomials all of whose terms have degree ≥ 2 in x. Since $f_0, ..., f_k$ generate M, there is a polynomial matrix P such that $Pf = x^t$. Then if P_0 is the evaluation of P at the origin, $P_0 J_0$ is the $n \times n$ identity matrix. So J_0 has rank n. $\square$

noisolat-edpt **5.2.12. Proposition.** *In the classical topology, a curve, smooth or not, contains no isolated point. .*

This was proved before for plane curves (Proposition 1.3.19).
proof. Let q be a point of a curve Y. Part **(i)** of Lemma 5.2.10 allows us to replace Y by an affine neighborhood of q. Let Y' be the normalization of Y. Part **(ii)** of that lemma allows us to replace Y by Y'. So we may assume that Y is a smooth affine curve, say $Y = \operatorname{Spec} B$. We can still replace Y by an open neighborhood of q, so we may assume that the maximal ideal $\mathfrak{m}_q$ at q is a principal ideal (5.2.5).

Let J_0' be the matrix obtained by deleting the column with index 0 from J_0. This matrix has rank at least $n-1$, and we may arrange indices so that the submatrix with indices $1 \leq i, j \leq n-1$ is invertible. The Implicit Function Theorem says that the equations $f_1, ..., f_{n-1}$ can be solved for the variables $x_1, ..., x_{n-1}$ as analytic functions of x_n. The locus Z of zeros of $f_1, ..., f_{n-1}$ is locally homeomorphic to the affine x_n-line (1.4.18), and it contains Y. Since Y has dimension 1, the component of Z that contains q must be equal to Y. So Y is locally homeomorphic to $\mathbb{A}^1$, which has no isolated point. Therefore q isn't an isolated point of Y. $\square$

Section 5.3 Constructible Sets

construct

In this section, X will denote a noetherian topological space. Every closed subset of X is a finite union of irreducible closed sets (2.2.18).

The intersection $L = Z \cap U$ of a closed set Z and an open set U is called a *locally closed* set. For instance, open sets and closed sets are locally closed.

5.3.1. Lemma. *The following conditions on a subset L of X are equivalent.* *lclosed-cond*

- *L is locally closed.*
- *L is a closed subset of an open subset U of X.*
- *L is an open subset of a closed subset Z of X.* □

A *constructible set* is a set that is the union of finitely many locally closed sets.

5.3.2. Examples. constrin-curve

(i) A subset S of a curve X is constructible if and only if it is either a finite set or the complement of a finite set. Thus S is constructible if and only if it is either closed or open.

(ii) In the affine plane $X = \operatorname{Spec} \mathbb{C}[x, y]$, let Z be the line $\{y = 0\}$, let $U = X - Z$ be its open complement, and let $p = (0, 0)$. The union $U \cup \{p\}$ is constructible, but not locally closed. □

We use this notation: Z will denote a closed set, U will denote an open set. and L will denote a locally closed set, such as $Z \cap U$.

5.3.3. Theorem. *The set $\mathbb{S}$ of constructible subsets of a noetherian topological space X is the smallest family of subsets that contains the open sets and is closed under the three operations of finite union, finite intersection, and complementation.* *defloc-closed*

proof. Let $\mathbb{S}_1$ denote the family of subsets obtained from the open sets by the three operations mentioned in the statement. Open sets are constructible, and using those three operations, one can make any constructible set from the open sets. So $\mathbb{S} \subset \mathbb{S}_1$. To show that $\mathbb{S} = \mathbb{S}_1$, we show that the family of constructible sets is closed under those operations.

It is obvious that a finite union of constructible sets is constructible. The intersection of two locally closed sets $L_1 = Z_1 \cap U_1$ and $L_2 = Z_2 \cap U_2$ is locally closed because $L_1 \cap L_2 = (Z_1 \cap Z_2) \cap (U_1 \cap U_2)$. If $S = L_1 \cup \cdots \cup L_k$ and $S' = L'_1 \cup \cdots \cup L'_r$ are constructible sets, the intersection $S \cap S'$ is the union of the locally closed intersections $(L_i \cap L'_j)$, so it is constructible.

Let S be the constructible set $L_1 \cup \cdots \cup L_k$. Its complement S^c is the intersection of the complements L_i^c of L_i: $S^c = L_1^c \cap \cdots \cap L_k^c$. We have shown that intersections of constructible sets are constructible. So to show that the complement S^c is constructible, it suffices to show that the complement of a locally closed set is constructible. Let L be the locally closed set $Z \cap U$, and let Z^c and U^c be the complements of Z and U, respectively. Then Z^c is open and U^c is closed. The complement L^c of L is the union $Z^c \cup U^c$ of two constructible sets, so it is constructible. □

5.3.4. Proposition. *In a noetherian topological space X, every constructible subset is a union $L_1 \cup \cdots \cup L_k$ of locally closed sets, $L_i = Z_i \cap U_i$, in which the closed sets Z_i are irreducible and distinct.* *contain-sopen*

proof. Let $L = Z \cap U$ be a locally closed set, and let $Z = Z_1 \cup \cdots \cup Z_r$ be the decomposition of Z into irreducible components. Then $L = (Z_1 \cap U) \cup \cdots \cup (Z_r \cap U)$, which is constructible. So every constructible set S is a union of locally closed sets $L_i = Z_i \cap U_i$ in which Z_i are irreducible. Next, suppose that two of the irreducible closed sets are equal, say $Z_1 = Z_2$. Then $L_1 \cup L_2 = (Z_1 \cap U_1) \cup (Z_1 \cap U_2) = Z_1 \cap (U_1 \cup U_2)$ is locally closed. So we can find an expression in which the closed sets are also distinct. □

5.3.5. Lemma. *sinx*

(i) *Let X_1 be a closed subset of a variety X, and let X_2 be its open complement. A subset S of X is constructible if and only if $S \cap X_1$ and $S \cap X_2$ are constructible.*

(ii) *Let X' be an open or a closed subvariety of a variety X.*

a) *If S is a constructible subset of X, then $S' = S \cap X'$ is a constructible subset of X'.*

b) *A subset S' of X' is a constructible subset of X' if and only if it is a constructible subset of X.*

proof. **(i)** This follows from Theorem 5.3.3.

(iia) It suffices to prove that, if L is a locally closed subset of X, the intersection $L' = L \cap X'$ is a locally closed subset of X'. If $L = Z \cap U$, then $Z' = Z \cap X'$ is closed in X', and $U' = U \cap X'$ is open in X'. So $L' = Z' \cap U'$ is locally closed.

(iib) It follows from **a)** that if a subset S' of X' is contructible in X, then it is constructible in X'. To show that a constructible subset of X' is contructible in X, it suffices to show that a locally closed subset $L' = Z' \cap U'$ of X' is locally closed in X. If X' is a closed subset of X, then Z' is a closed subset of X, and $U' = X \cap U$ for some open subset U of X. Since $Z' \subset X'$, $L' = Z' \cap U' = Z' \cap X' \cap U = Z' \cap U$, which is locally closed in X. If X' is open in X, then U' is open in X. Let Z be the closure of Z' in X. Then $L' = Z \cap U' = Z \cap X' \cap U' = Z' \cap U'$. Again, L' is locally closed in X. □

The next theorem illustrates a general fact, that sets arising in algebraic geometry are often constructible.

imagecon-str

5.3.6. Theorem. *Let $Y \xrightarrow{f} X$ be a morphism of varieties. The inverse image of a constructible subset of X is a constructible subset of Y. The image of a constructible subset of Y is a constructible subset of X.*

proof. The fact that a morphism is continuous implies that the inverse image of a constructible set is constructible. It is less obvious that the image of a constructible set is constructible. To prove that, we keep reducing the problem until there is nothing left to do.

Let S be a constructible subset of Y. Lemma 5.3.5 and Noetherian induction allow us to assume that the theorem is true when S is contained in a proper closed subvariety of Y, and also when its image $f(S)$ is contained in a proper closed subvariety of X.

Suppose that X is the union of a proper closed subvariety X_1 and its open complement X_2. The inverse image $Y_1 = f^{-1}(X_1)$ will be closed in Y, and its open complement Y_2 will be the inverse image of X_2. The constructible subset S of Y is the union of the constructible sets $S_1 = S \cap Y_1$ and $S_2 = S \cap Y_2$, and $f(S) = f(S_1) \cup f(S_2)$. It suffices to show that $f(S_1)$ and $f(S_2)$ are constructible, and to show this, it suffices to show that $f(S_i)$ is a constructible subset of X_i for $i = 1, 2$ (5.3.5) **(iib)**. Moreover, noetherian induction applies to X_1. So we need only show that $f(S_2)$ is a constructible subset of X_2. This means that we can replace X by X_2, which can be any nonempty open subset of X, and Y by its open inverse image.

Next, suppose that Y is the union of a proper closed subvariety Y_1 and its open complement Y_2, and let $S_i = S \cap Y_i$. It suffices to show that $f(S_i)$ is constructible, for $i = 1, 2$, and induction applies to S_1. So we may replace Y by any nonempty open subvariety.

Summing up, we can replace X by any nonempty open subset X', and Y by any nonempty open subset of its inverse image. We can do this finitely often.

Since a constructible set S is a finite union of locally closed sets, it suffices to show that the image of a locally closed subset of Y is constructible. Moreover, we may suppose that S has the form $Z \cap U$, where U is open and Z is closed and irreducible. Then Y is the union of the closed set $Z = Y_1$ and its complement $Y_2 = (Y - Z)$, and $S \cap Y_2 = \emptyset$. We may replace Y by $Y_1 = Z$. Then $S = Y \cap U = U$, and we may replace Y by U. We are thus reduced to the case that $S = Y$.

We may still replace X and Y by nonempty open subsets, so we may assume that they are affine, say $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$. Then the morphism $Y \to X$ corresponds to an algebra homomorphism $A \xrightarrow{\varphi} B$. If the kernel I of φ were nonzero, the image of Y would be contained in the proper closed subset $\operatorname{Spec} A/I$ of X, to which induction would apply. So we may assume that φ is injective.

Corollary 4.2.11 tells us that, for suitable nonzero s in A, the localization B_s will be a finite module over a polynomial subring $A_s[y_1, ..., y_k]$. We replace Y and X by the open subsets $Y_s = \operatorname{Spec} B_s$ and $X_s = \operatorname{Spec} A_s$. Then the maps $Y \to \operatorname{Spec} A[y]$ and $\operatorname{Spec} A[y] \to X$ are both surjective, so Y maps surjectively to X. □

Section 5.4 Closed Sets

us-ingcurves

Limits of sequences are often used to analyze subsets of a topological space. In the classical topology, a subset Y of $\mathbb{C}^n$ is closed if, whenever a sequence of points in Y has a limit in $\mathbb{C}^n$, the limit is in Y. In algebraic geometry, curves can be used as substitutes for sequences.

We use the following notation:

Cwith-point

5.4.1. *C is a smooth affine curve, q is a point of C, and C' is the complement of q in C.*

The closure of C' will be C, and we think of q as a limit point. In fact, the closure of C' is C in the classical topology as well as in the Zariski topology, because C has no isolated point (5.2.12). Theorem 5.4.3, which is below, characterizes constructible subset of a variety in terms of such limit points.

The next theorem tells us that there are enough curves to do the job.

5.4.2. Theorem. *(enough curves) Let Y be a constructible subset of a variety X, and let p be a point of its closure $\overline{Y}$. There exists a morphism $C \xrightarrow{f} X$ from a smooth affine curve to X and a point q of C, such that the image of $C' = C - \{q\}$ is contained in Y and $f(q) = p$.* *enoughcurves*

proof. If $X = p$, then $Y = p$ too. In this case, we may take for f the constant morphism from any curve C to p. So we may assume that X has dimension at least one. Next, we may replace X by an affine open subset X' that contains p, and Y by $Y' = Y \cap X'$. The closure $\overline{Y}'$ of Y' in X' will be the intersection $\overline{Y} \cap X'$, and it will contain p. So we may assume that X is affine, say $X = \operatorname{Spec} A$.

Since Y is constructible, it is a union $L_1 \cup \cdots \cup L_k$ of locally closed sets, say $L_i = Z_i \cap U_i$ where Z_i are irreducible closed sets and U_i are open. The closure of Y is the union $Z_1 \cup \cdots \cup Z_k$, and p will be in at least one of those closed sets, say $p \in Z_i$. We replace X by Z_i and Y by L_i. This reduces us to the case that Y is a nonempty open subset of X.

We use Krull's Theorem to slice X down to dimension one. Suppose that the dimension n of X is at least two. Let $D = X - Y$ be the closed complement of the open set Y. The components of D have dimension at most $n-1$. We choose an element α of the coordinate algebra A of X that is zero at p and isn't identically zero on any component of D, except at p itself, if p happens to be a component. Krull's Theorem tells us that every component of the zero locus of α has dimension $n-1$, and at least one of those components, call it V, contains p. If V were contained in D, it would be a component of D because $\dim V = n-1$ and $\dim D \le n-1$. By our choice of α, this isn't the case. So $V \not\subset D$, and therefore $V \cap Y \neq \emptyset$. Let $W = V \cap Y$. Because V is irreducible and Y is open, W is a dense open subset of V, its closure is V, and p is a point of V. We replace X by V and Y by W. The dimension of X is thereby reduced to $n-1$.

Thus it suffices to treat the case that X has dimension one. Then X will be a curve that contains p and Y will be a nonempty open subset of X. The normalization of X will be a smooth curve $X^{\#}$ that comes with an integral, and therefore surjective, morphism to X. Finitely many points of $X^{\#}$ will map to p. We choose for C an affine open subvariety of $X^{\#}$ that contains just one of those points, and we call that point q. $\square$

5.4.3. Theorem *(curve criterion for a closed set) Let Y be a constructible subset of a variety X. The following conditions are equivalent:* *closed-crittwo*

(a) *Y is closed.*

(b) *For every morphism $C \xrightarrow{f} X$ from a smooth affine curve to X, the inverse image $f^{-1}Y$ is closed in C.*

(c) *Let q be a point of a smooth affine curve C, let $C' = C - \{q\}$, and let $C \xrightarrow{f} X$ be a morphism. If $f(C') \subset Y$, then $f(C) \subset Y$.*

The hypothesis that Y be constructible is necessary. For example, in the affine line X, the set W of points with integer coordinates isn't constructible, but it satisfies condition **(b)**. Any morphism $C' \to X$ whose image is in W will map C' to a single point, and therefore it will extend to C.

proof of Theorem 5.4.3. The implications **(a)** $\Rightarrow$ **(b)** $\Rightarrow$ **(c)** are obvious. We prove the contrapositive of the implication **(c)** $\Rightarrow$ **(a)**. Suppose that Y isn't closed. We choose a point p of the closure $\overline{Y}$ that isn't in Y, and we apply Theorem 5.4.2. There exists a morphism $C \xrightarrow{f} X$ from a smooth curve to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. Since $q \notin Y$, this morphism shows that **(c)** doesn't hold either. $\square$

5.4.4. Theorem. *A constructible subset Y of a variety X is closed in the Zariski topology if and only if it is closed in the classical topology.* *class-closed*

proof. A Zariski closed set is closed in the classical topology because the classical topology is finer than the Zariski topology. Suppose that a constructible subset Y of X is closed in the classical topology. To show that Y is closed in the Zariski topology, we choose a point p of the Zariski closure $\overline{Y}$ of Y, and we show that p is a point of Y.

We use the notation (5.4.1). Theorem 5.4.2 tells us that there is a map $C \xrightarrow{f} X$ from a smooth curve C to X and a point q of C such that $f(q) = p$ and $f(C') \subset Y$. Let C_1 denote the inverse image $f^{-1}(Y)$ of Y. Because C_1 contains C', either $C_1 = C'$ or $C_1 = C$.

In the classical topology, a morphism is continuous. Since Y is closed, its inverse image C_1 is closed in C. If C_1 were C', then C' would closed as well as open. Its complement $\{q\}$ will be an isolated point of C.

Because a curve contains no isolated point, the inverse image of Y is C, which means that $f(C) \subset Y$. In particular, p is in Y.

Therefore Y is closed in the Zariski topology. $\square$

Section 5.5 Projective Varieties are Proper

proper

As has been noted before, an important property of projective space is that, in the classical topology, it is a compact space. A variety isn't compact in the Zariski topology unless it is a single point. However, in the Zariski topology, projective varieties have a property closely related to compactness: They are *proper*.

Before defining the concept of a proper variety, we explain an analogous property of compact spaces.

proper-compact **5.5.1. Proposition.** *Let X be a compact space, let Z be a Hausdorff space, and let V be a closed subset of $Z \times X$. The image of V via the projection $Z \times X \to Z$ is closed in Z.*

proof. Let W be the image of V in Z. We show that if a sequence of points z_i of the image W has a limit $\underline{z}$ in Z, then that limit is in W. For each i, we choose a point p_i of V that lies over z_i. So p_i is a pair (z_i, x_i), x_i being a point of X. Since X is compact, there is a subsequence of the sequence x_i that has a limit $\underline{x}$ in X. Passing to a subsequence of $\{p_i\}$, we may suppose that x_i has limit $\underline{x}$. Then p_i will have the limit $\underline{p} = (\underline{z}, \underline{x})$. Since V is closed, $\underline{p}$ is in V. Therefore $\underline{z}$ is in its image W. $\square$

defproper **5.5.2. Definition.** A variety X is *proper* if it has the following property: Let $Z \times X$ be the product of X with another variety Z, let π_Z denote the projection $Z \times X \longrightarrow Z$, and let V be a closed subvariety of $Z \times X$. Then the image $W = \pi_Z(V)$ is a closed subvariety of Z.

imclos (5.5.3)
$$\begin{array}{ccc} V & \xrightarrow{\subset} & Z \times X \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_Z} \\ W & \xrightarrow{\subset} & Z \end{array}$$

If X is proper, then because every closed set is a finite union of closed subvarieties, the image of any closed subset of $Z \times X$ will be a closed subset of Z,

propim-closed **5.5.4. Proposition.** *Let X be a proper variety, let V be a closed subvariety of X, and let $X \xrightarrow{f} Y$ be a morphism. The image $f(V)$ of V is a closed subvariety of Y.*

proof. In $X \times Y$, the graph Γ_f of f is a closed set isomorphic to X, and V corresponds to a subset V' of Γ_f that is closed in Γ_f and in $X \times Y$. The points of V' are pairs (x, y) such that $x \in V$ and $y = f(x)$. The image of V' via the projection to $X \times Y \to Y$ is the same as the image of V. Since X is proper, the image of V' is closed. $\square$

The next theorem is the most important application of the use of curves to characterize closed sets.

pnproper **5.5.5. Theorem.** *Projective varieties are proper. Therefore, if X is projective and $X \to Y$ is a morphism, the image in Y of a closed subvariety of X is a closed subvariety of X.*

proof. Let X be a projective variety. With notation as in 5.5.3, suppose we are given a closed subvariety V of the product $Z \times X$. We must show that its image W in Z is a closed subvariety of Z. Since V is irreducible, its image is irreducible, so it suffices to show that W is closed. Theorem 5.3.6 tells us that W is a constructible set, and since X is closed in projective space, it is compact in the classical topology. Proposition 5.5.1 tells us that W is closed in the classical topology, and 5.4.4 tells us that W is closed in the Zariski topology too. $\square$

needcover **5.5.6. Note.** Since Theorem 5.5.5 is about algebra, an algebraic proof would be preferable. To make an algebraic proof, one could attempt to use the curve criterion, proceeding as follows: Given a closed subset V of $Z \times X$ with image W and a point p in the closure of W, one might choose a map $C \xrightarrow{f} Z$ from an affine curve C to Z such that $f(q) = p$ and $f(C') \subset W$, C' being the complement of q in C. Then one would try to lift this map by finding a morphism $C \xrightarrow{g} Z \times X$ such that $g(C') \subset V$ and $f = \pi \circ g$. Since V is closed, it would contain $g(q)$, and therefore $f(q) = \pi g(q)$ would be in $\pi(V) = W$. Unfortunately, to find g, it may be necessary to replace C by a suitable covering. It isn't very difficult to make this method work, but it takes longer. That is why we resorted to the classical topology. $\square$

The next examples show how Theorem 5.5.5 can be used.

5.5.7. Example. (*singular curves*) We parametrize the plane curves of a given degree d. The number of monomials $x_0^i x_1^j x_2^k$ of degree $d = i+j+k$ is the binomial coefficient $\binom{d+2}{2}$. We label those monomials as $m_0, ..., m_r$, ordered arbitrarily, with $r = \binom{d+2}{2} - 1$. A homogeneous polynomial of degree d will be a combination $\sum z_i m_i$ with complex coefficients z_i, so the homogeneous polynomials f of degree d in x, taken up to scalar factors, are parametrized by the projective space of dimension r with coordinates z. Let's denote that projective space by Z. Points of Z correspond bijectively to divisors of degree d in the projective plane, as defined in (1.3.13). properex

The product variety $Z \times \mathbb{P}^2$ represents pairs (D, p), where D is a divisor of degree d and p is a point of $\mathbb{P}^2$. A variable homogeneous polynomial of degree d in x will be a bihomogeneous polynomial $f(z, x)$ of degree 1 in z and degree d in x. For example, in degree 2, f might be

$$z_0x_0^2 + z_1x_1^2 + z_2x_2^2 + z_3x_0x_1 + z_4x_0x_2 + z_5x_1x_2$$

The locus Γ: $\{f(z, x) = 0\}$ is a closed subset of $Z \times \mathbb{P}^2$. Its points are the pairs (D, p) such that D is the divisor of f and p is a point of D.

Let Σ be the set of pairs (D, p) such that p is a singular point of D. This is also a closed set, because it is defined by the system of equations $f_0(z, x) = f_1(z, x) = f_2(z, x) = 0$, where f_i are the partial derivatives $\frac{\partial f}{\partial x_i}$. (Euler's Formula will show that $f(x, z) = 0$.) The partial derivatives f_i are bihomogeneous, of degree 1 in z and degree $d-1$ in x. $\square$

The next proposition isn't especially easy to verify directly, but the proof becomes easy when one uses the fact that projective space is proper.

5.5.8. Proposition *The singular divisors of degree d — those that contain at least one singular point, form a closed subset S of the projective space Z of all divisors of degree d.* *singclosed*

proof. The subset S is the projection of the closed subset Σ of $Z \times \mathbb{P}^2$. Since $\mathbb{P}^2$ is proper, the image of Σ is closed. $\square$

5.5.9. Example. (*surfaces that contain a line*) We go back to the discussion of lines in a surface. Let $\mathbb{S}$ denote the projective space that parametrizes surfaces of degree d in $\mathbb{P}^3$. surface-line

5.5.10. Proposition *In $\mathbb{P}^3$, the surfaces of degree d that contain a line form a closed subset of the space $\mathbb{S}$.* *surfaceswith-line*

The Grssmanian $\mathbb{G} = G(2, 4)$ of lines in $\mathbb{P}^3$ is a projective variety (Corollary 3.7.13). Let Ξ be the subset of $\mathbb{G} \times \mathbb{S}$ of pairs of pairs $[\ell], [S]$ such that $\ell \subset S$. Lemma 3.7.17 tells us that Ξ is a closed subset of $\mathbb{G} \times \mathbb{S}$. Therefore its image in $\mathbb{S}$ is closed. $\square$

Section 5.6 Fibre Dimension

semicont

A function $Y \xrightarrow{\delta} \mathbb{Z}$ from a variety to the integers is a *constructible* function if, for every integer n, the set of points of Y such that $\delta(p) = n$ is constructible, and δ is an *upper semicontinuous* functiion if for every n, the set of points such that $\delta(p) \geq n$ is closed. For brevity, we refer to an upper semicontinuous function as *semicontinuous*, though the term is ambiguous. A function might be lower semicontinuous.

A function δ on a curve C is semicontinuous if and only if there exists an integer n and a nonempty open subset C' of C such that $\delta(p) = n$ for all points p of C' and $\delta(p) \geq n$ for all points of C not in C'.

The next curve criterion for semicontinuous functions follows from the criterion for closed sets.

5.6.1. Proposition. *(curve criterion for semicontinuity) Let Y be a variety. A function $Y \xrightarrow{\delta} \mathbb{Z}$ is semicontinuous if and only if it is a constructible function, and for every morphism $C \xrightarrow{f} Y$ from a smooth curve C to Y, the composition $\delta \circ f$ is a semicontinuous function on C.* $\square$ *uppercrit*

Let $Y \xrightarrow{f} X$ be a morphism of varieties, let q be a point of Y, and let Y_p be the fibre of f over $p = f(q)$. The *fibre dimension* $\delta(q)$ of f at q is the maximum among the dimensions of the components of the fibre that contain q.

Note. One could define the fibre dimension as a function d on X defining $d(p)$ to be the dimension of the fibre over p. This seems simpler. However, it is possible that a fibre contains components of various dimensions, and if so, the fibre dimension, which is a function on Y, is more precise.

uppersemi **5.6.2. Theorem.** *(semicontinuity of fibre dimension) Let $Y \xrightarrow{u} X$ be a morphism of varieties, and let $\delta(q)$ denote the fibre dimension at a point q of Y.*

(i) *Suppose that X is a smooth curve, that Y has dimension n, and that u does not map Y to a single point. Then δ is constant — the fibres have constant dimension: $\delta(q) = n - 1$ for all $q \in Y$.*

(ii) *Suppose that the image of Y contains a nonempty open subset of X, and let the dimensions of X and Y be m and n, respectively. There is a nonempty open subset X' of X such that $\delta(q) = n-m$ for every point q in the inverse image of X'.*

(iii) *δ is a semicontinuous function on Y.*

The proof of this theorem is left as a long exercise. When you have done it, you will have understood the chapter.

Section 5.7 Exercises

chapfiveex

5.7.1. Let $X = \operatorname{Spec} A$ be an affine curve, with $A = \mathbb{C}[x_0, ..., x_n]/P$, and let x_i also denote the residues of the variables in A. Let p be a point of X. We adjust coordinate so that p is the origin $(0, ..., 0)$, and we form the subalgebra B of the function field K of A that is generated by x_0 and the ratios $z_i = x_i/x_0, \quad i = 1, ..., n$. So $B = \mathbb{C}[x_0, z_1, z_2, ..., z_n]$. Let $Y = \operatorname{Spec} B$. The inclusion $A \subset B$ defines a morphism $Y \to X$ called the *blowup* of p in X. There will be finitely many points of Y in the fibre over p, and if coordinates are generic, there will be at least one such point. We choose a point p_1 of the fibre, we replace X by Y and p by p_1 and repeat. Prove that this blowing up process yields a curve that is smooth above p in finitely many steps. xblowups-ing

5.7.2. Prove that the ring $k[[x, y]]$ of formal power series with coefficients in a field k is a local ring and a unique factorization domain. fseries

5.7.3. Let A be a normal finite-type domain. Prove that the localization A_P of A at a prime ideal P of codimension 1 is a valuation ring. xlocval-ring

5.7.4. Let $X = \operatorname{Spec} A$, where $A = \mathbb{C}[x, y, z]/(y^2 - xz^2)$. Identify the normalization of X. xnormaliz

5.7.5. Let A be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$, and let P be the principal ideal generated by an irreducible polynomial $f(x_1, ..., x_n)$. The local ring A_P consists of fractions g/h of polynomials in which g is arbitrary, and h can be any polynomial not divisible by f. Describe valuation v associated to this local ring.

5.7.6. In the space $\mathbb{A}^{n\times n}$ of $n \times n$ matrices, let X be the locus of idempotent matrices: $A^2 = A$. The general linear group GL_n operates on X by conjugation.

(a) Decompose X into orbits for the operation of GL_n, and prove that the orbits are closed subsets of $\mathbb{A}^{n\times n}$.

(b) Determine the dimensions of the orbits.

5.7.7. Prove that, if a variety X is covered by countably many constructible sets, a finite number of those sets will cover X. xcount-cover

5.7.8. Show that if $f(x, y)$ is polynomial and if d divides the partial derivatives f_x and f_y, then f is constant on the locus $d = 0$. xfconst

5.7.9. Let S be a multiplicative system in a finite-type domain R, and let A and B be finite-type domains that contain R as subring. Let R', A', B' be the rings of S-fractions of R, A, B, respectively. Prove: xsimple-loc

(i) If a set of elements $\alpha_1, ..., \alpha_k$ generates A as R-algebra, it also generates A' as R'-algebra.

(ii) Let $A' \xrightarrow{\varphi'} B'$ be a homomorphism. For suitable s in S, there is a homomorphism $A_s \xrightarrow{\varphi_s} B_s$ whose localization is φ'. If φ' is injective, so is φ_s. If φ' is surjective or bijective, there will be an s such that φ_s is surjective or bijective.

(iii) If A' is contained in B' and if B' is a finite A'-module, then for suitable s in S, A_s is contained in B_s, and B_s is a finite A_s-module.

5.7.10. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is a finite morphism. Do this by showing that the fibres cannot have positive dimension. xpt-woptwo

5.7.11. Let G denote the Grassmanian $G(2, 4)$ of lines in $\mathbb{P}^3$, and let $[\ell]$ denote the point of G that corresponds to the line ℓ. In the product variety $G \times G$ of pairs of lines, let Z denote the set of pairs $[\ell_1], [\ell_2]$ whose intersection isn't empty. Prove that Z is a closed subset of $G \times G$. xclosed-inGG

5.7.12. Is the constructibility hypothesis in 5.6.1 necessary? exupper

5.7.13. Prove Theorem 5.5.8 directly, without appealing to Theorem 5.5.5. singular-closed

5.7.14. With reference to Note 5.5.6, let $X = \mathbb{P}^1$ and $Z = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. Find a closed subset V of $Z \times X$ whose image is Z, such that the identity map $Z \to Z$ can't be lifted to a map $Z \to V$. cantlift

5.7.15. (*a part of Theorem 5.8.2.*) Let $f : Y \to X$ be a morphism of varieties. Suppose we know that the fibre dimension is a constructible function. Use the curve criterion to show that fibre dimension is semicontinuous. xfibdim

5.7.16. Let $Y \xrightarrow{f} X$ be a morphism with finite fibres, and for p in X, let $N(p)$ be the number of points in the fibre $f^{-1}(p)$. Prove that N is a constructible function on X. xnum-berof-points xsemicont **5.7.17.** Prove that fibre dimension is a semicontinuous function. I recommend this outline, but you may use any method you like.

(i) We may assume that Y ane X are affine, $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.

(ii) The theorem is true when $A \subset B$ and B is an integral extension of a polynomial subring $A[y_1, ..., y_d]$.

(iii) The fibre dimension is a constructible function.

(iv) The theorem is true when X is a smooth curve.

(v) The theorem is true for all X.

xtwistcubic **5.7.18.** ??? twisted cubic specializes to plane nodal cubic

Chapter 6 MODULES

We review a few facts about localization before going to modules. Recall that, if s is a nonzero element of a domain A, the symbol A_s stands for the (simple) localization $A[s^{-1}]$, and if $X = \operatorname{Spec} A$, then $X_s = \operatorname{Spec} A_s$. This is what we will mean by the word 'localization' in this chapter.

• Let s be a nonzero element of a domain A and let M be an A-module, the localized module M_s is the A_s-module whose elements are the equivalence classes of fractions ms^{-k}, with m in M. The localized module M_s becomes an A-module by restriction of scalars. A homomorphism of A-modules $N \to M_s$ extends in a natural way to a homomorphism of A_s-modules $N_s \to M_s$.

• Let $X = \operatorname{Spec} A$ be an affine variety. The intersection of two localizations $X_s = \operatorname{Spec} A_s$ and $X_t = \operatorname{Spec} A_t$ is the localization $X_{st} = \operatorname{Spec} A_{(st)}$.

• Let $W \subset V \subset U$ be affine open subsets of a variety X. If V is a localization of U and W is a localization of V, then W is a localization of U.

• The affine open subsets of a variety X form a basis for the topology on a variety X, and the localizations of an affine variety form a basis for its topology.

• If U and V are affine open subsets of X, the open sets W that are localizations of U as well as localizations of V, form a basis for the topology on $U \cap V$.

See Chapters 2) and 3 for these assertions. We will use them without further comment. We will also use the concepts of *category* and *functor*. If you aren't familiar with these concepts, please read about them. You won't need to know very much. Learn the definitions and look at a few examples.

Section 6.1 The Structure Sheaf.

strsh

We associate two categories to a variety X. The first is the category *(opens)*. Its objects are the open subsets of X, and its morphisms are inclusions. If U and V are open sets and if $V \subset U$, there is a unique morphism $V \to U$ in *(opens)*. If $V \not\subset U$ there is no morphism $V \to U$.

The other category, *(affines)*, is a subcategory of the category *(opens)*, and it is the most important one. The objects of *(affines)* are the affine open subsets of X, and its morphisms are localizations. A morphism $V \to U$ in *(opens)* is a morphism in *(affines)* if U and V are affine and V is a localization of U — a subset of the form U_s, where s is a nonzero element of the coordinate algebra of U.

The *structure sheaf* $\mathcal{O}_X$ on a variety X is the functor

strshdef (6.1.1)
$$\textit{(affines)}^\circ \xrightarrow{\mathcal{O}_X} \textit{(algebras)}$$

from affine open sets to algebras, that sends an affine open set $U = \operatorname{Spec} A$ to its coordinate algebra A. When speaking of the structure sheaf, the coordinate algebra of U is denoted by $\mathcal{O}_X(U)$. If it is clear which variety is being studied, we may write $\mathcal{O}$ for $\mathcal{O}_X$.

As has been noted, inclusions $V \to U$ of affine open subsets needn't be localizations. We focus attention on localizations because the relation between the coordinate algebras of an affine variety and a localization is easy to understand. However, the structure sheaf extends with little difficulty to the category *(opens)*, (See Corollary 6.1.2 below.)

A brief review about regular functions: The *function field* of a variety X is the field of fractions of the coordinate algebra of any one of its affine open subsets, and a *rational function* on X is an element of the function field. A rational function f is *regular* on an affine open set $U = \operatorname{Spec} A$ if it is an element of A, and f is regular on any open set U that can be covered by affine open sets on which it is regular. Thus the function field contains the regular functions on every nonempty open subset, and the regular functions on an open subset are governed by the regular functions on its affine open subsets.

An affine variety is determined by its regular functions, but the regular functions don't determine a variety that isn't affine. For instance, the only rational functions that are regular everywhere on the projective line $\mathbb{P}^1$ are the constant functions, which are useless. We will be interested in regular functions on non-affine open sets, especially in functions that are regular on the whole variety, but one should always work with the affine open sets, where the definition of a regular function is clear.

Let $V \subset U$ be nonempty open subsets of a variety X. If a rational function is regular on U, it is also regular on V. Thus if U and V are affine, say $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, then $A \subset B$. However, it won't be clear how to construct B from A unless B is a localization. If $V = U_s$, then $B = A[s^{-1}]$. When B isn't a localization of A, the exact relationship between A and B will remain obscure.

We extend the notation introduced for affine open sets to all open sets, denoting the algebra of regular functions on an open set U by $\mathcal{O}_X(U)$.

6.1.2. Corollary. *Let X be a variety. By defining $\mathcal{O}_X(U)$ to be the algebra of regular functions on the open subset U, the structure sheaf $\mathcal{O}_X$ on X extends to a functor* extendOone

$$\textit{(opens)}^\circ \xrightarrow{\mathcal{O}_X} \textit{(algebras)} \qquad \square$$

The regular functions on U, the elements of $\mathcal{O}_X(U)$, are the *sections* of the structure sheaf $\mathcal{O}_X$ on U, and the elements of $\mathcal{O}_X(X)$, the rational functions that are regular everywhere, are *global sections* .

When $V \to U$ is a morphism in *(opens)*, $\mathcal{O}_X(U)$ will be contained in $\mathcal{O}_X(V)$. This gives us the homomorphism, an inclusion,

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

that makes $\mathcal{O}_X$ into a functor. Note that arrows are reversed by $\mathcal{O}_X$. If $V \to U$, then $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A functor that reverses arrows is a *contravariant* functor. The superscript $^\circ$ in (6.1.1) and (6.1.2) is a customary notation to indicate that a functor is contravariant.

6.1.3. Proposition *The extended structure sheaf has the following* ***sheaf property****:* extendO

- *Let Y be an open subset of X, and let $U^i = \operatorname{Spec} A_i$ be affine open subsets that cover Y. Then*

$$\mathcal{O}_X(Y) = \bigcap \mathcal{O}_X(U^i) \quad \left(= \bigcap A_i \right)$$

The fact that regular functions are elements of the function field makes the statement of the sheaf property especially simple here.

By definition, f is a regular function on X if there is a covering by affine open sets U^i such that f is in $\mathcal{O}_X(U^i)$ for every i. Therefore the next lemma proves the proposition.

6.1.4. Lemma. *Let Y be an open subset of a variety X. The intersection $\bigcap \mathcal{O}_X(U^i)$ is the same for every affine open covering $\{U^i\}$ of Y.* capsame

We prove the lemma first in the case of a covering of an affine open set by localizations.

sheaffor-loc **6.1.5. Sublemma.** *Let* $U = \operatorname{Spec} A$ *be an affine variety, and let* $\{U^i\}$ *be a covering of* U *by localizations, say* $U^i = \operatorname{Spec} A_{s_i}$*. Then* $A = \bigcap A_{s_i}$*, i.e.,* $\mathcal{O}(U) = \bigcap \mathcal{O}(U^i)$.

proof. It is clear that $A \subset \bigcap A_{s_i}$. We prove the opposite inclusion. A finite subset of the set $\{U^i\}$ will cover U, so we may assume that the index set is finite. Let α be an element of $\bigcap A_{s_i}$: $\alpha = s_i^{-r} a_i$, or $s_i^r \alpha = a_i$ for some a_i in A and some integer r. We can use the same r for every i. Because $\{U^i\}$ covers U, the elements s_i generate the unit ideal in A, and so do their powers s_i^r. There are elements b_i in A such that $\sum b_i s_i^r = 1$. Then $\alpha = \sum b_i s_i^r \alpha = \sum b_i a_i$, which is in A. □

proof of Lemma 6.1.4. Say that Y is covered by affine open sets $\{U^i\}$ and also by affine open sets $\{V^j\}$. We cover the intersections $U^i \cap V^j$ by open sets $W^{ij\nu}$ that are localizations of U^i and also localizations of V^j. Fixing i and letting j and ν vary, the set $\{W^{ij\nu}\}_{j,\nu}$ will be a covering of U^i by localizations, and the sublemma shows that $\mathcal{O}(U^i) = \bigcap_{j,\nu} \mathcal{O}(W^{ij\nu})$. Then $\bigcap_i \mathcal{O}(U^i) = \bigcap_{i,j,\nu} \mathcal{O}(W^{ij\nu})$. Similarly, $\bigcap_j \mathcal{O}(V^j) = \bigcap_{i,j,\nu} \mathcal{O}(W^{ij\nu})$. □

Section 6.2 $\mathcal{O}$–Modules

module On an affine variety $\operatorname{Spec} A$, one can work with A-modules. There is no need to do anything else. However, one can't do this when a variety isn't affine. The best one can do is to study modules on its affine open subsets. An $\mathcal{O}_X$*-module* on a variety X associates a module to every affine open subset.

defO-modtwo **6.2.1. Definition.** An $\mathcal{O}$*-module* $\mathcal{M}$ on a variety X is a (contravariant) functor

$$\textit{(affines)}^\circ \xrightarrow{\mathcal{M}} \textit{(modules)}$$

such that, for every affine open set U, $\mathcal{M}(U)$ is an $\mathcal{O}(U)$-module, and when s is a nonzero element of $\mathcal{O}(U)$, the module $\mathcal{M}(U_s)$ is the localization of $\mathcal{M}(U)$:

$$\mathcal{M}(U_s) = \mathcal{M}(U)_s$$

A *section* of an $\mathcal{O}$-module $\mathcal{M}$ on an affine open set U is an element of $\mathcal{M}(U)$, and an element of $\mathcal{M}(X)$ is a *global section* . The *module of sections* of $\mathcal{M}$ on U is $\mathcal{M}(U)$.

• An $\mathcal{O}$-module $\mathcal{M}$ is a *finite* $\mathcal{O}$*-module* if $\mathcal{M}(U)$ is a finite $\mathcal{O}(U)$-module for every affine open set U.

• A *homomorphism* $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of $\mathcal{O}$-modules consists of homomorphisms of $\mathcal{O}(U)$-modules

$$\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)$$

for each affine open subset U, such that, when s is a nonzero element of $\mathcal{O}(U)$, the homomorphism $\varphi(U_s)$ is the localization of $\varphi(U)$.

• A sequence of homomorphisms

exse-qsheaves (6.2.2) $$\mathcal{M} \to \mathcal{N} \to \mathcal{P}$$

of $\mathcal{O}$-modules on a variety X is *exact* if, for every affine open subset U of X, the sequence of sections $\mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U)$ is exact. □

At first glance, this definition of an $\mathcal{O}$-module will seem complicated — too complicated for comfort. However, when a module has a natural definition, one doesn't need to worry. The data involved in the definition are taken care of automatically. This will become clear as we go along.

Note. When we say that $\mathcal{M}(U_s)$ **is** the localization of $\mathcal{M}(U)$, it would be more correct to say that $\mathcal{M}(U_s)$ and $\mathcal{M}(U)_s$ are canonically isomorphic. Let's not worry about this.

Section 6.3 Some $\mathcal{O}$-Modules

moduleex-amples freemod **6.3.1.** The *free module* $\mathcal{O}^k$ is a simple example of an $\mathcal{O}$-module. Its sections on an affine open set U are the elements of the free $\mathcal{O}(U)$-module $\mathcal{O}(U)^k$. In particular, $\mathcal{O}$ itself is an $\mathcal{O}$-module.

6.3.2. The *kernel, image*, and *cokernel* of a homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ are among the operations that can be made on $\mathcal{O}$-modules. The kernel $\mathcal{K}$ of φ is the $\mathcal{O}$-module defined by $\mathcal{K}(U) = \ker\left(\mathcal{M}(U) \xrightarrow{\varphi(U)} \mathcal{N}(U)\right)$ for every affine open set U, and the image and cokernel are defined analogously. The reason that we work with localizatons is that many operations, including these, are compatible with localization. kerim

6.3.3. modules on a point sheafon-point

Let's denote the affine variety $\operatorname{Spec}\mathbb{C}$, which is a point, by p. The point p has just one nonempty open set: the whole space p. It is an affine open set, and $\mathcal{O}_p(p) = \mathbb{C}$. Let $\mathcal{M}$ be an $\mathcal{O}_p$-module. To define $\mathcal{M}$, the vector space $\mathcal{M}(p)$ can be assigned arbitrarily. One may say that a module on the point **is** a complex vector space.

6.3.4. the residue field module κ_p. refldmod

Let p be a point of a variety X. The residue field module κ_p is defined as follows: If an affine open subset U of X contains p, then $\mathcal{O}(U)$ has a residue field $k(p)$ at p, and $\kappa_p(U) = k(p)$. If U doesn't contain p, then $\kappa_p(U) = 0$.

For example, when p is the point at infinity of $X = \mathbb{P}^1$, then $\kappa_p(\mathbb{U}^0) = 0$ and $\kappa_p(\mathbb{U}^1) = \mathbb{C}$.

6.3.5. torsion modules.

An $\mathcal{O}$-module $\mathcal{M}$ is a *torsion module* if $\mathcal{M}(U)$ is a torsion $\mathcal{O}(U)$-module for every affine open set U (see (**2.6.6**)).

6.3.6. ideals.

An ideal $\mathcal{I}$ of the structure sheaf is an $\mathcal{O}$-submodule of $\mathcal{O}$. If Y is a closed subvariety of a variety X, the *ideal of* Y is the submodule of $\mathcal{O}$ whose sections on an affine open subset U of X are the rational functions on X that are regular on U and that vanish on $Y \cap U$.

Let p be a point of a variety X. The *maximal ideal* at p, which we denote by $\mathfrak{m}_p$, is an ideal. If an affine open subset U contains p, its coordinate algebra $\mathcal{O}(U)$ will have a maximal ideal whose elements are the regular functions that vanish at p. That maximal ideal is the module of sections $\mathfrak{m}_p(U)$ on U. If U doesn't contain p, then $\mathfrak{m}_p(U) = \mathcal{O}(U)$.

We extend the notation $V(\mathcal{I})$ for the zero set of an ideal $\mathcal{I}$ in the structure sheaf. A point p is in the set $V(\mathcal{I})$ if, whenever U is an affine open subset of X that contains p, all elements of $\mathcal{I}(U)$ vanish at p. When $\mathcal{I}$ is the ideal of functions that vanish on a closed subvariety Y, $V(\mathcal{I}) = Y$.

6.3.7. some homomorphisms defker

(i) Let κ_p be the residue field module at a point p of X. There is a homomorphism of $\mathcal{O}$-modules $\mathcal{O} \to \kappa_p$ whose kernel is the maximal ideal $\mathfrak{m}_p$.

(ii) Homomorphisms $\mathcal{O}^n \to \mathcal{O}^m$ of free $\mathcal{O}$-modules correspond to $m \times n$-matrices of global sections of $\mathcal{O}$.

(iii) Scalar multiplication by a global section f of $\mathcal{O}$ defines a homomorphism $\mathcal{M} \xrightarrow{f} \mathcal{M}$.

(iv) Let $\mathcal{M}$ be an $\mathcal{O}$-module. $\mathcal{O}$-module homomorphisms $\mathcal{O} \xrightarrow{\varphi} \mathcal{M}$ correspond bijectively to global sections of $\mathcal{M}$. This is analogous to the fact that, when M is a module over a ring A, homomorphisms $A \to M$ correspond to elements of M. If m is a global section of $\mathcal{M}$, the homomorphism $\mathcal{O}(U) \xrightarrow{\varphi(U)} \mathcal{M}(U)$ is multiplication by the restriction of m to U: $\varphi(f) = fm$.

Section 6.4 The Sheaf Property

In this section, we extend an $\mathcal{O}$-module $\mathcal{M}$ on a variety X to a functor *(opens)*$^\circ \xrightarrow{\widetilde{\mathcal{M}}}$ *(modules)* on all open subsets of X with these properties: nonaffine-sections

- $\widetilde{\mathcal{M}}(Y)$ is an $\mathcal{O}(Y)$-module for every open subset Y.
- When U is an affine open set, $\widetilde{\mathcal{M}}(U) = \mathcal{M}(U)$.

The tilde $\sim$ is used for clarity here. When we have finished with the discussion, we will use the same notation for the functor on *(affines)* and for its extension to *(opens)*.

Terminology. Let *(opens)*$^\circ \xrightarrow{\widetilde{\mathcal{M}}}$ *(modules)* be a functor. If U is an open subset of X, an element of $\widetilde{\mathcal{M}}(U)$ is a *section* of $\widetilde{\mathcal{M}}$ on U. If $V \xrightarrow{j} U$ is an inclusion of open subsets, the associated homomorphism $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is the *restriction* from U to V.

When $V \xrightarrow{j} U$ is an inclusion of open sets, the restriction to V of a section m on U may be denoted by $j^\circ m$. However, the restriction operation occurs very often, and because of this, we usually abbreviate, using the same symbol m for a section and for its restriction. Also, if an open set V is contained in two open sets U and U', and if m and m' are sections of $\widetilde{\mathcal{M}}$ on U and U', respectively, we may say that m and m' are *equal on* V if their restrictions to V are equal. □

extendO-mod

6.4.1. Theorem. *An $\mathcal{O}$-module $\mathcal{M}$ extends uniquely to a functor*

$$\textit{(opens)}^\circ \xrightarrow{\widetilde{\mathcal{M}}} \textit{(modules)}$$

that has the sheaf property that is described below. Moreover, for every open set U, $\widetilde{\mathcal{M}}(U)$ is an $\mathcal{O}(U)$-module, and for every inclusion $V \to U$ of nonempty open sets, the map $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ is compatible with scalar multiplication.

Compatibility with salar multiplication means that, when $\widetilde{\mathcal{M}}(V)$ is made into an $\mathcal{O}(U)$-mdule by restriction of scalars, the map becomes a homomorphism of $\mathcal{O}(U)$-modules. To be specific: Let m be a section of $\widetilde{\mathcal{M}}$ on U, and let α be a regular function on U, an element of $\mathcal{O}(U)$. If m' and α' denote the restrictions of m and α to V, then the restriction of αm is $\alpha' m'$.

The proof of this theorem, though not especially difficult, is lengthy because there are several things to check. In order not to break up the discussion, we have put the proof into Section 6.9 at the end of the chapter.

sheafprop

(**6.4.2**) **the sheaf property**

The sheaf property is the key requirement that determines the extension of an $\mathcal{O}$-module $\mathcal{M}$ to a functor $\widetilde{\mathcal{M}}$ on *(opens)*.

Let Y be an open subset of X, and let $\{U^i\}$ be a covering of Y by affine open sets. The intersections $U^{ij} = U^i \cap U^j$ are also affine open sets, so $\mathcal{M}(U^i)$ and $\mathcal{M}(U^{ij})$ are defined. The sheaf property asserts that an element m of $\widetilde{\mathcal{M}}(Y)$ corresponds to a set of elements m_i in $\mathcal{M}(U^i)$ such that the restrictions of m_j and m_i to U^{ij} are equal.

If the affine open subsets U^i are indexed by $i = 1, ..., n$, the sheaf property asserts that an element of $\widetilde{\mathcal{M}}(Y)$ is determined by a vector $(m_1, ..., m_n)$ with m_i in $\mathcal{M}(U^i)$, such that the restrictions of m_i and m_j to U^{ij} are equal. This means that $\widetilde{\mathcal{M}}(Y)$ is the kernel of the map

sheafker (6.4.3)

$$\prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^{ij})$$

that sends the vector $(m_1, ..., m_n)$ to the $n \times n$ matrix (z_{ij}), where z_{ij} is the difference $m_j - m_i$ of the restrictions of m_j and m_i to U^{ij}. The analogous description is true when the index set is infinite.

In short, the sheaf property tells us that sections of $\widetilde{\mathcal{M}}$ are determined locally: A section on an open set Y is determined by its restrictions to the open subsets U^i of any affine covering of Y.

Note. Since U^{ij} is contained in U^i, there is a morphism $U^{ij} \to U^i$ in *(opens)*. However, this morphism isn't necessarily a localization. If it isn't a localization, it won't be a morphism in *(affines)*, and the restriction maps $\mathcal{M}(U^i) \to \mathcal{M}(U^{ij})$ won't be a part of the structure of an $\mathcal{O}$-module. We need a definition of the restriction map for an arbitrary inclusion $V \to U$ of affine open subsets. This point will be taken care of in the proof of Theorem 6.4.1. (See Step 2 in Section 6.9.) We won't worry about it here. □

We drop the tilde now, and denote by $\mathcal{M}$ also the extension of an $\mathcal{O}$-module $\mathcal{M}$ to all open sets. The sheaf property for $\mathcal{M}$ is the statement that, when $\{U^i\}$ is an affine open covering of an open set U, the sequence

sheaf-proptwo (6.4.4)

$$0 \to \mathcal{M}(U) \xrightarrow{\alpha} \prod_i \mathcal{M}(U^i) \xrightarrow{\beta} \prod_{i,j} \mathcal{M}(U^{ij})$$

is exact, where α is the product of the restriction maps, and β is the map described in (6.4.3). So $\mathcal{M}(U)$ is mapped isomorphically to the kernel of β. In particular, the composition $\beta\alpha$ is the zero map. Thus elements of $\mathcal{M}(U)$ correspond bijectively to vectors $(m_1, ..., m_n)$, with m_i in $\mathcal{M}(U^i)$, such that the restrictions of m_i and m_j to $\mathcal{M}(U^{ij})$ are equal.

One should always work with the affine open sets. We may sometimes want to look at sections of an $\mathcal{O}$-module on other open sets, but the non-affine open sets are just along for the ride most of the time.

The next corollary follows from Theorem 6.4.1.

6.4.5. Corollary. *Let $\{U^i\}$ be an affine open covering of a variety X.* *injoncover*
(i) *An $\mathcal{O}$-module $\mathcal{M}$ is the zero module if and only if $\mathcal{M}(U^i) = 0$ for every i.*
(ii) *A homomorphism $\mathcal{M} \xrightarrow{\varphi} \mathcal{N}$ of $\mathcal{O}$-modules is injective, surjective, or bijective if and only if the maps $\mathcal{M}(U^i) \xrightarrow{\varphi(U^i)} \mathcal{N}(U^i)$ are injective, surjective, or bijective, respectively, for every i.*

proof. **(i)** Let V be an open subset of X. We cover the intersections $V \cap U^i$ by affine open sets $V^{i\nu}$ that are localizations of U^i. These sets, taken together, cover V. If $\mathcal{M}(U^i) = 0$, then the localizations $\mathcal{M}(V^{i\nu})$ are zero too. The sheaf property shows that the map $\mathcal{M}(V) \to \prod \mathcal{M}(V^{i\nu})$ is injective, and therefore that $\mathcal{M}(V) = 0$.

(ii) This follows from **(i)** because a homomorphism φ is injective or surjective if and only if its kernel or its cokernel is zero. □

(6.4.6) families of open sets omod

It is convenient to have a more compact notation for the sheaf property. For this, one can introduce symbols to represent families of open sets. Say that $\mathbf{U}$ and $\mathbf{V}$ represent families of open sets $\{U^i\}$ and $\{V^\nu\}$, respectively. A *morphism* of families $\mathbf{V} \to \mathbf{U}$ consists of a morphism from each V^ν to one of the subsets U^i. Such a morphism will be given by a map $\nu \rightsquigarrow i_\nu$ of index sets, such that $V^\nu \subset U^{i_\nu}$.

There may be more than one morphism $\mathbf{V} \to \mathbf{U}$, because a subset V^ν may be contained in more than one of the subsets U^i. To define a morphism, one must make a choice among those subsets. For example, let $\mathbf{U} = \{U^i\}$ be a family of open sets, and let V be another open set. For each i such that $V \subset U^i$, there is a morphism $V \to \mathbf{U}$ that sends V to U^i. In the other direction, there is a unique morphism $\mathbf{U} \to V$ provided that $U^i \subset V$ for all i.

We extend a functor *(opens)*$^\circ \xrightarrow{\mathcal{M}}$ *(modules)* to families $\mathbf{U} = \{U^i\}$, defining

$$\mathcal{M}(\mathbf{U}) = \prod \mathcal{M}(U^i). \tag{6.4.7}$$ section-sonfamily

Then a morphism of families $\mathbf{V} \xrightarrow{f} \mathbf{U}$ defines a map $\mathcal{M}(\mathbf{V}) \xleftarrow{f^\circ} \mathcal{M}(\mathbf{U})$ in a way that is fairly obvious, though our notation for it is clumsy. Say that $\mathbf{V} = \{V^\nu\}$, and that f is given by a map $\nu \rightsquigarrow i_\nu$ of index sets, with $V^\nu \to U^{i_\nu}$. A section of $\mathcal{M}$ on $\mathbf{U}$, an element of $\mathcal{M}(\mathbf{U})$, can be thought of as a vector $u = (u_i)$ with $u_i \in \mathcal{M}(U^i)$, and a section of $\mathcal{M}(\mathbf{V})$ as a vector $v = (v_\nu)$ with $v_\nu \in \mathcal{M}(V^\nu)$. If v_ν is the restriction of u_{i_ν} to V^ν, the restriction $f^\circ(u)$ of u is v.

We write the sheaf property in terms of families of open sets: Let $\mathbf{U}_0 = \{U^i\}$ be an affine open covering of an open set Y, and let $\mathbf{U}_1$ denote the family $\{U^{ij}\}$ of intersections: $U^{ij} = U^i \cap U^j$. The intersections are also affine, and there are two sets of inclusions

$$U^{ij} \subset U^i \quad \text{and} \quad U^{ij} \subset U^j$$

They give us two morphisms of families $\mathbf{U}_1 \xrightarrow{d_0,d_1} \mathbf{U}_0$ of affine open sets: $U^{ij} \xrightarrow{d_0} U^j$ and $Uij \xrightarrow{d_1} U^i$. We also have a morphism $\mathbf{U}_0 \to Y$, and the composed morphisms $\mathbf{U}_1 \xrightarrow{d_i} \mathbf{U}_0 \to Y$ are equal. These maps form what we all a *covering diagram*

$$Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1 \tag{6.4.8}$$ covdiagr

When we apply a functor *(opens)* $\xrightarrow{\mathcal{M}}$ *(modules)* to this diagram, we obtain a sequence

defbeta (6.4.9)
$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha_{\mathbb{U}}} \mathcal{M}(\mathbf{U}_0) \xrightarrow{\beta_{\mathbb{U}}} \mathcal{M}(\mathbf{U}_1)$$

where $\alpha_{\mathbf{U}}$ is the restriction map and $\beta_{\mathbb{U}}$ is the difference $d_0 - d_1$ of the maps induced by the two morphisms $\mathbf{U}_1 \rightrightarrows \mathbf{U}_0$. The sheaf property for the covering $\mathbf{U}_0$ of Y (6.4.4) is the assertion that this sequence is exact, which means that $\alpha_{\mathbf{U}}$ is injective, and that its image is the kernel of $\beta_{\mathbf{U}}$.

6.4.10. Note. Let $\{U^i\}$ be an affine open covering of Y. Then $U^{ii} = U^i$ and $U^{ij} = U^{ji}$. These coincidences twoopensets lead to redundancy in the statement (6.4.9) of the sheaf property. If the indices are $i = 1, ..., k$, we only need to look at intersections U^{ij} with $i < j$. The product $\mathcal{M}(\mathbf{U}_1) = \prod_{i,j} \mathcal{M}(U^{ij})$ that appears in the sheaf property can be replaced by the product with increasing pairs of indices $\prod_{i<j} \mathcal{M}(U^{ij})$. For instance, if an open set Y is covered by two affine open sets U and V. The sheaf property for this covering is an exact sequence

$$0 \to \mathcal{M}(Y) \xrightarrow{\alpha} \big[\mathcal{M}(U)\times\mathcal{M}(V)\big] \xrightarrow{\beta} \big[\mathcal{M}(U\cap U)\times\mathcal{M}(U\cap V)\times\mathcal{M}(V\cap U)\times\mathcal{M}(V\cap V)\big]$$

The exact sequence

twoopens (6.4.11)
$$0 \to \mathcal{M}(Y) \to \big[\mathcal{M}(U)\times\mathcal{M}(V)\big] \xrightarrow{-\cdot+} \mathcal{M}(U\cap V)$$

is equivalent, and less redundant. □

strsheafPn **6.4.12. Example.**
Let A denote the polynomial ring $\mathbb{C}[x, y]$, and let V be The complement of a point p in affine space $X = \operatorname{Spec} A$ is an open set, but it isn't affine. We cover V by two localizations of X: $X_x = \operatorname{Spec} A[x^{-1}]$ and $X_y = \operatorname{Spec} A[y^{-1}]$. The sheaf property (6.4.11) for $\mathcal{O}_X$ and for this covering is equivalent to an exact sequence

$$0 \to \mathcal{O}_X(V) \to A[x^{-1}]\times A[y^{-1}] \to A[(xy)^{-1}]$$

It shows that a regular function on V is in the intersection $A[x^{-1}]\cap A[y^{-1}]$, which is A. Therefore the sections of the structure sheaf $\mathcal{O}_X$ on V are the elements of A, the same as the sections on X. □

We have been working (tacitly) with nonempty open sets. The next lemma takes care of the empty set.

emptyset **6.4.13. Lemma.** *The only section of an $\mathcal{O}$-module $\mathcal{M}$ on the empty set is the zero section: $\mathcal{M}(\emptyset) = \{0\}$. In particular, $\mathcal{O}(\emptyset)$ is the zero ring.*

proof. This follows from the sheaf property. The empty set $\emptyset$ is covered by the empty covering, the covering indexed by the empty set. Therefore $\mathcal{M}(\emptyset)$ is contained in an empty product. We want the empty product to be a module, and we have no choice but to define it to be zero. Then $\mathcal{M}(\emptyset)$ is zero too. □

If you find this reasoning pedantic, you can take $\mathcal{M}(\emptyset) = \{0\}$ as an axiom.

coherence **(6.4.14) the coherence property**

In addition to the sheaf property, an $\mathcal{O}$-module on a variety X has a property called *coherence*.

cohprop **6.4.15. Proposition.** *(coherence property.) Let Y be an open subset of a variety X, let s be a nonzero regular function on Y, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then $\mathcal{M}(Y_s)$ is the localization $\mathcal{M}(Y)_s$ of $\mathcal{M}(Y)$. In particular, $\mathcal{O}_X(U_s)$ is the localization $\mathcal{O}_X(U)_s$.*

When Y is affine, compatibility with localization is a part of the structure of an $\mathcal{O}$-module. The coherence property is the compatibility with localization for any open set.

proof of Proposition 6.4.15. Let $\mathbf{U}_0 = \{U^i\}$ be a family of affine open sets that covers an open set Y. The intersections U^{ij} will be affine open sets too. We inspect the covering diagram $Y \leftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$. If s is a nonzero regular function on Y, the localization of this diagram forms a covering diagram $Y_s \leftarrow \mathbf{U}_{0,s} \leftleftarrows \mathbf{U}_{1,s}$

in which $\mathbf{U}_{0,s} = \{U^i_s\}$ is an affine covering of Y_s. Therefore $\mathcal{M}(\mathbf{U}_0)_s \approx \mathcal{M}(\mathbf{U}_{0,s})$. The sheaf property for the two covering diagrams gives us exact sequences

$$0 \to \mathcal{M}(Y) \to \mathcal{M}(\mathbf{U}_0) \to \mathcal{M}(\mathbf{U}_1) \qquad \text{and} \qquad 0 \to \mathcal{M}(Y_s) \to \mathcal{M}(\mathbf{U}_{0,s}) \to \mathcal{M}(\mathbf{U}_{1,s})$$

The sequence on the left maps to the one on the right, and since s is invertible in sequence on the right, the localization of the left sequence maps to it:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}(Y)_s \arrow[r] \arrow[d, "a"'] & \mathcal{M}(\mathbf{U}_0)_s \arrow[r] \arrow[d, "b"'] & \mathcal{M}(\mathbf{U}_1)_s \arrow[d, "c"'] \\
0 \arrow[r] & \mathcal{M}(Y_s) \arrow[r] & \mathcal{M}(\mathbf{U}_{0,s}) \arrow[r] & \mathcal{M}(\mathbf{U}_{1,s})
\end{tikzcd}$$

Since localization is an exact operation, the top row is exact, and so is the bottom row. Since $\mathbf{U}_0$ and $\mathbf{U}_1$ are families of affine open sets, the vertical arrows b and c are bijections. It follows that a is a bijection. □

Section 6.5 More Modules

moremodules

6.5.1. kernel kerthm

As we have remarked, many operations that one makes on modules over a ring are compatible with localization, and therefore can be made on $\mathcal{O}$-modules. However, for sections over a non-affine open set one must use the sheaf property. The sections over a non-affine open set are almost never determined by an operation. The kernel of a homomorphism is among the few exceptions.

6.5.2. Proposition. *Let X be a variety, and let $\mathcal{K}$ be the kernel of a homomorphism of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{N}$, so that the there is an exact sequence $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{N}$. For every open subset Y of X, the sequence of sections* *leftexact-section*

$$0 \to \mathcal{K}(Y) \to \mathcal{M}(Y) \to \mathcal{N}(Y) \tag{6.5.3}$$ *lexsect*

is exact.

proof. We choose a covering diagram $Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$, and inspect the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(\mathbf{U}_0) \arrow[r] \arrow[d] & \mathcal{M}(\mathbf{U}_0) \arrow[r] \arrow[d] & \mathcal{N}(\mathbf{U}_0) \arrow[d] \\
0 \arrow[r] & \mathcal{K}(\mathbf{U}_1) \arrow[r] & \mathcal{M}(\mathbf{U}_1) \arrow[r] & \mathcal{N}(\mathbf{U}_1)
\end{tikzcd}$$

where the vertical maps are the maps $\beta_{\mathbf{U}}$ described in (6.4.9). The rows are exact because $\mathbf{U}_0$ and $\mathbf{U}_1$ are families of affines, and the sheaf property asserts that the kernels of the vertical maps form the sequence (6.5.3). That sequence is exact because taking kernels is a left exact operation (2.1.19). □

The section functor isn't right exact. When $\mathcal{M} \to \mathcal{N}$ is a surjective homomorphism of $\mathcal{O}$-modules and Y is a non-affine open set, the map $\mathcal{M}(Y) \to \mathcal{N}(Y)$ may fail to be surjective. There is an example below. Cohomology, which will be discussed in the next chapter, is a substitute for right exactness.

6.5.4. modules on the projective line coverpone

The projective line $\mathbb{P}^1$ is covered by the standard open sets $\mathbb{U}^0$ and $\mathbb{U}^1$, and the intersection $\mathbb{U}^{01} = \mathbb{U}^0 \cap \mathbb{U}^1$ is a localization, both of $\mathbb{U}^0$ and of $\mathbb{U}^1$. The coordinate algebras of these affine open sets are $\mathcal{O}(\mathbb{U}^0) = A_0 = \mathbb{C}[u]$, $\mathcal{O}(\mathbb{U}^1) = A_1 = \mathbb{C}[u^{-1}]$, and $\mathcal{O}(\mathbb{U}^{01}) = A_{01} = \mathbb{C}[u, u^{-1}]$ is the Laurent polynomial ring. The form (6.4.11) of the sheaf property asserts that a global section of $\mathcal{O}$ is determined by polynomials $f(x)$ and $g(x)$ such that $f(u) = g(u^{-1})$ in A_{01}. The only such polynomials f and g are the constants. So the constants are the only rational functions that are regular everywhere on $\mathbb{P}^1$. I think we knew this.

If $\mathcal{M}$ is an $\mathcal{O}$-module, $\mathcal{M}(\mathbb{U}^0) = M_0$ and $\mathcal{M}(\mathbb{U}^1) = M_1$ will be modules over the algebras A_0 and A_1, and the A_{01}-module $\mathcal{M}(\mathbb{U}^{01}) = M_{01}$ will be obtained by localizing M_0 and also by localizing M_1. Let $v = u^{-1}$. Then

$$M_0[u^{-1}] \approx M_{01} \approx M_1[v^{-1}]$$

As (6.4.11) tells us, a global section of $\mathcal{M}$ is determined by a pair of elements m_1, m_2 in M_1, M_2, respectively, that become equal in the common localization M_{01}. In fact, this data determines the module $\mathcal{M}$.

modonpone **6.5.5. Lemma.** *With notation as above, let M_0, M_1, and M_{01} be modules over the algebras A_0, A_1, and A_{01}, respectively, and let $M_0[u^{-1}] \xrightarrow{\varphi_0} M_{01}$ and $M_1[v^{-1}] \xrightarrow{\varphi_1} M_{01}$ be A_{01}-isomorphisms. There is an $\mathcal{O}_X$-module $\mathcal{M}$, unique up to isomorphism, such that $\mathcal{M}(\mathbb{U}^0)$ and $\mathcal{M}(\mathbb{U}^1)$ are isomorphic to M_0 and M_1, respectively, and such that the diagram below commutes.*

$$\begin{tikzcd}
\mathcal{M}(\mathbb{U}^0) \arrow[r] \arrow[d] & \mathcal{M}(\mathbb{U}^{01}) \arrow[d] & \mathcal{M}(\mathbb{U}^1) \arrow[l] \arrow[d] \\
M_0 \arrow[r, "\varphi_0"] & M_{01} & M_1 \arrow[l, "\varphi_1"']
\end{tikzcd}$$

The proof is at the end of this section.

Suppose that M_0 and M_1 are free modules of rank r over A_0 and A_1. Then M_{01} will be a free A_{01}-module of rank r. A basis $\mathbf{B}_0$ of the free A_0-module M_0 will also be a basis of the A_{01}-module M_{01}, and a basis $\mathbf{B}_1$ of M_1 will be a basis of M_{01}. When regarded as bases of M_{01}, $\mathbf{B_0}$ and $\mathbf{B_1}$ will be related by an invertible $r{\times}r$ A_{01}-matrix P, and as Lemma 6.5.5 tells us, that matrix determines $\mathcal{M}$ up to isomorphism. When $r = 1$, P will be an invertible $1{\times}1$ matrix in the Laurent polynomial ring A_{01} — a unit of that ring. The units in A_{01} are scalar multiples of powers of u. Since the scalar can be absorbed into one of the bases, an $\mathcal{O}$-module of rank 1 is determined, up to isomorphism, by a power of u. It is one of the twisting modules that will be described below, in Section 6.8.

The Birkhoff-Grothendieck Theorem, which will be proved in Chapter 8, describes the $\mathcal{O}$-modules on the projective line whose sections on $\mathbb{U}^0$ and on $\mathbb{U}^1$ are free, as direct sums of free $\mathcal{O}$-modules of rank one. This means that by changing the bases $\mathbf{B}_0$ and $\mathbf{B}_1$, one can diagonalize the matrix P. Such changes of basis will be given by an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1, respectively. In down-to-Earth terms, the Birkhoff-Grothendieck Theorem asserts that, for any invertible A_{01}-matrix P, there exist an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1 , such that $Q_0^{-1}PQ_1$ is diagonal. □

tensprodmodule **6.5.6. tensor products**

As Corollary 2.1.29 asserts, tensor products are compatible with localization. If M and N are modules over a domain A and s is a nonzero element of A, the canonical map $(M \otimes_A N)_s \to M_s \otimes_{A_s} N_s$ is an isomorphism. Therefore the tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ of $\mathcal{O}$-modules $\mathcal{M}$ and $\mathcal{N}$ is defined. On an affine open set U, $[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](U)$ is the tensor product $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules, let $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ be the tensor product module, and let V be an arbitrary open subset of X. There is a canonical map

tensprodmap (6.5.7) $$\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \to [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)$$

By definition of the tensor product module, this map is an equality when V is affine. To describe the map for arbitrary V, we cover V by a family $\mathbf{U}_0$ of affine open sets and form a diagram

$$\begin{tikzcd}
 & \mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \arrow[r, "f"] \arrow[d, "a"] & \mathcal{M}(\mathbf{U}_0) \otimes_{\mathcal{O}(\mathbf{U}_0)} \mathcal{N}(\mathbf{U}_0) \arrow[r, "g"] \arrow[d, "b"] & \mathcal{M}(\mathbf{U}_1) \otimes_{\mathcal{O}(\mathbf{U}_1)} \mathcal{N}(\mathbf{U}_1) \arrow[d, "c"] \\
0 \arrow[r] & {[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)} \arrow[r] & {[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](\mathbf{U}_0)} \arrow[r] & {[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](\mathbf{U}_1)}
\end{tikzcd}$$

The family $\mathbf{U}_1$ of intersections also consists of affine open sets, so by definition of the tensor product, the vertical maps b and c are equalities. The bottom row is exact, and the composition gf is zero. So f maps $\mathcal{M}(V) \otimes_{\mathcal{O}(V)} \mathcal{N}(V)$ to the kernel of g, which is equal to $[\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}](V)$. Thus (6.5.7) is the map a. When V isn't affine, this map needn't be either injective or surjective.

residuefieldmodule **6.5.8. Examples.** These examples illustrate the failure of bijectivity of (6.5.7).

(i) Let p and q be distinct points of the projective line X, and let κ_p and κ_q be the residue field modules on X. Then $\kappa_p(X) = \kappa_q(X) = \mathbb{C}$, so $\kappa_p(X) \otimes_{\mathcal{O}(X)} \kappa_q(X) \approx \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}$. But $\kappa_p \otimes_{\mathcal{O}} \kappa_q = 0$. The map (6.5.7) with $V = X$, which is

$$\kappa_p(X) \otimes_{\mathcal{O}(X)} \kappa_q(X) \to [\kappa_p \otimes_{\mathcal{O}} \kappa_q](X)$$

is the zero map. It isn't injective.

(ii) Let p a point of a variety X, and let $\mathfrak{m}_p$ and κ_p be the maximal ideal and residue field modules at p. There is an exact sequence of $\mathcal{O}$-modules

$$0 \to \mathfrak{m}_p \to \mathcal{O} \xrightarrow{\pi_p} \kappa_p \to 0 \tag{6.5.9}$$ ideal-sheafatp

The sequence of global sections is exact.

(iii) Let p and q be the points $(1,0)$ and $(0,1)$ of the projective line $\mathbb{P}^1$. We form a homomorphism

$$\mathfrak{m}_p \times \mathfrak{m}_q \xrightarrow{\varphi} \mathcal{O}$$

φ being the map $(a,b) \mapsto b-a$. On the open set $\mathbb{U}^0$, $\mathfrak{m}_q \to \mathcal{O}$ is bijective and therefore surjective. Similarly, $\mathfrak{m}_p \to \mathcal{O}$ is surjective on $\mathbb{U}^1$. Since $\mathbb{U}^0$ and $\mathbb{U}^1$ cover $\mathbb{P}^1$, φ is surjective. The only global sections of $\mathfrak{m}_p$, $\mathfrak{m}_q$, and $\mathfrak{m}_p \times \mathfrak{m}_q$ are the zero sections, while $\mathcal{O}$ has the nonzero global section 1. The map on global sections determined by φ isn't surjective. □

6.5.10. the function field module ffldmod

Let F be the function field of a variety X. The *function field module* $\mathcal{F}$ is defined as follows: The module of sections $\mathcal{F}(U)$ on any nonempty open set U is the field F. This is an $\mathcal{O}$-module. It is called a *constant* $\mathcal{O}$-module because the $\mathcal{F}(U)$ is the same for every nonempty U. It won't be a finite module unless X is a point.

Tensoring with the function field module: Let $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X, and let $\mathcal{F}$ be the function field module. We describe the tensor product module $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$.

If $U = \operatorname{Spec} A$ is an affine open set and $M = \mathcal{M}(U)$, the module of sections of on U is the F-vector space $M \otimes_A F$. If S is the multipicative system of nonzero elements of A, then $M \otimes_A F$ is the localization MS^{-1}. On a simple localization U_s, the module of sections will be $M_s \otimes_{A_s} F$, which is the same as $M \otimes_A F$ because s is invertible in F. Thus the vector space $M \otimes_A F$ is independent of the affine open set U. So $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$ is a constant $\mathcal{O}$-module.

If $\mathcal{M}$ is a torsion module, the tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}$ will be the zero module.

(6.5.11) $\mathcal{O}$-modules on affine varieties modaff

The next proposition shows that, on an affine variety $\operatorname{Spec} A$, $\mathcal{O}$-modules correspond bijectively to (ordinary) A-modules.

6.5.12. Proposition. *Let $X = \operatorname{Spec} A$ be an affine variety. Sending an $\mathcal{O}$-module $\mathcal{M}$ to the A-module $\mathcal{M}(X)$ of its global sections defines a bijective correspondence between $\mathcal{O}$-modules and A-modules.* *moduleon-affines*

proof. We must invert the functor *$\mathcal{O}$–(modules) $\to$ A–(modules)* that sends $\mathcal{M}$ to $\mathcal{M}(X)$. Given an A-module M, the corresponding $\mathcal{O}$-module $\mathcal{M}$ is defined as follows: Let $U = \operatorname{Spec} B$ be an affine open subset of X. The inclusion $U \subset X$ corresponds to an algebra homomorphism $A \to B$. We define $\mathcal{M}(U)$ to be the B-module $B \otimes_A M$. If s is a nonzero element of B, then $B_s \otimes_A M$ is canonically isomorphic to the localization $(B \otimes_A M)_s$ of $B \otimes_A M$. Therefore $\mathcal{M}$ is an $\mathcal{O}$-module, and $\mathcal{M}(X) = M$.

Conversely, let $\mathcal{M}$ be an $\mathcal{O}$-module such that $\mathcal{M}(X) = M$. Then, with notation as above, the map $M = \mathcal{M}(X) \to \mathcal{M}(U)$ induces a homomorphism of B-modules $M \otimes_A B \to \mathcal{M}(U)$. When U is a localization X_s of X, so that $B = A_s$, both $M \otimes_A A_s$ and $\mathcal{M}(X_s)$ are the localizations of M, so they are isomorphic. Therefore the module $\mathcal{M}$ is determined up to isomorphism. □

6.5.13. Example. delete-point

When an open set isn't affine, defining $\mathcal{M}(V) = B \otimes_A M$, as in Proposition 6.5.12, may be wrong, as this example shows. Let X be the affine plane $\operatorname{Spec} A$, $A = \mathbb{C}[x,y]$, let V be the complement of the origin in X, and let M be the A-module A/yA. This module can be identified with $\mathbb{C}[x]$, which becomes an A-module when scalar multiplication by y is defined to be zero. Here $\mathcal{O}(V) = \mathcal{O}(X) = A$ (6.4.12). If we followed the method used for affine open sets, we would set $\mathcal{M}(V) = A \otimes_A M = \mathbb{C}[x]$.

To identify $\mathcal{M}(V)$ correctly, we cover V by the two affine open sets $X_x = \operatorname{Spec} A[x^{-1}]$ and $X_y = \operatorname{Spec} A[y^{-1}]$. Then $\mathcal{M}(X_x) = M[x^{-1}]$ while $\mathcal{M}(X_y) = 0$. The sheaf property of $\mathcal{M}$ shows that $\mathcal{M}(V) \approx \mathcal{M}(X_x) = M[x^{-1}] = \mathbb{C}[x, x^{-1}]$. $\square$

omodlim **(6.5.14) limits of $\mathcal{O}$-modules**

limits **6.5.15.** A *directed set* $M_\bullet$ of modules over a ring R is a sequence of homomorphisms of R-modules $M_0 \to M_1 \to M_2 \to \cdots$. Its *limit* $\varinjlim M_\bullet$ is the R-module whose elements are equivalence classes on the union $\bigcup M_k$, the equivalence relation being that elements m in M_i and m' in M_j are equivalent if they have the same image in M_n when n is sufficiently large. An element of $\varinjlim M_\bullet$ will be represented by an element of M_i for some i.

limexample **6.5.16. Example.** Let $R = \mathbb{C}[x]$ and let $\mathfrak{m}$ be the maximal ideal xR. Repeated multiplication by x defines a directed set

$$R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \cdots$$

whose limit is isomorphic to the Laurent polynomial ring $R[x^{-1}] = \mathbb{C}[x, x^{-1}]$. Proving this is an exercise. $\square$

A *directed set of $\mathcal{O}$-modules* on a variety X is a sequence $\mathcal{M}_\bullet = \{\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \cdots\}$ of homomorphisms of $\mathcal{O}$-modules. So, for every affine open set U, the $\mathcal{O}(U)$-modules $\mathcal{M}_n(U)$ form a directed set, as defined in (6.5.15). The *direct limit* $\varinjlim \mathcal{M}_\bullet$ is defined simply, by taking the limit for each affine open set: $[\varinjlim \mathcal{M}_\bullet](U) = \varinjlim [\mathcal{M}_\bullet(U)]$. This limit operation is compatible with localization, so $\varinjlim \mathcal{M}_\bullet$ is an $\mathcal{O}$-module. In fact, the equality $[\varinjlim \mathcal{M}_\bullet](U) = \varinjlim [\mathcal{M}_\bullet(U)]$ will be true for every open set.

A map $\mathcal{M}_\bullet \to \mathcal{N}_\bullet$ of directed sets of $\mathcal{O}$-modules is a diagram

$$\begin{array}{ccccc} \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \mathcal{N}_1 & \longrightarrow & \mathcal{N}_2 & \longrightarrow & \cdots \end{array}$$

A sequence $\mathcal{M}_\bullet \to \mathcal{N}_\bullet \to \mathcal{P}_\bullet$ of maps of directed sets is exact if the sequences $\mathcal{M}_i \to \mathcal{N}_i \to \mathcal{P}_i$ are exact for every i.

jstarlimit **6.5.17. Lemma. (i)** *The limit operation is exact. If $\mathcal{M}_\bullet \to \mathcal{N}_\bullet \to \mathcal{P}_\bullet$ is an exact sequence of directed sets of $\mathcal{O}$-modules, the limits form an exact sequence.*

(ii) *Tensor products are compatible with limits: If $\mathcal{N}_\bullet$ is a directed set of $\mathcal{O}$-modules and $\mathcal{M}$ is another $\mathcal{O}$-module, then $\varinjlim [\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}_\bullet] \approx \mathcal{M} \otimes_{\mathcal{O}} [\varinjlim \mathcal{N}_\bullet]$.* $\square$

moduleHom **(6.5.18) the module of homomorphisms**

We look at homomorphisms of modules over a ring before going to $\mathcal{O}$-modules.

Let M and N be modules over a ring A. The set of homomorphisms $M \to N$ is often denoted by $\operatorname{Hom}_A(M, N)$. It becomes an A-module with some fairly obvious laws of composition: If φ and ψ are homomorphisms and a is an element of A, then $\varphi+\psi$ and $a\varphi$ are defined by

homismodule (6.5.19) $$[\varphi+\psi](m) = \varphi(m) + \psi(m) \quad \text{and} \quad [a\varphi](m) = a\varphi(m)$$

Because φ is a module homomorphism, it is also true that $\varphi(m_1+m_2) = \varphi(m_1) + \varphi(m_2)$ and that $a\varphi(m) = \varphi(am)$.

NequalsHom **6.5.20. Lemma. (i)** *An A-module N is canonically isomorphic to $\operatorname{Hom}_A(A, N)$. The homomorphism $A \xrightarrow{\varphi} N$ that corresponds to an element x of N is multiplication by x: $\varphi(a) = ax$. The element of N that corresponds to a homomorphism $A \xrightarrow{\varphi} N$ is $x = \varphi(1)$.*

(ii) *$\operatorname{Hom}_A(A^k, N)$ is isomorphic to N^k, and $\operatorname{Hom}_A(A^k, A^\ell)$ is isomorphic to the module $A^{\ell \times k}$ of $k \times \ell$ A-matrices.* $\square$

6.5.21. Lemma. *The functor* Hom_A *is left exact and **contravariant** in the first variable. For any A-module N, an exact sequence* $M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_3 \to 0$ *of A-modules induces an exact sequence* homfinite

$$0 \to \operatorname{Hom}_A(M_3, N) \xrightarrow{\circ b} \operatorname{Hom}_A(M_2, N) \xrightarrow{\circ a} \operatorname{Hom}_A(M_1, N) \qquad \square$$

The functor Hom_A is **covariant** in the second variable, and it is also left exact in that variable.

6.5.22. Corollary. *If* M *and* N *are finite A-modules over a notherian ring A, then* $\operatorname{Hom}_A(M, N)$ *is a finite A-module.* homfinitetwo

proof. Let $A^k \to M \to 0$ be a surjective map. Then $\operatorname{Hom}_A(A^k, N) = N^k$. Lemma 6.5.20**(i)** gives us an injective map $\operatorname{Hom}_A(M, N) \to N^k$. So $\operatorname{Hom}_A(M, N)$ is isomorphic to a submodule of the finite module N^k. $\square$

The module Hom is compatible with localization:

6.5.23. Lemma. *Let* M *and* N *be modules over a noetherian domain A, and suppose that* M *is a finite module. Let S be a multiplicative system in A. The localization* $S^{-1}\operatorname{Hom}_A(M, N)$ *is canonically isomorphic to* $\operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$. localize-Hom

proof. Since $\operatorname{Hom}_A(A, M) = M$, it is true that $S^{-1}\operatorname{Hom}_A(A, M) = S^{-1}M = \operatorname{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}M)$ and that $S^{-1}\operatorname{Hom}_A(A^k, M) = (S^{-1}M)^k = \operatorname{Hom}_{S^{-1}A}(S^{-1}A^k, S^{-1}M)$.

We choose a presentation $A^\ell \to A^k \to M \to 0$ of the A-module M (**2.1.20**). Its localization, which is $(S^{-1}A)^\ell \to (S^{-1}A)^k \to S^{-1}M \to 0$, is a presentation of the $S^{-1}A$-module $S^{-1}M$. The sequence

$$0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(A^k, N) \to \operatorname{Hom}_A(A^\ell, N)$$

is exact, and so is its localization. So the lemma follows from the case that $M = A^k$. $\square$

The lemma shows that, when $\mathcal{M}$ and $\mathcal{N}$ are finite $\mathcal{O}$-modules on a variety X, there is an $\mathcal{O}$-module of homomorphisms $\mathcal{M} \to \mathcal{N}$. This $\mathcal{O}$-module may be denoted by $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$. When $U = \operatorname{Spec} A$ is an affine open set, $M = \mathcal{M}(U)$, and $N = \mathcal{N}(U)$, the module of sections of $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ on U is the A-module $\operatorname{Hom}_A(M, N)$.

The analogues of Lemma 6.5.20 and lemma 6.5.21 are true for $\underline{\operatorname{Hom}}$:

6.5.24. Corollary. (i) *An* $\mathcal{O}$*-module* $\mathcal{M}$ *on a smooth curve Y is isomorphic to* $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{O}, \mathcal{M})$. leftexco

(ii) *The functor* $\underline{\operatorname{Hom}}$ *is left exact and **contravariant** in the first variable, and it is left exact and **covariant** in the first variable.* $\square$

Notation. The notation Hom and $\underline{\operatorname{Hom}}$ is cumbersome as well as confusing. It seems permissible to drop the symbol Hom, and to write ${}_A(M, N)$ for $\operatorname{Hom}_A(M, N)$. Similarly, if $\mathcal{M}$ and $\mathcal{M}$ are $\mathcal{O}$-modules on a variety X, we will write ${}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ or ${}_X(\mathcal{M}, \mathcal{N})$ for $\underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$. dropHom

(6.5.25) the dual module dualmod

Let $\mathcal{M}$ be a locally free $\mathcal{O}$-modules on a variety X. The *dual module* $\mathcal{M}^*$ is the $\mathcal{O}$-module ${}_{\mathcal{O}}(\mathcal{M}, \mathcal{O})$ of homomorphisms $\mathcal{M} \to \mathcal{O}$. A section of $\mathcal{M}^*$ on an affine open set U is an $\mathcal{O}(U)$-module homomorphism $\mathcal{M}(U) \to \mathcal{O}(U)$.

The dualizing operation is contravariant. A homomorphism $\mathcal{M} \to \mathcal{N}$ of locally free $\mathcal{O}$-modules induces a homomorphism $\mathcal{M}^* \leftarrow \mathcal{N}^*$.

If $\mathcal{M}$ is a free module with basis $v_1, ..., v_k$, then $\mathcal{M}^*$ will also be free, with the dual basis $v_1^*, ..., v_k^*$. The dual basis is defined by

$$v_i^*(v_i) = 1 \quad \text{and} \quad v_i^*(v_j) = 0 \text{ if } i \neq j$$

So when $\mathcal{M}$ is locally free, $\mathcal{M}^*$ is also locally free.

The dual $\mathcal{O}^*$ of the structure sheaf $\mathcal{O}$ is $\mathcal{O}$. If $\mathcal{M}$ and $\mathcal{N}$ are locally free $\mathcal{O}$-modules, the dual of the tensor product $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ is isomorphic to the tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}^*$.

bidualm **6.5.26. Corollary.** **(i)** *Let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules, and let $\mathcal{M}^*$ be the dual of $\mathcal{M}$. The module ${}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ of homomorphisms is isomorphic to the tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}$.*
(ii) *A locally free $\mathcal{O}$-module $\mathcal{M}$ is canonically isomorphic to its bidual: $(\mathcal{M}^*)^* \approx \mathcal{M}$.*
(iii) *If $\mathcal{M}$ and $\mathcal{M}'$ are locally free $\mathcal{O}$-modules, the tensor product $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{M}'^*$ is isomorphic to $(\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}')^*$.*

*proof.***(i)** We identify $\mathcal{N}$ with the module ${}_{\mathcal{O}}(\mathcal{O},\mathcal{N})$ (6.5.24). Given sections φ of $\mathcal{M}^* = {}_{\mathcal{O}}(\mathcal{M},\mathcal{O})$ and γ of $\mathcal{N} =_{\mathcal{O}} (\mathcal{O},\mathcal{N})$, the composition $\gamma\varphi$ is a map $\mathcal{M} \to \mathcal{N}$, a section of ${}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. This composition is bilinear, so it defines a map $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N} \to {}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$. To show that this map is an isomorphism is a local problem, so we may assume that $Y = \operatorname{Spec} A$ is affine and that $\mathcal{M}$ and $\mathcal{N}$ are free modules of ranks k an ℓ, respectively. Then both $\mathcal{M}^* \otimes_{\mathcal{O}} \mathcal{N}$ and ${}_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ are the modules of $k\times\ell$ A-matrices. □

dualseq **6.5.27. Proposition.** *Let X be a variety.*

(i) *Let $\mathcal{P} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P}$ be homomorphisms of $\mathcal{O}$-modules whose composition gf is the identity map on $\mathcal{P}$. So f is injective and g is surjective. Then $\mathcal{N}$ is the direct sum of the image $\mathcal{P}$ of f and the kernel $\mathcal{K}$ of g: $\mathcal{N} \approx \mathcal{P} \oplus \mathcal{K}$.*
(ii) *Let $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ be an exact sequence of $\mathcal{O}$-modules. If $\mathcal{P}$ is locally free, the dual modules form an exact sequence $0 \to \mathcal{P}^* \to \mathcal{N}^* \to \mathcal{M}^* \to 0$.*

proof. **(i)** This follows from the analogous statement about modules over a ring.

(ii) The sequence $0 \to \mathcal{P}^* \to \mathcal{N}^* \to \mathcal{M}^*$ is exact whether or not the modules are locally free (6.5.21) **(ii)**. The zero on the right comes from the fact that, when $\mathcal{P}$ is locally free, it is free on some affine covering. The given sequence splits locally, and therefore the map $\mathcal{N}^* \to \mathcal{M}^*$ is locally surjective. □

6.5.28. proof of Proposition 6.5.5.

With notation as in the statement of the proposition, we suppose given the modules M_0, M_1 and an isomorphism $M_0[u^{-1}] \to M_1[v^{-1}]$, and we are to show that this data comes from an $\mathcal{O}$-module $\mathcal{M}$. Proposition 6.5.12 shows that M_i defines $\mathcal{O}$-modules $\mathcal{M}_i$ on $\mathbb{U}^i$ for $i = 0, 1$, and the restrictions of $\mathcal{M}_0$ and $\mathcal{M}_1$ to $\mathbb{U}^{01}$ are isomorphic. Let's denote all of these modules by $\mathcal{M}$. Then $\mathcal{M}$ is defined on any open set that is contained, either in $\mathbb{U}^0$, or in $\mathbb{U}^1$.

Let V be an arbitrary open set V, and let $V^i = V \cap \mathbb{U}^i$, $i = 0, 1, 01$. We define $\mathcal{M}(V)$ to be the kernel of the map $[\mathcal{M}(V^0)\times\mathcal{M}(V^1)] \to \mathcal{M}(V^{01})$. With this definition, $\mathcal{M}$ becomes a functor. We must verify the sheaf property, and the notation can get confusing. We suppose given an open covering $\{V^\nu\}$ of V, and to avoid confusion with V^0 and V^1, we denote $\{V^\nu\}$ by $\mathbf{W}_0$, and $\{V^\nu \cap V^\mu\}$ by $\mathbf{W}_1$, so that the corresponding covering diagram is $V \leftarrow \mathbf{W}_0 \leftleftarrows \mathbf{W}_1$. We form a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \longrightarrow & \mathcal{M}(\mathbf{W}_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V^0)\times\mathcal{M}(V^1) & \longrightarrow & * & \longrightarrow & * \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M}(V^{01}) & \longrightarrow & * & \longrightarrow & *
\end{array}$$

in which the first asterisk in the second row stands for $\mathcal{M}(\mathbf{W}_0 \cap \mathbb{U}^0)\times\mathcal{M}(\mathbf{W}_0 \cap \mathbb{U}^1)$, etc. The columns are exact by our definition of $\mathcal{M}$, and the second and third rows are exact because the open sets involved are contained in $\mathbb{U}^0$ or $\mathbb{U}^1$. Since kernel is a left exact operation, the top row is exact too. This is the sheaf property. □

Section 6.6 Direct Image

directim-age

Let $Y \xrightarrow{f} X$ be a morphism of varieties, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The *direct image* $f_*\mathcal{N}$ is the $\mathcal{O}_X$-module defined as follows: Its sections on an affine open subset U of X are the sections of $\mathcal{N}$ on the inverse image $f^{-1}U$ in Y:

$$[f_*\mathcal{N}](U) = \mathcal{N}(f^{-1}U)$$

For example, the direct image $f_*\mathcal{O}_Y$ of the structure sheaf $\mathcal{O}_Y$ is the co_X-module defined by $[f_*\mathcal{O}_Y](U) = \mathcal{O}_Y(f^{-1}U)$.

A morphism is a continuous map, so when U is open in X, its inverse image $f^{-1}(U)$ will be open in Y. However, if U is an affine open subset of X, $f^{-1}(U)$ needn't be affine. Since $\mathcal{O}$-modules are defined in terms of affine open subsets, one must be careful.

The direct image generalizes restriction of scalars in modules over rings. Recall (**2.1.31**) that, if $A \xrightarrow{\varphi} B$ is an algebra homomorphism and ${}_BN$ is a B-module, one can *restrict scalars* to make N into an A-module. Scalar multiplication by an element a of A on the restricted module ${}_AN$ is defined to be scalar multiplication by its image $\varphi(a)$. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $Y \xrightarrow{f} X$ be the morphism of affine varieties defined by an algebra homomorphism $A \xrightarrow{\varphi} B$. If an $\mathcal{O}_Y$-module $\mathcal{N}$ is determined by the B-module ${}_BN$, the direct image $f_*\mathcal{N}$ is the $\mathcal{O}_X$-module determined by the A-module ${}_AN$.

6.6.1. Examples. dirimex

(i) Let p be the point $\operatorname{Spec}\mathbb{C}$, let $Y = \mathbb{P}^1$, and let π be the morphism $Y \to p$. By definition, $[\pi_*\mathcal{M}](p) = \mathcal{M}(X)$. A module on p is equivalent with the vector space of its global sections (6.3.3). So in this situation, $\pi_*\mathcal{M}$ is simply the space of global sections of $\mathcal{M}$. In particular, $\pi_*\mathcal{O}_Y = \mathcal{O}_p = \mathbb{C}$.

(ii) When the fibres of a morphism $Y \xrightarrow{f} X$ are projetive varieties, regular functions on the fibres will be constant functions, and because of this the direct image $f_*\mathcal{O}_Y$ or the structure sheaf on Y is likely to be the structure sheaf $\mathcal{O}_X$ on X.

For instance, let $X = \operatorname{Spec}\mathbb{C}[x]$, and let Y be the family of plane curves defined by the equation $y_0^3 + y_1^3 + y_2^3 + ty_0y_1y_2 = 0$ in $\mathbb{P}^2 \times X$. There is a morphism $Y \xrightarrow{f} X$, whose fibre over a point $x = a$ is the cubic curve obtained by substituting $x = a$ into the equation. The direct image $f_*\mathcal{O}_Y$ of the structure sheaf on Y is simply the structure sheaf $\mathcal{O}_X$ on X. It doesn't seem worthwhile to prove this here. But there will be $\mathcal{O}_Y$-modules whose direct images are more interesting.

(iii) Let $Y \xrightarrow{f} X$ be an integral morphism, let $U = \operatorname{Spec} A$ be an affine open subset of X, and let $V = f^{-1}U$. Then V is affine, $V = \operatorname{Spec} B$, and B is a finite A-module. If $\mathcal{N}$ is an $\mathcal{O}_Y$-module, $N = \mathcal{N}(V)$ is a B-module. Then $[\pi_*\mathcal{N}](U)$ is the A-module ${}_AN$ obtained from ${}_BN$ by restriction of scalars.

6.6.2. Lemma. *Let $Y \xrightarrow{f} X$ be a morphism of varieties. The direct image $f_*\mathcal{N}$ of an $\mathcal{O}_Y$-module $\mathcal{N}$ is an $\mathcal{O}_X$-module. Moreover, for all open subsets U of X, not only for affine open subsets,* *fstarcoh*

$$[f_*\mathcal{N}](U) = \mathcal{N}(f^{-1}U)$$

proof. Let $U' \to U$ be an inclusion of affine open subsets of X, and let $V = f^{-1}U$ and $V' = f^{-1}U'$. These inverse images are open subsets of Y, but they aren't necessarily affine open subsets. Nevertheless, the inclusion $V' \to V$ gives us a homomorphism $\mathcal{N}(V) \to \mathcal{N}(V')$, and therefore a homomorphism $f_*\mathcal{N}(U) \to f_*\mathcal{N}(U')$. So $f_*\mathcal{N}$ is a functor whose $\mathcal{O}_X$-module structure is explained as follows: Composition with f defines a homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$, and $\mathcal{N}(V)$ is an $\mathcal{O}_Y(V)$-module. Restriction of scalars to $\mathcal{O}_X(U)$ makes $[f_*\mathcal{N}](U) = \mathcal{N}(V)$ into an $\mathcal{O}_X(U)$-module.

To show that $f_*\mathcal{N}$ is an $\mathcal{O}_X$-module, we must show that if s is a nonzero element of $\mathcal{O}_X(U)$, then $[f_*\mathcal{N}](U_s)$ is obtained by localizing $[f_*\mathcal{N}](U)$. Let s' be the image of s in $\mathcal{O}_Y(V)$. Scalar multiplication by s on $[f_*\mathcal{N}](U)$ is given by restriction of scalars, so it is the same as scalar multiplication by s' on $\mathcal{N}(V)$. If $s' \neq 0$, the localization $V_{s'}$ is the inverse image of U_s. So $[f_*\mathcal{N}](U_s) = \mathcal{N}(V_{s'})$. The coherence property (**6.4.14**) tells us that $\mathcal{N}(V_{s'}) = \mathcal{N}(V)_{s'}$. Then $[f_*\mathcal{N}](U_s) = \mathcal{N}(V_{s'}) = \mathcal{N}(V)_{s'} = [f_*\mathcal{N}(U)]_s$. If $s' = 0$, then $\mathcal{N}(V)_{s'} = 0$. In this case, because scalar multiplication is defined by restricting scalars, s annihilates $[f_*\mathcal{N}](U)$, and therefore $[f_*\mathcal{N}](U)_s$ is zero too. $\square$

The concept of direct image is important for any morphism f, but for us the most important cases will be that f is the inclusion of a closed subvariety or an open subvariety. We discuss those special cases now.

dirimlim **6.6.3. Lemma.** *Direct images are compatible with limits: If $\mathcal{M}_\bullet$ is a directed set of $\mathcal{O}$-modules, then* $\varinjlim (f_*\mathcal{M}_\bullet) \approx f_*(\varinjlim \mathcal{M}_\bullet)$. □

extbyzero **(6.6.4) extension by zero**

When $Y \xrightarrow{i} X$ is the inclusion of a **closed** subvariety into a variety X and $\mathcal{N}$ is an $\mathcal{O}_Y$-module, the direct image $i_*\mathcal{N}$ is also called the *extension* of $\mathcal{N}$ *by zero.*. If U is an open subset of X then, because i is an inclusion map, $i^{-1}U = U \cap Y$. Therefore

$$[i_*\mathcal{N}](U) = \mathcal{N}(U \cap Y)$$

The term "extension by zero" refers to the fact that, when an open set U of X doesn't meet Y, the intersection $U \cap Y$ is empty, and the module of sections of $[i_*\mathcal{N}](U)$ is zero. So $i_*\mathcal{N}$ is zero outside of the closed set Y.

extfrom-point **6.6.5. Examples.**

(i) Let $p \xrightarrow{i} X$ be the inclusion of a point into a variety. When we view the residue field $k(p)$ as a module on the point p, its extension by zero is the residue field module κ_p.

(ii) Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety, and let $\mathcal{I}$ be the ideal of Y in $\mathcal{O}_Y$. The extension by zero $i_*\mathcal{O}_Y$ of the structure sheaf on Y fits into an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

So $i_*\mathcal{O}_Y$ is isomorphic to the quotient module $\mathcal{O}_X/\mathcal{I}$. □

izero **6.6.6. Proposition.** *Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X, and let $\mathcal{I}$ be the ideal of Y. Let $\mathscr{M}$ denote the subcategory of the category of $\mathcal{O}_X$-modules that are annihilated by $\mathcal{I}$. Extension by zero defines an equivalence of categories*

$$\mathcal{O}_Y - modules \xrightarrow{i_*} \mathscr{M}$$

proof. Let U be an affine open subset of X. The intersection $U \cap Y = V$ is a closed subvariety of U, and $[i_*\mathcal{N}](U) = \mathcal{N}(V)$. Let α be a section of $i_*\mathcal{N}(U) = \mathcal{N}(V)$. Scalar multiplication on $i_*\mathcal{N}$ is defined by restriction of scalars from $\mathcal{O}_X$ to $\mathcal{O}_Y$. If f is a section of $\mathcal{O}_X$ on U and $\overline{f}$ is its restriction to V, then $f\alpha = \overline{f}\alpha$. If f is in $\mathcal{I}(U)$, then $\overline{f} = 0$ and therefore $f\alpha = \overline{f}\alpha = 0$. So the extension by zero of an $\mathcal{O}_Y$-module is annihilated by $\mathcal{I}$. The direct image $i_*\mathcal{N}$ is an object of $\mathscr{M}$.

To complete the proof, we construct an inverse to the direct image. Starting with an $\mathcal{O}_X$-module $\mathcal{M}$ that is annihilated by $\mathcal{I}$, we construct an $\mathcal{O}_Y$-module $\mathcal{N}$ such that $i_*\mathcal{N}$ is isomorphic to $\mathcal{M}$.

Let V be an open subset of Y. The topology on Y is induced from the topology on X, so $V = X_1 \cap Y$ for some open subset X_1 of X. We try to set $\mathcal{N}(V) = \mathcal{M}(X_1)$. To show that this is well-defined, we show that if X_2 is another open subset of X and if $V = X_2 \cap Y$, then $\mathcal{M}(X_2)$ is isomorphic to $\mathcal{M}(X_1)$. Let $X_3 = X_1 \cap X_2$. Then it is also true that $V = X_3 \cap Y$. Since $X_3 \subset X_1$, we have a map $\mathcal{M}(X_1) \to \mathcal{M}(X_3)$, and it suffices to show that this map is an isomorphism. The same reasoning will give us an isomorphism $\mathcal{M}(X_2) \to \mathcal{M}(X_3)$.

The complement $U = X_1 - V$ of V in X_1 is an open subset of X_1 and of X, and $U \cap Y = \emptyset$. We cover U by a set $\{U^i\}$ of affine open sets. Then X_1 is covered by the open sets $\{U^i\}$ together with X_3. The restriction of $\mathcal{I}$ to each of the sets U^i is the unit ideal, and since $\mathcal{I}$ annihilates $\mathcal{M}$, $\mathcal{M}(U^i) = 0$. The sheaf property shows that $\mathcal{M}(X_1)$ is isomorphic to $\mathcal{M}(X_3)$.

The rest of the proof, checking localization and verifying that $\mathcal{N}$ is determined up to isomorphism, is boring. □

in-cludeopen **(6.6.7) inclusion of an open set**

Let $Y \xrightarrow{j} X$ be the inclusion of an **open** subvariety Y into a variety X.

Before going to the direct image, we introduce a trivial operation, *restriction* from X to Y. Since open subsets of Y are also open subsets of X, we can *restrict* an $\mathcal{O}$-module $\mathcal{M}$ from X to Y. By definition, the sections of the restricted module on a subset U of Y are simply the elements of $\mathcal{M}(U)$. For example, the restriction of the structure sheaf $\mathcal{O}_X$ to the open set Y is the structure sheaf $\mathcal{O}_Y$ on Y. We use subscript notation for restriction, writing $\mathcal{M}_Y$ for the restriction of an $\mathcal{O}_X$-module $\mathcal{M}$ to Y, and denoting the given module $\mathcal{M}$ by $\mathcal{M}_X$ when that seems advisable for clarity. If U is an open subset of Y,

(6.6.8) $$\mathcal{M}_Y(U) = \mathcal{M}_X(U)$$ restr-toopen

Now the direct image: Let $Y \xrightarrow{j} X$ be the inclusion of an open subvariety Y, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The inverse image of an open subset U of X is the intersection $Y \cap U$. So the direct image of $\mathcal{N}$ is, by definition,

$$[j_*\mathcal{N}](U) = \mathcal{N}(Y \cap U)$$

For example, $[j_*\mathcal{O}_Y](U)$ is the algebra of rational functions on X that are regular on $Y \cap U$.

6.6.9. Example. Let $X_s \xrightarrow{j} X$ be the inclusion of a localization into an affine variety $X = \operatorname{Spec} A$. Modules on X correspond to their global sections, which are A-modules. Similarly, modules on X_s correspond to A_s-modules. When wef restrict the $\mathcal{O}_X$-module $\mathcal{M}_X$ that corresponds to an A-module M to the open set X_s, we obtain the $\mathcal{O}_{X_s}$-module $\mathcal{M}_{X_s}$ that corresponds to the A_s-module M_s. The module M_s is also the module of global sections of $j_*\mathcal{M}_{X_s}$ on X: exam-pledirec-timage

$$[j_*\mathcal{M}_{X_s}](X) = \mathcal{M}_{X_s}(X_s) = M_s$$

Here, the localization M_s is made into an A-module by restriction of scalars. $\square$

6.6.10. Proposition. *Let $Y \xrightarrow{j} X$ be the inclusion of an open subvariety Y into a variety X.* *fstarexact*

(i) *The restriction $\mathcal{O}_X$-modules $\to \mathcal{O}_Y$-modules is an exact operation.*

(ii) *If Y is an **affine** open subvariety of X, the direct image functor j_* is exact.*

(iii) *Let $\mathcal{M} = \mathcal{M}_X$ be an $\mathcal{O}_X$-module. There is a canonical homomorphism $\mathcal{M}_X \to j_*[\mathcal{M}_Y]$.*

(iv) *Let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. The restriction of the direct image $j_*\mathcal{N}$ to Y is equal to $\mathcal{N}$, i.e., $[j_*\mathcal{N}]_Y = \mathcal{N}$.*

proof. **(ii)** Let U be an affine open subset of X, and let $\mathcal{M} \to \mathcal{N} \to \mathcal{P}$ be an exact sequence of $\mathcal{O}_Y$-modules. The sequence $j_*\mathcal{M}(U) \to j_*\mathcal{N}(U) \to j_*\mathcal{P}(U)$ is the same as the sequence $\mathcal{M}(U \cap Y) \to \mathcal{N}(U \cap Y) \to \mathcal{P}(U \cap Y)$, though the scalars have changed. Since U and Y are affine, $U \cap Y$ is affine. By definition of exactness, this sequence is exact.

(iii) Let U be open in X. Then $j_*\mathcal{M}_Y(U) = \mathcal{M}(U \cap Y)$. Since $U \cap Y \subset U$, $\mathcal{M}(U)$ maps to $\mathcal{M}(U \cap Y)$.

(iv) An open subset V of Y is also open in X, and $[j_*\mathcal{N}]_Y(V) = \mathcal{N}(V \cap Y) = \mathcal{N}(V)$. $\square$

6.6.11. Example. Let $X = \mathbb{P}^n$ and let j denote the inclusion $\mathbb{U}^0 \subset X$ of the standard affine open subset into X. The direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ is the algebra of rational functions that are allowed to have poles on the hyperplane at infinity. The inverse image of an open subset W of X is its intersection with $\mathbb{U}^0$: $j^{-1}W = W \cap \mathbb{U}^0$. So the sections of the direct image $j_*\mathcal{O}_{\mathbb{U}^0}$ on an open subset W of X are the regular functions on $W \cap \mathbb{U}^0$: dirim-standaffine

$$[j_*\mathcal{O}_{\mathbb{U}^0}](W) \;=\; \mathcal{O}_{\mathbb{U}^0}(W \cap \mathbb{U}^0) \;=\; \mathcal{O}_X(W \cap \mathbb{U}^0)$$

Say that we write a rational function α on X as a fraction g/h of relatively prime polynomials. Then α is a section of $\mathcal{O}_X$ on W if h doesn't vanish at any point of W, and α is a section of $[j_*\mathcal{O}_{\mathbb{U}^0}]$ pn W if h doesn't vanish on $W \cap \mathbb{U}^0$. Arbitrary powers of x_0 can appear in the denominator of a section of $j_*\mathcal{O}_{\mathbb{U}^0}$. $\square$

Section 6.7 Support

annand-supp

Annihilators. Let M be a module over a ring A. The *annihilator I of an element m* of M is the set of elements α of A such that $\alpha m = 0$. It is an ideal of A that is often denoted by $\operatorname{ann}(m)$.

The *annihilator of an A-module M* is the set of elements of A such that $aM = 0$. It is an ideal too.

Support. Let A be a finite-type domain and let $X = \operatorname{Spec} A$. The *support* of a finite A-module M is the locus $C = V(I)$ of zeros of its annihilator I in X, the set of points p of X such that $I \subset \mathfrak{m}_p$ (2.4.2). The support of a finite module will be a closed subset of X.

localize-support **6.7.1. Lemma.** *Let $X = \operatorname{Spec} A$ be an affine variety, let I be the annihilator of an element m of an A-module M, and let s be a nonzero element of A. The annihilator of the image of m in the localized module M_s is the localized ideal I_s. If M is a finite module with support C, the support of M_s is the intersection $C_s = C \cap X_s$.* □

This lemma allows us to extend the concepts of annihilator and support to finite $\mathcal{O}$-modules on a variety X.

When $\mathcal{I}$ is an ideal of $\mathcal{O}$, we will denote by $V(\mathcal{I})$ the closed set of points p such that $\mathcal{I} \subset \mathfrak{m}_p$ — the set of points at which all elements of $\mathcal{I}$ vanish.

Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a variety X, and let $\mathcal{I}$ be its annihilator. The *support* of $\mathcal{M}$ is the closed subset $V(\mathcal{I})$ of points such that $\mathcal{I} \subset \mathfrak{m}_p$. For example, the support of the residue field module κ_p is the point p. The support of the maximal ideal $\mathfrak{m}_p$ at p is the whole variety X.

support-dimzero

(6.7.2) $\mathcal{O}$-modules with support of dimension zero

suppfinite **6.7.3. Proposition.** *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a variety X.*
(i) *Suppose that the support of $\mathcal{M}$ is a single point p, let $M = \mathcal{M}(X)$, and let U be an affine open subset of X. If U contains p, then $\mathcal{M}(U) = M$, and if U doesn't contain p, then $\mathcal{M}(U) = 0$.*
(ii) *(Chinese Remainder Theorem) If the support of $\mathcal{M}$ is a finite set $\{p_1, ..., p_k\}$, then $\mathcal{M}$ is the direct sum $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ of $\mathcal{O}$-modules supported at the points p_i.*

proof. **(i)** Let $\mathcal{I}$ be the annihilator of $\mathcal{M}$. The locus $V(\mathcal{I})$ is the support p. If p isn't contained in U, then when we restrict $\mathcal{M}$ to U, we obtain an $\mathcal{O}_U$-module whose support is empty. Therefore $\mathcal{I}(U)$ is the unit ideal, and the restriction of $\mathcal{M}$ to U is the zero module.

Next, suppose that p is contained in U, and let V denote the complement of p in X. We cover X by a set $\{U^i\}$ of affine open sets with $U = U^1$, and such that $U^i \subset V$ if $i > 1$. By what has been shown, $\mathcal{M}(U^i) = 0$ if $i > 0$ and $\mathcal{M}(U^{ij}) = 0$ if $j \neq i$. The sheaf axiom for this covering shows that $\mathcal{M}(X) \approx \mathcal{M}(U)$. □

Note. If i denotes the inclusion of a point p into a variety X, it is natural to suppose that an $\mathcal{O}$-module $\mathcal{M}$ supported at p will be the extension by zero of a module on the point p — a vector space. However, this won't be true unless cm is annihilated by the maximal ideal $\mathfrak{m}_p$. □

Section 6.8 Twisting

twisting-modules The twisting modules that we define here are among the most important modules on projective space.

As before, a homogeneous fraction of degree n in $x_0, ..., x_n$ is a fraction g/h of homogeneous polynomials with $\deg\, g - \deg\, h = d$. When g and h are relatively prime, the fraction g/h is regular on an open subset V of $\mathbb{P}^n_x$ if h isn't zero at any point of V.

The definition of the twisting module $\mathcal{O}(n)$ is this: Its sections of $\mathcal{O}(n)$ on an open subset V of $\mathbb{P}^n$ are the homogeneous fractions of degree n that are regular on V. In particular, $\mathcal{O}(0) = \mathcal{O}$.

Odismodule **6.8.1. Proposition.**
(i) *Let V be an affine open subset of $\mathbb{P}^n$ that is contained in the standard affine open set $\mathbb{U}^0$. The sections of the twisting module $\mathcal{O}(n)$ on V form a free module of rank one with basis x_0^n, over the coordinate algebra $\mathcal{O}(V)$.*
(ii) *The twisting module $\mathcal{O}(n)$ is an $\mathcal{O}$-module.*

proof. **(i)** Let V be an open set contained in $\mathbb{U}^0$, and let α be a section of $\mathcal{O}(n)$ on V. Then $f = \alpha x_0^{-n}$ has degree zero. It is a rational function. Since $V \subset \mathbb{U}^0$, x_0 doesn't vanish at any point of V. Since α is regular on V, f is a regular function on V, and $\alpha = f x_0^n$.

(ii) It is clear that $\mathcal{O}(n)$ is a contravariant functor. We verify compatibility with localization. Let $V = \operatorname{Spec} A$ be an affine open subset of X and let s be a nonzero element of A. We must show that $[\mathcal{O}(n)](V_s)$ is the localization of $[\mathcal{O}(n)](V)$. We already know that $[\mathcal{O}(n)](V)$ is a subset of $[\mathcal{O}(n)](V_s)$. What has to be shown is that if β is a section of $\mathcal{O}(n)$ on V_s, then $s^k\beta$ is a section on V when k is sufficiently large.

We cover V by the affine open sets $V^i = V \cap \mathbb{U}^i$. To show that $s^k\beta$ is a section on V, it suffices to show that it is a section on $V \cap \mathbb{U}^i$ for every i. This follows from the sheaf property. We apply **(i)** to the open subset V^0_s of V^0. Since V^0_s is contained in $\mathbb{U}^0$, β can be written uniquely in the form fx_0^n, where f is a rational function that is regular on V^0_s and n is an integer. The coherence property fo $\mathcal{O}$-modules shows that s^kf is a regular function on V^0 when k is large, and then $s^k\alpha = s^kfx_0^n$ is a section of $\mathcal{O}(n)$ on V^0. The analogous statement is true for every index i. □

As part **(i)** of the proposition shows, $\mathcal{O}(n)$ is quite similar to the structure sheaf. However, $\mathcal{O}(n)$ is only *locally* free. Its sections on the standard open set $\mathbb{U}^1$ form a free $\mathcal{O}(\mathbb{U}^1)$-module with basis x_1^n. That basis is related to the basis x_0^n on $\mathbb{U}^0$ by the factor $(x_0/x_1)^n$, a rational function that isn't invertible, either on on $\mathbb{U}^0$ or on $\mathbb{U}^1$.

6.8.2. Proposition. *When $d \geq 0$, the global sections of the twisting module $\mathcal{O}(n)$ on $\mathbb{P}^n$ are the homogeneous polynomials of degree n. When $n < 0$, the only global section of $\mathcal{O}(n)$ is zero.* *sectionsofOn*

proof. A nonzero global section u of $\mathcal{O}(n)$ will restrict to a section on the standard affine open set $\mathbb{U}^0$. Since elements of $\mathcal{O}(\mathbb{U}^0)$ are homogeneous fractions of degree zero whose denominators are powers of x_0, and since $[\mathcal{O}(n)](\mathbb{U}^0)$ is a free module over $\mathcal{O}(\mathbb{U}^0)$ with basis x_0^d, we will have $u = g/x_0^k$ for some some homogeneous polynomial g and some k. Similarly, restriction to $\mathbb{U}^1$ shows that u has the form g_1/x_1^ℓ. It follows that $k = \ell = 0$ and that $u = g$. Since u has degree n, so does g. □

6.8.3. Examples. prodhomfr

(i) The product uv of homogeneous fractions of degrees r and s is a homogeneous fraction of degree $r+s$, and if u and v are regular on an open set V, so is their product uv. So multiplication defines a homomorphism of $\mathcal{O}$-modules

(6.8.4) $$\mathcal{O}(r) \times \mathcal{O}(s) \to \mathcal{O}(r+s)$$ oros

(ii) Multiplication by a homogeneous polynomial f of degree n defines an injective homomorphism

(6.8.5) $$\mathcal{O}(k) \xrightarrow{f} \mathcal{O}(k+n).$$ multbyfone

When $k = -n$, this becomes a homomorphism $\mathcal{O}(-n) \xrightarrow{f} \mathcal{O}$. □

The twisting modules $\mathcal{O}(n)$ have a second interpretation. They are isomorphic to the modules that we denote by $\mathcal{O}(nH)$, of rational functions on projective space $\mathbb{P}^d$ with poles of order at most n on the hyperplane $H : \{x_0 = 0\}$ at infinity.

By definition, the sections of $\mathcal{O}(nH)$ on an open set V are the rational functions f such that x_0^nf is a section of $\mathcal{O}(n)$ on V. Thus multiplication by x_0^n defines an isomorphism

(6.8.6) $$\mathcal{O}(nH) \xrightarrow{x_0^n} \mathcal{O}(n)$$ OnOnH

If f is a section of $\mathcal{O}(nH)$ on an open set V, and if we write f as a homogeneous fraction g/h of degree zero, with g, h relatively prime, the denominator h may have x_0^k, with $k \leq n$, as factor. The other factors of h cannot vanish anywhere on V. If $f = g/h$ is a *global* section of $\mathcal{O}(nH)$, then $h = cx_0^k$, with $c \in \mathbb{C}$ and $k \leq n$. A global section of $\mathcal{O}(nH)$ can be represented as a homogeneous fraction g/x_0^k of degree zero.

Since x_0 doesn't vanish at any point of the standard affine open set $\mathbb{U}^0$, the sections of $\mathcal{O}(nH)$ on an open subset V of $\mathbb{U}^0$ are simply the regular functions on V. So the restrictions of $\mathcal{O}(nH)$ and of $\mathcal{O}$ to $\mathbb{U}^0$ are equal. Using the subscript notation (**6.6.7**) for restriction to an open set,

(6.8.7) $$\mathcal{O}(nH)_{\mathbb{U}^0} = \mathcal{O}_{\mathbb{U}^0}$$ jstarOH

Let V be an open subset of another standard affine open set, say of $\mathbb{U}^1$. The ideal of $H \cap \mathbb{U}^1$ in $\mathbb{U}^1$ is the principal ideal generated by $v_0 = x_0/x_1$, and v_0 generates the ideal of $H \cap V$ in V too. If f is a rational function, then because x_1 doesn't vanish on $\mathbb{U}^1$, the function fv_0^n will be regular on V if and only if the homogeneous fraction fx_0^n is regular there. So f will be a section of $\mathcal{O}(nH)$ on V if and only if fv_0^n is a regular function. Because v_0 generates the ideal of H in V, we say that such a function f has a pole of order at most n on H.

The isomorphic $\mathcal{O}$-modules $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ are interchangeable. The twisting module $\mathcal{O}(n)$ is often better because its definition is independent of coordinates. On the other hand, $\mathcal{O}(nH)$ can be convenient because its restriction to $\mathbb{U}^0$ is the structure sheaf $\mathcal{O}_{\mathbb{U}^0}$.

curveideal **6.8.8. Proposition.** *Let Y be a hypersurface of degree n in $\mathbb{P}^n$, the zero locus of an irreducible homogeneous polynomial f of degree n, let $\mathcal{I}$ be the ideal of Y, and let $\mathcal{O}(-n)$ be the twisting module on X. Multiplication by f defines an isomorphism $\mathcal{O}(-n) \xrightarrow{f} \mathcal{I}$.*

proof. We may choose coordinates so that f isn't isn't divisible by any of the coordinate veriables x_i.

If α is a section of $\mathcal{O}(-n)$ on an open set V, then $f\alpha$ will be a regular function on V that vanishes on $Y \cap V$. Therefore the image of the multiplication map $\mathcal{O}(-n) \xrightarrow{f} \mathcal{O}$ is contained in $\mathcal{I}$. This map is injective because $\mathbb{C}[x_0, ..., x_n]$ is a domain. To show that the multiplication map is an isomorphism, it suffices to show that its restrictions to the standard affine open sets $\mathbb{U}^i$ are surjective. (6.4.5). We work with $\mathbb{U}^0$, as usual.

Because x_0 desn't divide f, $Y \cap \mathbb{U}^0$ will be a nonempty and therefore dense open subset of Y. The sections of $\mathcal{O}$ on $\mathbb{U}^0$ are the homogeneous fractions g/x_0^k of degree zero. Such a fraction is a section of $\mathcal{I}$ on $\mathbb{U}^0$ if and only if g vanishes on $Y \cap \mathbb{U}^0$. If so, then since $Y \cap \mathbb{U}^0$ is dense in Y and since the zero set of g is closed, g will vanish on Y, and therefore it will be divisible by f: $g = fq$. The sections of $\mathcal{I}$ on $\mathbb{U}^0$ have the form fq/x_0^k. They are in the image of the map $\mathcal{O}(-n) \to \mathcal{I}$. □

The proposition has an interesting corollary:

idealsisom **6.8.9. Corollary.** *When regarded as $\mathcal{O}$-modules, the ideals of all hypersurfaces of degree n are isomorphic.*

twistmodule **(6.8.10) twisting a module**

deftwistm **6.8.11. Definition** Let $\mathcal{M}$ be an $\mathcal{O}$-module on projective space $\mathbb{P}^d$, and let $\mathcal{O}(n)$ be the twisting module. The nth *twist* of $\mathcal{M}$ is defined to be the tensor product $\mathcal{M}(n) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(n)$. Similarly, $\mathcal{M}(nH) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(nH)$. Twisting is a functor on $\mathcal{O}$-modules.

If X is a closed subvariety of $\mathbb{P}^d$ and $\mathcal{M}$ is an $\mathcal{O}_X$-module, $\mathcal{M}(n)$ and $\mathcal{M}(nH)$ are obtained by twisting the extension of $\mathcal{M}$ by zero. (See the equivalence of categories (6.6.6)).

Since x_0^n is a basis of $\mathcal{O}(n)$ on $\mathbb{U}^0$, a section μ of $\mathcal{M}(n)$ on an open subset V of $\mathbb{U}^0$ can be written in the form $\mu = m \otimes gx_0^n$, where g is a regular function on V and m is a section of $\mathcal{M}$ on V (6.8.1). The function g can be moved over to m, so μ can also be written in the form $\mu = m \otimes x_0^n$. This expression for μ is unique.

The modules $\mathcal{O}(n)$ and $\mathcal{O}(nH)$ form directed sets that are related by a diagram

Oninclusions (6.8.12)
$$\begin{array}{ccccccc}
\mathcal{O} & \xrightarrow{\subset} & \mathcal{O}(H) & \xrightarrow{\subset} & \mathcal{O}(2H) & \xrightarrow{\subset} & \cdots \\
\Big\| & & \Big\downarrow{\scriptstyle x_0} & & \Big\downarrow{\scriptstyle x_0^2} & & \\
\mathcal{O} & \xrightarrow{x_0} & \mathcal{O}(1) & \xrightarrow{x_0} & \mathcal{O}(2) & \longrightarrow & \cdots
\end{array}$$

In this diagram, the vertical arrows are bijections and the horizontal arrows are injections. The limit of the upper directed set is the module whose sections on an open set V are rational functions that can have arbitrary poles on $H \cap V$, and are otherwise regular. This is also the module $j_*\mathcal{O}_{\mathbb{U}^0}$, where j denotes the inclusion of the standard affine open set $\mathbb{U}^0$ into X (see (6.6.10) **(iii)**):

limequalsjstar (6.8.13)
$$\varinjlim \mathcal{O}(nH) = j_*\mathcal{O}_{\mathbb{U}^0}$$

The next diagram is obtained by tensoring (6.8.12) with $\mathcal{M}$.

(6.8.14) $$\begin{array}{ccccccc} \mathcal{M} & \longrightarrow & \mathcal{M}(H) & \longrightarrow & \mathcal{M}(2H) & \longrightarrow & \cdots \\ \| & & \downarrow{\scriptstyle x_0} & & \downarrow{\scriptstyle x_0^2} & & \\ \mathcal{M} & \xrightarrow{1\otimes x_0} & \mathcal{M}(1) & \xrightarrow{1\otimes x_0} & \mathcal{M}(2) & \longrightarrow & \cdots \end{array}$$ Mnmaps

The vertical maps are bijective, but because $\mathcal{M}$ may have torsion, the horizontal maps needn't be injective.

Since tensor products are compatible with limits,

(6.8.15) $$\varinjlim \mathcal{M}(nH) = \varinjlim \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(nH) = \mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0} \approx j_*\mathcal{M}_{\mathbb{U}^0}$$ limjstar

The last isomorphism needs explanation. In the next lemma, we denote the standard affine open subset $\mathbb{U}^0$ of $\mathbb{P}^n$ by $\mathbb{U}$, and H is the hyperplane at infinity, as before.

6.8.16. Lemma. *Let $\mathcal{M}$ be an $\mathcal{O}$-module on projective space $\mathbb{P}^n$, and let j be the inclusion of the standard affine open set $\mathbb{U}$ into $\mathbb{P}^n$.* *Mjstarequaljs-tarM*

(i) *For every k, the restriction of $\mathcal{M}(kH)$ to the open set $\mathbb{U}$ is the same as the restriction $\mathcal{M}_{\mathbb{U}}$ of $\mathcal{M}$ to $\mathbb{U}$. The restriction of $j_*\mathcal{M}_{\mathbb{U}}$ to $\mathbb{U}$ is $\mathcal{M}_{\mathbb{U}}$ too, and the restriction of the map $\mathcal{M}(kH) \to j_*\mathcal{M}_{\mathbb{U}}$ to $\mathbb{U}$ is the identity map.*

(ii) *The direct image $j_*\mathcal{M}_{\mathbb{U}}$ is isomorphic to $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}}$.*

proof. **(i)** Because the intersection $H \cap \mathbb{U}$ is empty, the restrictions of $\mathcal{M}(kH)$ and $\mathcal{M}$ to $\mathbb{U}$ are equal. The fact that the restriction of $j_*\mathcal{M}_{\mathbb{U}}$ is also equal to $\mathcal{M}_{\mathbb{U}}$ follows from Proposition 6.6.10 **(iv)**.

(ii) Suppose given a section of $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}}$ on an open set V, that has the form $\alpha \otimes f$, where α is a section of $\mathcal{M}$ on V and and f is a section of $j_*\mathcal{O}_{\mathbb{U}}$ on V, i.e., a regular function on $V \cap \mathbb{U}$. We denote the restriction of α to $V \cap \mathbb{U}$ by the same symbol α. Then αf will be a section of $\mathcal{M}$ on $V \cap \mathbb{U}$ and therefore a section of $j_*\mathcal{M}_{\mathbb{U}}$ on V. The map $(\alpha, f) \to \alpha f$ is bilinear, so sending $\alpha \otimes f$ to αf defines a homomorphism $\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}} \to j_*\mathcal{M}_{\mathbb{U}}$. To show that this map is an isomorphism, it suffices to verify that it defines a bijective map on each of the standard affine open sets $\mathbb{U}^i$. We omit the trivial case $i = 0$, and look at $\mathbb{U}^1$. On that open set, $[j_*\mathcal{M}_{\mathbb{U}}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^{01})$. Also, $[j_*\mathcal{O}_{\mathbb{U}}](\mathbb{U}^1) = \mathcal{O}(\mathbb{U} \cap \mathbb{U}^1) = \mathcal{O}(\mathbb{U}^1)[v_0^{-1}],\ \ v_0 = x_0/x_1$. By definition of the tensor product, $[\mathcal{M} \otimes_{\mathcal{O}} j_*\mathcal{O}_{\mathbb{U}^0}](\mathbb{U}^1) = \mathcal{M}(\mathbb{U}^1) \otimes_{\mathcal{O}(\mathbb{U}^1)} \mathcal{O}(\mathbb{U}^0 \cap \mathbb{U}^1) = \mathcal{M}(\mathbb{U}^1)[v_0^{-1}]$, and $\mathcal{M}(\mathbb{U}^1)[v_0^{-1}] = \mathcal{M}(\mathbb{U}^{01})$. $\square$

(6.8.17) generating an $\mathcal{O}$-module generators

Le $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X. A set of global sections $m = (m_1, ..., m_k)$ of $\mathcal{M}$ defines a map

(6.8.18) $$\mathcal{O}^k \xrightarrow{m} \mathcal{M}$$ gen

n that sends a section $(\alpha_1, ..., \alpha_k)$ of $\mathcal{O}^k$ on an open set to the combination $\sum \alpha_i m_i$. The set of global sections $m_1, ..., m_k$ *generates* $\mathcal{M}$ if this map is surjective. If the sections generate $\mathcal{M}$, then they (more precisely, their restrictions), generate the $\mathcal{O}(U)$-module $\mathcal{M}(U)$ for every affine open set U. When U isn't affine, they may fail to generate $\mathcal{M}(U)$.

6.8.19. Example. Let $X = \mathbb{P}^1$. For $n \geq 0$, the global sections of the twisting module $\mathcal{O}(n)$ are the polynomials of degree n in the coordinate variables x_0, x_1 (6.8.2). Consider the map $\mathcal{O} \oplus \mathcal{O} \xrightarrow{(x_0^n, x_1^n)} \mathcal{O}(n)$. On $\mathbb{U}^0$, $\mathcal{O}(n)$ has basis x_0^n. Therefore this map is surjective on $\mathbb{U}^0$. Similarly, it is surjective on $\mathbb{U}^1$. So it is a surjective map on all of X (6.4.5). The global sections x_0^n, x_1^n generate $\mathcal{O}(n)$. However, the global sections of $\mathcal{O}(n)$ are the homogeneous polynomials of degree n. When $n > 1$, the two sections x_0^n, x_1^n don't span the space of global sections, and the map $\mathcal{O}^2(X) \xrightarrow{(x_0^n, x_1^n)} [\mathcal{O}(n)](X)$ isn't surjective. $\square$ generate-example

The next theorem explains the importance of the twisting operation.

6.8.20. Theorem. *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. For large k, the twist $\mathcal{M}(k)$ is generated by global sections.* *gentwist* *proof.* We may assume that X is projective space $\mathbb{P}^n$. We are to show that if $\mathcal{M}$ is a finite $\mathcal{O}$-module, the global sections generate $\mathcal{M}(k)$ when k is large, and it suffices to show that for each $i = 0, ..., n$, the restrictions of those global sections to $\mathbb{U}^i$ generate the $\mathcal{O}(\mathbb{U}^i)$-module $[\mathcal{M}(k)](\mathbb{U}^i)$ (6.4.5). We work with the index $i = 0$.

We replace $\mathcal{M}(k)$ by the isomorphic module $\mathcal{M}(kH)$. We recall that $\varinjlim \mathcal{M}(kH) = j_*\mathcal{M}_{\mathbb{U}^0}$ (6.8.15) and that the restrictions of the maps $\mathcal{M}(kH) \to j_*\mathcal{M}_{\mathbb{U}^0}$ to $\mathbb{U}^0$ are bijective for every k (6.8.16 **(i)**).

Let $A_0 = \mathcal{O}(\mathbb{U}^0)$ and $M_0 = \mathcal{M}(\mathbb{U}^0)$. We choose a finite set of generators $m_1, ..., m_r$ for the finite A_0-module M_0. The elements of M_0, in particular, the generators, are also global sections of $j_*\mathcal{M}_{\mathbb{U}^0}$. Since $\varinjlim \mathcal{M}(kH) = j_*\mathcal{M}_{\mathbb{U}^0}$, they are represented by global sections $m_1', ..., m_r'$ of $\mathcal{M}(kH)$ when k is large. The restrictions of m_i and m_i' to $\mathbb{U}^0$ are equal, so the restrictions of $m_1', ..., m_r'$ generate M_0 too. Then M_0 is generated by global sections of $\mathcal{M}(kH)$, as was to be shown. $\square$

Section 6.9 Extending a Module: proof

proveex-
tension

We prove Theorem 6.4.1 here. The statement to be proved is that an $\mathcal{O}$-module $\mathcal{M}$ on a variety X has a unique extension to a functor

$$\textit{(opens)} \xrightarrow{\widetilde{\mathcal{M}}} \textit{(modules)}$$

with the sheaf property (6.4.4), and that a homomorphism of $\mathcal{O}$-modules $\mathcal{M} \to \mathcal{N}$ has a unique extension to a homomorphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$.

The proof has the following steps:

1. Verification of the sheaf property for a covering of an affine open set by localizations.
2. Extension of the functor $\mathcal{M}$ to all morphisms between affine open sets.
3. Definition of $\widetilde{\mathcal{M}}$.

Step 1. (*the sheaf property for a covering of an affine open set by localizations*)

This point has been mentioned before, in Section **6.4.2**. Suppose that an affine open subset $Y = \operatorname{Spec} A$ of X is covered by a family of localizations $\mathbf{U}_0 = \{U_{s_i}\}$, and let $\mathcal{M}$ be an $\mathcal{O}$-module. Let M, M_i, and M_{ij} denote the mocules of sections $\mathcal{M}(Y), \mathcal{M}(U_{s_i})$, and $\mathcal{M}(U_{s_i s_j})$, respectively. The exact sequence that expresses the sheaf property for the covering diagram $Y \longleftarrow \mathbf{U}_0 \leftleftarrows \mathbf{U}_1$ becomes

localize-
module

(6.9.1)
$$0 \to M \xrightarrow{\alpha} \prod M_i \xrightarrow{\beta} \prod M_{ij}$$

where α sends an element m of M to the vector $(m, ..., m)$ of its images in $\prod_i M_i$, and β sends a vector $(m_1, ..., m_k)$ in $\prod_i M_i$ to the matrix (z_{ij}), with $z_{ij} = m_j - m_i$ in M_{ij}. We must show that the sequence (6.9.1) is exact.

exactness at M: Since the open sets U^i cover Y, the elements $s_1, ..., s_k$ generate the unit ideal. Let m be an element of M that maps to zero in every M_i. Then there exists an n such that $s_i^n m = 0$, and we can use the same exponent n for all i. The elements s_i^n generate the unit ideal. Writing $\sum a_i s_i^n = 1$, we have $m = \sum a_i s_i^n m = \sum a_i 0 = 0$.

exactness at $\prod M_i$: Let m_i be elements of M_i such that $m_j = m_i$ in M_{ij} for all i, j. We must find an element w in M that maps to m_j in M_j for every j.

We write m_i as a fraction: $m_i = s_i^{-n} x_i$, or $x_i = s_i^n m_i$, with x_i in M, using the same integer n for all i. The equation $m_j = m_i$ in M_{ij} tells us that $s_i^n x_j = s_j^n x_i$ in M_{ij}. Since M_{ij} is the localization $M[(s_i s_j)^{-1}]$ $(s_i s_j)^r s_i^n x_j = (s_i s_j)^r s_j^n x_i$ will be true in M, if r is large.

We adjust the notation. Let $\widetilde{x}_i = s_i^r x_i$, and $\widetilde{s}_i = s_i^{r+n}$. Then in M, $\widetilde{x}_i = \widetilde{s}_i m_i$ and $\widetilde{s}_j \widetilde{x}_i = \widetilde{s}_i \widetilde{x}_j$. The elements $\widetilde{s}_i$ generate the unit ideal. So there is an equation in A, of the form $\sum a_i \widetilde{s}_i = 1$. Let $w = \sum a_i \widetilde{x}_i$. This is an element of M, and

$$\widetilde{x}_j = \big(\sum_i a_i \widetilde{s}_i\big)\widetilde{x}_j = \sum_i a_i \widetilde{s}_j \widetilde{x}_i = \widetilde{s}_j w$$

Since $m_j = \widetilde{s}_j^{-1} \widetilde{x}_j$, $m_j = w$ is true in M_j. Since j is arbitrary, w is the required element of M. $\square$

Step 2. (*extending an $\mathcal{O}$-module to all morphisms between affine open sets*)

The $\mathcal{O}$-module $\mathcal{M}$ comes with localization maps $\mathcal{M}(U) \to \mathcal{M}(U_s)$. It doesn't come with homomorphisms $\mathcal{M}(U) \to \mathcal{M}(V)$ when $V \to U$ is an arbitrary inclusion of affine open sets. We define those maps here.

Let $\mathcal{M}$ be an $\mathcal{O}$-module and let $V \to U$ be an inclusion of affine open sets. To describe the homomorphism $\mathcal{M}(U) \to \mathcal{M}(V)$, we cover V by a family $\mathbf{V}_0 = \{V^i\}$ of open sets that are localizations of U and of V. We inspect the covering diagram $V \leftarrow \mathbf{V}_0 \leftleftarrows \mathbf{V}_1$ and the corresponding exact sequence $0 \to \mathcal{M}(V) \xrightarrow{\alpha} \mathcal{M}(\mathbf{V}_0) \xrightarrow{\beta} \mathcal{M}(\mathbf{V}_1)$. When we add the map $V \to U$ to the covering diagram:

$$U \leftarrow V \leftarrow \mathbf{V}_0 \leftleftarrows \mathbf{V}_1$$

the two composed maps $U \leftarrow V \leftleftarrows \mathbf{V}_1$ are equal.
Since V^i are localizations of U and V^{ij} are localizations of V^i and of V^j, the $\mathcal{O}$-module $\mathcal{M}$ comes with maps $\mathcal{M}(U) \xrightarrow{\psi} \mathcal{M}(\mathbf{V}_0) \rightrightarrows \mathcal{M}(\mathbf{V}_1)$. Here the two composed maps $\mathcal{M}(U) \rightrightarrows \mathcal{M}(\mathbf{V}_1)$ are equal. Their difference, which is $\beta\psi$, is the zero map. Therefore ψ maps $\mathcal{M}(U)$ to the kernel of β which, according to Step 1, is $\mathcal{M}(V)$. This gives us a map $\mathcal{M}(U) \xrightarrow{\eta} \mathcal{M}(V)$ that makes a diagram

(6.9.2)
$$\begin{array}{ccc} \mathcal{M}(U) & \xrightarrow{\eta} & \mathcal{M}(V) \\ \| & & \downarrow{\scriptstyle\alpha} \\ \mathcal{M}(U) & \xrightarrow{\psi} & \mathcal{M}(\mathbf{V}_0) \end{array}$$
UtoV

Both ψ and α are compatible with multiplication by a regular function f on U, and α is injective. So η is also compatible with multiplication by f.

We must check that η is independent of the covering $\mathbf{V}_0$. Let $\mathbf{V}'_0 = \{V'^j\}$ be another covering of V by localizations of U. We cover each of the open sets $V^i \cap V'^j$ by localizations $W^{ij\nu}$ of U. Taken together, these open sets form a covering $\mathbf{W}_0$ of V. We have a map $\mathbf{W}_0 \xrightarrow{\epsilon} \mathbf{V}_0$ that gives us a map of covering diagrams

$$[V \to \mathbf{W}_0 \rightrightarrows \mathbf{W}_1] \quad \longrightarrow \quad [V \to \mathbf{V}_0 \rightrightarrows \mathbf{V}_1]$$

and therefore a diagram

(6.9.3)
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(V) & \longrightarrow & \mathcal{M}(\mathbf{V}_0) & \xrightarrow{\beta_{\mathbf{V}}} & \mathcal{M}(\mathbf{V}_1) \\ & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{M}(V) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \xrightarrow{\beta_{\mathbf{W}}} & \mathcal{M}(\mathbf{W}_1) \end{array}$$
compare-cov

whose rows are exact sequences.

Also, the composed maps $\mathbf{V}_1 \rightrightarrows \mathbf{V}_0 \to U$ are equal. Similarly, the composed maps $\mathbf{W}_1 \rightrightarrows \mathbf{W}_0 \to U$ are equal. This gives us a diagram

(6.9.4)
$$\begin{array}{ccccc} \mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{V}_0) & \xrightarrow{\beta_{\mathbf{V}}} & \mathcal{M}(\mathbf{V}_1) \\ \| & & \downarrow & & \downarrow \\ \mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{W}_0) & \xrightarrow{\beta_{\mathbf{W}}} & \mathcal{M}(\mathbf{W}_1) \end{array}$$
compare-covtwo

in which $\mathcal{M}(U)$ is mapped to the kernels of $\beta_{\mathbf{V}}$ and $\beta_{\mathbf{W}}$, both of which are equal to $\mathcal{M}(V)$. Looking at the diagram, one sees that the maps $\mathcal{M}(U) \to \mathcal{M}(V)$ defined using the two coverings $\mathbf{V}_0$ and $\mathbf{W}_0$ are the same.

We show that this extended functor has the sheaf property for an affine covering $\mathbf{U}_0 = \{U^i\}$ of an affine variety U. Let $\mathbf{V}_0$ be the affine covering of U that is obtained by covering each U^i by localizations of U. The sheaf property to be verified is that the top row of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{U}_0) & \longrightarrow & \mathcal{M}(\mathbf{U}_1) \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & \mathcal{M}(U) & \longrightarrow & \mathcal{M}(\mathbf{V}_0) & \longrightarrow & \mathcal{M}(\mathbf{V}_1) \end{array}$$

is exact, and because $\mathbf{V}_0$ is a covering of U by affines, the bottom row is exact. Because $\mathbf{V}_0$ covers $\mathbf{U}_0$, $\mathbf{V}_1$ covers $\mathbf{U}_1$ as well. So the maps β and γ are injective. It follows that the top row is exact.

Step 3. (*definition of* $\widetilde{\mathcal{M}}$)

Let Y be an open subset of X. We use the sheaf property to define $\widetilde{\mathcal{M}}(Y)$. We choose a (finite) covering $\mathbf{U}_0 = \{U^i\}$ of Y by affine open sets, and we define $\widetilde{\mathcal{M}}(Y)$ to be the kernel $K_\mathbf{U}$ of the map $\mathcal{M}(\mathbf{U}_0) \xrightarrow{\beta_\mathbf{U}} \mathcal{M}(\mathbf{U}_1)$, where $\beta_\mathbf{U}$ is the map described in (6.4.9). When we show that this kernel is independent of the covering $\mathbf{U}_0$, it will follow that $\widetilde{\mathcal{M}}$ is well-defined, and that it has the sheaf property.

Let $\mathbf{V}_0 = \{V^\nu\}$ be another covering of Y by affine open sets. One can go from $\mathbf{U}_0$ to $\mathbf{V}_0$ and back in a finite number of steps, each of which changes a covering by adding or deleting a single affine open set. So we consider a family $\mathbf{W}_0 = \{U^i, V\}$ obtained by adding one affine open subset V of Y to $\mathbf{U}_0$, and we let $\mathbf{W}_1$ be the family of intersections of pairs of elements of $\mathbf{W}_0$. Then with notation as above, we have a map $K_\mathbf{W} \to K_\mathbf{U}$. We show that, for any element (u_i) in the kernel $K_\mathbf{U}$, there is a unique element v in $\mathcal{M}(V)$ such that $((u_i), v)$ is in the kernel $K_\mathbf{W}$. This will show that $K_\mathbf{W} = K_\mathbf{U}$.

To define the element v, we let $V^i = U^i \cap V$. Since $\mathbf{U}_0 = \{U^i\}$ is a covering of Y, $\mathbf{V}_0 = \{V^i\}$ is a covering of V by affine open sets. Let v_i be the restriction of the section u_i to V^i. Since (u_i) is in the kernel of $\beta_\mathbf{U}$, $u_i = u_j$ on U^{ij}. Then it is also true that $v_i = v_j$ on the smaller open set V^{ij}. So (v_i) is in the kernel of the map $\mathcal{M}(\mathbf{V}_0) \xrightarrow{\beta_\mathbf{V}} \mathcal{M}(\mathbf{V}_1)$, and since $\mathbf{V}_0$ is a covering of the affine variety V by affine open sets, Step 2 tells us that the kernel of $\beta_\mathbf{V}$ is $\mathcal{M}(V)$. So there is a unique element v in $\mathcal{M}(V)$ that restricts to v_i on V^i for each i. We show that, with this element v, (u_i, v) is in the kernel of $\beta_\mathbf{W}$.

When the subsets in the family $\mathbf{W}_1$ are listed in the order

$$\mathbf{W}_1 = \{U^i \cap U^j\}, \{V \cap U^j\}, \{U^i \cap V\}, \{V \cap V\}$$

the map $\beta_\mathbf{W}$ sends $((u_i), v)$ to $\big[(u_j - u_i), (u_j - v), (v - u_i), 0\big]$, restricted appropriately. Here $u_i = u_j$ on $U^i \cap U^j$ because (u_i) is in the kernel of $\beta_\mathbf{U}$, and $u_j = v_j = v$ on $U^j \cap V = V^j$ by definition.

To prove that $\widetilde{\mathcal{M}}$ is a functor, we must define the restriction map $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$ for an arbitrary inclusion $V \subset U$ of open sets. Let $\mathbf{U}_0 = \{U^i\}$ be an affine cover of U. We cover the open sets $V \cap U^i$ by affine opens $V^{i\nu}$. Then $\mathbf{V}_0 = \{V^{i\nu}_{i,\nu}\}$ is an affine cover of V, an we have maps $\mathbf{V}_0 \to \mathbf{U}_0$ and $\mathbf{V}_1 \to \mathbf{U}_1$. This gives us a diagram

mapcov (6.9.5)
$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathcal{M}}(U) \arrow[r] & \widetilde{\mathcal{M}}(\mathbf{U}_0) \arrow[r, "\beta"] \arrow[d] & \widetilde{\mathcal{M}}(\mathbf{U}_1) \arrow[d] \\
0 \arrow[r] & \widetilde{\mathcal{M}}(V) \arrow[r] & \widetilde{\mathcal{M}}(\mathbf{V}_0) \arrow[r, "\beta"] & \widetilde{\mathcal{M}}(\mathbf{V}_1)
\end{tikzcd}$$

that induces the required map $\widetilde{\mathcal{M}}(U) \to \widetilde{\mathcal{M}}(V)$. Here one must show that this map is independent of the choices of $\mathbf{U}$ and $\mathbf{V}$, but that is boring.

This completes the proof of Theorem 6.4.1. $\square$

Section 6.10 Exercises

6.10.1. Let X be the affine plane $\operatorname{Spec} A$, where $A = \mathbb{C}[x,y]$, and let U be the complement of the origin in X. chapsixex xsect snotfin

(i) Let $\mathcal{M}$ be the $\mathcal{O}_X$-module that correponds to the A-module $M = A/yA$. Show that $\mathcal{M}$ is a finite $\mathcal{O}$-module, but that $\mathcal{M}(U)$ isn't a finite module over the ring $\mathcal{O}(U)$.

(ii) Show that, for any $k \geq 1$, the homomorphism

$$\mathcal{O}\times \xrightarrow{(x,y)^t} \mathcal{O}$$

is surjective on U, though the associated map of sections on U isn't surjective.

6.10.2. Prove that a simple module over a finite type $\mathbb{C}$-algebra has dimension 1. xsimple-mod

6.10.3. Prove that if an $\mathcal{O}$-module has the coherence property for affine open sets U, then it has the sheaf property for affine open coverings of affine open sets. xcohprop-sheaf

6.10.4. Let U be the complement of a finite set in $\mathbb{P}^d$. Determine $H^0(U, \mathcal{O}_U)$. xcomplvin

6.10.5. Let V be the complement of a point in projective space $\mathbb{P}^n$ Detemine $\mathcal{O}_{\mathbb{P}}(V)$. regfnconst

6.10.6. Let $U' \subset U$ be affine open sets in a variety X, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Say that $\mathcal{O}(U) = A$, $\mathcal{O}(U') = A'$, $\mathcal{M}(U) = M$, and $\mathcal{M}(U') = M'$. Prove that $M' = M \otimes_A A'$. xMisten-sor

6.10.7. Prove that, to define an $\mathcal{O}$-module $\mathcal{M}$ on $\mathbb{P}^1$, it is enough to give modules M_0, M_1, and M_{01} over the rings $\mathbb{C}[u]$, $\mathbb{C}[u^{-1}]$, and $\mathbb{C}[u,u^{-1}]$, respectively, together with isomorphisms $M_0[u^{-1}] \approx M_{01} \approx M_1[u]$. xMze-roMoneenuf

6.10.8. Let s be an element of a domain A, and let M be an A-module. Prove that the limit of the directed set $M \xrightarrow{s} M \xrightarrow{s} \cdots$ is isomorphic to the localization M_s. xlimmult

6.10.9. Show that if $\mathcal{I}$ and $\mathcal{J}$ are (quasicoherent) ideals of $\mathcal{O}$, so is $\mathcal{I} \cap \mathcal{J}$. xinterscoh

6.10.10. Let $R = \mathbb{C}[x,y]$. Determine the limit of the directed set $R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \cdots$. limmult

6.10.11. Let s be an element of a domain A, and let M be an A-module. Prove that the limit of the directed set $M \xrightarrow{s} M \xrightarrow{s} \cdots$ is isomorphic to the localization M_s. xlimM

6.10.12. Give an example of a finite $\mathcal{O}$-module $\mathcal{M}$ and an open set U such that $\mathcal{M}(U)$ isn't a finite $\mathcal{O}(U)$-module. Hint: The reason that this might occur is that there might not be rational functions that are regular on X, though $\mathcal{M}$ has global sections. xOUnotfi-nite

6.10.13. Let $\mathcal{R} = \mathbb{C}[\S_{\prime}, \S_{\infty}, \S_{\in}]$, and let $f = x_0^2 - x_1x_2$. xgenrel

(a) Determine generators and defining relations for the ring $\mathcal{R}_{\{f\}}$ of homogeneous fractions of degree zero whose denominator is a power of f.

(b) Prove that the twisting module $\mathcal{O}(1)$ isn't a free module on the open subset $\mathbb{U}_{\{}f\}$ of $\mathbb{P}^2$ at which $f \neq 0$.

6.10.14. Let X be a variety. Prove that every strictly ascending chain of submodules of a finite $\mathcal{O}$-module $\mathcal{M}$ is finite. xascchcd

6.10.15. Let $s = z^2 - xy$. Determine the degree one part of $k[x,y,z]_s$, and proof that $\mathcal{O}(1)$ is not free there. notfree

6.10.16. What are the sections of $\mathcal{O}(nH)$ on an open set V that isn't contained in any $\mathbf{U}^i$. xsectOn

6.10.17. Describe the kernel of multiplication by a homogeneous polynomial of degree d xkermult

$$\mathcal{M}(k) \xrightarrow{f} \mathcal{M}(k+d)$$

6.10.18. Let $\mathcal{M}$ be an $\mathcal{O}$-module on $\mathbb{P}^n$. Prove that if coordinates x of $\mathbb{P}^n$ are in general position, multiplication by x_i defines an injective map $\mathcal{M} \to \mathcal{M}(1)$. xmultinj

6.10.19. In the description (6.5.4) of modules over the projective line, we considered the standard affine open sets U^0 and U^1. Interchanging these open sets changes the variable t to t^{-1}, and it changes the matrix P accordingly. Does it follow, when the rank is 1, that the $\mathcal{O}$-modules defined by t^k and by t^{-k} are isomorphic? xtwistsi-som xmultf **6.10.20.** Describe the kernel and cokernel of multiplication by a homogeneous polynomial f of degree d: $\mathcal{M}(k) \xrightarrow{f} \mathcal{M}(k+d)$

sectwist **6.10.21.** Let $X = \mathbb{P}^2$. What are the sections of the twisting module $\mathcal{O}_X(n)$ on the open complement of the line $\{x_1 + x_2 = 0\}$?

xmultinject **6.10.22.** Let M be a finite module over a finite-type domain A, and let α be a nonzero element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s = \alpha - c$ is an injective map $M \xrightarrow{s} M$.

Chapter 7 COHOMOLOGY

cohomology

Section 7.1 Cohomology

cohqcoh

In a classic 1956 paper "Faisceaux Algébriques Cohérents", Serre showed how the Zariski topology could be used to define cohomology of $\mathcal{O}$-modules. That cohomology is the topic of the chapter.

To save time, we define cohomology only for $\mathcal{O}$-modules. Anyhow, the Zariski topology has limited use for cohomology with other coefficients. In the Zariski topology, the constant coefficient cohomology $H^q(X,\mathbb{Z})$ is zero for all $q>0$.

Let $\mathcal{M}$ be an $\mathcal{O}$-module on a variety X. The *zero-dimensional cohomology* of $\mathcal{M}$ is the space $\mathcal{M}(X)$ of its global sections. When speaking of cohomology, one denotes the space $\mathcal{M}(X)$ by $H^0(X,\mathcal{M})$.

The functor

$$\textit{(}\mathcal{O}\textit{-modules)} \xrightarrow{H^0} \textit{(vector spaces)}$$

that carries an $\mathcal{O}$-module $\mathcal{M}$ to $H^0(X,\mathcal{M})$ is left exact: If

SS (7.1.1)
$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$$

is a short exact sequence of $\mathcal{O}$-modules, the associated sequence of global sections

globalsections (7.1.2)
$$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{N}) \to H^0(X,\mathcal{P})$$

is exact. But unless X is affine, the map $H^0(X,\mathcal{N}) \to H^0(X,\mathcal{P})$ needn't be surjective. The *cohomology* is a sequence of functors *(*$\mathcal{O}$*-modules)* $\xrightarrow{H^q}$ *(vector spaces)*,

$$H^0,\ H^1,\ H^2,\ \dots$$

beginning with H^0, one for each *dimension*, that compensates for the lack of exactness in the following way:

(a) To every short exact sequence (7.1.1) of $\mathcal{O}$-modules, there is an associated long exact *cohomology sequence*

HSS (7.1.3)
$$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{N}) \to H^0(X,\mathcal{P}) \xrightarrow{\delta^0}$$

$$\xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to H^1(X,\mathcal{N}) \to H^1(X,\mathcal{P}) \xrightarrow{\delta^1} ...$$

$$... \xrightarrow{\delta^{q-1}} H^q(X,\mathcal{M}) \to H^q(X,\mathcal{N}) \to H^q(X,\mathcal{P}) \xrightarrow{\delta^q} \cdots$$

The maps δ^q in this sequence are the *coboundary maps*.

(b) A diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}' & \longrightarrow & \mathcal{N}' & \longrightarrow & \mathcal{P}' & \longrightarrow & 0 \end{array}$$

whose rows are short exact sequences of $\mathcal{O}$-modules induces a map of cohomology sequences

(7.1.4)

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^q(X,\mathcal{N}) & \longrightarrow & H^q(X,\mathcal{P}) & \xrightarrow{\delta^q} & H^{q+1}(X,\mathcal{M}) & \longrightarrow & H^{q+1}(X,\mathcal{N}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^q(X,\mathcal{N}') & \longrightarrow & H^q(X,\mathcal{P}') & \xrightarrow{\delta^q} & H^{q+1}(X,\mathcal{M}') & \longrightarrow & H^{q+1}(X,\mathcal{N}) & \longrightarrow & \cdots \end{array}$$

deltadia-
gram

Thus a map of exact sequences of $\mathcal{O}$-modules induces a map of cohomology sequences. These properties make the sequence of functors $H^0, H^1, ...$ into a *cohomological functor*.

A sequence H^q, $q = 0, 1, ...$ of functors from $\mathcal{O}$-modules to vector spaces that comes with long cohomology sequences for every short exact sequence of $\mathcal{O}$-modules is called a *cohomological functor*.

Most of Diagram 7.1.4 arises from the fact that the H^q are functors. The only additional property is that the squares

(7.1.5)
$$\begin{array}{ccc} H^q(X,\mathcal{P}) & \xrightarrow{\delta^q} & H^{q+1}(X,\mathcal{M}) \\ \downarrow & & \downarrow \\ H^q(X,\mathcal{P}') & \xrightarrow{\delta^q} & H^{q+1}(X,\mathcal{M}') \end{array}$$

deltadia-
gramtwo

that involve the coboundary maps δ commute.

Unfortunately, there is no natural construction of cohomology. We present a construction in Section 7.4, but it isn't canonical. One needs to look at an explicit construction at times, but it is usually best to work with the characteristic properties that are described below, in Section 7.3.

The cohomology in dimension one, H^1, has an interesting interpretation that you can read about if you like. We won't use it. The cohomology in dimension greater than one has no useful direct interpretation.

Section 7.2 Complexes

complexes

We need complexes because they are used in the construction of cohomology.

A *complex* of vector spaces is a sequence of homomorphisms of vector spaces

(7.2.1)
$$\cdots \to V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1} \xrightarrow{d^{n+1}} \cdots$$

complex-
one

indexed by the integers, such that the composition $d^n d^{n-1}$ of adjacent maps is zero —such that the image of d^{n-1} is contained in the kernel of d^n. This complex may be denoted by $V^\bullet$.

The q-dimensional *cohomology* of a complex $V^\bullet$ is the quotient

(7.2.2)
$$\mathbf{C}^q(V^\bullet) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1}).$$

cohcoplx-
one

A complex whose cohomology is zero is an exact sequence.

A finite sequence of homomorphisms $V^k \xrightarrow{d^k} V^{k+1} \to \cdots \xrightarrow{d^{\ell-1}} V^\ell$ such that the compositions $d^n d^{n-1}$ are zero can be made into a complex by defining $V^n = 0$ for all other integers n. In all of our complexes, V^q will be zero when $q < 0$. For example, a homomorphism of vector spaces $V^0 \xrightarrow{d^0} V^1$ can be made into the complex

$$\cdots \to 0 \to V^0 \xrightarrow{d^0} V^1 \to 0 \to \cdots$$

The cohomology $\mathbf{C}^0$ of this complex is the kernel of d^0, $\mathbf{C}^1$ is its cokernel, and $\mathbf{C}^q$ is zero for all other q.

A *map* $V^\bullet \xrightarrow{\varphi} V'^\bullet$ of complexes is a collection of homomorphisms $V^n \xrightarrow{\varphi^n} V'^n$ making a diagram

$$\begin{tikzcd}
{} \arrow[r] & V^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "\varphi^{n-1}"'] & V^n \arrow[r, "d^n"] \arrow[d, "\varphi^n"'] & V^{n+1} \arrow[r] \arrow[d, "\varphi^{n+1}"'] & \cdots \\
{} \arrow[r] & V'^{n-1} \arrow[r, "d'^{n-1}"] & V'^n \arrow[r, "d'^n"] & V'^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

A map of complexes induces maps on the cohomology

$$\mathbf{C}^q(V^\bullet) \to \mathbf{C}^q(V'^\bullet)$$

because $\ker d^q$ maps to $\ker d'^q$ and $\operatorname{im} d^q$ maps to $\operatorname{im} d'^q$.

A sequence of maps of complexes

exseqcplx (7.2.3) $$\cdots \to V^\bullet \xrightarrow{\varphi} V'^\bullet \xrightarrow{\psi} V''^\bullet \to \cdots$$

is *exact* if the sequences

(7.2.4) $$\cdots \to V^q \xrightarrow{\varphi^q} V'^q \xrightarrow{\psi^q} V''^q \to \cdots$$

are exact for every q.

cohcplx **7.2.5. Proposition.**
Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ be a short exact sequence of complexes. For every q, there are maps $\mathbf{C}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{C}^{q+1}(V^\bullet)$ such that the sequence

$$0 \to \mathbf{C}^0(V^\bullet) \to \mathbf{C}^0(V'^\bullet) \to \mathbf{C}^0(V''^\bullet) \xrightarrow{\delta^0} \mathbf{C}^1(V^\bullet) \to \mathbf{C}^1(V'^\bullet) \to \mathbf{C}^1(V''^\bullet) \xrightarrow{\delta^1} \mathbf{C}^2(V^\bullet) \to \quad \cdots$$

is exact.

The proof of the proposition is below.

This long exact sequence is the *cohomology sequence* associated to the short exact sequence of complexes. Thus the set of functors $\{\mathbf{C}^q\}$ is a cohomological functor on the category of complexes.

snakeco-homology **7.2.6. Example.** We make the Snake Lemma 2.1.19 into a cohomology sequence. Suppose given a diagram

$$\begin{tikzcd}
 & V \arrow[r, "u"] \arrow[d, "f"'] & V' \arrow[r] \arrow[d, "f'"'] & V'' \arrow[r] \arrow[d, "f''"'] & 0 \\
0 \arrow[r] & W \arrow[r] & W' \arrow[r, "v"'] & W'' &
\end{tikzcd}$$

with exact rows. We form the complex $0 \to V \xrightarrow{f} W \to 0$ with V in degree zero, so that $\mathbf{C}^0(V^\bullet) = \ker f$ and $\mathbf{C}^1(V^\bullet) = \operatorname{coker} f$, and we do the analogous thing for the maps f' and f''. Then the Snake Lemma becomes an exact sequence

$$\mathbf{C}^0(V^\bullet) \to \mathbf{C}^0(V'^\bullet) \to \mathbf{C}^0(V''^\bullet) \to \mathbf{C}^1(V^\bullet) \to \mathbf{C}^1(V'^\bullet) \to \mathbf{C}^1(V''^\bullet) \qquad \square$$

proof of Proposition 7.2.5. Let

$$V^\bullet = \{\cdots \to V^{q-1} \xrightarrow{d^{q-1}} V^q \xrightarrow{d^q} V^{q+1} \xrightarrow{d^{q+1}} \cdots\}$$

be a complex. We use this notation:
D^q is the cokernel of d^{q-1},
B^q is the image of d^{q-1}, and
Z^q is the kernel of d^q.
So we have exact an sequence $0 \to B^q \to V^q \to D^q \to 0$, and $B^q \subset Z^q \subset V^q$. The cohomology $\mathbf{C}^q(V^\bullet)$ of the complex, which is Z^q/B^q, is a subspace of D^q.

The map $V^q \xrightarrow{d^q} V^{q+1}$ factors through D^q, and its image is contained in Z^{q+1}. This gives us a map $D^q \xrightarrow{f^q} Z^{q+1}$. Then d^q is the composition of three maps:

$$V^q \xrightarrow{\pi^q} D^q \xrightarrow{f^q} Z^{q+1} \xrightarrow{i^{q+1}} V^{q+1}$$

where π^q is the projection from V^q to D^q and i^{q+1} is the inclusion of Z^{q+1} into V^{q+1}. Studying these maps, one finds that

$$\mathbf{C}^q(V^\bullet) = \ker\, f^q \quad \text{and} \quad \mathbf{C}^{q+1}(V^\bullet) = \operatorname{coker} f^q. \tag{7.2.7}$$ hiskerand-coker

Let $0 \to V^\bullet \to V'^\bullet \to V''^\bullet \to 0$ be a short exact sequence of complexes, as in Proposition 7.2.5. In the diagram below, the top row is exact because D^q is a cokernel, and cokernel is a right exact operation, and the bottom row is exact because Z^q is a kernel, and kernel is left exact:

$$\begin{tikzcd}
 & D^q \arrow[r] \arrow[d, "f^q"'] & D'^q \arrow[r] \arrow[d, "f'^q"'] & D''^q \arrow[r] \arrow[d, "f''^q"'] & 0 \\
0 \arrow[r] & Z^{q+1} \arrow[r] & Z'^{q+1} \arrow[r] & Z''^{q+1} &
\end{tikzcd}$$

When we apply (7.2.7) to this diagram, the Snake Lemma gives us an exact sequence

$$\mathbf{C}^q(V^\bullet) \to \mathbf{C}^q(V'^\bullet) \to \mathbf{C}^q(V''^\bullet) \xrightarrow{\delta^q} \mathbf{C}^{q+1}(V^\bullet) \to \mathbf{C}^{q+1}(V'^\bullet) \to \mathbf{C}^{q+1}(V''^\bullet)$$

The cohomology sequence associated to the short exact sequence of complexes is obtained by splicing these sequences together. □

The coboundary maps δ^q in cohomology sequences are related in a natural way. If

$$\begin{tikzcd}
0 \arrow[r] & U^\bullet \arrow[r] \arrow[d] & U'^\bullet \arrow[r] \arrow[d] & U''^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V^\bullet \arrow[r] & V'^\bullet \arrow[r] & V''^\bullet \arrow[r] & 0
\end{tikzcd}$$

is a diagram of maps of complexes whose rows are short exact sequences, the diagrams

$$\begin{tikzcd}
\mathbf{C}^q(U''^\bullet) \arrow[r, "\delta^q"] \arrow[d] & \mathbf{C}^{q+1}(U^\bullet) \arrow[d] \\
\mathbf{C}^q(V''^\bullet) \arrow[r, "\delta^q"] & \mathbf{C}^{q+1}(V^\bullet)
\end{tikzcd}$$

commute. It isn't difficult to check this. Thus a map of short exact sequences induces a map of cohomology sequences.

Section 7.3 Characteristic Properties of Cohomology

charprop

The cohomology $H^q(X, \cdot\,)$ of $\mathcal{O}$-modules, which is a sequence of functors $H^0,\ H^1,\ H^2,\ \cdots$

$$(\mathcal{O}\text{-modules}) \xrightarrow{H^q} (\text{vector spaces})$$

is characterized by the three properties below. The first two have already been mentioned.

charpropone

(7.3.1) Characteristic Properties of Cohomology

1. $H^0(X, \mathcal{M})$ is the space $\mathcal{M}(X)$ of global sections of $\mathcal{M}$.

2. The sequence $H^0, H^1, H^2, \cdots$ is a cohomological functor on $\mathcal{O}$-modules: A short exact sequence of $\mathcal{O}$-modules produces a long exact cohomology sequence.

3. Let $Y \xrightarrow{f} X$ be the inclusion of an *affine* open subset Y into X, let $\mathcal{N}$ be an $\mathcal{O}_Y$-module, and let $f_*\mathcal{N}$ be its direct image on X. The cohomology $H^q(X, f_*\mathcal{N})$ is zero for all $q > 0$.

When Y is an affine variety, the global section functor is exact: If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ is a short exact sequence of $\mathcal{O}$-modules on Y, the sequence

$$0 \to H^0(Y, \mathcal{M}) \to H^0(Y, \mathcal{N}) \to H^0(Y, \mathcal{P}) \to 0$$

is exact (6.2.2). There is no need for the higher cohomology H^q when Y is affine. One may as well define $H^q(Y, \cdot) = 0$ and $q > 0$. The third characteristic property is based on this observation.

Intuitively, the third property tells us that allowing poles on the complement of an affine open set kills cohomology in positive dimension.

existcohom

7.3.2. Theorem. *There exists a cohomology theory with the properties (**7.3.1**), and it is unique up to unique isomorphism.*

The proof is in the next section.

cohzeroaffine

7.3.3. Corollary. *If X is an affine variety, $H^q(X, \mathcal{M}) = 0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q > 0$.*

This follows when one applies the third characteristic property to the identity map $X \to X$. □

7.3.4. Example. This example shows how the third characteristic property can be used. let j be inclusion of standard affine $\mathbb{U}^0$ into $X = \mathbb{P}$. Then $\varinjlim \mathcal{M}(nH) \approx j_*\mathcal{M}_{\mathbb{U}^0}$, where $\mathcal{M}_{\mathbb{U}^0}$ is the restriction of $\mathcal{M}$ to $\mathbb{U}^0$. In particular, $\varinjlim \mathcal{O}(dH) = j_*\mathcal{O}_{\mathbb{U}^0}$. The third property tells us that the cohomology $H^q(X, j_*\mathcal{O}_{\mathbb{U}^0})$ of the direct image of $\mathcal{O}^0_{\mathbb{U}}$ is zero when $q > 0$. The direct image is isomorphic to the limit $\varinjlim \mathcal{O}_X(nH)$ (6.8.12). As we will see below (7.4.28), cohomology commutes with direct limits. Therefore $\varinjlim H^q(X, \mathcal{O}_X(nH))$ and $\varinjlim H^q(X, \mathcal{O}_X(n))$ are zero when $q > 0$. □

cohdirsum

7.3.5. Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules on a variety X. The cohomology $H^q(X, \mathcal{M} \oplus \mathcal{N})$ of the direct sum is $\mathcal{M} \oplus \mathcal{N}$ is canonically isomorphic to the direct sum $H^q(X, \mathcal{M}) \oplus H^q(X, \mathcal{N})$.*

In this statement, one could substitute just about any functor for H^q.

proof. Recall that $M \oplus N = M \times N$ (**2.1.21**). We have homomorphisms of $\mathcal{O}$-modules $\mathcal{M} \xrightarrow{i_1} \mathcal{M} \oplus \mathcal{N} \xrightarrow{\pi_1} \mathcal{M}$ and analogous homomorphisms $\mathcal{N} \xrightarrow{i_2} \mathcal{M} \oplus \mathcal{N} \xrightarrow{\pi_2} \mathcal{N}$. The direct sum can be characterized by these maps, together with the relations $\pi_1 i_1 = id_{\mathcal{M}}$, $\pi_2 i_2 = id_{\mathcal{N}}$, $\pi_2 i_1 = 0$, $\pi_1 i_2 = 0$, and $i_1\pi_1 + i_2\pi_2 = id_{\mathcal{M}\oplus\mathcal{N}}$. The proof of this is an exercise. Applying the functor H^q, gives analogous homomorphisms relating $H^q(\mathcal{M})$, $H^q(\mathcal{N})$, and $H^q(\mathcal{M} \oplus \mathcal{N})$. □

Section 7.4 Existence of Cohomology

constrcoh

The proof of existence and uniqueness of cohomology are based on the following facts:

• The intersection of two affine open subsets of a variety is an affine open set. (Theorem 3.6.9.)

• A sequence $\cdots \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to \cdots$ of $\mathcal{O}$-modules on a variety X is exact if and only if, for every affine open subset U, the sequence of sections $\cdots \to \mathcal{M}(U) \to \mathcal{N}(U) \to \mathcal{P}(U) \to \cdots$ is exact. (This is the definition of exactness.)

We begin by choosing an arbitrary affine covering $\mathbf{U} = \{U^\nu\}$ of our variety X by finitely many affine open sets U^ν, and we use this covering to describe the cohomology. When we have shown that cohomology is unique, we will know that it is independent of our choice of covering.

Let $\mathbf{U} \xrightarrow{j} X$ denote the family of inclusions $U^\nu \xrightarrow{j^\nu} X$ of our chosen affine open sets into X. If $\mathcal{M}$ is an $\mathcal{O}$-module, $\mathcal{R}_\mathcal{M}$ will denote the $\mathcal{O}$-module $\prod j^\nu_* \mathcal{M}_{U^\nu}$, which we write as $j_*\mathcal{M}_\mathbf{U}$, where $\mathcal{M}_{U^\nu}$ is the restriction of $\mathcal{M}$ to U^ν. As has been noted, there is a canonical map $\mathcal{M} \to j^\nu_*\mathcal{M}_{U^\nu}$, and therefore a canonical map $\mathcal{M} \to \mathcal{R}_\mathcal{M}$ (6.6.10).

7.4.1. Lemma. **(i)** *Let X' be an open subset of X. The module $\mathcal{R}_\mathcal{M}(X')$ of sections of $\mathcal{R}_\mathcal{M}$ on X' is is the product $\prod_\nu \mathcal{M}(X' \cap U^\nu)$. In particular, the space of global sections $\mathcal{R}_\mathcal{M}(X)$, which is $H^0(X, \mathcal{R}_\mathcal{M})$, is the product $\prod_\nu \mathcal{M}(U^\nu)$.* defcalr
(ii) *The canonical map $\mathcal{M} \to \mathcal{R}_\mathcal{M}$ is injective. Thus, if $\mathcal{S}_\mathcal{M}$ denotes the cokernel of that map, there is a short exact sequence of $\mathcal{O}$-modules*

(7.4.2) $$0 \to \mathcal{M} \to \mathcal{R}_\mathcal{M} \to \mathcal{S}_\mathcal{M} \to 0$$ MRzeroMone

(iii) *For any cohomology theory with the characteristic properties and for any $q > 0$, $H^q(X, \mathcal{R}_\mathcal{M}) = 0$.*

proof. **(i)** This is seen by going through the definitions:

$$\mathcal{R}(X') \;=\; \textstyle\prod_\nu [j^\nu_* \mathcal{M}_{U^\nu}](X') = \prod_\nu \mathcal{M}_{U^\nu}(X' \cap U^\nu) = \prod_\nu \mathcal{M}(X' \cap U^\nu).$$

(ii) Let X' be an open subset of X. The map $\mathcal{M}(X') \to \mathcal{R}_\mathcal{M}(X')$ is the product of the restriction maps $\mathcal{M}(X') \to \mathcal{M}(X' \cap U^\nu)$. Because the open sets U^ν cover X, the intersections $X' \cap U^\nu$ cover X'. The sheaf property of $\mathcal{M}$ tells us that the map $\mathcal{M}(X') \to \prod_\nu \mathcal{M}(X' \cap U^\nu)$ is injective.

(iii) This follows from the third characteristic property. $\square$

7.4.3. Lemma. (i) *A short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ of $\mathcal{O}$-modules embeds into a diagram* Rcminjective

(7.4.4) $$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_\mathcal{M} & \longrightarrow & \mathcal{R}_\mathcal{N} & \longrightarrow & \mathcal{R}_\mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S}_\mathcal{M} & \longrightarrow & \mathcal{S}_\mathcal{N} & \longrightarrow & \mathcal{S}_\mathcal{P} \end{array}$$ ttdiagr

whose rows and columns are short exact sequences. (We have suppressed the surrounding zeros.)
(ii) *The sequence of global sections $0 \to \mathcal{R}_\mathcal{M}(X) \to \mathcal{R}_\mathcal{N}(X) \to \mathcal{R}_\mathcal{P}(X) \to 0$ is exact.*

proof. **(i)** We are given that the top row of the diagram is a short exact sequence, and we have seen that the columns are short exact sequences. To show that the middle row

(7.4.5) $$0 \to \mathcal{R}_\mathcal{M} \to \mathcal{R}_\mathcal{N} \to \mathcal{R}_\mathcal{P} \to 0$$ Rsequence

is exact, we must show that if X' is an affine open subset, the sections on X' form a short exact sequence. The sections are explained in Lemma 7.4.1 **(i)**. Since products of exact sequences are exact, we must show that the sequence

$$0 \to \mathcal{M}(X' \cap U^\nu) \to \mathcal{N}(X' \cap U^\nu) \to \mathcal{P}(X' \cap U^\nu) \to 0$$

is exact. This is true because $X' \cap U^\nu$ is an intersection of affine opens, and is therefore affine.

Now that we know that the first two rows of the diagram are short exact sequences, the Snake Lemma tells us that the bottom row is a short exact sequence.

(ii) The sequence of of global sections is the product of the sequences

$$0 \to \mathcal{M}(U^\nu) \to \mathcal{N}(U^\nu) \to \mathcal{P}(U^\nu) \to 0$$

These sequences are exact because the open sets U^ν are affine. □

uniquecoh **(7.4.6) uniqueness of cohomology**

Suppose that a cohomology with the characteristic properties (**7.3.1**) is given, and let $\mathcal{M}$ be an $\mathcal{O}$-module. The cohomology sequence associated to the sequence $0 \to \mathcal{M} \to \mathcal{R}_\mathcal{M} \to \mathcal{S}_\mathcal{M} \to 0$ is

$$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{R}_\mathcal{M}) \to H^0(X,\mathcal{S}_\mathcal{M}) \xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to H^1(X,\mathcal{R}_\mathcal{M}) \to \cdots$$

Lemma 7.4.1 **(iii)** tells us that $H^q(X,\mathcal{R}_\mathcal{M}) = 0$ when $q > 0$. So this cohomology sequence breaks up into an exact sequence

hone (7.4.7) $$0 \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{R}_\mathcal{M}) \to H^0(X,\mathcal{S}_\mathcal{M}) \xrightarrow{\delta^0} H^1(X,\mathcal{M}) \to 0$$

and isomorphisms

hq (7.4.8) $$0 \to H^q(X,\mathcal{S}_\mathcal{M}) \xrightarrow{\delta^q} H^{q+1}(X,\mathcal{M}) \to 0$$

for every $q > 0$. The first three terms of the sequence (7.4.7), and the arrows connecting them, depend on our choice of covering of X, but the important point is that they don't depend on the cohomology. So that sequence determines $H^1(X,\mathcal{M})$ up to unique isomorphism as the cokernel of a map that is independent of the cohomology, and this is true for every $\mathcal{O}$-module $\mathcal{M}$, including for the module $\mathcal{S}_\mathcal{M}$. Therefore it is also true that $H^1(X,\mathcal{S}_\mathcal{M})$ is determined uniquely. This being so, $H^2(X,\mathcal{M})$ is determined uniquely for every $\mathcal{M}$, by the isomorphism (7.4.8), with $q = 1$. The isomorphisms (7.4.8) determine the rest of the cohomology up to unique isomorphism by induction on q. □

existcoh **(7.4.9) construction of cohomology**

One can use the sequence (7.4.2) and induction to construct cohomology, but it seems clearer to proceed by iterating the construction of $\mathcal{R}_\mathcal{M}$.

Let $\mathcal{M}$ be an $\mathcal{O}$-module. We rewrite the exact sequence (7.4.2), labeling $\mathcal{R}_\mathcal{M}$ as $\mathcal{R}^0_\mathcal{M}$, and $\mathcal{S}_\mathcal{M}$ as $\mathcal{M}^1$:

MtoR-toMone (7.4.10) $$0 \to \mathcal{M} \to \mathcal{R}^0_\mathcal{M} \to \mathcal{M}^1 \to 0$$

and we repeat the construction with $\mathcal{M}^1$. Let $\mathcal{R}^1_\mathcal{M} = \mathcal{R}^0_{\mathcal{M}^1}$ $(= j_*\mathcal{M}^1_{\mathbb{U}})$, so that there is an exact sequence

cmonesequence (7.4.11) $$0 \to \mathcal{M}^1 \to \mathcal{R}^1_\mathcal{M} \to \mathcal{M}^2 \to 0$$

analogous to the sequence (7.4.10), with $\mathcal{M}^2 = \mathcal{R}^1_\mathcal{M}/\mathcal{M}^1$. We combine the sequences (7.4.10) and (7.4.11) into an exact sequence

RMone (7.4.12) $$0 \to \mathcal{M} \to \mathcal{R}^0_\mathcal{M} \to \mathcal{R}^1_\mathcal{M} \to \mathcal{M}^2 \to 0$$

Then we let $\mathcal{R}^2_\mathcal{M} = \mathcal{R}^0_{\mathcal{M}^2}$. We continue in this way, to construct modules $\mathcal{R}^k_\mathcal{M}$ that form an exact sequence

RM (7.4.13) $$0 \to \mathcal{M} \to \mathcal{R}^0_\mathcal{M} \to \mathcal{R}^1_\mathcal{M} \to \mathcal{R}^2_\mathcal{M} \to \cdots$$

The next lemma follows by induction from Lemma 7.4.1 **(iii)** and Lemma 7.4.3 **(i,ii)**.

7.4.14. Lemma. *Rcmexact*

(i) *Let* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *be a short exact sequence of* $\mathcal{O}$*-modules. For every* n*, the sequences*

$$0 \to \mathcal{R}^n_{\mathcal{M}} \to \mathcal{R}^n_{\mathcal{N}} \to \mathcal{R}^n_{\mathcal{P}} \to 0$$

are exact, and so are the sequences of global sections

$$0 \to \mathcal{R}^n_{\mathcal{M}}(X) \to \mathcal{R}^n_{\mathcal{N}}(X) \to \mathcal{R}^n_{\mathcal{P}}(X) \to 0$$

(ii) *If* $H^0,\ H^1,\ \ldots$ *is a cohomology theory, then* $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ *for all* n *and all* $q > 0$. □

An exact sequence such as (7.4.13) is called a *resolution* of $\mathcal{M}$, and because $H^q(X, \mathcal{R}^n_{\mathcal{M}}) = 0$ when $q > 0$, it is an *acyclic resolution*.

Continuing with the proof of existence, we consider the complex of $\mathcal{O}$-modules that is obtained by omitting the term $\mathcal{M}$ from (7.4.13). Let $\mathcal{R}^\bullet_{\mathcal{M}}$ denote that complex:

(7.4.15) $$\mathcal{R}^\bullet_{\mathcal{M}} \;=\; 0 \to \mathcal{R}^0_{\mathcal{M}} \to \mathcal{R}^1_{\mathcal{M}} \to \mathcal{R}^2_{\mathcal{M}} \to \cdots$$ Rsequencetwo

The complex $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ of its global sections:

(7.4.16) $$\mathcal{R}^\bullet_{\mathcal{M}}(X) \;=\; 0 \to \mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X) \to \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$ RMX

which we can also write as

$$0 \to H^0(X, \mathcal{R}^0_{\mathcal{M}}) \to H^0(X, \mathcal{R}^1_{\mathcal{M}}) \to H^0(X, \mathcal{R}^2_{\mathcal{M}}) \to \cdots$$

The complex $\mathcal{R}^\bullet_{\mathcal{M}}$ becomes the resolution (7.4.13) when the module $\mathcal{M}$ is inserted. So it is an exact sequence except at $\mathcal{R}^0_{\mathcal{M}}$. But the global section functor is only left exact. The sequence (7.4.16) of global sections $\mathcal{R}^\bullet_{\mathcal{M}}(X)$ needn't be exact anywhere. However, the sequence of global sections is a complex because $\mathcal{R}^\bullet_{\mathcal{M}}$ is a complex. The composition of adjacent maps is zero.

Recall that the cohomology of a complex $0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots$ of vector spaces is defined to be $\mathbf{C}^q(V^\bullet) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1})$, and that $\{\mathbf{C}^q\}$ is a cohomological functor on complexes (7.2.5).

7.4.17. Definition. The cohomology of an $\mathcal{O}$-module $\mathcal{M}$ is the cohomology of the complex $\mathcal{R}^\bullet_{\mathcal{M}}(X)$: definecoh

$$H^q(X, \mathcal{M}) = \mathbf{C}^q(\mathcal{R}^\bullet_{\mathcal{M}}(X))$$

Thus if we denote the maps in the complex (7.4.16) by d^q:

$$0 \to \mathcal{R}^0_{\mathcal{M}}(X) \xrightarrow{d^0} \mathcal{R}^1_{\mathcal{M}}(X) \xrightarrow{d^1} \mathcal{R}^2_{\mathcal{M}}(X) \to \cdots$$

then $H^q(X, \mathcal{M}) = (\ker\, d^q)/(\mathrm{im}\, d^{q-1})$.

7.4.18. Lemma. *Let* X *be an affine variety. With cohomology defined as above,* $H^q(X, \mathcal{M}) = 0$ *for all* $\mathcal{O}$*-modules* $\mathcal{M}$ *and all* $q > 0$. *affinecohzero*

proof. When X is affine, the sequence of global sections of the exact sequence (7.4.13) is exact. □

To show that our definition gives the unique cohomology, we verify the three characteristic properties. Since the sequence (7.4.13) is exact and since the global section functor is left exact, $\mathcal{M}(X)$ is the kernel of the map $\mathcal{R}^0_{\mathcal{M}}(X) \to \mathcal{R}^1_{\mathcal{M}}(X)$. This kernel is also equal to $\mathbf{C}^0(\mathcal{R}^\bullet_{\mathcal{M}}(X))$, so our cohomology has the first property: $H^0(X, \mathcal{M}) = \mathcal{M}(X)$.

To show that we obtain a cohomological functor, we apply Lemma 7.4.14 to conclude that, for a short exact sequence $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$, the spaces of global sections

(7.4.19) $$0 \to \mathcal{R}^\bullet_{\mathcal{M}}(X) \to \mathcal{R}^\bullet_{\mathcal{N}}(X) \to \mathcal{R}^\bullet_{\mathcal{P}}(X) \to 0,$$ crtwo

form an exact sequence of complexes. The cohomology $H^q(X, \cdot\,)$ is a cohomological functor because cohomology of complexes is a cohomological functor. This is the second characteristic property.

We make a digression before verifying the third characteristic property.

(7.4.20) affine morphisms

affinemorph

Let $Y \xrightarrow{f} X$ be a morphism of varieties. Let $U \xrightarrow{j} X$ be the inclusion of an open subvariety into X and let V be the inverse image $f^{-1}U$, which is an open subvariety of Y. These varieties and maps form a diagram

jstarfstar (7.4.21)
$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

As before, the notation $\mathcal{M}_U$ stands for the restriction of $\mathcal{M}$ to the open subset U.

With notation as in the diagram above, let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. When we restrict the direct image $f_*\mathcal{N}$ of $\mathcal{N}$ to U, we obtain an $\mathcal{O}_U$-module $[f_*\mathcal{N}]_U$. We can obtain an $\mathcal{O}_U$-module in a second way: First restrict the module $\mathcal{N}$ to the open subset V of Y, and then take its direct image. This gives us the $\mathcal{O}_U$-module $g_*[\mathcal{N}_V]$.

jstarfstartwo **7.4.22. Lemma.** *The $\mathcal{O}_U$-modules $g_*[\mathcal{N}_V]$ and $[f_*\mathcal{N}]_U$ are equal.*

proof. Let U' be an open subset of U, and let $V' = g^{-1}U'$. Then

$$[f_*\mathcal{N}]_U(U') = [f_*\mathcal{N}](U') = \mathcal{N}(V') = \mathcal{N}_V(V') = [g_*[\mathcal{N}_V]](U') \qquad \square$$

defaffmorph **7.4.23. Definition.** An *affine morphism* is a morphism $Y \xrightarrow{f} X$ of varieties that has this property: The inverse image $f^{-1}(U)$ of every affine open subset U of X is an affine open subset of Y. $\square$

The following are examples of affine morphisms:

- the inclusion of an affine open subset Y into X,
- the inclusion of a closed subvariety Y into X,
- a finite morphism, or an integral morphism.

The inclusion of a non-affine open set needn't be an affine morphism. For example, if Y is the complement of a point of the projective plane X, the inclusion $Y \to X$ isn't an affine morphism.

fstaraffmorph **7.4.24. Lemma.** *Let $Y \xrightarrow{f} X$ be an affine morphism and let $\mathcal{N} \to \mathcal{N}' \to \mathcal{N}''$ be an exact sequence of $\mathcal{O}_Y$-modules. The direct images form an exact sequence of $\mathcal{O}_X$-modules $f_*\mathcal{N} \to f_*\mathcal{N}' \to f_*\mathcal{N}''$.* $\square$

Let $Y \xrightarrow{f} X$ be an affine morphism, let j be the map from our chosen affine covering $\mathbf{U} = \{U^\nu\}$ to X, and let $\mathbf{V}$ denote the family $\{V^\nu\} = \{f^{-1}U^\nu\}$ of inverse images. Then $\mathbf{V}$ is an affine covering of Y, and there is a morphism $\mathbf{V} \xrightarrow{g} \mathbf{U}$. We form a diagram analogous to (7.4.21), in which $\mathbf{V}$ and $\mathbf{U}$ replace V and U, respectively:

VtoU (7.4.25)
$$\begin{tikzcd} \mathbf{V} \arrow[r, "i"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ \mathbb{U} \arrow[r, "j"] & X \end{tikzcd}$$

affinedirectimage **7.4.26. Proposition.** *Let $Y \xrightarrow{f} X$ be an affine morphism, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Let $H^q(X, \cdot\,)$ be cohomology defined as in (7.4.17), and let $H^q(Y, \cdot\,)$ be cohomology defined in the analogous way, using the covering $\mathbf{V}$ of Y. Then $H^q(X, f_*\mathcal{N})$ is isomorphic to $H^q(Y, \mathcal{N})$.*

cohextbyO **7.4.27. Corollary.** *Let $Y \xrightarrow{i} X$ be the inclusion of a closed subvariety Y into a variety X, and let $\mathcal{N}$ be an $\mathcal{O}_Y$-module. Then, for every q, $H^q(Y, \mathcal{N})$ and $H^q(X, i_*\mathcal{N})$ are isomorphic.* $\square$

proof of Proposition 7.4.26. This proof is easy, once one has untangled the notation. To compute the cohomology of $f_*\mathcal{N}$ on X, we substitute $\mathcal{M} = f_*\mathcal{N}$ into (7.4.17):

$$H^q(X, f_*\mathcal{N}) = \mathbf{C}^q(\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)).$$

To compute the cohomology of $\mathcal{N}$ on Y, we let

$$\mathcal{R}'^0_{\mathcal{N}} = i_*[\mathcal{N}_V]$$

where i is as in Diagram 7.4.25, and we continue, to construct a resolution $0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{R}'^1_{\mathcal{N}} \to \cdots$ and the complex $\mathcal{R}'^\bullet_{\mathcal{N}}$ obtained by replacing the term $\mathcal{N}$ by zero. (The prime is there to remind us that $\mathcal{R}'$ is defined using the covering $\mathbf{V}$ of Y.) Then

$$H^q(Y, \mathcal{N}) = \mathbf{C}^q(\mathcal{R}'^\bullet_{\mathcal{N}}(Y)).$$

It suffices to show that the complexes $\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)$ and $\mathcal{R}'^\bullet_{\mathcal{N}}(Y)$ are isomorphic. If so, we will have

$$H^q(X, f_*\mathcal{N}) = \mathbf{C}^q(\mathcal{R}^\bullet_{f_*\mathcal{N}}(X)) \approx \mathbf{C}^q(\mathcal{R}'^\bullet_{\mathcal{N}}(Y)) = H^q(Y, \mathcal{N})$$

as required.

By definition of the direct image, $[f_*\mathcal{R}'^q_{\mathcal{N}}](X) = \mathcal{R}'^q_{\mathcal{N}}(Y)$. So we must show that $[\mathcal{R}^q_{f_*\mathcal{N}}](X) \approx f_*\mathcal{R}'^q_{\mathcal{N}}(X)$, and it suffices to show that $\mathcal{R}^q_{f_*\mathcal{N}} \approx f_*\mathcal{R}'^q_{\mathcal{N}}$. We look back at the definition (7.4.11) of the modules $\mathcal{R}^0$ in its rewritten form (7.4.10). We refer to Diagram 7.4.25. On Y, the analogous sequence for $\mathcal{N}$ is

$$0 \to \mathcal{N} \to \mathcal{R}'^0_{\mathcal{N}} \to \mathcal{N}^1 \to 0$$

where $\mathcal{R}'^0_{\mathcal{N}} = i_*\mathcal{N}_{\mathbf{V}}$. When f is an affine morphism, the direct image of this sequence is exact:

$$0 \to f_*\mathcal{N} \to f_*\mathcal{R}'^0_{\mathcal{N}} \to f_*\mathcal{N}^1 \to 0$$

Here

$$f_*\mathcal{R}'^0_{\mathcal{N}} = f_*i_*\mathcal{N}_{\mathbf{V}} = j_*g_*\mathcal{N}_{\mathbf{V}} = j_*[f_*\mathcal{N}]_{\mathbb{U}} = \mathcal{R}^0_{f_*\mathcal{N}}$$

the third equality being Lemma 7.4.22. So $f_*\mathcal{R}'^0_{\mathcal{N}} = \mathcal{R}^0_{f_cn}$. Now induction on q applies. □

We go back to the proof of existence of cohomology to verify the third characteristic property, which is that when $Y \xrightarrow{f} X$ is the inclusion of an affine open subset, $H^q(X, f_*\mathcal{N}) = 0$ for all $\mathcal{O}_Y$-modules $\mathcal{N}$ and all $q > 0$. The inclusion of an affine open set is an affine morphism, so $H^q(Y, \mathcal{N}) = H^q(X, f_*\mathcal{N})$ (7.4.26), and since Y is affine, $H^q(Y, \mathcal{N}) = 0$ for all $q > 0$ (7.4.18). □

Proposition 7.4.26 is one of the places where a specific construction of cohomology is used. The characteristic properties don't apply directly. The next proposition is another such place.

7.4.28. Lemma. *Cohomology is compatible with limits of directed sets of $\mathcal{O}$-modules: $H^q(X, \varinjlim \mathcal{M}_\bullet) \approx \varinjlim H^q(X, \mathcal{M}_\bullet)$ for all q.* *cohlimit*

proof. The direct and inverse image functors and the global section functor are all compatible with $\varinjlim$, and $\varinjlim$ is exact (6.5.17). So the module $\mathcal{R}^q_{\varinjlim \mathcal{M}_\bullet}$ that is used to compute the cohomology of $\varinjlim \mathcal{M}_\bullet$ is isomorphic to $\varinjlim [\mathcal{R}^q_{\mathcal{M}_\bullet}]$, and $\mathcal{R}^q_{\varinjlim \mathcal{M}_\bullet}(X)$ is isomorphic to $\varinjlim [\mathcal{R}^q_{\mathcal{M}_\bullet}](X)$. □

Section 7.5 Cohomology of the Twisting Modules

cohprojsp

The cohomology of the twisting modules $\mathcal{O}(d)$ on $\mathbb{P}^n$ helps to determine the cohomology of other modules. As we will see, $H^q(\mathbb{P}^n, \mathcal{O}(d))$ is zero for most values of q.

Lemma 7.4.18 about vanishing of cohomology on an affine variety, and Lemma 7.4.26 about the direct image via an affine morphism, were stated using a particular affine covering. Since we know that cohomology is unique, that particular covering is irrelevant. Though it isn't necessary, we restate those lemmas here as a corollary:

affineco-hzerotwo **7.5.1. Corollary. (i)** *On an affine variety X, $H^q(X,\mathcal{M})=0$ for all $\mathcal{O}$-modules $\mathcal{M}$ and all $q>0$.*

(ii) *Let $Y \xrightarrow{f} X$ be an affine morphism. If $\mathcal{N}$ is an $\mathcal{O}_Y$-module, then $H^q(X,f_*\mathcal{N})$ and $H^q(Y,\mathcal{N})$ are isomorphic. If Y is an affine variety, $H^q(X,f_*\mathcal{N})=0$ for all $q>0$.* □

One case in which **(ii)** applies is that f is the inclusion of a closed subvariety Y into a variety X.

cohXcohP **7.5.2. Corollary** *Let $X \xrightarrow{i} \mathbb{P}^n$ be the embedding of a projective variety into projective space and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. For all q, $H^q(X,\mathcal{M})$ is isomorphic to the cohomology $H^q(\mathbb{P}^n,i_*\mathcal{M})$ of its extension by zero $i_*\mathcal{M}$.* □

Recall also that if $\mathcal{M}$ is an $\mathcal{O}_X$-module on a projective variety X, the twist $\mathcal{M}(d)$ of $\mathcal{M}$ is defined as the $\mathcal{O}_X$-module that corresponds to the twist of its extension by zero $i_*\mathcal{M}$, which is $[i_*\mathcal{M}]\otimes_{\mathcal{O}}\mathcal{O}(d)$.

Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on projective space $\mathbb{P}^n$. Recall also that the twisting modules $\mathcal{O}(d)$ and the twists $\mathcal{M}(d)=\mathcal{M}\otimes_{\mathcal{O}}\mathcal{O}(d)$ are isomorphic to $\mathcal{O}(dH)$ and $\mathcal{M}(dH)$, respectively (6.8.11) and that there are maps of directed sets

$$\begin{array}{ccccccc}
\mathcal{O} & \xrightarrow{\subset} & \mathcal{O}(H) & \xrightarrow{\subset} & \mathcal{O}(2H) & \xrightarrow{\subset} & \cdots \\
{\scriptstyle 1}\big\downarrow & & {\scriptstyle x_0}\big\downarrow & & {\scriptstyle x_0^2}\big\downarrow & & \\
\mathcal{O} & \xrightarrow{x_0} & \mathcal{O}(1) & \xrightarrow{x_0} & \mathcal{O}(2) & \xrightarrow{x_0} & \cdots
\end{array}
\quad , \quad
\begin{array}{ccccccc}
\mathcal{M} & \longrightarrow & \mathcal{M}(H) & \longrightarrow & \mathcal{M}(2H) & \longrightarrow & \cdots \\
{\scriptstyle 1}\big\downarrow & & {\scriptstyle x_0}\big\downarrow & & {\scriptstyle x_0^2}\big\downarrow & & \\
\mathcal{M} & \xrightarrow{x_0} & \mathcal{M}(1) & \xrightarrow{x_0} & \mathcal{M}(2) & \xrightarrow{x_0} & \cdots
\end{array}$$

where the second diagram is obtained from the first one by tensoring with $\mathcal{M}$ (7.4.11). Let $\mathbb{U}$ denote the standard affine open subset $\mathbb{U}^0$ of $\mathbb{P}^n$, and let j be the inclusion of $\mathbb{U}$ into $\mathbb{P}^n$. Then $\varinjlim \mathcal{O}(dH)\approx j_*\mathcal{O}_{\mathbb{U}}$ and $\varinjlim \mathcal{M}(dH)\approx j_*\mathcal{M}_{\mathbb{U}}$ (6.8.15). Since the inclusion $\mathbb{U}\xrightarrow{j}\mathbb{P}^n$ is an affine morphism and $\mathbb{U}$ is affine, $H^q(\mathbb{P}^n,j_*\mathcal{O}_{\mathbb{U}})=0$ and $H^q(\mathbb{P}^n,j_*\mathcal{M}_{\mathbb{U}})=0$ for all $q>0$.

The next corollary follows from the facts that $\mathcal{M}(d)$ is isomorphic to $\mathcal{M}(dH)$, and that cohomology is compatible with direct limits (7.4.28).

Onse-quencetwo **7.5.3. Corollary.** *For all projective varieties X, for all $\mathcal{O}$-modules $\mathcal{M}$ and for all $q>0$, $\varinjlim_d H^q(X,\mathcal{O}(d))=0$ and $\varinjlim_d H^q(X,\mathcal{M}(d))=0$.* □

dimnota-tion **7.5.4. Notation.** If $\mathcal{M}$ is an $\mathcal{O}$-module, we denote the dimension of $H^q(X,\mathcal{M})$ by $\mathbf{h}^q(X,\mathcal{M})$ or by $\mathbf{h}^q\mathcal{M}$. We can write $\mathbf{h}^q\mathcal{M}=\infty$ if the dimension is infinite. However, in Section 7.7, we will see that when $\mathcal{M}$ is a finite $\mathcal{O}$-module on a projective variety X, the dimension of $H^q(X,\mathcal{M})$ is finite for every q. □

cohOd **7.5.5. Theorem.** *Let $X=\mathbb{P}^n$.*

(i) *For $d\ge 0$, $\mathbf{h}^0(X,\mathcal{O}(d))=\binom{d+n}{n}$ and $\mathbf{h}^q(X,\mathcal{O}(d))=0$ if $q\neq 0$.*

(ii) *For $r>0$, $\mathbf{h}^n(X,\mathcal{O}(-r))=\binom{r-1}{n}$ and $\mathbf{h}^q(X,\mathcal{O}(-r))=0$ if $q\neq n$.*

Note that the case $d=0$ asserts that $\mathbf{h}^0(\mathbb{P}^n,\mathcal{O})=1$ and $\mathbf{h}^q(\mathbb{P}^n,\mathcal{O})=0$ for all $q>0$, and the case $r=1$ asserts that $\mathbf{h}^q(\mathbb{P}^n,\mathcal{O}(-1))=0$ for all q.

proof. We have described the global sections of $\mathcal{O}(d)$ before: If $d\ge 0$, $H^0(X,\mathcal{O}(d))$ is the space of homogeneous polynomials of degree d in the coordinate variables. Its dimension is $\binom{d+n}{n}$, and $H^0(X,\mathcal{O}(d))=0$ if $d<0$. (See (6.8.2).)

Let Y be the hyperplane at infinity $\{x-0=0\}$, and let $Y\xrightarrow{i}X$ be the inclusion of Y into X.

(i) *the case $d\ge 0$.*

By induction on n, we may assume that the theorem has been proved for Y, which is a projective space of dimension $n-1$. We consider the exact sequence

basecase (7.5.6)
$$0\to\mathcal{O}_X(-1)\xrightarrow{x_0}\mathcal{O}_X\to i_*\mathcal{O}_Y\to 0$$

and its twists

Od (7.5.7)
$$0\to\mathcal{O}_X(d-1)\xrightarrow{x_0}\mathcal{O}_X(d)\to i_*\mathcal{O}_Y(d)\to 0$$

The twisted sequences are exact because they are obtained by tensoring (7.5.6) with the invertible $\mathcal{O}$-modules $\mathcal{O}(d)$. Because the inclusion i is an affine morphism, $H^q(X, i_*\mathcal{O}_Y(d)) \approx H^q(Y, \mathcal{O}_Y(d))$.

The monomials of degree d in $n{+}1$ variables form a basis of the space of global sections of $\mathcal{O}_X(d)$. Setting $x_0 = 0$ and deleting terms that become zero gives us a basis of $\mathcal{O}_Y(d)$. Every global section of $\mathcal{O}_Y(d)$ is the restriction of a global section of $\mathcal{O}_X(d)$. The sequence of global sections

$$0 \to H^0(X, \mathcal{O}_X(d{-}1)) \xrightarrow{x_0} H^0(X, \mathcal{O}_X(d)) \to H^0(Y, \mathcal{O}_Y(d)) \to 0$$

is exact. The cohomology sequence associated to (7.5.7) tells us that the map $H^1(X, \mathcal{O}_X(d-1)) \longrightarrow H^1(X, \mathcal{O}_X(d))$ is injective.

By induction on the dimension n, $H^q(Y, \mathcal{O}_Y(d)) = 0$ for $d \geq 0$ and $q \geq 0$. When combined with the injectivity noted above, the cohomology sequence of (7.5.7) shows that the maps $H^q(X, \mathcal{O}_X(d{-}1)) \to H^q(X, \mathcal{O}_X(d))$ are bijective for every $q > 0$. Since the limits are zero (7.5.3), $H^q(X, \mathcal{O}_X(d)) = 0$ for all $d \geq 0$ and all $q > 0$.

(ii) *the case* $d < 0$*, or* $r > 0$.

We use induction on the integers r and n. We suppose the theorem proved for a given r, and we substitute $d = -r$ into the sequence (7.5.7):

(7.5.8) $$0 \to \mathcal{O}_X(-(r{+}1)) \xrightarrow{x_0} \mathcal{O}_X(-r) \to i_*\mathcal{O}_Y(-r) \to 0$$ Or

The base case $r = 0$ is the exact sequence (7.5.6). In the cohomology sequence associated to that sequence, the terms $H^q(X, \mathcal{O}_X)$ and $H^q(Y, \mathcal{O}_Y)$ are zero when $q > 0$, and $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = \mathbb{C}$. Therefore $H^q(X, \mathcal{O}_X(-1)) = 0$ for every q. This proves **(ii)** for $r = 1$.

Our induction hypothesis is that, $\mathbf{h}^n(X, \mathcal{O}(-r)) = \binom{r-1}{n}$ and $\mathbf{h}^q = 0$ if $q \neq n$. By induction on n, we may suppose that $\mathbf{h}^{n-1}(Y, \mathcal{O}(-r)) = \binom{r-1}{n-1}$ and that $\mathbf{h}^q = 0$ if $q \neq n-1$.

Instead of displaying the cohomology sequence associated to (7.5.8), we assemble the dimensions of cohomology into a table in which the asterisks stand for entries that are to be determined:

(7.5.9) $$\begin{array}{cc|ccc} & & \mathcal{O}_X(-(r{+}1)) & \mathcal{O}_X(-r) & i_*\mathcal{O}_Y(-r) \\ \hline \mathbf{h}^0 & : & * & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \mathbf{h}^{n-2} & : & * & 0 & 0 \\ \mathbf{h}^{n-1} & : & * & 0 & \binom{r-1}{n-1} \\ \mathbf{h}^n & : & * & \binom{r-1}{n} & 0 \end{array}$$ co-hdimstwo

The second column is determined by induction on r and the third column is determined by induction on n. The cohomology sequence shows that that

$$\mathbf{h}^n(X, \mathcal{O}(-(r{+}1))) = \binom{r-1}{n-1} + \binom{r-1}{n}$$

and that the other entries labeled with an asterisk are zero. The right side of this equation is equal to $\binom{r}{n}$. $\square$

Section 7.6 Cohomology of Hypersurfaces

cohhyper

We begin by determining the cohomology of a plane projective curve. Let X be the projective plane $\mathbb{P}^2$ and let $C \xrightarrow{i} X$ denote the inclusion of a plane curve of degree k. The ideal $\mathcal{I}$ of functions that vanish on C is isomorphic to the twisting module $\mathcal{O}_X(-k)$ (6.8.8 so one has an exact sequence

(7.6.1) $$0 \to \mathcal{O}_X(-k) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0$$ coh-planecurvetwo

The table below shows dimensions of the cohomology. Theorem 7.5.5 determines the first two columns, and the cohomology sequence determines the last column.

cohdims (7.6.2)

	$\mathcal{O}_X(-k)$	$\mathcal{O}_X$	$i_*\mathcal{O}_C$
$\mathbf{h}^0$:	0	1	1
$\mathbf{h}^1$:	0	0	$\binom{k-1}{2}$
$\mathbf{h}^2$:	$\binom{k-1}{2}$	0	0

Since the inclusion of the curve C into the projective plane X is an affine morphism, $\mathbf{h}^q(C, \mathcal{O}_C) = \mathbf{h}^q(X, i_*\mathcal{O}_C)$. Therefore

coh-planecurve (7.6.3) $$\mathbf{h}^0(C, \mathcal{O}_C) = 1,\ \ \mathbf{h}^1(C, \mathcal{O}_C) = \binom{k-1}{2},\ \text{ and }\ \mathbf{h}^q = 0\ \text{ when }\ q > 1$$

The dimension of $H^1(C, \mathcal{O}_C)$, which is $\binom{k-1}{2}$, is called the *arithmetic genus* of C. It is denoted by p_a or $p_a(C)$. As we will see later (8.8.2), the arithmetic genus of smooth curve is equal to its topological genus: $p_a = g$. But the arithmetic genus of a plane curve of degree k is $\binom{k-1}{2}$ when the curve C is singular too.

We restate the results as a corollary.

coh-planecurve **7.6.4. Corollary.** *Let C be a plane curve of degree k. Then* $\mathbf{h}^0\mathcal{O}_C = 1,\ \mathbf{h}^1\mathcal{O}_C = \binom{k-1}{2} = p_a$, *and* $\mathbf{h}^q = 0$ *if* $q \neq 0, 1$. □

The fact that $\mathbf{h}^0\mathcal{O}_C = 1$ tells us that the only rational functions that are regular everywhere on C are the constants. This reflects a fact that will be proved later, that a plane curve is compact and connected in the classical topology, but it isn't a proof of that fact.

In the next section we will see that the cohomology on any projective curve is zero except in dimensions 0 and 1. To determine cohomology of a projective curve in a higher dimensional projective space, we will need to know that its cohomology is finite-dimensional, which is Theorem 7.7.3 below, and that it is zero in dimension greater than one, which is Theorem 7.7.1. Cohomology of projective curves is the topic of Chapter 8.

One can make a similar computation for the hypersurface Y in $X = \mathbb{P}^n$ defined by an irreducible homogeneous polynomial f of degree k. The ideal of Y is isomorphic to $\mathcal{O}_X(-k)$ (6.8.8), so there is an exact sequence

$$0 \to \mathcal{O}_X(-k) \xrightarrow{f} \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

Since we know the cohomology of $\mathcal{O}_X(-k)$ and of $\mathcal{O}_X$, and since $H^q(X, i_*\mathcal{O}_Y) \approx H^q(Y, \mathcal{O}_Y)$, we can use this sequence to compute the dimensions of the cohomology of $\mathcal{O}_Y$.

cohhyper-surface **7.6.5. Corollary.** *Let Y be a hypersurface of dimension d and degree k in a projective space of dimension $d+1$. Then* $\mathbf{h}^0(Y, \mathcal{O}_Y) = 1,\ \mathbf{h}^d(Y, \mathcal{O}_Y) = \binom{k-1}{d+1}$, *and* $\mathbf{h}^q(Y, \mathcal{O}_Y) = 0$ *for all other* q. □

In particular, when S is the *surface* in $\mathbb{P}^3$ defined by an irreducible polynomial of degree k, $\mathbf{h}^0(S, \mathcal{O}_S) = 1$, $\mathbf{h}^1(S, \mathcal{O}_S) = 0$, $\mathbf{h}^2(S, \mathcal{O}_S) = \binom{k-1}{3}$, and $\mathbf{h}^q = 0$ if $q > 2$. For a projective surface S that isn't embedded into $\mathbb{P}^3$, it is still true that $\mathbf{h}^q = 0$ when $q > 2$, but $\mathbf{h}^1(S, \mathcal{O}_S)$ may be nonzero. The dimensions $\mathbf{h}^1(S, \mathcal{O}_S)$ and $\mathbf{h}^2(S, \mathcal{O}_S)$ are invariants of the surface S that are somewhat analogous to the genus of a curve. In classical terminology, $\mathbf{h}^2(S, \mathcal{O}_S)$ is the *geometric genus* p_g and $\mathbf{h}^1(S, \mathcal{O}_S)$ is the *irregularity* q . The *arithmetic genus* p_a of S is defined to be

patwo (7.6.6) $$p_a \ = \ \mathbf{h}^2(S, \mathcal{O}_S) - \mathbf{h}^1(S, \mathcal{O}_S) = p_g - q$$

Therefore the irregularity is $q = p_g - p_a$. When S is a surface in $\mathbb{P}^3$, $q = 0$ and $p_g = p_a$.

In modern terminology, it would be more natural to replace the arithmetic genus by the Euler characteristic of the structure sheaf $\chi(\mathcal{O}_S)$, which is defined to be $\sum_q (-1)^q \mathbf{h}^q\mathcal{O}_S$ (see (**7.7.7**) below). The Euler characteristic of the structure sheaf on a curve is

$$\chi(\mathcal{O}_C) = \mathbf{h}^0(C, \mathcal{O}_C) - \mathbf{h}^1(C, \mathcal{O}_C) = 1 - p_a$$

and on a surface S it is

$$\chi(\mathcal{O}_S) = \mathbf{h}^0(S, \mathcal{O}_S) - \mathbf{h}^1(S, \mathcal{O}_S) + \mathbf{h}^2(S, \mathcal{O}_S) = 1 + p_a$$

But because of tradition, the arithmetic genus is still used quite often.

Section 7.7 Three Theorems about Cohomology

threethms

These theorems are taken from Serre's paper.

7.7.1. Theorem. *Let X be a projective variety, and let $\mathcal{M}$ be a finite $\mathcal{O}_X$-module. If the support of $\mathcal{M}$ has dimension k, then $H^q(X, \mathcal{M}) = 0$ for all $q > k$. In particular, if X has dimension n, then $H^q(X, \mathcal{M}) = 0$ for all $q > n$.* *cohsupport*

See Section 6.7 for the definition of support.

7.7.2. Theorem. *Let $\mathcal{M}(d)$ be the twist of a finite $\mathcal{O}_X$-module $\mathcal{M}$ on a projective variety X. For sufficiently large d, $H^q(X, \mathcal{M}(d)) = 0$ for all $q > 0$.* *largetwist*

7.7.3. Theorem. *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. For every q, the cohomology $H^q(X, \mathcal{M})$ is a finite-dimensional vector space.* *findim*

7.7.4. Notes. (a) As the first theorem asserts, the highest dimension in which cohomology of an $\mathcal{O}_X$-module on a projective variety X can be nonzero is the dimension of X. It is also true that, on projective variety X of dimension n, there will be $\mathcal{O}_X$-modules $\mathcal{M}$ such that $H^n(X, \mathcal{M}) \neq 0$. In contrast, in the classical topology on a projective variety X of dimnsion n, the constant coefficient cohomology $H^{2n}(X_{class}, \mathbb{Z})$ isn't zero. As we have mentioned, the constant coefficient cohomology $H^q(X_{zar}, \mathbb{Z})$ in the Zariski topology is zero for every $q > 0$, and when X is affine, the cohomology of any $\mathcal{O}_X$-module is zero when $q > 0$. descind

(b) The third theorem tells us that the space of global sections $H^0(X, \mathcal{M})$ of a finite $\mathcal{O}$-module $\mathcal{M}$ on a projective variety X is finite-dimensional. This is one of the most important consequences of the theorem, and it isn't easy to prove directly. Cohomology needn't be finite-dimensional on a variety that isn't projective. For example, on an affine variety $X = \operatorname{Spec} A$, $H^0(X, \mathcal{O}) = A$ isn't finite-dimensional unless X is a point. When X is the complement of a point in $\mathbb{P}^2$, $H^1(X, \mathcal{O})$ isn't finite-dimensional.

(c) The proofs have an interesting structure. The first theorem allows us to use *descending* induction to prove the second and third theorems, beginning with the fact that $H^k(X, \mathcal{M}) = 0$ when k is greater than the dimension of X. □

In these theorems, we are given that X is a closed subvariety of a projective space $\mathbb{P}^n$. We can replace an $\mathcal{O}_X$-module by its extension by zero to $\mathbb{P}^n$ (7.5.1), since this doesn't change the cohomology or the dimension of support. The twist $\mathcal{M}(d)$ of an $\mathcal{O}_X$-module that is referred to in the second theorem is defined in terms of the extension by zero. So we may assume that X is a projective space.

The proofs are based on the cohomology of the twisting modules (7.5.5) and on the vanishing of the limit $\varinjlim H^q(X, \mathcal{M}(d))$ for $q > 0$ (7.5.3).

proof of Theorem 7.7.1 (*vanishing in large dimension*)
Here $\mathcal{M}$ is a finite $\mathcal{O}$-module whose support S has dimension at most k. We are to show that $H^q(X, \mathcal{M}) = 0$ when $q > k$. We choose coordinates so that the hyperplane $H : x_0 = 0$ doesn't contain any component of the support S. Then $H \cap S$ has dimension at most $k-1$. We inspect the multiplication map $\mathcal{M}(-1) \xrightarrow{x_0} \mathcal{M}$. The kernel $\mathcal{K}$ and cokernel $\mathcal{Q}$ are annihilated by x_0, so the supports of $\mathcal{K}$ and $\mathcal{Q}$ are contained in H. Since they are also in S, the supports have dimension at most $k - 1$. We can apply induction on k to them. In the base case $k = 0$, the supports of $\mathcal{K}$ and $\mathcal{Q}$ will be empty, and their cohomology will be zero.

We break the exact sequence $0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{M} \to \mathcal{Q} \to 0$ into two short exact sequences by introducing the kernel $\mathcal{N}$ of the map $\mathcal{M} \to \mathcal{Q}$:

(7.7.5) $$0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{N} \to 0 \quad \text{and} \quad 0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{Q} \to 0$$ kercoker

The induction hypothesis applies to $\mathcal{K}$ and to $\mathcal{Q}$. It tells us that $H^q(X,\mathcal{K}) = 0$ and $H^q(X,\mathcal{Q}) = 0$, when $q \geq k$. For $q > k$, the relevant parts of the cohomology sequences associated to the two exact sequences become

$$0 \to H^q(X,\mathcal{M}(-1)) \to H^q(X,\mathcal{N}) \to 0 \quad \text{and} \quad 0 \to H^q(X,\mathcal{N}) \to H^q(X,\mathcal{M}) \to 0$$

respectively. Therefore the maps $H^q(X,\mathcal{M}(-1)) \to H^q(X,\mathcal{N}) \to H^q(X,\mathcal{M})$ are bijective, and this is true for every $\mathcal{O}$-module whose support has dimension $\leq k$, including for the $\mathcal{O}$-module $\mathcal{M}(d)$. For every $\mathcal{O}$-module whose support has dimension at most k, every d, and every $q > k$, the canonical map $H^q(X,\mathcal{M}(d-1)) \to H^q(X,\mathcal{M}(d))$ is bijective.

According to (7.5.3), the limit $\varinjlim H^q(X,\mathcal{M}(d))$ is zero. It follows that, when $q > k$, $H^q(X,\mathcal{M}(d)) = 0$ for all d, and in particular, $H^q(X,\mathcal{M}) = 0$.

proof of Theorem 7.7.2 (*vanishing for a large twist*)

Let $\mathcal{M}$ be a finite $\mathcal{O}$-module on a projective variety X. We recall that $\mathcal{M}(r)$ is generated by global sections when r is sufficiently large (6.8.20). Choosing generators gives us a surjective map $\mathcal{O}^r \to \mathcal{M}(r)$. Let $\mathcal{N}$ be the kernel of this map. When we twist the sequence $0 \to \mathcal{N} \to \mathcal{O}^r \to \mathcal{M}(r) \to 0$, we obtain short exact sequences

threetwos (7.7.6) $$0 \to \mathcal{N}(d) \to \mathcal{O}(d)^r \to \mathcal{M}(d{+}r) \to 0$$

for every $d \geq 0$. These sequences are useful because $H^q(X,\mathcal{O}(d)) = 0$ when $q > 0$ (7.5.5).

To prove Theorem 7.7.2, we must show this:

(*) *Let $\mathcal{M}$ be a finite $\mathcal{O}$-module. For sufficiently large d and for all $q > 0$, $H^q(X,\mathcal{M}(d)) = 0$.*

Let n be the dimnsion of X. By Theorem 7.7.1, $H^q(X,\mathcal{M}) = 0$ for any $\mathcal{O}$-module $\mathcal{M}$, when $q > n$, In particular, $H^q(X,\mathcal{M}(d)) = 0$ when $q > n$. This leaves a finite set of integers $q = 1,...,n$ to consider, and it suffices to consider them one at a time. If (*) is true for each individual q it will be true for the finite set of integers $q = 1,...,n$ at the same time, and therefore for all positive integers q, as the theorem asserts.

We use descending induction on q, the base case being $q = n+1$, for which (*) is true. We suppose that (*) is true for every finite $\mathcal{O}$-module $\mathcal{M}$ when $q = p+1$, and that $p > 0$, and we show that (*) is true for every finite $\mathcal{O}$-module $\mathcal{M}$ when $q = p$.

We substitute $q = p$ into the cohomology sequence associated to the sequence (7.7.6). The relevant part of that sequence is

$$\to H^p(X,\mathcal{O}(d)^m) \to H^p(X,\mathcal{M}(d{+}r)) \xrightarrow{\delta^p} H^{p+1}(X,\mathcal{N}(d)) \to$$

Since p is positive, $H^p(X,\mathcal{O}(d)) = 0$ for all $d \geq 0$, and therefore the map δ^p is injective. We note that $\mathcal{N}$ is a finite $\mathcal{O}$-module. So our induction hypothesis applies to it. The induction hypothesis tells us that, when d is large, $H^{p+1}(X,\mathcal{N}(d)) = 0$ and therefore that $H^p(X,\mathcal{M}(d{+}r)) = 0$. The particular integer $d{+}r$ isn't useful. Our conclusion is that, for every finite $\mathcal{O}$-module $\mathcal{M}$, $H^p(X,\mathcal{M}(k)) = 0$ when k is large enough. □

proof of Theorem 7.7.3 (*finiteness of cohomology*)

This proof uses ascending induction on the dimension of support as well as descending induction on the degree d of a twist. As was mentioned above, it isn't easy to prove directly that the space $H^0(X,\mathcal{M})$ of global sections is finite-dimensional.

Let $\mathcal{M}$ be an $\mathcal{O}$-module whose support has dimension at most k. We go back to the sequences (7.7.5) and their cohomology sequences, in which the supports of $\mathcal{K}$ and $\mathcal{Q}$ have dimension $\leq k{-}1$. Ascending induction on the dimension of the support of $\mathcal{M}$ allows us to assume that $H^r(X,\mathcal{K})$ and $H^r(X,\mathcal{Q})$ are finite-dimensional for all r. Denoting finite-dimensional spaces ambiguously by $finite$, the two cohomology sequences become

$$\cdots \to finite \to H^q(X,\mathcal{M}(-1)) \to H^q(X,\mathcal{N}) \to finite \to \cdots$$

and

$$\cdots \to finite \to H^q(X,\mathcal{N}) \to H^q(X,\mathcal{M}) \to finite \to \cdots$$

The first of these sequences shows that if $H^q(X,\mathcal{M}(-1))$ has infinite dimension, then $H^q(X,\mathcal{N})$ has infinite dimension too, and the second sequence shows that if $H^q(X,\mathcal{N})$ has infinite dimension, then $H^q(X,\mathcal{M})$ has infinite dimenson. Therefore $H^q(X, \mathcal{M}(-1))$ and $H^q(X, \mathcal{M})$ are either both finite-dimensional, or else they are both infinite-dimensional. This applies to the twisted modules $\mathcal{M}(d)$ as well as to $\mathcal{M}$: $H^q(X, \mathcal{M}(d-1))$ and $H^q(X, \mathcal{M}(d))$ are both finite-dimensional or both infinite-dimensional.

Suppose that $q > 0$. Then $H^q(X, \mathcal{M}(d)) = 0$ when d is large enough (Theorem 7.7.2). Since the zero space is finite-dimensional, we can use the sequence together with descending induction on d, to conclude that $H^q(X, \mathcal{M}(d))$ is finite-dimensional for every finite module $\mathcal{M}$ and every d. In particular, $H^q(X, \mathcal{M})$ is finite-dimensional.

This leaves the case that $q = 0$. To prove that $H^0(X, \mathcal{M})$ is finite-dimensional, we put $d = -r$ with $r > 0$ into the sequence (7.7.6):

$$0 \to \mathcal{N}(-r) \to \mathcal{O}(-r)^m \to \mathcal{M} \to 0$$

The corresponding cohomology sequence is

$$0 \to H^0(X, \mathcal{N}(-r)) \to H^0(X, \mathcal{O}(-r))^m \to H^0(X, \mathcal{M}) \xrightarrow{\delta^0} H^1(X, \mathcal{N}(-r)) \to \cdots .$$

Here $H^0(X, \mathcal{O}(-r))^m = 0$, and we've shown that $H^1(X, \mathcal{N}(-r))$ is finite-dimensional. It follows that $H^0(X, \mathcal{M})$ is finite-dimensional, and this completes the proof. □

Notice that the finiteness of H^0 comes out only at the end. The higher cohomology is essential for the proof.

(7.7.7) Euler characteristic eulerchar

Theorems 7.7.1 and 7.7.3 allow us to define the Euler characteristic of a finite module on projective variety.

7.7.8. Definition. Let X be a projective variety. The *Euler characteristic* of a finite $\mathcal{O}$-module $\mathcal{M}$ is the alternating sum of the dimensions of its cohomology: defeuler

(7.7.9)
$$\chi(\mathcal{M}) = \sum (-1)^q \mathbf{h}^q(X, \mathcal{M}).$$
chi

This makes sense because $\mathbf{h}^q(X, \mathcal{M})$ is finite for every q, and is zero when q is large.

Try not to confuse the Euler characterstic of an $\mathcal{O}$-module with the topological Euler characteristic of the variety X.

7.7.10. Proposition. (i) *If* $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \to 0$ *is a short exact sequence of finite* $\mathcal{O}$*-modules on a projective variety* X*, then* $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P}) = 0$.
(ii) *If* $0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \cdots \to \mathcal{M}_n \to 0$ *is an exact sequence of finite* $\mathcal{O}$*-modules on* X*, the alternating sum* $\sum (-1)^i \chi(\mathcal{M}_i)$ *is zero.*

7.7.11. Lemma. *Let* $0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ *be an exact sequence of finite dimensional vector spaces. The alternating sum* $\sum (-1)^q \dim V^q$ *is zero.* □ *altsum*

proof of Proposition 7.7.10. **(i)** Let n be the dimension of X. The cohomology sequence associated to the given sequence is

$$0 \to H^0(\mathcal{M}) \to H^0(\mathcal{N}) \to H^0(\mathcal{P}) \to H^1(\mathcal{M}) \to \cdots \to H^n(\mathcal{N}) \to H^n(\mathcal{P}) \to 0$$

and the lemma tells us that the alternating sum of its dimensions is zero. That alternating sum is also equal to $\chi(\mathcal{M}) - \chi(\mathcal{N}) + \chi(\mathcal{P})$.
(ii) Let 's denote the given sequence by $\mathbb{S}_0$ and the alternating sum $\sum_i (-1)^i \chi(\mathcal{M}_i)$ by $\chi(\mathbb{S}_0)$.

Let $\mathcal{N} = \mathcal{M}_1/\mathcal{M}_0$. The sequence $\mathbb{S}_0$ decomposes into the two exact sequences

$$\mathbb{S}_1 : \ 0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{N} \to 0 \quad \text{and} \quad \mathbb{S}_2 : \ 0 \to \mathcal{N} \to \mathcal{M}_2 \to \cdots \to \mathcal{M}_k \to 0$$

One sees directly that $\chi(\mathbb{S}_0) = \chi(\mathbb{S}_1) - \chi(\mathbb{S}_2)$, and then the assertion follows from **(i)** by induction on n. □

Section 7.8 Bézout's Theorem

bezout

As an application of cohomology, we use it to prove Bézout's Theorem.

We restate the theorem to be proved:

bezoutrestated **7.8.1. Bézout's Theorem.** *Let Y and Z be distinct curves, of degrees m and n, respectively, in the projective plane X. The number of intersection points $Y \cap Z$, when counted with an appropriate multiplicity, is equal to mn. Moreover, the multiplicity is 1 at a point at which Y and Z intersect transversally.*

The definition of the multiplicity will emerge during the proof.

Note. Let f and g be relatively prime homogeneous polynomials. When one replaces Y and Z by their divisors of zeros (1.3.13), the theorem remains true whether or not they are irreducible. However, though the proof isn't signifiantly different from the one we give here, it requires setting up some notation.

intersectlines **7.8.2. Example.** Suppose that f and g are products of linear polynomials, so that Y is the union of m lines and Z is the union of n lines, and suppose that those lines are distinct. Since distinct lines intersect transversally in a single point, there are mn intersection points of multiplicity 1. $\square$

proof of Bézout's Theorem. We suppress notation for the extension by zero from Y or Z to the plane X. Let f and g be the irreducible homogeneous polynomials whose zero loci are Y and Z. Multiplication by f defines a short exact sequence

$$0 \to \mathcal{O}_X(-m) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

where $\mathcal{O}_Y$ stands for its extension by zero. This exact sequence describes $\mathcal{O}_X(-m)$ as the ideal $\mathcal{I}$ of regular functions that vanish on Y, and there is a similar sequence describing the module $\mathcal{O}_X(-n)$ as the ideal $\mathcal{J}$ of Z. The zero locus of the ideal $\mathcal{I}+\mathcal{J}$ is the intersection $Y \cap Z$, which is a finite set of points $\{p_1, ..., p_k\}$.

Let $\overline{\mathcal{O}}$ denote the quotient $\mathcal{O}_X/(\mathcal{I}+\mathcal{J})$. Its support is the finite set $Y \cap Z$, and therefore $\overline{\mathcal{O}}$ is isomorphic to a direct sum $\bigoplus \overline{\mathcal{O}}_i$, where each $\overline{\mathcal{O}}_i$ is a finite-dimensional algebra whose support is p_i (**6.7.2**). We define the *intersection multiplicity* of Y and Z at p_i to be the dimension of $\overline{\mathcal{O}}_i$, which is also the dimension $\mathbf{h}^0(X, \overline{\mathcal{O}}_i)$ of its space of its global sections. Let's denote that intersection multiplicity by μ_i. The dimension of $H^0(X, \overline{\mathcal{O}})$ is the sum $\mu_1+\cdots+\mu_k$, and $H^q(X, \overline{\mathcal{O}}) = 0$ for all $q > 0$ (Theorem 7.7.1). The Euler characteristic $\chi(\overline{\mathcal{O}})$ is equal to $\mathbf{h}^0(X, \overline{\mathcal{O}})$. We'll show that $\chi(\overline{\mathcal{O}}) = mn$, and therefore that $\mu_1+\cdots+\mu_k = mn$. This will prove Bézout's Theorem.

We form a sequence of $\mathcal{O}$-modules, in which $\mathcal{O}$ stands for $\mathcal{O}_X$:

bezoutresolution (7.8.3)
$$0 \to \mathcal{O}(-m-n) \xrightarrow{(g,f)^t} \mathcal{O}(-m) \times \mathcal{O}(-n) \xrightarrow{(-f,g)} \mathcal{O} \xrightarrow{\pi} \overline{\mathcal{O}} \to 0$$

In order to interpret the maps in this sequence as matrix multiplication with homomorphisms acting on the left, sections of $\mathcal{O}(-m) \times \mathcal{O}(-n)$ should be represented as column vectors $(u, v)^t$, u and v being sections of $\mathcal{O}(-m)$ and $\mathcal{O}(-n)$, respectively.

bresolexact **7.8.4. Lemma.** *The sequence (7.8.3) is exact.*

proof. We may suppose that coordinates have been chosen so that none of the points making up $Y \cap Z$ lie on the coordinate axes.

To prove exactness, it suffices to show that the sequence of sections on each of the standard affine open sets is exact. We look at theindex 0 as usual, denoting $\mathbb{U}^0$ by $\mathbb{U}$. Let A be the algebra of regular functions on $\mathbb{U}$, which is the polynomial algebra $\mathbb{C}[u_1, u_2]$, with $u_i = x_i/x_0$. We identify $\mathcal{O}(k)$ with $\mathcal{O}(kH)$, H being the hyperplane at infinity. The restriction of the module $\mathcal{O}(kH)$ to $\mathbb{U}$ is isomorphic to $\mathcal{O}_{\mathbb{U}}$. Its sections on $\mathbb{U}$ are the elements of A. Let $\overline{A}$ be the algebra of sections of $\overline{\mathcal{O}}$ on $\mathbb{U}$. Since f and g are relatively prime, so are their dehomogenizations $F = f(1, u_1, u_2)$ and $G = g(1, u_1, u_2)$. The sequence of sections of (7.8.3) on $\mathbb{U}$ is

$$0 \to A \xrightarrow{(G,F)^t} A \times A \xrightarrow{(-F,G)} A \to \overline{A} \to 0$$

and the only place at which exactness of this sequence isn't obvious is at $A \times A$. Suppose that $(u, v)^t$ is in the kernel of the map $(F, -G)$, i.e., that $Fu = Gv$. Since F and G are relatively prime, F divides v, G divides u, and $v/F = u/G$. Let $w = v/F = u/G$. Then $(u, v)^t = (G, F)^t w$. So $(u, v)^t$ is the image of w. $\square$

We go back to the proof of Bézout's Theorem. Proposition 7.7.10 **(ii)**, applied to the exact sequence (7.8.3), tells us that the alternating sum

(7.8.5) $$\chi(\mathcal{O}(-m-n)) \,-\, \chi(\mathcal{O}(-m)) \,+\, \big(\chi(\mathcal{O}(-n)\times\mathcal{O}(-n)\big) \,-\, \chi(\overline{\mathcal{O}})$$ chiforbe-zout

is zero. Since cohomology is compatible with products, $\chi(\mathcal{M}\times\mathcal{N}) = \chi(\mathcal{M}) + \chi(\mathcal{N})$ for any $\mathcal{O}$-modules $\mathcal{M}$ and $\mathcal{N}$. Solving for $\chi(\overline{\mathcal{O}})$ and applying Theorem 7.5.5,

$$\chi(\overline{\mathcal{O}}) \;=\; \binom{n+m-1}{2} - \binom{m-1}{2} - \binom{n-1}{2} + 1$$

The right side of this equation evaluates to mn. This completes the proof. □

We still need to explain the assertion that the mutiplicity at a transversal intersection p is equal to 1. This will be true if and only if $\mathcal{I}+\mathcal{J}$ generates the maximal ideal $\mathfrak{m}$ of $A = \mathbb{C}[y,z]$ at p locally, and it is obvious when Y and Z are lines. In that case we may choose affine coordinates so that p is the origin in the plane $X = \operatorname{Spec} A$ and the curves are the coordinate axes $\{z = 0\}$ and $\{y = 0\}$. The variables y, z generate the maximal ideal at the origin.

Suppose that Y and Z intersect transverally at p, but that they aren't lines. We choose affine coordinates so that p is the origin and that the tangent directions of Y and Z at p are the coordinate axes. The affine equations of Y and Z will have the form $y' = 0$ and $z' = 0$, where $y' = y + g(y,z)$ and $z' = z + h(y,z)$, g and h being polynomials all of whose terms have degree at least 2. Because Y and Z may intersect at points other than p, the elements y' and z' may not generate the maximal ideal $\mathfrak{m}$ at p. However, they do generate the maximal ideal locally. To show this, it suffices to show that they generate the maximal ideal M in the local ring R at p. By Corollary 5.1.2, it suffices to show that y' and z' generate M/M^2, and this is true because y' and z' are congruent to y and z modulo M^2. □

Section 7.9 Uniqueness of the Coboundary Maps

unique-cobound

In Section 7.4, we constructed a cohomology $\{H^q\}$ that has the characteristic properties, and we showed that the functors H^q are unique. We didn't show that the coboundary maps δ^q that appear in the cohomology sequences are unique. We go back to fill this gap now.

To make it clear that there is something to show, we note that the cohomology sequence (7.1.3) remains exact when a coboundary map δ^q is multiplied by a nonzero constant such as -1. Why can't we define a new collection of coboundary maps by changing some signs? The reason we can't do this is that we used the coboundary maps δ^q in (7.4.7) and (7.4.8), to identify $H^q(X,\mathcal{M})$. Having done that, we aren't allowed to change δ^q for the particular short exact sequences (7.4.2). We show that the coboundary maps for those sequences determine the coboundary maps for every short exact sequence of $\mathcal{O}$-modules

$$(A) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \to 0$$

The sequences (7.4.2) were rewritten as (7.4.10). We will use that form.

To show that the coboundaries for the sequence (A) are determined uniquely, we relate it to the sequence (7.4.10), for which the coboundary maps are fixed:

$$(B) \qquad 0 \to \mathcal{M} \longrightarrow \mathcal{R}^0_{\mathcal{M}} \longrightarrow \mathcal{M}^1 \to 0$$

We map the sequnces (A) and (B) to a third short exact sequence

$$(C) \qquad 0 \to \mathcal{M} \xrightarrow{\psi} \mathcal{R}^0_{\mathcal{N}} \longrightarrow \mathcal{Q} \to 0$$

where ψ is the composition of the injective maps $\mathcal{M} \to \mathcal{N} \to \mathcal{R}^0_{\mathcal{N}}$ and $\mathcal{Q}$ is the cokernel of ψ.

First, we inspect the diagram

$$\begin{array}{cccccc}
(A) & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{P} \\
 & \| & & \downarrow & & \downarrow \\
(C) & \mathcal{M} & \xrightarrow{\psi} & \mathcal{R}^0_{\mathcal{N}} & \longrightarrow & \mathcal{Q}
\end{array}$$

and its diagram of coboundary maps

$$\begin{array}{cccc} (A) & H^q(X,\mathcal{P}) & \xrightarrow{\delta^q_A} & H^{q+1}(X,\mathcal{M}) \\ & \downarrow & & \| \\ (C) & H^q(X,\mathcal{Q}) & \xrightarrow{\delta^q_C} & H^{q+1}(X,\mathcal{M}) \end{array}$$

This diagram shows that the coboundary map δ^q_A for the sequence (A) is determined by the coboundary map δ^q_C for (C).

Next, we inspect the diagram

BtoC (7.9.1)
$$\begin{array}{cccccc} (B) & \mathcal{M} & \longrightarrow & \mathcal{R}^0_{\mathcal{M}} & \longrightarrow & \mathcal{M}^1 \\ & \| & & u\downarrow & & v\downarrow \\ (C) & \mathcal{M} & \xrightarrow{\psi} & \mathcal{R}^0_{\mathcal{N}} & \longrightarrow & \mathcal{Q} \end{array}$$

and its diagram of coboundary maps

$$\begin{array}{cccc} (B) & H^q(X,\mathcal{M}^1) & \xrightarrow{\delta^q_B} & H^{q+1}(X,\mathcal{M}) \\ & \downarrow & & \| \\ (C) & H^q(X,\mathcal{Q}) & \xrightarrow{\delta^q_C} & H^{q+1}(X,\mathcal{M}) \end{array}$$

When $q > 0$, δ^q_C and δ^q_B are bijective because the cohomology of $\mathcal{R}^0_{\mathcal{M}}$ and $\mathcal{R}^0_{\mathcal{N}}$ is zero in positive dimension. Then δ^q_C is determined by δ^q_B, and so is δ^q_A.

We have to look more closely to settle the case $q = 0$. The map labeled u in (7.9.1) is injective. The Snake Lemma shows that v is injective, and that the cokernels of u and v are isomorphic. We write both of those cokernels as $\mathcal{R}^0_{\mathcal{P}}$. When we add the cokernels to the diagram, and pass to cohomology, we obtain a diagram whose relevant part is

$$\begin{array}{cccccc} (B) & H^0(X,\mathcal{R}^0_{\mathcal{M}}) & \longrightarrow & H^0(X,\mathcal{M}^1) & \xrightarrow{\delta^0_B} & H^1(X,\mathcal{M}) \\ & u\downarrow & & \downarrow v & & \| \\ (C) & H^0(X,\mathcal{R}^0_{\mathcal{N}}) & \xrightarrow{\beta} & H^0(X,\mathcal{Q}) & \xrightarrow{\delta^0_C} & H^1(X,\mathcal{M}) \\ & \downarrow\gamma & & \downarrow & & \\ & H^0(X,\mathcal{R}^0_{\mathcal{P}}) & = & H^0(X,\mathcal{R}^0_{\mathcal{P}}) & & \end{array}$$

Its rows and columns are exact. We want to show that the map δ^0_C is determined uniquely by δ^0_B. It is determined by δ^0_B on the image of v and it is zero on the image of β. To show that δ^0_C is determined by δ^0_B, it suffices to show that the images of v and β together span $H^0(X,\mathcal{Q})$. This follows from the fact that γ is surjective. Thus δ^0_C is determined by δ^0_B, and so is δ^0_A. $\square$

Section 7.10 Exercises

7.10.1. Let X be the complement of the point $(0,0,1)$ in $\mathbb{P}^2$. Use the covering of X by the two standard affine open sets U^0, U^1 to compute the cohomology $H^q(X, \mathcal{O}_X)$. chapsev- enex xcohcom- plpt

7.10.2. Let $0 \to V_0 \to \cdots \to V_n \to 0$ be a complex of finite-dimensional vector spaces. Prove that $\sum_i (-1)^i \dim V_i = \sum (-1)^q \mathbf{C}^q(V^\bullet\}$. xeuler- complex

7.10.3. Let M, N, and P be abelian groups, and let $M \xrightarrow{i_1} M \oplus N \xrightarrow{\pi_1} M$ and $N \xrightarrow{i_2} M \oplus N \xrightarrow{\pi_2} N$ be homomorphisms. Suppose that $\pi_1 i_1 = id_M$, $\pi_2 i_2 = id_N$, $\pi_2 i_1 = 0$, $\pi_1 i_2 = 0$, and $i_1\pi_1 + i_2\pi_2 = id_P$. Prove that P is isomorphic to $M \oplus N$. xprodprop

7.10.4. Let $0 \to \mathcal{M}_0 \to \cdots \to \mathcal{M}_k \to 0$ be an exact sequence of $\mathcal{O}$-modules on a variety X. Prove that if $H^q(\mathcal{M}_i) = 0$ for all $q > 0$ and all i, the sequence of global sections is exact. xglobsec- sexact

7.10.5. *the Cousin Problem.* Let X be a projective variety. xcousin

(i) Detemine the cohomology of the function field module $\mathcal{F}$ (6.5.10).

(ii) Suppose that X is a projective space. Leet $\{U^i\}$, $i = 1, ..., k$ be an open covering of X. Suppose that rational functions f_i are given, such that $f_i - f_j$ is a regular function on $U^i \cap U^j$ for all i and j. The Cousin Problem asks for a rational function $\widetilde{f}$ such that $\widetilde{f} - f_i$ is a regular function on U^i for every i. Analyze this problem making use of the exact sequence $0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{Q} \to 0$, where $\mathcal{Q}$ is the quotient $\mathcal{F}/\mathcal{O}$. What can one say for other varieties X?

7.10.6. Prove that if a variety X is covered by two affine open sets, then $H^q(X, \mathcal{M}) = 0$ for every $\mathcal{O}$-module $\mathcal{M}$ and every $q > 1$. xcohdi- mone

7.10.7. Let C be a plane curve of degree d with δ nodes and κ cusps, and let C' be its normalization. Determine the genus of C'. nodes- cusps

7.10.8. Let $f(x_0, x_1, x_2)$ be an irreduible homogeneous polynomial of degree $2d$, and let Y be the projective double plane $y^2 = f(x_0, x_1, x_2)$. Compute the cohomology $H^q(Y, \mathcal{O}_Y)$. xcohd- blplane

7.10.9. Let A, B be 2×2 variable matrices, let P be the polynomial ring $\mathbb{C}[a_{ij}, b_{ij}]$. and let R be the algebra $P/(AB - BA)$. Show that R has a resolution as P-module of the form $0 \to P^2 \to P^3 \to P \to R \to 0$. (Hint: Write the equations in terms of $a_{11} - a_{22}$ and $b_{11} - b_{22}$.) xABBA

7.10.10. Prove that a regular function on a projective variety is constant. xregfn- const

7.10.11. *algebraic version of Bézout's Theorem.* Let f and g be homogeneous polynomials of degrees m and n, respectively, in x, y, z. The algebra $A = \mathbb{C}[x, y, z/(f, g)$ inherits a grading by degree: $A = A_0 \oplus A_1 \oplus \cdots$, where A_n is the image of the space of homogeneous polynomials of degree n, together with 0. xalgbez

(i) Show that the sequence

$$0 \to R \overset{(-g,f)}{\longrightarrow} R^2 \overset{>}{(f,g)^t} \longrightarrow R \to A \to 0$$

is exact.

(ii) Prove that $\dim A_k = mn$ for all sufficiently large k.

(iii) Explain in what way this is an algebraic version of Bézout's Theorem.

7.10.12. Let p_1, p_2, p_3 and q_1, q_2, q_3 be distinct points on a conic C, and let L_{ij} be the line through p_i and q_j. xpascal

(i) Let g and h be the homogeneous cubic polynomials whose zero loci are $L_{12} \cup L_{13} \cup L_{23}$ and $L_{21} \cup L_{31} \cup L_{32}$, respectively, and let x be another point on C. Show that for some scalar c, the cubic $f = g + ch$ vanishes at x as well as at the six given points p_i and q_i. What does Bézout's Theorem tell us about this cubic f?

(ii) *Pascal's Theorem.* asserts that the three intersection points $r_1 = L_{23} \cap L_{32}$, $r_2 = L_{31} \cap L_{13}$, and $r_3 = L_{12} \cap L_{21}$ lie on a line. Prove Pascal's Theorem.

(iii) Let six lines $Z_1, ..., Z_6$ be given, and suppose that a conic C is tangent to each of those lines. Let $p_{12} = Z_1 \cap Z_2$, $p_{23} = Z_2 \cap Z_3$, $p_{34} = Z_3 \cap Z_4$, $p_{45} = Z_4 \cap Z_5$, $p_{56} = Z_5 \cap Z_6$, and $p_{61} = Z_6 \cap Z_1$. We think of the six lines as sides of a 'hexagon', whose vertices are $p_{ij} = L_i \cap L_j$ for $ij = 12, 23, 34, 45, 56$, and 61. The 'main diagonals' are the lines D_1, D_2, and D_3, through p_{12} and p_{45}, p_{23} and p_{56}, and p_{61} and p_{34}, respectively. *Brianchon's Theorem* asserts that the main diagonals have a common point. Prove Brianchon's Theorem by studying the dual configuration in $\mathbb{P}^*$.

xfiniteproj **7.10.13.** Let $Y \xrightarrow{\pi} \mathbb{P}^d$ be a finite morphism of varieties. Prove that Y is a projective variety. Do this by showing that the global sections of $\mathcal{O}_Y(nH) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(nH)$ define a map to projective space whose image is isomorphic to Y.

xmod-fandg **7.10.14.** (a) Let $f(x,y,z)$ and $g(x,y,z)$ be homogeneous polynomials of degrees m and n, and with no common factor. Let R be the polynomial ring $\mathbb{C}[x,y,z]$, and let $A = R/(f,g)$. Show that the sequence

$$0 \to R \xrightarrow{(-g,f)} R^2 \xrightarrow{(f,g)^t} R \to A \to 0$$

is exact.

(b) Let Y be an affine variety with integrally closed coordinate ring B. Let I be an ideal of B generated by two elements u, v, and let X be the locus $V(I)$ in Y. Suppose that $\dim X \le \dim Y - 2$. Use the fact that $B = \bigcup B_Q$ where Q ranges over prime ideals of codimension 1 to prove that this sequence is exact:

$$0 \to B \xrightarrow{(v,-u)^t} B^2 \xrightarrow{(u,v)} B \to B/I \to 0.$$

xmodfg **7.10.15.** Let I be the ideal of $\mathbb{C}[x_0, x_1, x_2, x_3]$ generated by two homogeneous polynomials f, g, of dimensions d, e respectively, and assume that the locus $X = V(I)$ in $\mathbb{P}^3$ has dimension 1. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$. Multiplication by f and g defines a map $\mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O}$. Let $\mathcal{A}$ be the cokernel of this map.

(i) Construct an exact sequence

$$0 \to \mathcal{O}(-d-e) \to \mathcal{O}(-d) \oplus \mathcal{O}(-e) \to \mathcal{O} \to \mathcal{A} \to 0.$$

(ii) Show that X is a connected subset of $\mathbb{P}^3$ in the Zariski topology, i.e., that it is not the union of two proper disjoint Zariski-closed subsets.

(iii) Determine the genus of X in the case that X is a smooth curve.

xcplt-inttwo **7.10.16.** A curve in $\mathbb{P}^3$ that is the zero locus of a homogeneous prime ideal generated by two elements is a *complete intersection*. Determine the genus of a smooth complete intersection when the generators have degrees r and s.

xafplusbg **7.10.17.** *a theorem of Max Noether.* Max Noether was Emmy Noether's father. He was also a major figure in algebraic geometry.

Let f and g homogeneous polynomials in $x_0, ..., x_k$, of degrees r and s, respectively, with $k \ge 2$. Suppose that the locus $X : \{f = g = 0\}$ in $\mathbb{P}^k$ consists of rs distinct points if $k = 2$, or is a closed subvariety if $k > 2$. (So f and g are relatively prime.) A theorem that is called the AF+BG Theorem, asserts that, if a homogeneous polynomial p of degree n vanishes on X, there are polynomials a and b of degrees $n-r$ and $n-s$, respectively, such that $p = af + bg$. Prove this theorem.

xmi-norstwo **7.10.18.** Let

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix}$$

be a 3×2 matrix whose entries are homogeneous quadratic polynomials in four variables $x_0, ..., x_3$. Let $M = (m_1, m_2, m_3)$ be the 1×3 matrix of minors

$$m_1 = u_{21}u_{32} - u_{22}u_{31}, \quad m_2 = -u_{11}u_{32} + u_{12}u_{31}, \quad m_3 = u_{11}u_{22} - u_{12}u_{21}$$

The matrices U and M give us a sequence

$$0 \to \mathcal{O}(-6)^2 \xrightarrow{U} \mathcal{O}(-4)^3 \xrightarrow{M} \mathcal{O} \to \mathcal{O}/I \to 0$$

where I is the ideal generated by the minors.

(i) Suppose that the above sequence is exact, and that the locus of zeros of I in $\mathbb{P}^3$ is a curve. Determine the genus of thatq curve.

(ii) Prove that, if the locus is a curve, the sequence is exact.

Chapter 8 THE RIEMANN-ROCH THEOREM FOR CURVES

rrcurves

The main topic of this chapter is the analysis of a classical problem of algebraic geometry, which is to determine the rational functions on a smooth projective curve with given poles. This can be difficult. The rational functions whose poles have orders at most r_i at p_i, for $i = 1, ..., k$, form a vector space, and one is usually happy if one can determine the dimension of that space. The most important tool for determining the dimension is the Riemann-Roch Theorem.

Section 8.1 Divisors

divtwo

Before discussing divisors, we take a brief look at modules on a smooth curve. Smooth affine curves were discussed in Chapter 5. An affine curve is smooth if its local rings are valuation rings, or if its coordinate ring is a normal domain. An arbitrary curve is smooth if it has an open covering by smooth affine curves.

Recall that a module over a domain A is called torsion-free if its only torsion element is zero (**2.6.6**). This definition is extended to $\mathcal{O}$-modules by applying it to affine open subsets.

lfree **8.1.1. Lemma.** *Let Y be a smooth curve.*
(i) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is locally free if and only if it is torsion-free.*
(ii) *An $\mathcal{O}_Y$-module $\mathcal{M}$ that isn't torsion-free has a nonzero global section.*

proof. **(i)** We may assume that Y is affine, $Y = \operatorname{Spec} B$, and that $\mathcal{M}$ is the $\mathcal{O}$-module associated to a B-module M. Let $\widetilde{B}$ and $\widetilde{M}$ be the localizations of B and M at a point q, respectively. Then $\widetilde{M}$ is a finite, torsion-free module over the valuation ring $\widetilde{B}$. It suffices to show that, for every point q of Y, $\widetilde{M}$ is a free $\widetilde{B}$-module (**2.6.13**). The next sublemma does this.

vringpid **8.1.2. Sublemma.** *A finite, torsion-free module $\widetilde{M}$ over a valuation ring $\widetilde{B}$ is a free module.*

proof. It is easy to prove this directly, or, one can use the fact that a valuation ring is a principal ideal domain. Its nonzero ideals are powers of the maximal ideal $\widetilde{\mathfrak{m}}$, which is a principal ideal. Every finite, torsion-free module over a principal ideal domain is free. □

proof of Lemma 8.1.1(**ii**)**.** If the torsion submodule of $\mathcal{M}$ isn't zero, then for some affine open subset U of Y, there will be nonzero elements m in $\mathcal{M}(U)$ and a in $\mathcal{O}(U)$, such that $am = 0$. Let Z be the finite set of zeros of a in U, and let $V = Y - Z$ be the open complement of Z in Y. Then a is invertible on the intersection $W = U \cap V$, and since $am = 0$, the restriction of m to W is zero.

The open sets U and V cover Y, and the sheaf property for this covering can be written as an exact sequence

$$0 \to \mathcal{M}(Y) \to \mathcal{M}(U) \times \mathcal{M}(V) \xrightarrow{-,+} \mathcal{M}(W)$$

(see Lemma 6.4.10). In this sequence, the section $(m, 0)$ of $\mathcal{M}(U) \times \mathcal{M}(V)$ maps to zero in $\mathcal{M}(W)$. Therefore it is the image of a nonzero global section of $\mathcal{M}$. $\square$

8.1.3. Lemma. *Let Y be a smooth curve. Every nonzero ideal $\mathcal{I}$ of $\mathcal{O}_Y$ is a product $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$ of powers of maximal ideals.* *idealprod-max*

proof. This follows for any smooth curve from the case that Y is affine, which is Proposition 5.2.7. $\square$

We come to divisors now.

A *divisor* on a smooth curve Y is a finite combination

$$D = r_1 q_1 + \cdots + r_k q_k$$

where r_i are integers and q_i are points. The terms whose integer coefficients r_i are zero can be omitted or not, as desired.

The *support* of the divisor D is the set of points q_i of Y such that $r_i \neq 0$. The *degree* of D is the sum $r_1 + \cdots + r_k$ of the coefficients.

Let Y' be an open subset of Y. The *restriction* of a divisor $D = r_1 q_1 + \cdots + r_k q_k$ to Y' is the divisor obtained from D by deleting points that aren't in Y'. For example, say that $D = q$. The restriction to Y' is q if q is in Y', and is zero if q is not in Y'.

A divisor $D = \sum r_i q_i$ is *effective* if all of its coefficients r_i are non-negative, and if Y' is an open subset of Y, D is *effective on* Y' if its restriction to Y' is effective — if $r_i \geq 0$ for every i such that q_i is a point of Y'. Let $D = \sum r_i p_i$ and $E = \sum s_i p_i$ be divisors. We my write $E \geq D$ if $s_i \geq r_i$ for all i, or if $E - D$ is effctive. With this notation, $D \geq 0$ if D is effective — if $r_i \geq 0$ for all i.

(8.1.4) the divisor of a function divfn

Let f be a nonzero rational function on a smooth curve Y. The *divisor of f* is

$$\operatorname{div}(f) = \sum_{q \in Y} \mathrm{v}_q(f)\, q$$

where, as usual, v_q denotes the valuation of K that corresponds to the point q of Y. The divisor of the zero function is the zero divisor.

The divisor of f is written here as a sum over all points q, but it becomes a finite sum when we disregard terms with coefficient zero, because f has finitely many zeros and poles. The coefficients will be zero at all other points.

The map

(8.1.5) $$K^\times \xrightarrow{\operatorname{div}} (divisors)^+$$ Ktodiv

that sends a nonzero rational function to its divisor is a homomorphism from the multiplicative group $K^\times$ of nonzero elements of K to the additive group of divisors:

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$$

As before, if r is a positive integer, a nonzero rational function f has a *zero of order* r at q if $\mathrm{v}_q(f) = r$, and it has a *pole of order* r at q if $\mathrm{v}_q(f) = -r$. Thus the divisor of f is the difference of two effective divisors:

$$\operatorname{div}(f) = zeros(f) - poles(f)$$

A rational function f is regular on Y if and only if $\operatorname{div}(f)$ is effective — if and only if $poles(f) = 0$.

The divisor of a rational function is called a *principal divisor*. The image of the map (8.1.5) is the set of principal divisors.

Two divisors D and E are *linearly equivalent* if their difference $D - E$ is a principal divisor. For instance, the divisors $zeros(f)$ and $poles(f)$ of a rational function f are linearly equivalent.

levelsets **8.1.6. Lemma.** *Let f be a rational function on a smooth curve Y. For all complex numbers c, the divisors of zeros of $f-c$, the level sets of f, are linearly equivalent.*

proof. The functions $f-c$ have the same poles as f. $\square$

moduleOD **(8.1.7) the module $\mathcal{O}(D)$**

To analyze functions with given poles on a smooth curve Y, we associate an $\mathcal{O}$-module $\mathcal{O}(D)$ to a divisor D. The nonzero sections of $\mathcal{O}(D)$ on an open subset V of Y are the nonzero rational functions f such that the the divisor $\operatorname{div}(f)+D$ is effective on V — such that its restriction to V is effective.

ODV (8.1.8) $[\mathcal{O}(D)](V) \;=\; \{f \mid \operatorname{div}(f)+D \text{ is effective on } V\} \cup \{0\} \;=\; \{f \mid poles(f) \leq D \text{ on } V\} \cup \{0\}$

Points that aren't in the open set V impose no conditions on the sections of $\mathcal{O}(D)$ on V. A section on V can have arbitrary zeros or poles at points not in V.

When D is effective, the global sections of $\mathcal{O}(D)$ are the solutions of the classical problem that was mentioned at the beginning of the chapter, to determine the rational functions with given poles.

Let D be the divisor $\sum r_i q_i$. If q_i is a point of an open set V and if $r_i > 0$, a section of $\mathcal{O}(D)$ on V may have a pole of order at most r_i at q_i, and if $r_i < 0$ a section must have a zero of order at least $-r_i$ at q_i. For example, the module $\mathcal{O}(-q)$ is the maximal ideal $\mathfrak{m}_q$. The sections of $\mathcal{O}(-q)$ on an open set V that contains q are the regular functions on V that are zero at q. Similarly, the sections of $\mathcal{O}(q)$ on an open set V that contains q are the rational functions that have a pole of order at most 1 at q and are regular at every other point of V. The sections of $\mathcal{O}(-q)$ and of $\mathcal{O}(q)$ on an open set V that doesn't contain p are the regular functions on V.

The fact that a section of $\mathcal{O}(D)$ is allowed to have a pole at q_i when $r_i > 0$ contrasts with the divisor of a function. If $\operatorname{div}(f) = \sum r_i q_i$, then $r_i > 0$ means that f has a zero at q_i. If $\operatorname{div}(f) = D$, then f will be a global section of $\mathcal{O}(-D)$.

ODinvert **8.1.9. Lemma.** *For any divisor D on a smooth curve, $\mathcal{O}(D)$ is a locally free module of rank one.*

proof. If D is a principal divisor, say $D = \operatorname{div}(g)$, then $\mathcal{O}(D) = \{f \mid \operatorname{div}(f) + \operatorname{div}(g) \geq 0\} = \{f \mid \operatorname{div}(fg) \geq 0\}$. This is the module $g^{-1}\mathcal{O}$. It is a free module of rank one. Since Y is a smooth curve, every divisor is locally principal, because the maximal ideal $\mathfrak{m}_q = \mathcal{O}(-q)$ is a locally prinicipal ideal. $\square$

LisOD **8.1.10. Proposition.** *Let D and E be divisors on a smooth curve Y.*
(i) *The map $\mathcal{O}(D) \otimes_{\mathcal{O}} \mathcal{O}(E) \to \mathcal{O}(D+E)$ that sends $f \otimes g$ to the product fg is an isomorphism.*
(ii) *$\mathcal{O}(D) \subset \mathcal{O}(E)$ if and only if $E \geq D$, or $E-D$ is effective.*

proof. We may assume that Y is affine and that the supports of D and E contain at most one point: $D = rp$ and $E = sp$. We may also assume that the maximal ideal at p is a principal ideal, generated by an element x. Then $\mathcal{O}(D)$, $\mathcal{O}(E)$, and $\mathcal{O}(D+E)$ have bases x^r, x^s and x^{r+s}, respectively. $\square$

The function field module $\mathcal{K}$ is the union of the modules $\mathcal{O}(D)$.

idealOD **8.1.11. Proposition.** *Let Y be a smooth curve.*
(i) *The nonzero ideals of $\mathcal{O}_Y$ are the modules $\mathcal{O}(-E)$, where E is an effective effective divisor.*
(ii) *The finite $\mathcal{O}$-submodules of the function field module $\mathcal{K}$ of Y are the modules $\mathcal{O}(D)$, where D can be any divisor.*

proof. **(i)** Say that $E = r_1 q_1 + \cdots + r_k q_k$, with $r_i \geq 0$. A rational function f is a section of $\mathcal{O}(-E)$ if $\operatorname{div}(f) - E$ is effective, which means that $poles(f) = 0$, and $zeros(f) \geq E$. This also describes the elements of the ideal $\mathcal{I} = \mathfrak{m}_1^{r_1} \cdots \mathfrak{m}_k^{r_k}$.

(ii) Let $\mathcal{L}$ be a finite $\mathcal{O}$-submodule of $\mathcal{K}$. For any section α of $\mathcal{L}$, there will be a nonzero rational function g such that $g\alpha$ is a regular function, a section of $\mathcal{O}$. Since $\mathcal{L}$ is a finite $\mathcal{O}$-module, there will be a nonzero rational function g such that $g\alpha$ is regular for every α in $\mathcal{L}$. Then $g\mathcal{L} \subset \mathcal{O}$, so $g\mathcal{L}$ is an ideal. It is equal to $\mathcal{O}(-E)$ for some effective divisor D, and we will have $\mathcal{L} = \mathcal{O}(D)$, where $D = -E + \operatorname{div}(g)$. $\square$

8.1.12. Proposition. *Let D and E be divisors on a smooth curve Y. Multiplication by a rational function f such that $\operatorname{div}(f)+E-D \geq 0$ defines a homomorphism of $\mathcal{O}$-modules $\mathcal{O}(D) \to \mathcal{O}(E)$, and every homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ is multiplication by such a function.* *mapOD-toOE*

proof. For any $\mathcal{O}$-module $\mathcal{M}$, a homomorphism $\mathcal{O} \to \mathcal{M}$ is multiplication by a global section of $\mathcal{M}$ (6.3.7). So a homomorphism $\mathcal{O} \to \mathcal{O}(E-D)$ will be multiplication by a rational function f such that $\operatorname{div}(f)+E-D \geq 0$. If f is such a function, one obtains a homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ by tensoring with $\mathcal{O}(D)$. $\square$

8.1.13. Corollary. *Let D and E be divisors on a smooth curve Y.* *ODOE*

(i) *The modules $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are isomorphic if and only if the divisors D and E are linearly equivalent.*

(ii) *Let f be a rational function on Y, and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$.* $\square$

(**8.1.14**) **invertible modules** invertmod

An *invertible* $\mathcal{O}$-module is a locally free module of rank one — a module that is isomorphic to the free module $\mathcal{O}$ in a neighborhood of any point. The tensor product $\mathcal{L}\otimes_{\mathcal{O}}\mathcal{M}$ of invertible modules is invertible. The dual $\mathcal{L}^*$ of an invertible module $\mathcal{L}$ is invertible.

For any divisor D on a smooth curve Y, $\mathcal{O}(D)$ is an invertible module (8.1.9). Its dual is the module $\mathcal{O}(-D)$.

8.1.15. Lemma. *Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module.* *inverse-mod*

(i) *Let $\mathcal{L}^*$ be the dual module. The canonical map $\mathcal{L}^*\otimes_{\mathcal{O}}\mathcal{L} \to \mathcal{O}$ defined by $\gamma\otimes\alpha \mapsto \gamma(\alpha)$ is an isomorphism. Thus $\mathcal{L}^*$ may be thought of as an inverse to $\mathcal{L}$. (This is the reason for the term 'invertible'.)*

(ii) *The map $\mathcal{O} \to {}_{\mathcal{O}}(\mathcal{L},\mathcal{L})$ that sends a regular function α to the homomorphism of multiplication by α is an isomorphism.*

(iii) *Every nonzero homomorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ to a locally free module $\mathcal{M}$ is injective.*

proof. **(i,ii)** It is enough to verify these assertions in the case that $\mathcal{L}$ is free, isomorphic to $\mathcal{O}$, in which case they are clear.

(iii) The problem is local, so we may assume that the variety is affine, say $Y = \operatorname{Spec} A$, and that $\mathcal{L}$ and $\mathcal{M}$ are free. Then φ becomes a nonzero homomorphism $A \to A^k$, which is injective because A is a domain. $\square$

8.1.16. Lemma. *Every invertible $\mathcal{O}$-module $\mathcal{L}$ on a smooth curve Y is isomorphic to one of the form $\mathcal{O}(D)$.* *Lequal-sOD*

If $\mathcal{L}$ is an invertible module and D is a divisor, we denote by $\mathcal{L}(D)$ the invertible $\mathcal{O}$-module $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}(D)$. Its sections on an affine open set U are products αf $(= \alpha \otimes f)$, where α is a section of $\mathcal{L}$ and f is a section of $\mathcal{O}(D)$.

proof of Lemma 8.1.16 When Y is affine, this has been proved before (8.1.11).

We choose a nonzero section α of $\mathcal{L}$ on an open set U. This section won't extend to X. Since $\mathcal{L}$ is locally isomorphic to $\mathcal{O}$, the section α can be realized, locally at a point p, as an element of the function field K of X. Thinking of $\mathcal{L}$ locally as $\mathcal{O}$, α will have a pole of some order r at p. Locally at p, it will be a section of $\mathcal{O}(rp)$. Going back to $\mathcal{L}$, α will extend locally to a section of $\mathcal{L}(rp)$. There are finitely many points of X not in the open set U, so α will extend to a global section of $\mathcal{L}(D)$ for some divisor $D = \sum r_i p_i$. This gives us a nonzero map $\mathcal{O} \to \mathcal{L}(D)$. Passing to duals, we obtain a nonzero map $\mathcal{L}^*(-D) \to \mathcal{O}$. Propositioin8.1.11 tells us that $\mathcal{L}^*(-D)$ is equal to $\mathcal{O}(-E)$ for some effective divisor E. Then $\mathcal{L}^* \approx \mathcal{O}(D-E)$, and $\mathcal{L} \approx \mathcal{O}(E-D)$. $\square$

We will use the next lemma in the proof of Theorem 8.6.15. Let $\mathcal{L} \subset \mathcal{M}$ be an inclusion of invertible modules $\mathcal{L}$ and $\mathcal{M}$ on a smooth curve Y, let q be a point in the support of $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{L}$, and let V be an affine open subset of Y that contains q.

8.1.17. Lemma. *With notation as above, suppose that a rational function f has a simple pole at q and is regular at all other points of V. If α is a section of $\mathcal{L}$ on V, then $f^{-1}\alpha$ is a section of $\mathcal{M}$ on V.* *thetaspole* *proof.* Working locally, we may assume that $\mathcal{L} = \mathcal{O}$, and therefore $\mathcal{L}^* = \mathcal{O}$. Since $\mathcal{O} = \mathcal{L} \subset \mathcal{M}$, we have $\mathcal{M}^* \subset \mathcal{O}$. So $\mathcal{M}^*$ is an ideal, equal to $\mathcal{O}(-D)$ for some effective divisor D, and $\mathcal{M} = \mathcal{O}(D)$. Since q is in the support of $\overline{\mathcal{M}}$, the coefficient of q in D is positive. Therefore $\mathcal{L} = \mathcal{O} \subset \mathcal{O}(q) \subset \mathcal{O}(D) = \mathcal{M}$. With this notation, α will be a section of $\mathcal{O}$ and f^{-1} will be a section of $\mathcal{O}(q)$. Then $f^{-1}\alpha$ will be a section of $\mathcal{O}(q)$, and therefore a section of $\mathcal{O}(D)$, which is $\mathcal{M}$. $\square$

Section 8.2 The Riemann-Roch Theorem I

rrone

Let Y be a smooth projective curve. In Chapter 7, we learned that, when $\mathcal{M}$ is a finite $\mathcal{O}_Y$-module, the cohomology $H^q(Y, \mathcal{M})$ is a finite-dimensional vector space for $q = 0, 1$, and is zero when $q > 1$. As before, we denote the dimension of the space $H^q(Y, \mathcal{M})$ by $\mathbf{h}^q\mathcal{M}$ or, if there is ambiguity about the variety, by $\mathbf{h}^q(Y, \mathcal{M})$.

The *Euler characteristic* (7.6.6) of a finite $\mathcal{O}$-module $\mathcal{M}$ is

chicurve (8.2.1)
$$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^1\mathcal{M}$$

In particular,
$$\chi(\mathcal{O}_Y) \;=\; \mathbf{h}^0\mathcal{O}_Y - \mathbf{h}^1\mathcal{O}_Y$$

The dimension $\mathbf{h}^1\mathcal{O}_Y$ is called the *arithmetic genus* of Y. It is denoted by p_a. This is the notation that was used for plane curves (7.6.4). We will see below, in (8.2.9)**(iv)**, that $\mathbf{h}^0\mathcal{O}_Y = 1$. So

chicurvetwo (8.2.2)
$$\chi(\mathcal{O}) \;=\; 1 - p_a$$

RRcurve **8.2.3. Riemann-Roch Theorem (version 1).** *Let $D = \sum r_i p_i$ be a divisor on a smooth projective curve Y. Then*
$$\chi(\mathcal{O}(D)) \;=\; \chi(\mathcal{O}) + \deg\, D \qquad \big(= \deg\, D + 1 - p_a\big)$$

proof. We analyze the effect on cohomology when a divisor is changed by adding or subtracting a point, by inspecting the inclusion $\mathcal{O}(D-p) \subset \mathcal{O}(D)$. Let ϵ be the cokernel of that inclusion map, so that there is a short exact sequence

addpoint (8.2.4)
$$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \epsilon \to 0$$

in which ϵ is a one-dimensional vector space supported at p. Because $\mathfrak{m}_p$ is isomorphic to $\mathcal{O}(-p)$, this sequence can be obtained by tensoring the sequence

kappaseq (8.2.5)
$$0 \to \mathfrak{m}_p \to \mathcal{O} \to \kappa_p \to 0$$

with the module $\mathcal{O}(D)$, which is locally free of rank one.

Since ϵ is a one-dimensional module supported at p, $\mathbf{h}^0\epsilon = 1$, and $\mathbf{h}^1\epsilon = 0$. Let's denote the one-dimensional vector space $H^0(Y, \epsilon)$ by $[1]$. Then the cohomology sequence associated to (8.2.4) is

addpointtwo (8.2.6)
$$0 \to H^0(Y, \mathcal{O}(D-p)) \to H^0(Y, \mathcal{O}(D)) \xrightarrow{\gamma} [1] \xrightarrow{\delta} H^1(Y, \mathcal{O}(D-p)) \to H^1(Y, \mathcal{O}(D)) \to 0$$

In this exact sequence, one of the two maps, γ or δ, must be zero. Either

(1) γ is zero and δ is injective. In this case
$$\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D) + 1, \quad \text{or}$$

(2) δ is zero and γ is surjective, in which case
$$\mathbf{h}^0\mathcal{O}(D)-p) = \mathbf{h}^0\mathcal{O}(D) - 1 \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D)$$

In either case,

chichange (8.2.7)
$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D-p)) + 1$$

The Riemann-Roch theorem follows, because $\deg D = \deg(D-p)+1$, and we can get from $\mathcal{O}$ to $\mathcal{O}(D)$ by a finite number of operations, each of which changes the divisor by adding or subtracting a point. $\square$

Because $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$, this version of the Riemann-Roch Theorem gives reasonably good control of H^0. It is less useful for controlling H^1. One wants the full Riemann-Roch Theorem for that. Because the full theorem requires some preparation, we have put it into Section 8.7. However, version 1 has important consequences:

8.2.8. Corollary. *Let p be a point of a smooth projective curve Y. The dimension $\mathbf{h}^0(Y, \mathcal{O}(np))$ tends to infinity with n. Therefore there exist rational functions with a pole of large order at p and no other poles.* *onepole*

proof. When we go from $\mathcal{O}(np)$ to $\mathcal{O}((n+1)p)$, either $\mathbf{h}^0$ increases or else $\mathbf{h}^1$ decreases. Since $H^1(Y, \mathcal{O}(np))$ is finite-dimensional, the second possibility can occur only finitely many times. $\square$

8.2.9. Corollary. *Let Y be a smooth projective curve.* *RRcor*

(i) *The divisor of a rational function on Y has degree zero: The number of zeros is equal to the number of poles.*

(ii) *Linearly equivalent divisors on Y have equal degrees.*

(iii) *A nonconstant rational function on Y takes every value, including infinity, the same number of times.*

(iv) *A rational function on Y that is regular at every point of Y is a constant: $H^0(Y, \mathcal{O}) = \mathbb{C}$.*

proof. **(i)** Let f be a nonzero rational function and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$, so $\chi(\mathcal{O}(D)) = \chi(\mathcal{O})$. On the other hand, by Riemann-Roch, $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D$. Therefore $\deg D = 0$.

(ii) If D and E are linearly equivalent divisors, say $D - E = \operatorname{div}(f)$, then $D - E$ has degree zero, and $\deg D = \deg E$.

(iii) The zeros of the functions $f - c$ are linearly equivalent to the poles of f.

(iv) According to **(iii)**, a nonconstant function must have a pole. $\square$

8.2.10. Corollary. *Let D be a divisor on Y.* *RRcorwo*

If $\deg D \geq p_a$*, then* $\mathbf{h}^0\mathcal{O}(D) > 0$*. If* $\mathbf{h}^0\mathcal{O}(D) > 0$*, then* $\deg D \geq 0$.

proof. If $\deg D \geq p_a$, then $\chi(\mathcal{O}(D)) = \deg D + 1 - p_a \geq 1$, and $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$. If $\mathcal{O}(D)$ has a nonzero global section f, a rational function such that $\operatorname{div}(f) + D = E$ is effective, then $\deg E \geq 0$, and because the degree of $\operatorname{div}(f)$ is zero, $\deg D \geq 0$. $\square$

8.2.11. Theorem. *With its classical topology, a smooth projective curve Y is a connected, compact, orientable two-dimensional manifold.* *curveconn*

proof. We prove connectedness here. The other points have been discussed before (see Theorem 1.8.19).

A nonempty topological space is connected if it isn't the union of two disjoint, nonempty, closed subsets. Suppose that, in the classical topology, Y is the union of disjoint, nonempty closed subsets Y_1 and Y_2. Both Y_1 and Y_2 will be compact manifolds. Let p be a point of of Y_1. Corollary 8.2.8 shows that there is a nonconstant rational function f whose only pole is at p. Then f will be a regular function on the complement of p, and therefore on the entire compact manifold Y_2.

For review: Any point q of the smooth curve Y has a neighborhood V that is analytically equivalent to an open subset U of the affine line X. If a function g on V is analytic, the function on U that corresponds to g is an analytic function of one variable on U. The maximum principle for analytic functions asserts that a nonconstant analytic function on an open region of the complex plane has no maximal absolute value in the region. This applies to the open set U and therefore also to the neighborhood V of q. Since q can be any point of Y_2, a nonconstant function g that is analytic on Y_2 cannot have a maximum on Y_2. On the other hand, since Y_2 is compact, a continuous function does have a maximum. So an analytic function g on Y_2 must be a constant.

Going back to the rational function f with a single pole p, The function f with pole p will be analytic, and therefore constant, on Y_2. When we subtract that constant from f, we obtain a nonconstant rational function on Y that is zero on Y_2. But since Y has dimension 1, the zero locus of a rational function is finite. This is a contradiction. $\square$

Section 8.3 The Birkhoff-Grothendieck Theorem

birkgroth

This theorem describes finite, torsion-free modules on the projective line. We'll denote $\mathbb{P}^1$ by X here.

BGtheorem **8.3.1. Birkhoff-Grothendieck Theorem.** *A finite, torsion-free $\mathcal{O}$-module on the projective line X is isomorphic to a direct sum of twisting modules: $\mathcal{M} \approx \bigoplus \mathcal{O}(n_i)$.*

This theorem was proved by Grothendieck in 1957 using cohomology. It had been proved by Birkhoff in 1909, in the following equivalent form:

Birkhoff Factorization Theorem. Let $A_0 = \mathbb{C}[u]$, $A_1 = \mathbb{C}[u^{-1}]$, and $A_{01} = \mathbb{C}[u, u^{-1}]$. Let P be an invertible A_{01}-matrix. There exist an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1 such that $Q_0^{-1} P Q_1$ is diagonal, and its diagonal entries are integer powers of u.

proof of the Birkhoff-Grothendieck Theorem.

This is Grothendieck's proof. According to Theorem 7.5.5, the cohomology of the twisting modules on the projective line X is $\mathbf{h}^0\mathcal{O} = 1$, $\mathbf{h}^1\mathcal{O} = 0$, and if r is a positive integer, then

$$\mathbf{h}^0\mathcal{O}(r) = r+1, \;\; \mathbf{h}^1\mathcal{O}(r) = 0, \;\; \mathbf{h}^0\mathcal{O}(-r) = 0, \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(-r) = r-1$$

mapstoMbounded **8.3.2. Lemma.** *Let $\mathcal{M}$ be a finite, torsion-free $\mathcal{O}$-module on the projective line X. For sufficiently large r,*
(i) *the only module homomorphism $\mathcal{O}(r) \to \mathcal{M}$ is the zero map, and*
(ii) $\mathbf{h}^0(X, \mathcal{M}(-r)) = 0$.

proof. **(i)** Let $\mathcal{O}(r) \xrightarrow{\varphi} \mathcal{M}$ be a nonzero homomorphism from the twisting module $\mathcal{O}(r)$ to a locally free module $\mathcal{M}$. Then φ will be injective (8.1.15), and the associated map $H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{M})$ will also be injective. So $\mathbf{h}^0(X, \mathcal{O}(r)) \le \mathbf{h}^0(X, \mathcal{M})$. Since $\mathbf{h}^0(X, \mathcal{O}(r)) = r+1$ and since $\mathbf{h}^0(X, \mathcal{M})$ is finite, r is bounded.

(ii) A global section of $\mathcal{M}(-r)$ defines a map $\mathcal{O} \to \mathcal{M}(-r)$. Its twist by r will be a map $\mathcal{O}(r) \to \mathcal{M}$. By **(i)**, r is bounded. □

We go to the proof now.

Lemma 8.1.1 tells us that $\mathcal{M}$ is locally free. We use induction on the rank of $\mathcal{M}$. We suppose that $\mathcal{M}$ has rank $r > 0$, and that the theorem has been proved for locally free $\mathcal{O}$-modules of rank less than r. The plan is to show that $\mathcal{M}$ has a twisting module as a direct summand, so that $\mathcal{M} = \mathcal{W} \oplus \mathcal{O}(n)$ for some $\mathcal{W}$. Then induction on the rank, applied to $\mathcal{W}$, proves the theorem.

Since twisting is compatible with direct sums, we may replace $\mathcal{M}$ by a twist $\mathcal{M}(n)$. Instead of showing that $\mathcal{M}$ has a twisting module $\mathcal{O}(n)$ as a direct summand, we show that, after we replace $\mathcal{M}$ by a suitable twist, the structure sheaf $\mathcal{O}$ will be a direct summand.

As we know (6.8.20), the twist $\mathcal{M}(n)$ will have a nonzero global section when n is sufficiently large, and it will have no nonzero global section when n is sufficiently negative (Lemma 8.3.2 **(ii)**). Therefore, when we replace $\mathcal{M}$ by a suitable twist, we will have $H^0(X, \mathcal{M}) \neq 0$ but $H^0(X, \mathcal{M}(-1)) = 0$. We assume that this is true for $\mathcal{M}$.

We choose a nonzero global section m of $\mathcal{M}$ and consider the injective multiplication map $\mathcal{O} \xrightarrow{m} \mathcal{M}$. Let $\mathcal{W}$ be its cokernel, so that we have a short exact sequence

cvcwsequence (8.3.3)
$$0 \to \mathcal{O} \xrightarrow{m} \mathcal{M} \to \mathcal{W} \to 0$$

sectionbasic **8.3.4. Lemma.** *Let $\mathcal{W}$ be the $\mathcal{O}$-module that appears in the sequence (8.3.3).*
(i) $H^0(X, \mathcal{W}(-1)) = 0$.
(ii) *$\mathcal{W}$ is torsion-free, and therefore locally free.*
(iii) *$\mathcal{W}$ is a direct sum $\bigoplus_{i=1}^{r-1} \mathcal{O}(n_i)$ of twisting modules on $\mathbb{P}^1$, with $n_i \le 0$.*

proof. **(i)** This follows from the cohomology sequence associated to the twisted sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{M}(-1) \to \mathcal{W}(-1) \to 0$$

because $H^0(X, \mathcal{M}(-1)) = 0$ and $H^1(X, \mathcal{O}(-1)) = 0$.

(ii) If the torsion submodule of $\mathcal{W}$ were nonzero, the torsion submodule of $\mathcal{W}(-1)$ would also be nonzero, and then $\mathcal{W}(-1)$ would have a nonzero global section (8.1.1).

(iii) The fact that $\mathcal{W}$ is a direct sum of twisting modules follows by induction on the rank: $\mathcal{W} \approx \bigoplus \mathcal{O}(n_i)$. Since $H^0(X, \mathcal{W}(-1)) = 0$, we must have $H^0(X, \mathcal{O}(n_i-1)) = 0$. Therefore $n_i - 1 < 0$, and $n_i \le 0$. $\square$

We go back to the proof of Theorem 8.3.1. Because $\mathcal{O}^* = \mathcal{O}$, the dual of the sequence (8.3.3) is an exact sequence

$$0 \to \mathcal{W}^* \xleftarrow{m} \mathcal{M}^* \xleftarrow{\pi} \mathcal{O} \to 0$$

and $\mathcal{W}^* \approx \bigoplus \mathcal{O}(-n_i)$ with $-n_i \ge 0$. Therefore $\mathbf{h}^1\mathcal{W}^* = 0$. The map $H^0(\mathcal{M}) \to H^0(\mathcal{O})$ is surjective. Let α be a global section of $\mathcal{M}^*$ whose image $\pi(\alpha)$ in $\mathcal{O}$ is 1. Multiplication by α defines a map $\mathcal{O} \xrightarrow{\alpha} \mathcal{M}^*$ such that $\pi \circ \alpha = id$. Then $\mathcal{M}^* = \mathcal{W}^* \oplus im(\alpha) \approx \mathcal{W}^* \oplus \mathcal{O}$. $\square$

Section 8.4 Differentials

diff

We now introduce some terminology that will be used in version II of the Riemann-Roch theorem: differentials and branched coverings. Differentials enter into the Riemann-Roch Theorem, though why they do is a mystery.

Try not to get bogged down in these preliminary disussions. Give the next pages a quick read to learn the terminology. You can look back as needed. Begin to read more carefully when you get to Section 8.6.

Let A be an algebra and let M be an A-module. A *derivation* $A \xrightarrow{\delta} M$ is a $\mathbb{C}$-linear map that satisfies the product rule for differentiation — a map that has these properties:

(8.4.1) $$\delta(ab) = a\,\delta b + b\,\delta a\ , \quad \delta(a{+}b) = \delta a + \delta b\ , \quad \text{and} \quad \delta c = 0$$ deriv

for all a, b in A and all c in $\mathbb{C}$. The fact that δ is $\mathbb{C}$-linear, i.e., that it is a homomorphism of vector spaces, follows: Since $dc = 0$, $\delta(cb) = c\,\delta b$.

For example, differentiation $\frac{d}{dt}$ is a derivation $\mathbb{C}[t] \to \mathbb{C}[t]$.

8.4.2. Lemma. *Let $A \xrightarrow{\varphi} B$ be an algebra homomorphism, and let $M \xrightarrow{g} N$ be a homomorphism of B-modules.* *compde*

(i) *Let $B \xrightarrow{\delta} M$ be a derivation. The composed maps $A \xrightarrow{\delta\varphi} M$ and $B \xrightarrow{g\delta} N$ are derivations.*

(ii) *Suppose that φ is surjective. Let $B \xrightarrow{h} M$ be a map, and let $d = h \circ \varphi$. If $A \xrightarrow{d} M$ is a derivation, then h is a derivation.* $\square$

The *module of differentials* Ω_A of an algebra A is an A-module that is generated by elements denoted by da, one for each element a of A. Its elements are (finite) combinations $\sum b_i\, da_i$, with a_i and b_i in A. The defining relations among the generators da are the ones that make the map $A \xrightarrow{d} \Omega_A$ that sends a to da a derivation: For all a, b in A and all c in $\mathbb{C}$,

(8.4.3) $$d(ab) = a\,db + b\,da\ , \quad d(a{+}b) = da + db\ , \quad \text{and} \quad dc = 0$$ defdiff

The elements of Ω_A are called *differentials*.

8.4.4. Lemma. *ho-momderiv*

(i) *When we compose a homomorphism $\Omega_A \xrightarrow{f} M$ of $\mathcal{O}$-modules with the derivation $A \xrightarrow{d} \Omega_A$, we obtain a derivation $A \xrightarrow{fd} M$. Composition with d defines a bijection between homomorphisms $\Omega_A \to M$ and derivations $A \xrightarrow{\delta} M$.*

(ii) *Ω is a functor: An algebra homomorphism $A \xrightarrow{u} B$ induces a homomorphism $\Omega_A \xrightarrow{v} \Omega_B$ that is compatible with the ring homomorphism u, and that makes a diagram*

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_B \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} \\ A & \xrightarrow{d} & \Omega_A \end{array}$$

By compatibility of v with u we mean that, if ω is an element of Ω_A and α is in A, then $v(\alpha\omega) = u(\alpha)v(\omega)$.

proof. **(i)** The composition $\delta = f \circ d$ is a derivation $A \to M$. In the other direction, given a derivation $A \xrightarrow{\delta} M$, we define a map $\Omega_A \xrightarrow{f} M$ by $f(\,da) = \delta(a)$. It follows from the defining relations for Ω_A that f is a well-defined homomorphism of A-modules.

(ii) When Ω_B is made into an A-module by restriction of scalars, the composed map $A \xrightarrow{u} B \xrightarrow{d} \Omega_B$ will be a derivation to which **(i)** applies. $\square$

omegafree **8.4.5. Lemma.** *Let R be the polynomial ring $\mathbb{C}[x_1, ..., x_n]$. The module of differentials Ω_R is a free R-module with basis $dx_1, ..., dx_n$.*

proof. The formula $df = \sum \frac{df}{dx_i} dx_i$ follows from the defining relations. It shows that the elements $dx_1, ..., dx_n$ generate the R-module Ω_R. Let $v_1, ..., v_n$ be a basis of a free R-module V. The product rule for derivatives shows that the map $\delta : R \to V$ defined by $\delta(f) = \frac{\partial f}{\partial x_i} v_i$ is a derivation. It induces a module homomorphism $\Omega_A \xrightarrow{\varphi} V$ that sends dx_i to v_i. Since $dx_1, ..., dx_n$ generate Ω_R and since $v_1, ..., v_n$ is a basis of V, φ is an isomorphism. $\square$

omegafreetwo **8.4.6. Proposition.** *Let I be an ideal of an algebra R, and let A be the quotient algebra R/I. Let dI denote the set of differentials df with f in I. The subset $N = dI + I\Omega_R$ is a submodule of Ω_R, and Ω_A is isomorphic to the quotient module Ω_R/N.*

The proposition can be interpreted this way: Suppose that the ideal I is generated by elements $f_1, ..., f_r$ of R. Then Ω_A is the quotient of Ω_R that is obtained from Ω_R by introducing these two rules:

- $df_i = 0$, and
- multiplication by f_i is zero.

For example, let R be the polynomial ring $\mathbb{C}[y]$ in one variable, let I be the principal ideal (y^2), and let A be the quotient R/I. Then $2y\,dy$ generates dI, and y^2dy generates $I\Omega_A$. The R-module N is generated by ydy. If $\overline{y}$ denotes the residue of y in A, Ω_A is generated by an element $d\overline{y}$, with the relation $\overline{y}\,d\overline{y} = 0$. In particular, Ω_A isn't the zero module.

proof of Proposition 8.4.6. First, $I\Omega_R$ is a submodule of Ω_R, and dI is an additive subgroup of Ω_R. To show that N is a submodule, we must show that scalar multiplication by an element of R maps dI to N, i.e., that if g is in R and f is in I, then $g\,df$ is in N. By the product rule, $g\,df = d(fg) - f\,dg$. Since I is an ideal, fg is in I. Then $d(fg)$ is in dI, and $f\,dg$ is in $I\Omega_R$. So $g\,df$ is in N.

The two rules shown above hold in Ω_A because the generators f_i of I are zero in A. Therefore N is in the kernel of the surjective map $\Omega_R \xrightarrow{v} \Omega_A$ defined by the homomorphism $R \to A$. Let $\overline{\Omega}$ denote the quotient module Ω_R/N. This is an A-module and, because $N \subset \ker v$, v defines a surjective map of A-modules $\overline{\Omega} \xrightarrow{\overline{v}} \Omega_A$. We show that $\overline{v}$ is bijective. Let r be an element of R, let a be its image in A, and let $\overline{dr}$ be its image in $\overline{\Omega}$. The composed map $R \xrightarrow{d} \Omega_R \xrightarrow{\pi} \overline{\Omega}$ is a derivation that sends r to $\overline{dr}$, and I is in its kernel. So it defines a derivation $R/I = A \xrightarrow{\delta} \overline{\Omega}$ that sends a to $\overline{dr}$. This derivation corresponds to a homomorphism of A-modules $\Omega_A \to \overline{\Omega}$ that sends da to $\overline{dr}$, and that inverts $\overline{v}$. $\square$

Omegafinite **8.4.7. Corollary.** *If A is a finite-type algebra, then Ω_A is a finite A-module.*

This follows from Proposition 8.4.6 because the module of differentials on the polynomial ring $\mathbb{C}[x_1, ..., x_n]$ is a finite module. $\square$

localizeomega **8.4.8. Lemma.** *Let S be a multiplicative system in a domain A. The module $\Omega_{S^{-1}A}$ of differentials of $S^{-1}A$ is canonically isomorphic to the module of fractions $S^{-1}\Omega_A$. In particular, if K is the field of fractions of A, then $K \otimes_A \Omega_A \approx \Omega_K$.*

We have moved the symbol S^{-1} to the left for clarity.

proof of Lemma 8.4.8. The composed map $A \to S^{-1}A \xrightarrow{d} \Omega_{S^{-1}A}$ is a derivation. It defines an A-module homomorphism $\Omega_A \to \Omega_{S^{-1}A}$ which extends to an $S^{-1}A$-homomorphism $S^{-1}\Omega_A \xrightarrow{\varphi} \Omega_{S^{-1}A}$ because scalar multiplication by the elements of S is invertible in $\Omega_{S^{-1}A}$. The relation $ds^{-k} = -ks^{k-1}ds$ follows from the definition of a differential, and it shows that φ is surjective. The quotient rule

$$\delta(s^{-k}a) = -ks^{-k-1}a\,ds + s^{-k}da$$

defines a derivation $S^{-1}A \xrightarrow{\delta} S^{-1}\Omega_A$, which corresponds to a homomorphism $\Omega_{S^{-1}A} \to S^{-1}\Omega_A$ that inverts φ. Here, one must show that δ is well-defined, that $\delta(s_1^{-k}a_1) = \delta(s_2^{-\ell}a_2)$ if $s_1^{-\ell}a_1 = s_2^{-k}a_2$, and that δ is a derivation. You will be able to do this. $\square$

Lemma 8.4.8 shows that a finite $\mathcal{O}$-module Ω_Y of differentials on a variety Y is defined such that, when $U = \operatorname{Spec} A$ is an affine open subset of Y, $\Omega_Y(U) = \Omega_A$.

8.4.9. Proposition. *If y is a local generator for the maximal ideal at a point q of a smooth curve Y, then in a suitable neighborhood of q, the module Ω_Y of differentials will be a free $\mathcal{O}$-module with basis dy. Therefore Ω_Y is an invertible module.* *omega-funct*

proof. We may assume that Y is affine, say $Y = \operatorname{Spec} B$. Let q be a point of Y, and let y be an element of B with $\mathrm{v}_q(y) = 1$. To show that dy generates Ω_B locally, we may localize, so we may suppose that y generates the maximal ideal $\mathfrak{m}$ at q. We must show that after we localize B once more, every differential df with f in B will be a multiple of dy. Let $c = f(q)$. Then $f = c + yg$ for some g in B, and because $dc = 0$, $df = g\,dy + y\,dg$. Here $g\,dy$ is in $B\,dy$ and $y\,dy$ is in $\mathfrak{m}\Omega_B$, so

$$\Omega_B = B\,dy + \mathfrak{m}\Omega_B$$

An element α of Ω_B can be written as $\alpha = b\,dy + \gamma$, with b in B and γ in $\mathfrak{m}\Omega_B$. This means that, if W denotes the quotient module $\Omega_B/(B\,dy)$, then $W = \mathfrak{m}W$. The Nakayama Lemma tells us that there is an element z in $\mathfrak{m}$ such that $s = 1 - z$ annihilates W. When we replace B by its localization B_s, we will have $W = 0$ and $\Omega_B = B\,dy$, as required.

We must still verify that the generator dy of Ω_B isn't a torsion element. If it were, say $b\,dy = 0$ with $b \neq 0$, then Ω_B would be zero except at the finite set of zeros of b in Y. If that were the case, we could replace the point q by a point at which Ω_B is zero, keeping the rest of the notation unchanged. Let $R = \mathbb{C}[y]$ and $A = R/I$. The module Ω_R is free, with basis dy, and as noted above, if $\overline{y}$ is the residue of y in A, the A-module Ω_A is generated by $d\overline{y}$, with the relation $\overline{y}\,d\overline{y} = 0$. It isn't the zero module. Proposition 5.2.7 tells us that, at our point q, the algebra $B/\mathfrak{m}_q^2$ is isomorphic to A, and Proposition 8.4.6 tells us that Ω_A is a quotient of Ω_B. Since Ω_A isn't zero, neither is Ω_B. $\square$

Section 8.5 Branched Coverings

By a *branched covering*, we mean an integral morphism $Y \xrightarrow{\pi} X$ of smooth curves. Chevalley's Finiteness Theorem (see (4.6.7)) shows that, when Y is projective, every such morphism is a branched covering, unless it maps Y to a point. cover-curve

Let $Y \to X$ be a branched covering. The function field K of Y will be a finite extension of the function field F of X. The *degree* of the covering is defined to be the degree $[K:F]$ of that field extension. The degree will be denoted by $[Y:X]$. If $X' = \operatorname{Spec} A$ is an affine open subset of X, its inverse image Y' will be an affine open subset $Y' = \operatorname{Spec} B$ of Y, and B will be a locally free A-module whose rank is equal to the degree $[Y:X]$.

To describe the fibre of a branched covering $Y \xrightarrow{\pi} X$ over a point p of X, we may localize. So we may assume that X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that the maximal ideal $\mathfrak{m}_p$ of A at a point p is a principal ideal, generated by an element x of A.

If a point q of Y lies over p, the *ramification index* at q, which we denote by e, is defined to be $\mathrm{v}_q(x)$, where v_q is the valuation of the function field K that corresponds to q. We usually denote the ramification index by e. Then, if y is a local generator for the maximal ideal $\mathfrak{m}_q$ of B at q, we will have

$$x = uy^e$$

where u is a *local unit* — a rational function on Y that is invertible on some open neighborhood of q.

Points of Y whose ramification indices are greater than one are called *branch points*. We will also call a point p of X a *branch point* of the covering Y if there is a branch point of Y that lies over p.

mostpts **8.5.1. Lemma. (i)** *A branched covering $Y \to X$ has finitely many branch points.*

(ii) *Let n denote the degree $[Y:X]$. If a point p of X isn't a branch point, the fibre over p consists of n points with ramification indices equal to 1.*

proof. This is very simple. We can delete finite sets of points, so we may suppose that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then B is a finite A-module of rank n. Let F and K be the fraction fields of A and B, respectively, and let β be an element of B that generates the field extension K/F. Then $A[\beta] \subset B$, and since these two rings have the same fraction field, there will be a nonzero element s in A such that $A_s[\beta] = B_s$. We may replace A and B by A_s and B_s, so that $B = A[\beta]$. Let g be the monic irreducible polynomial for β over A. The discriminant of g isn't the zero ideal (1.7.21). So for all but finitely many points p of X, the discriminant will be nonzero, and there will be n points of Y over p with ramification indices equal to 1. $\square$

degoneisom **8.5.2. Corollary.** *A branched covering $Y \xrightarrow{\pi} X$ of degree one is an isomorphism.*

proof. When $[Y:X] = 1$, the function fields of Y and X are equal. Then, because $Y \to X$ is an integral morphism and X is normal, $Y = X$. $\square$

The next lemma follows from Lemma 8.1.3 and the Chinese Remainder Theorem.

extendedidealisprod **8.5.3. Lemma.** *Let $Y \xrightarrow{\pi} X$ be a branched covering, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Suppose that the maximal ideal $\mathfrak{m}_p$ at p is a principal ideal, generated by an element x. Let $q_1, ..., q_k$ be the points of Y that lie over a point p of X and let $\mathfrak{m}_i$ and e_i be the maximal ideal and ramification index at q_i, respectively.*

(i) *The extended ideal $\mathfrak{m}_p B = xB$ is the product ideal $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.*

(ii) *Let $\overline{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\overline{B} = B/xB$ is isomorphic to the product $\overline{B}_1 \times \cdots \times \overline{B}_k$.*

(iii) *The degree $[Y:X]$ of the covering is the sum $e_1 + \cdots + e_k$ of the ramification indices at the points q_i.* $\square$

locanstr **(8.5.4) local analytic structure**

The local analytic structure of a branched covering $Y \xrightarrow{\pi} X$ in the classical topology is very simple. We explain it here because it is helpful for intuition as well as useful.

yethroottwo **8.5.5. Proposition.** *In the classical topology, Y is locally isomorphic to an e-th root covering $y^e = x$.*

proof. Let q be a point of Y, let p be its image in X, let x and y be local generators for the maximal ideals $\mathfrak{m}_p$ of $\mathcal{O}_X$ at p, and $\mathfrak{m}_q$ of $\mathcal{O}_Y$ at q, respectively. Let $e = \mathrm{v}_q(x)$ be the ramification index at q. So $x = uy^e$, where u is a local unit at q. In a neighborhood of q in the classical topology, u will have an analytic e-th root w. The element wy also generates $\mathfrak{m}_q$ locally, and $x = (wy)^e$. We replace y by wy. Then the implicit function theorem tells us that that x and y are local analytic coordinate functions on X and Y (see (1.4.18)). $\square$

pprimeapproachesp **8.5.6. Corollary.** *Let $Y \xrightarrow{\pi} X$ be a branched covering, let $\{q_1, ..., q_k\}$ be the fibre over a point p of X, and let e_i be the ramification index at q_i. As a point p' of X approaches p, e_i of the points that lie over p' approach q_i.* $\square$

workonX **(8.5.7) supressing the notation for the direct image**

When considering a branched covering $Y \xrightarrow{\pi} X$ of smooth curves, we will often pass between an $\mathcal{O}_Y$-module $\mathcal{M}$ and its direct image $\pi_*\mathcal{M}$, and it will be convenient to work primarily on X. Recall that if X' is an open subset X' of X and Y' is its invere image, then

$$[\pi_*\mathcal{M}](X') = \mathcal{M}(Y')$$

One can think of the direct image $\pi_*\mathcal{M}$ as working with $\mathcal{M}$, but looking only at the open subsets Y' of Y that are inverse images of open subsets of X. If we look only at such subsets, the only significant difference between $\mathcal{M}$ and its direct image will be that, when X' is open in X and $Y' = \pi^{-1}X'$, the $\mathcal{O}_Y(Y')$-module $\mathcal{M}(Y')$ is made into an $\mathcal{O}_X(X')$-module by restriction of scalars.

To simplify notation, we will often drop the symbol π_*, and write $\mathcal{M}$ instead of $\pi_*\mathcal{M}$. If X' is an open subset of X, $\mathcal{M}(X')$ will stand for $[\pi_*\mathcal{M}](X') = \mathcal{M}(\pi^{-1}X')$. When denoting the direct image of an $\mathcal{O}_Y$-module $\mathcal{M}$ by the same symbol $\mathcal{M}$, we may refer to it as an $\mathcal{O}_X$-module. In accordance with this convention, we may also write $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$, but we must be careful to include the subscript Y.

This abbreviation is analogous to the one used for restriction of scalars in a module. When $A \to B$ is an algebra homomorphism and M is a B-module, the B-module B_M and the A-module ${}_AM$ obtained by restriction of scalars are usually denoted by the same letter M.

8.5.8. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, of degree $n = [Y:X]$. With notation as above,* *BrankArank*

(i) *The direct image of $\mathcal{O}_Y$, which we also denote by $\mathcal{O}_Y$, is a locally free $\mathcal{O}_X$-module and of rank n.*

(ii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a torsion $\mathcal{O}_Y$-module if and only if its direct image, also denoted by $\mathcal{M}$, is a torsion $\mathcal{O}_X$-module.*

(iii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module if and only if its direct image is a locally free $\mathcal{O}_X$-module. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module of rank r, then its direct image is an co_X-module of rank nr.* $\square$

Section 8.6 Trace of a Differential

tracediff

(8.6.1) trace of a function

tracefn

Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, and let F and K be the function fields of X and Y, respectively.

The trace map $K \xrightarrow{\mathrm{tr}} F$ for a field extension of finite degree has been defined before (4.3.11). If α is an element of K, multiplication by α on the F-vector space K is an F-linear operator, and $\mathrm{tr}(k)$ is the trace of that operator. The trace is F-linear: If f_i are in F and α_i are in K, then $\mathrm{tr}(\sum f_i\alpha_i) = \sum f_i\,\mathrm{tr}(\alpha_i)$. Moreover, the trace carries regular functions to regular functions: If $X' = \operatorname{Spec} A'$ is an affine open subset of X, with inverse image $Y' = \operatorname{Spec} B'$, then because A' is a normal algebra, the trace of an element of B' will be in A' (4.3.7). Using our abbreviated notation $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$ (**8.5.7**), the trace defines a homomorphism of $\mathcal{O}_X$-modules

(8.6.2) $$\mathcal{O}_Y \xrightarrow{\mathrm{tr}} \mathcal{O}_X$$ trace-global

Analytically, the trace can be described as a sum over the sheets of the covering. Let $n = [Y:X]$. Over a point p of X that isn't a branch point, there will be n points $q_1, ..., q_n$ of Y. If U is a small neighborhood of p in X in the classical topology, its inverse image V will consist of disjoint neighborhoods V_i of q_i, each of which maps bijectively to U. On V_i, the ring of analytic functions will be isomorphic to the ring $\mathcal{A}$ of analytic functions on U. So the ring of analytic funtions on V is isomorphic to the direct sum $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ of n copies of $\mathcal{A}$. If a rational function g on Y is regular on V, its restriction to V can be written as $g = g_1 \oplus \cdots \oplus g_n$, with g_i in $\mathcal{A}_i$. The matrix of left multiplication by g on $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ is the diagonal matrix with entries g_i, so

(8.6.3) $$\mathrm{tr}(g) = g_1 + \cdots + g_n$$ trsum

8.6.4. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, let p be a point of X, let $q_1, ..., q_k$ be the fibre over p, and let e_i be the ramification index at q_i. If a rational function g on Y is regular at the points $q_1, ..., q_k$, its trace is regular at p, and its value at p is $[\mathrm{tr}(g)](p) = e_1g(q_1) + \cdots + e_kg(q_k)$.* *tracesum*

proof. The regularity was discussed above. If p isn't a branch point, we will have $k = n$ and $e_i = 1$ for all i. In this case, the lemma follows by evaluating (8.6.3). It follows by continuity for any point p. As a point p' approaches p, e_i points q' of Y approach q_i (8.5.6). For each point q' that approaches q_i, the limit of $g(q')$ will be $g(q_i)$. $\square$

traced **(8.6.5) trace of a differential**

The structure sheaf is naturally contravariant. A branched covering $Y \xrightarrow{\pi} X$ corresponds to an $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to \mathcal{O}_Y$. The trace map for functions is a homomorphism of $\mathcal{O}_X$-modules in the opposite direction: $\mathcal{O}_Y \xrightarrow{\text{tr}} \mathcal{O}_X$.

Differentials are also naturally contravariant. A morphism $Y \xrightarrow{\pi} X$ induces an $\mathcal{O}_X$-module homomorphism $\Omega_X \to \Omega_Y$ that sends a differential dx on X to a differential on Y that we denote by dx too (8.4.4) **(ii)**. As is true for functions, there is a trace map for differentials in the opposite direction. It is defined below, in (8.6.7), and it will be denoted by τ:

$$\Omega_Y \xrightarrow{\tau} \Omega_X$$

First, a lemma about the natural contravariant map $\Omega_X \to \Omega_Y$:

dxydy **8.6.6. Lemma.** *Let $Y \to X$ be a branched covering.*
(i) *Let p be the image in X of a point q of Y, let x and y be local generators for the maximal ideals of X and Y at p and q, respectively, and let e be the ramification index at q. There is a local unit v at q such that $dx = vy^{e-1}dy$.*
(ii) *The canonical homomorphism $\Omega_X \to \Omega_Y$ is injective.*

proof. **(i)** As we have noted before, $x = uy^e$, for some local unit u. Since dy generates Ω_Y locally, there is a rational function z that is regular at q, such that $du = zdy$. Then

$$dx \;=\; d(uy^e) \;=\; y^e z\, dy + ey^{e-1}u\, dy \;=\; vy^{e-1}dy$$

where $v = yz + eu$. Since yz is zero at q and eu is a local unit, v is a local unit.
(ii) See ((8.1.15) **(iv)**). □

To define the trace for differentials, we begin with differentials of the functions fields F and K of X and Y, respectively. As Proposition 8.4.9 shows, the $\mathcal{O}_Y$-module Ω_Y is invertible. So the module Ω_K of K-differentials, which is a localization of Ω_Y, is a free K-module of rank one. Any nonzero differential will form a K-basis. We choose as basis a nonzero F-differential α. Its image in Ω_K, which we denote by α too, will be a K-basis for Ω_K. We could, for instance, take $\alpha = dx$, where x is some local coordinate function on X.

Since α is a basis, any element β of Ω_K can be written uniquely, as

$$\beta = g\alpha$$

where g is an element of K. The *trace* $\Omega_K \xrightarrow{\tau} \Omega_F$ is defined by

deftrdif (8.6.7)
$$\tau(\beta) = \text{tr}(g)\alpha$$

where $\text{tr}(g)$ is the trace of the function g. Since the trace for functions is F-linear, τ is also an F-linear map.

We need to check that τ is independent of the choice of α. If α' is another nonzero F-differential, then $f\alpha' = \alpha$ for some nonzero element f of F, and $gf\alpha' = g\alpha$. Since $f \in F$ and tr is F-linear, $\text{tr}(gf) = f\,\text{tr}\,g$. Then

$$\tau(gf\alpha') = \text{tr}(gf)\alpha' = f\,\text{tr}(g)\alpha' = \text{tr}(g)f\alpha' = \text{tr}(g)\alpha = \tau(g\alpha)$$

Using α' in place of α gives the same value for the trace.

For example, let x be a local generator for the maximal ideal $\mathfrak{m}_p$ at a point p of X. If the degree $[Y:X]$ of Y over X is n, then when we regard dx as a differential on Y,

tracedx (8.6.8)
$$\tau(dx) = n\,dx$$

A differenial of the function field K will be called a *rational differential*. A rational differential β is *regular* at a point q of Y if there is an affine open neighborhood $Y' = \operatorname{Spec} B$ of q such that β is an element of Ω_B. If y is a local generator for the maximal ideal $\mathfrak{m}_q$ and $\beta = g\,dy$, the differential β is regular at q if and only if the rational function g is regular at q.

Let p be a point of X. Working locally at p, we may suppose that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, that the maximal ideal at p is a principal ideal, generated by an element x of A, and that the differential dx generates Ω_A. Let $q_1, ..., q_k$ be the points of Y that lie over p, and let e_i be the ramification index at q_i.

8.6.9. Corollary. *With notation as above,* *poleorder*

(i) *When viewed as a differential on Y, dx has a zero of order $e_i - 1$ at q_i.*

(ii) *When a differential β on Y that is regular at q_i is written as $\beta = g\,dx$, the rational function g has a pole of order at most $e_i - 1$ at q_i.*

This follows from Lemma 8.6.6 **(i)**. □

8.6.10. Main Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering, let p be a point of X, let $q_1, ..., q_k$ be the points of Y that lie over p, and let β be a rational differential on Y.* *tracereg*

(i) *If β is regular at the points $q_1, ..., q_k$, its trace $\tau(\beta)$ is regular at p.*

(ii) *If β has a simple pole at q_i and is regular at q_j for all $j \neq i$, then $\tau(\beta)$ is not regular at p.*

proof. **(i)** Corollary 8.6.9 tells us that $\beta = g\,dx$, where g has poles of orders at most $e_i - 1$ at the points q_i. Since x has a zero of order e_i at q_i, the function xg is regular at q_i, and its value there is zero. Then $\operatorname{tr}(xg)$ is regular at p, and its value at p is zero (8.6.4). So $x^{-1}\operatorname{tr}(xg)$ is a regular function at p. Since tr is F-linear and x is in F, $x^{-1}\operatorname{tr}(xg) = \operatorname{tr}(g)$. Therefore $\operatorname{tr}(g)$ and $\tau(\beta) = \operatorname{tr}(g)dx$ are regular at p.

(ii) With $\beta = g\,dx$, the function xg will be regular at p. Its value at q_j will be zero when $j \neq i$, and not zero when $j = i$. Then $\operatorname{tr}(xg)$ will be regular at p, but not zero there (8.6.4). Therefore $\tau(\beta) = x^{-1}\operatorname{tr}(xg)dx$ won't be regular at p. □

8.6.11. Corollary. *The trace map is a homomorphism of $\mathcal{O}_X$-modules $\Omega_Y \xrightarrow{\tau} \Omega_X$.* □ *omegaYandX*

8.6.12. Example. Let Y be the locus $y^e = x$ in $\mathbb{A}^2_{x,y}$. Multiplication by $\zeta = e^{2\pi i/e}$ permutes the sheets of Y over X. The trace of a power y^k is sumroot-cover

(8.6.13) $$\operatorname{tr}(y^k) \;=\; \sum_{j=0}^{e-1} \zeta^{jk} y^k$$ sumzeta

The sum $\sum_j \zeta^{kj}$ is zero unless $k \equiv 0$ modulo e. So $dy = y^{1-e}dx/e$, and $\tau(dy) = 0$. But $\tau(dy/y) = dx/x$, so dy/y isn't regular at $x = 0$. □

Let $Y \xrightarrow{\pi} X$ be a branched covering. As is true for any $\mathcal{O}_Y$-module, the module of differentials Ω_Y is isomorphic to the module of homomorphisms ${}_{\mathcal{O}_Y}(\mathcal{O}_Y, \Omega_Y)$. The homomorphism $\mathcal{O}_Y \to \Omega_Y$ that corresponds to a section β of Ω_Y on an open set U sends a regular function f on U to $f\beta$. We denote that homomorphism by β too: $\mathcal{O}_Y \xrightarrow{\beta} \Omega_Y$.

8.6.14. Lemma. *Composition with the trace defines a homomorphism of $\mathcal{O}_X$-modules* *compwtau*

$$\Omega_Y \approx {}_{\mathcal{O}_Y}(\mathcal{O}_Y, \Omega_Y) \xrightarrow{\tau} {}_{\mathcal{O}_X}(\mathcal{O}_Y, \Omega_X)$$

This is true because τ is $\mathcal{O}_X$-linear. An $\mathcal{O}_Y$-linear map becomes an $\mathcal{O}_X$-linear map by restriction of scalars. So when we compose an $\mathcal{O}_Y$-linear map β with τ, the result will be $\mathcal{O}_X$-linear. It will be a homomorphism of $\mathcal{O}_X$-modules. □

8.6.15. Theorem. **(i)** *The map (8.6.14) is bijective.*

(ii) *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. Composition with the trace defines a bijection* *traceomega*

(8.6.16) $${}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \xrightarrow{\tau\circ} {}_{\mathcal{O}_X}(\mathcal{M}, \Omega_{\mathcal{O}_X})$$ *compw-trace*

This theorem will follow from the Main Lemma, when one looks carefully.

Note. The domain and range of the map (8.6.16) are to be interpreted as modules on X. When we put the symbols $\underline{\mathrm{Hom}}$ and π_* that we suppress into the notation, it becomes a bijection

$$\pi_*\big(\underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_Y)\big) \xrightarrow{\tau\circ} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\pi_*\mathcal{M}, \Omega_X)$$

Because the theorem is about modules on X, we can verify it locally on X. In particular, we may suppose that X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. When we state the theorem in terms of algebras and modules, it becomes this:

traceomegaalgebra

8.6.17. Theorem. *Let $Y \to X$ be a branched covering, with $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.*
(i) *The trace map $\Omega_B = {}_B(B, \Omega_B) \xrightarrow{\tau\circ} {}_A(B, \Omega_A)$ is bijective.*
(ii) *For any locally free B-module M, composition with the trace defines a bijection ${}_B(M, \Omega_B) \xrightarrow{\tau\circ} {}_A(M, \Omega_A)$.*

Here, when we write ${}_A(M, \Omega_A)$, we are interpreting the B-module M that appears there as an A-module by restriction of scalars.

homisBmod

8.6.18. Lemma. *Let $A \subset B$ be rings, let M be a B-module, and let N be an A-module. Then ${}_A(M, N)$ has the structure of a B-module.*

proof. We must define scalar multiplication of a homomorphism $M \xrightarrow{\varphi} N$ of A-modules by an element b of B. The definition of $b\varphi$ is: $[b\varphi](m) = \varphi(bm)$. Here one must show that this map $[b\varphi]$ is a homomorphism of A-modules $M \to N$, and that the axioms for a B-module are true for ${}_A(M, N)$. You will be able to check these things. $\square$

proof of Theorem 8.6.15 **(i)**. Since the theorem is local, we are still allowed to localize. We use the algebra notation of Theorem 8.6.17. As A-modules, both B and Ω_B are torsion-free, and therefore locally free. Localizing as needed, we may assume that they are free A-modules, and that Ω_A is a free module of rank one with basis dx. Then ${}_A(B, \Omega_A)$ will be a free A-module too.

Let's denote ${}_A(B, \Omega_A)$ by Θ. Lemma 8.6.18 tells us that Θ is a B-module. Because B and Ω_A are free A-modules, Θ is a free A-module and a locally free B-module. Since Ω_A has A-rank 1, the A-rank of Θ is the same as the A-rank of B. So the B-rank of Θ is 1, the same as the B-rank of B (see (8.5.8)). Therefore Θ is an invertible B-module.

If x is a local coordinate on X, then $\tau\, dx \neq 0$ (8.6.8). The trace map $\Omega_B \xrightarrow{\tau} \Theta$ isn't the zero map. Since domain and range are invertible B-modules, it is an injective homomorphism. Its image, which is isomorphic to Ω_B, is an invertible submodule of the invertible B-module Θ.

To show that $\Omega_B = \Theta$, we apply Lemma 8.1.17 to show that the quotient $\overline{\Theta} = \Theta/\Omega_B$ is the zero module. Suppose not, and let q be a point in the support of $\overline{\Theta}$. Let p be the image of q in X and let $q_1, ..., q_k$ be the fibre over p, with $q = q_1$.

We choose a differential α that is regular at all of the points q_i. If y is a local generator for the maximal ideal at q_1, then $\alpha = g\, dy$, where g is a regular function there. We assume also that α has been chosen so that that $g(q_1) \neq 0$.

Let f be a rational function that is regular on an affine open set V containing $q_1, ..., q_k$, such that $f(q_1) = 0$ and $f(q_i) \neq 0$ when $i > 1$. Lemma 8.1.17 tells us that $\beta = f^{-1}\alpha$ is a section of Θ on V, but the Main Lemma 8.6.10 tells us that $\tau(\beta)$ isn't regular at p. This contradicion proves the theorem. $\square$

proof of Theorem 8.6.15 **(ii)**. We go back to the statement in terms of $\mathcal{O}$-modules. We are to show that if $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, composition with the trace defines a bijective map ${}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \to {}_{\mathcal{O}_X}(\mathcal{M}, \Omega_{\mathcal{O}_X})$. Part **(i)** of the theorem tells us that this is true in when $\mathcal{M} = \mathcal{O}_Y$. Therefore it is also true when $\mathcal{M}$ is a free module $\mathcal{O}_Y^k$. And, since **(ii)** is a statement about $\mathcal{O}_X$-modules, it suffices to prove it locally on X.

locallyfreeonX

8.6.19. Lemma. *Let $q_1, ..., q_k$ be points of a smooth curve Y, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. There is an open set V that contains the points $q_1, ..., q_k$, such that $\mathcal{M}$ is free on V.*

We assume the lemma and complete the proof of the theorem. Let $\{q_1, ..., q_k\}$ be the fibre over a point p of X and let be V as in the lemma. The complement $D = Y - V$ is a finite set whose image C in X is also finite and it doesn't contain p. If U is the complement of C in X, its inverse image W will be a subset of V that contains the points of the fibre, on which $\mathcal{M}$ is free. □

proof of the lemma. We may assume that Y is affine, $Y = \operatorname{Spec} B$, and that the $\mathcal{O}$-module $\mathcal{M}$ corresponds to a locally free B-module M.

We go back to Lemma 8.5.3. Let $\mathfrak{m}_i$ be the maximal ideal of B at q_i, and let $\overline{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\overline{B} = B/xB$ is isomorphic to the product $\overline{B}_1 \times \cdots \times \overline{B}_k$. Since $\mathcal{M}$ is locally free, $M/\mathfrak{m}_i M = \overline{M}_i$ is a free $\overline{B}_i$-module. Its dimension is the rank r of $\mathcal{M}$.

If M has rank r, there will be a set of elements $m = (m_1, ..., m_r)$ in M whose residues form a basis of $\overline{M}_i$ for every i. This follows from the Chinese Remainder Theorem. Therefore m generates M locally at each of the points. Let M' be the B-submodule of M generated by m. The cokernel of the map $M' \to M$ is zero at the points $q_1, ..., q_k$, and therefore it's support, which is a finite set, is disjoint from those points. When we localize to delete this finite set from X, the set m ia a basis for M. □

Note. Theorem 8.6.15 is subtle. Unfortunately the proof, though understandable, doesn't give an intuitive explanation of the fact that Ω_B is isomorphic to ${}_A(B, \Omega_A)$. To get more insight into that, we would need a better understanding of differentials. My father Emil Artin said: "One doesn't really understand differentials, but one can learn to work with them."

Section 8.7 The Riemann-Roch Theorem II

rroch

(8.7.1) the Serre dual serredual

Let Y be a smooth projective curve, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. The *Serre dual* of $\mathcal{M}$, is the module

(8.7.2) $$\mathcal{M}^S \;=\; {}_Y(\mathcal{M}, \Omega_Y)$$ defSerredual

Its sections on an open subset U are the homomorphismsof $\mathcal{O}_Y(U)$-modules $\mathcal{M}(U) \to \Omega_Y(U)$.

The Serre dual can also be written as $\mathcal{M}^S = \mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y$, where $\mathcal{M}^*$ is the ordinary dual ${}_Y(\mathcal{M}, \mathcal{O}_Y)$. Since the module Ω_Y is invertible, it is locally isomorphic to $\mathcal{O}_Y$. So the Serre dual $\mathcal{M}^S$ will be locally isomorphic to the ordinary dual $\mathcal{M}^*$. It will be a locally free module of the same rank as $\mathcal{M}$, and the bidual $(\mathcal{M}^S)^S$ will be isomorphic to $\mathcal{M}$:

$$(\mathcal{M}^S)^S \approx (\mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y)^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \otimes_{\mathcal{O}} \Omega_Y^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \approx \mathcal{M}$$

For example, $\mathcal{O}_Y^S = \Omega_Y$ and $\Omega_Y^S = \mathcal{O}_Y$.

8.7.3. Riemann-Roch Theorem, version 2. *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module on a smooth projective curve Y, and let $\mathcal{M}^S$ be its Serre dual. Then $\mathbf{h}^0\mathcal{M} = \mathbf{h}^1\mathcal{M}^S$ and $\mathbf{h}^1\mathcal{M} = \mathbf{h}^0\mathcal{M}^S$.* *dualcohom*

Because $\mathcal{M}$ and $(\mathcal{M}^S)^S$ are isomorphic, the two assertions are equivalent. The second one follows from the first when one replaces $\mathcal{M}$ by $\mathcal{M}^S$. For example, $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$ and $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y$, which is the arithmetic genus p_a.

If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then

(8.7.4) $$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^0\mathcal{M}^S$$ chitwo

A more precise statement of the Riemann-Roch Theorem is that $H^1(Y, \mathcal{M})$ and $H^0(Y, \mathcal{M}^S)$ are dual vector spaces in a canonical way. This becomes important when one wants to apply the theorem to a cohomology sequence, but we omit the proof. The fact that the dimensions are equal is enough for many applications.

Our plan is to prove Theorem 8.7.3 directly for the projective line $\mathbb{P}^1$. This will be easy, because the structure of locally free modules on $\mathbb{P}^1$ is very simple. We derive it for an arbitrary smooth projective curve Y by projection to $\mathbb{P}^1$. Projection to $\mathbb{P}$ is a method that was used by Grothendieck.

Let Y be a smooth projective curve, let $X = \mathbb{P}^1$, and let $Y \xrightarrow{\pi} X$ be a branched covering. Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module, and let the Serre dual of $\mathcal{M}$, as defined in (8.7.2), be

$$\mathcal{M}_1^S = {}_Y(\mathcal{M}, \Omega_Y)$$

The direct image of $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module that we are denoting by $\mathcal{M}$ too, and we can form the Serre dual on X. Let

$$\mathcal{M}_2^S = {}_X(\mathcal{M}, \Omega_X)$$

abouthom **8.7.5. Corollary.** *The direct image $\pi_*\mathcal{M}_1^S$, which we denote by $\mathcal{M}_1^S$, is isomorphic to $\mathcal{M}_2^S$.*

proof. This is Theorem 8.6.15. □

The corollary allows us to drop the subscripts from $\mathcal{M}^S$. Because a branched covering $Y \xrightarrow{\pi} X$ is an affine morphism, the cohomology of $\mathcal{M}$ and of its Serre dual $\mathcal{M}^S$ can be computed, either on Y or on X. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then $H^q(Y, \mathcal{M}) \approx H^q(X, \mathcal{M})$ and $H^q(Y, \mathcal{M}^S) \approx H^q(X, \mathcal{M}^S)$ (7.4.26).

Thus it is enough to prove Riemann-Roch for the projective line.

dualityforpone

(8.7.6) Riemann-Roch for the projective line

The Riemann-Roch Theorem for the projective line $X = \mathbb{P}^1$ is a consequence of the Birkhoff-Grothendieck Theorem, which tells us that a locally free $\mathcal{O}_X$-module $\mathcal{M}$ on X is a direct sum of twisting modules. To prove Riemann-Roch for the projective line X, it suffices to prove it for the twisting modules $\mathcal{O}_X(k)$

8.7.7. Lemma. *The module of differentials Ω_X on X is isomorphic to the twisting module $\mathcal{O}_X(-2)$.*

omegapone

proof. Since Ω_X is invertible, the Birkhoff-Grothendieck Theorem tells us that it is isomorphic to a twisting module $\mathcal{O}_X(k)$ for some k. We need only identify the integer k.

Let $\mathbb{U}^0 = \operatorname{Spec} \mathbb{C}[x]$, and $\mathbb{U}^1 = \operatorname{Spec} \mathbb{C}[z]$ be the standard open subsets of $\mathbb{P}^1$, with $z = x^{-1}$. On $\mathbb{U}^0$, the module of differentials is free, with basis dx, and $dx = d(z^{-1}) = -z^{-2}dz$ describes the differential dx on $\mathbb{U}^1$. Since the point p at infinity is $\{z = 0\}$, dx has a pole of order 2 at p. It is a global section of $\Omega_X(2p)$, and as a section of that module, it isn't zero anywhere. So multiplication by dx defines an isomorphism $\mathcal{O} \to \Omega_X(2p)$ that sends 1 to dx. Tensoring with $\mathcal{O}(-2p)$, we find that $\mathcal{O}(-2p)$ is isomorphic to Ω_X. □

twisthom **8.7.8. Lemma.** *Let let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules on the projective line X. Then ${}_X(\mathcal{M}(r), \mathcal{N})$ is canonically isomorphic to ${}_X(\mathcal{M}, \mathcal{N}(-r))$.*

proof. When we tensor a homomorphism $\mathcal{M}(r) \xrightarrow{\varphi} \mathcal{N}$ with $\mathcal{O}(-r)$, we obtain a homomorphism $\mathcal{M} \to \mathcal{N}(-r)$. Tensoring with $\mathcal{O}(r)$ is the inverse operation. □

The Serre dual $\mathcal{O}(n)^S$ of $\mathcal{O}(n)$ is therefore

$$\mathcal{O}(n)^S = {}_X(\mathcal{O}(n), \mathcal{O}(-2)) \approx \mathcal{O}(-2-n)$$

To prove Riemann-Roch for $X = \mathbb{P}^1$, we must show that

$$\mathbf{h}^0\mathcal{O}(n) = \mathbf{h}^1 X, \mathcal{O}(-2-n) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(n) = \mathbf{h}^0\mathcal{O}(-2-n)$$

This follows from Theorem 7.5.5, which computes the cohomology of the twisting modules. As we've noted before, the two assertions are equivalent, so it suffices to verify the first one. If $n < 0$, then $-2-n > 0$. In this case $\mathbf{h}^0\mathcal{O}(n) = \mathbf{h}^1\mathcal{O}(-2-n) = 0$. If $n \geq 0$, Theorem 7.5.5 asserts that $\mathbf{h}^0\mathcal{O}(n) = n+1$ and that $\mathbf{h}^1\mathcal{O}(-2-n) = (2+n) - 1 = n+1$.

Section 8.8 Using Riemann-Roch

appl genussm-curve

(8.8.1) genus

There are three closely related numbers associated to a smooth projective curve Y: its *topological genus* g, its *arithmetic genus* $p_a = \mathbf{h}^1\mathcal{O}_Y$, and the *degree* δ of the module of differentials Ω_Y.

8.8.2. Theorem. *Let Y be a smooth projective curve. The topological genus g and the arithmetic genus p_a of Y are equal, and the degree δ of the module Ω_Y is $2p_a - 2$, which is equal to $2g - 2$.* *genus-genus*

Thus the Riemann-Roch Theorem 8.2.3 can we written as

$$\chi(\mathcal{O}(D)) = \deg\, D + 1 - g$$

We'll write it this way, once the theorem is proved.
proof. Let $Y \xrightarrow{\pi} X$ be a branched covering with $X = \mathbb{P}^1$. The topological Euler characteristic $e(Y)$, which is $2 - 2g$, can be computed in terms of the branching data for the covering, as in (1.8.22). Let q_i be the ramification points in Y, and let e_i be the ramification index at q_i. Then e_i sheets of the covering come together at q_i. One might say that $e_i - 1$ points are lacking in Y. If the degree of Y over X is n, then since $e(X) = 2$,

(8.8.3) $$2 - 2g = e(Y) \;=\; ne(X) - \sum(e_i - 1) \;=\; 2n - \sum(e_i - 1)$$ degdelta

We compute the degree δ of Ω_Y in two ways. First, the Riemann-Roch Theorem tells us that $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = p_a$ and $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$. So $\chi(\Omega_Y) = -\chi(\mathcal{O}_Y) = p_a - 1$. The Riemann-Roch Theorem also tells us that $\chi(\Omega_Y) = \delta + 1 - p_a$. Therefore

(8.8.4) $$\delta = 2p_a - 2$$ delta

Next, we compute δ by computing the divisor of the differential dx on Y, x being a coordinate in X. Let q_i be one of the ramification points in Y, and let e_i be the ramification index at q_i. Then dx has a zero of order $e_i - 1$ at q_i. At the point of X at infinity, dx has a pole of order 2. Let's suppose that the point at infinity isn't a branch point. Then there will be n points of Y at which dx has a pole of order 2, n being the degree of Y over X. The degree of Ω_Y is therefore

(8.8.5) $$\delta = \;zeros - \;poles = \sum(e_i - 1) - 2n$$ deltatwo

Combining (8.8.5) with (8.8.3), one sees that $\delta = 2g - 2$. Since we also have $\delta = 2p_a - 2$, we conclude that $g = p_a$. $\square$

(8.8.6) canonical divisors canondiv

Because the module Ω_Y of differentials on a smooth curve Y is invertible, it is isomorphic to $\mathcal{O}(K)$ for some divisor K (Proposition 8.1.11). Such a divisor K is called a *canonical divisor*. The degree of K is $2g - 2$ (8.8.2). It is often convenient to represent Ω_Y as a module $\mathcal{O}(K)$, though the canonical divisor K isn't unique. It is determined only up to linear equivalence. (See (8.1.13).)

When written in terms of a canonical divisor K, the Serre dual of an invertible module $\mathcal{O}(D)$ will be $\mathcal{O}(D)^S = {}_{\mathcal{O}}(\mathcal{O}(D), \mathcal{O}(K)) \approx \mathcal{O}(K - D)$ (see (**8.1.14**) and (8.1.10)). With this notation, the Riemann-Roch Theorem for $\mathcal{O}(D)$ becomes

(8.8.7) $$\mathbf{h}^0\mathcal{O}(D) = \mathbf{h}^1\mathcal{O}(K - D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(K - D)$$ $\square$ RRforOD

(8.8.8) curves of genus zero genuszero

Let Y be a smooth projective curve Y of genus g zero, and let p be a point of Y. The exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(p) \to \epsilon \to 0$$

where ϵ is a one-dimensional module supported at p, gives us an exact cohomology sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y(p)) \to H^0(Y, \epsilon) \to 0$$

The zero on the right is due to the fact that $\mathbf{h}^1\mathcal{O}_Y = g = 0$. We also have $\mathbf{h}^0\mathcal{O}_Y = 1$ and $\mathbf{h}^0\epsilon = 1$, so when Y has genus zero, $\mathbf{h}^0\mathcal{O}_Y(p) = 2$. We choose a basis $(1, x)$ for $H^0(Y, \mathcal{O}_Y(p))$, 1 being the constant function and x being a nonconstant function with a single pole of order 1 at p. This basis defines a point of $\mathbb{P}^1$ with values in the function field K of Y, and therefore a morphism $Y \xrightarrow{\varphi} \mathbb{P}^1$. Because x has just one pole of order 1, it takes every value exactly once. Therefore φ is bijective. It is a map of degree 1, and therefore an isomorphism (8.5.2).

gzero **8.8.9. Corollary.** *Every smooth projective curve of genus zero is isomorphic to the projective line* $\mathbb{P}^1$. $\square$

A curve, smooth or not, whose function field is isomorphic to the field $\mathbb{C}(t)$ of rational functions in one variable is called a *rational curve*. A smooth projective curve of genus zero is a rational curve.

genusone **(8.8.10) curves of genus one**

A smooth projective curve of genus $g = 1$ is called an *elliptic curve*. The Riemann-Roch Theorem tells us that on an elliptic curve Y,

$$\chi(\mathcal{O}(D)) = \deg D$$

Since $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = 1$, Ω_Y has a nonzero global section ω. Since Ω_Y has degree zero (8.8.2), ω doesn't vanish anywhere. Multiplication by ω defines an isomorphism $\mathcal{O} \to \Omega_Y$. So Ω_Y is a free module of rank one.

hrpforelliptic **8.8.11. Lemma.** *Let D be a divisor of degree $r > 0$ an elliptic curve Y. Then $\mathbf{h}^0\mathcal{O}(D) = r$, and $\mathbf{h}^1\mathcal{O}(D) = 0$.*

This follows from Riemann-Roch. The Serre dual of $\mathcal{O}(D)$ is $\mathcal{O}(-D)$, so $\mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(-D)$, which is zero because the degree of $-D$ is negative. $\square$

Now, since $H^0(Y, \mathcal{O}_Y) \subset H^0(Y, \mathcal{O}_Y(p))$, and since both spaces have dimension one, they are equal. So (1) is a basis for $H^0(Y, \mathcal{O}_Y(p))$. We choose a basis $(1, x)$ for the two-dimensional space $H^1(Y, \mathcal{O}_Y(2p))$. Then x isn't a section of $\mathcal{O}(p)$. It has a pole of order precisely 2 at p and no other pole. Next, we choose a basis $(1, x, y)$ for $H^1(Y, \mathcal{O}_Y(3p))$. So y has a pole of order 3 at p, and no other pole. The point $(1, x, y)$ of $\mathbb{P}^2$ with values in K determines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^2$.

Let u, v, w be coordinates in $\mathbb{P}^2$. The map φ sends a point q distinct from p to $(u, v, w) = (1, x(q), y(q))$. Since Y has dimension one, φ is a finite morphism. Its image Y' will be a closed subvariety of $\mathbb{P}^2$ of dimension one — a plane curve.

To determine the image of the point p, we multiply $(1, x, y)$ by $\lambda = y^{-1}$, obtaining the equivalent vector $(y^{-1}, xy^{-1}, 1)$. The rational function y^{-1} has a zero of order 3 at p, and xy^{-1} has a simple zero there. Evaluating at p, we see that the image of p is the point $(0, 0, 1)$.

The map $Y \to \mathbb{P}^2$ restricts to an integral morphism $Y \to Y'$, where Y' is the image. Let ℓ be a generic line $\{au+bv+cw = 0\}$ in $\mathbb{P}^2$. The rational function $a+bx+cy$ on Y has a pole of order 3 at p and no other pole. It takes every value, including zero, three times, and the set of three points of Y at which $a + bx + cy$ is zero is the inverse image of the intersection $Y' \cap \ell$. The only possibilities for the degree of Y' are 1 and 3. Since $1, x, y$ are independent, they don't satisfy a homogeneous linear equation. So Y' isn't a line. The image Y' is a cubic curve (see Corollary 1.3.10).

To determine the image, we look for a cubic relation among the functions $1, x, y$ on Y. The seven monomials $1, x, y, x^2, xy, x^3, y^2$ have poles at p, of orders $0, 2, 3, 4, 5, 6, 6$, respectively, and no other poles. They are sections of $\mathcal{O}_Y(6p)$. Riemann-Roch tells us that $\mathbf{h}^0\mathcal{O}_Y(6p) = 6$. So those seven functions are linearly dependent. The dependency relation gives us a cubic equation among x and y, which we may write in the form

$$cy^2 + (a_1x + a_3)y + (a_0x^3 + a_2x^2 + a_4x + a_6) = 0$$

There can be no linear relation among functions whose orders of pole at p are distinct. So when we delete either x^3 or y^2 from the list of monomials, we obtain an independent set of six functions. They form a basis for the six-dimensional space $H^0(Y, \mathcal{O}(6p))$. In the cubic relation, the coefficients c and a_0 aren't zero. We normalize

c and a_0 to 1. Next, we eliminate the linear term in y from the relation by substituting $y - \frac{1}{2}(a_1x + a_3)$ for y, and we eliminate the quadratic term in x. by substituting $x - \frac{1}{3}a_2$ for x. Bringing the terms in x to the other side of the equation, we are left with a cubic relation of the form

$$y^2 = x^3 + a_4x + a_6$$

The coefficients a_4 and a_6 have been changed, of course.

The cubic curve Y' defined by the homogenized equation $y^2z = x^3 + a_4xz^2 + a_6z^3$ is the image of Y. This curve meets a generic line $ax + by + cz = 0$ in three points and, as we saw above, its inverse image in Y consists of three points too. Therefore the morphism $Y \xrightarrow{\varphi} Y'$ is generically injective, and Y is the normalization of Y'. Corollary 7.6.4 computes the cohomology of Y': $\mathbf{h}^0\mathcal{O}_{Y'} = \mathbf{h}^1\mathcal{O}_{Y'} = 1$. This tells us that $\mathbf{h}^q\mathcal{O}_{Y'} = \mathbf{h}^q\mathcal{O}_Y$ for all q. Let's denote the direct image $\varphi_*(\mathcal{O}_Y)$ by $\mathcal{O}_Y$, and let $\mathcal{F}$ be the $\mathcal{O}_{Y'}$-module $\mathcal{O}_Y/\mathcal{O}_{Y'}$. Since Y is the normalization of Y', $\mathcal{F}$ is a torsion module. The exact sequence $0 \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to \mathcal{F} \to 0$ shows that $\mathbf{h}^0\mathcal{F} = 0$. So $\mathcal{F}$ is torsion module with no global sections. Therefore $\mathcal{F} = 0$, and $Y \approx Y'$.

8.8.12. Corollary. *Every elliptic curve is isomorphic to a cubic curve in $\mathbb{P}^2$.* □ *genusonecubic*

(8.8.13) the group law on an elliptic curve

genusonegroup

The points of an elliptic curve Y form an abelian group, once one chooses a point to be the identity element.

We choose a point of an elliptic curve Y and label it o. We'll write the law of composition as $p \oplus q$, to make a clear distinction between the product in the group, which is a point of Y, and the divisor $p + q$.

Let p and q be points of Y. To define $p \oplus q$, we compute the cohomology of $\mathcal{O}_Y(p + q - o)$. Lemma hrpforelliptic **(ii)** shows that $\mathbf{h}^0\mathcal{O}_Y(p + q - o) = 1$ and that $\mathbf{h}^1\mathcal{O}_Y(p + q - o) = 0$. So there is a nonzero function f, unique up to scalar factor, with simple poles at p and q and a simple zero at o. This function has exactly one additional zero. That zero is defined to be the sum $p \oplus q$ in the group. In terms of linearly equivalent divisors, $p \oplus q$ is the unique point s such that the divisor $p + q$ is linearly equivalent to $o + s$.

8.8.14. Proposition. *The law of composition $\oplus$ defined above makes an elliptic curve into an abelian group.* *grplaw*

The proof is left as an exercise. □

(8.8.15) interlude: maps to projective space

maptoP

Let Y be a smooth projective curve. We have seen that any set $(f_0, ..., f_n)$ of rational functions on Y, not all zero, defines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ (5.2.3). As a reminder, let q be a point of Y and let $g_j = f_j/f_i$, where i is an index such that the value $\mathrm{v}_q(f_i)$ is a minimum among $\mathrm{v}_q(f_k)$, for $k = 0, ..., n$. The rational functions g_j are regular at q for all j, and the morphism φ sends the point q to $(g_0(q), ..., g_n(q))$. For example, the inverse image $\varphi^{-1}(\mathbb{U}^0)$ of the standard open set $\mathbb{U}^0$ is the set of points of Y at which the functions $g_j = f_j/f_0$ are regular. If q is such a point, then $\varphi(q) = (1, g_1(q), ..., g_n(q))$.

8.8.16. Lemma. *Let Y be a smooth projective curve, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism to projective space defined by a set $(f_0, ..., f_n)$ of rational functions on Y (not all zero).* *mapcurve*

(i) *If the space of rtionl functions that is spanned by $\{f_0, ..., f_n\}$ has dimension at least two, then φ isn't a constant morphism to a point.*

(ii) *If $f_0, ..., f_n$ are linearly independent, the image of Y isn't contained in a hyperplane.* □

The *degree* d of a nonconstant morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ from a projective curve Y (smooth or not) to projective space is defined to be the number of points of the inverse image $\varphi^{-1}H$ of a generic hyperplane H in $\mathbb{P}^n$. You will be able to check that this number is well-defined.

basept **(8.8.17) base points**

Let D be a divisor on the smooth projective curve Y, and suppose that $\mathbf{h}^0\mathcal{O}(D) = k > 1$. A basis $(f_0, ..., f_k)$ of global sections of $\mathcal{O}(D)$ defines a morphism $Y \to \mathbb{P}^{k-1}$. This is the most common way to construct a morphism to projective space, though one could use any set of rational functions that aren't all zero.

If a global section of $\mathcal{O}(D)$ vanishes at a point p of Y, it is a global section of $\mathcal{O}(D-p)$ too. A *base point* of $\mathcal{O}(D)$ is a point of Y at which every global section of $\mathcal{O}(D)$ vanishes. A base point can be described in terms of the usual exact sequence

$$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \epsilon \to 0$$

The point p is a base point if $\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D)$, or if $\mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D) - 1$.

degnobasept **8.8.18. Lemma.** *Let D be a divisor on a smooth projective curve Y with $\mathbf{h}^0\mathcal{O}(D) > 1$, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism defined by a basis of global sections.*

(i) *The image of φ isn't contained in any hyperplane.*

(ii) *If $\mathcal{O}(D)$ has no base point, the degree r of the morphism φ is equal to degree of D. If there are base points, the degree is lower.* $\square$

exbasepoints **8.8.19. Proposition.** *Let K be a canonical divisor on a smooth projective curve Y of positive genus.*

(i) *$\mathcal{O}(K)$ has no base point.*

(ii) *Every point p of Y is a base point of $\mathcal{O}(K+p)$.*

proof. **(i)** Let p be a point of Y. We apply Riemann-Roch to the usual exact sequence

$$0 \to \mathcal{O}(K-p) \to \mathcal{O}(K) \to \epsilon \to 0$$

where ϵ is a one-dimensional module whose support is p. The Serre duals of $\mathcal{O}(K)$ and $\mathcal{O}(K-p)$ are $\mathcal{O}$ and $\mathcal{O}(p)$, respectively. They form an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p) \to \epsilon' \to 0$$

Because Y has genus $g > 0$, there is no rational function on Y with just one simple pole. So $\mathbf{h}^0\mathcal{O} = \mathbf{h}^0\mathcal{O}(p) = 1$. Riemann-Roch tells us that $\mathbf{h}^1\mathcal{O}(K-p) = \mathbf{h}^1\mathcal{O}(K) = 1$. The cohomology sequence

$$0 \to H^0(\mathcal{O}(K-p)) \to H^0(\mathcal{O}(K)) \to [1] \to H^1(\mathcal{O}(K-p)) \to H^1(\mathcal{O}(K)) \to 0$$

shows that $\mathbf{h}^0\mathcal{O}(K-p) = \mathbf{h}^0\mathcal{O}(K) - 1$. So p is not a base point.

(ii) Here, the relevant sequence is

$$0 \to \mathcal{O}(K) \to \mathcal{O}(K+p) \to \epsilon'' \to 0$$

The Serre dual of $\mathcal{O}(K+p)$ is $\mathcal{O}(-p)$, which has no global section. Therefore $\mathbf{h}^1\mathcal{O}(K+p) = 0$, while $\mathbf{h}^1\mathcal{O}(K) = 1$. The cohomology sequence

$$0 \to \mathbf{h}^0\mathcal{O}(K) \to \mathbf{h}^0\mathcal{O}(K+p) \to [1] \to \mathbf{h}^1\mathcal{O}(K) \to \mathbf{h}^1\mathcal{O}(K+p) \to 0$$

shows that $H^0(\mathcal{O}(K+p)) = H^0(\mathcal{O}(K))$. So p is a base point of $\mathcal{O}(K+p)$. $\square$

hyper **(8.8.20) hyperelliptic curves**

A *hyperelliptic curve* Y is a smooth projective curve of genus $g \geq 2$ that can be represented as a branched double covering of the projective line. So a smooth projective curve Y is hyperelliptic if there exists a morphism $Y \xrightarrow{\pi} X = \mathbb{P}^1$ of degree two. The justification for the term 'hyperelliptic' is that every elliptic curve can be represented (in many ways) as a double cover of $\mathbb{P}^1$. The global sections of $\mathcal{O}(2p)$, where p can be any point of Y define a map of degree 2 to $\mathbb{P}^1$.

The topological Euler characteristic of a hyperelliptic curve Y can be computed in terms of the covering $Y \to X$, which will be branched at a finite set, say of n points of Y. Since π has degree two, the ramification index at a branch point will be 2. The Euler characteristic is therefore $e(Y) = 2e(X) - n = 4 - n$. Since we know that $e(Y) = 2 - 2g$, the number of branch points is $n = 2g + 2$. When $g = 3$, $n = 8$.

It would take some experimentation to guess that the next remarkable theorem might be true, and to find a proof.

8.8.21. Theorem. *Let Y be a hyperelliptic curve, let $Y \xrightarrow{\pi} X = \mathbb{P}^1$ be a branched covering of degree 2, and let $Y \xrightarrow{\kappa} \mathbb{P}^{g-1}$ be the morphism defined by the global sections of $\Omega_Y = \mathcal{O}(K)$. The morphism κ factors through π. There is a morphism $X \xrightarrow{u} \mathbb{P}^{g-1}$ such that κ is the composition $Y \xrightarrow{\pi} X \xrightarrow{u} \mathbb{P}^{g-1}$:* *hyper-canon*

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "\kappa"'] & X \arrow[d, "u"'] \\ \mathbb{P}^{g-1} \arrow[r, equal] & \mathbb{P}^{g-1} \end{tikzcd}$$

proof. Let x be an affine coordinate in X, so that the standard affine open subset $\mathbb{U}^0$ of X is $\operatorname{Spec} \mathbb{C}[x]$. We suppose that the point of X at infinity isn't a branch point of the covering π. The open set $Y^0 = \pi^{-1}\mathbb{U}^0$ will be described by an equation of the form $y^2 = f(x)$, where f is a polynomial of degree $n = 2g + 2$ with simple roots, and there will be two points of Y above the point of X at infinity. They are interchanged by the automorphism $y \to -y$. Let's call those points q_1 and q_2.

We start with the differential dx, which we view as a differential on Y. Then $2y\,dy = f'(x)dx$. Since f has simple roots, f' doesn't vanish at any of them. Therefore dx has simple zeros on Y above the roots of f. We also have a regular function on Y^0 with simple roots at those points, namely the function y. Therefore the differential $\omega = \frac{dx}{y}$ is regular and nowhere zero on Y^0. Because the degree of a differential on Y is $2g - 2$, ω has a total of $2g - 2$ zeros at infinity. By symmetry, ω has zeros of order $g - 1$ at q_1 and at q_2. So $K = (g-1)q_1 + (g-1)q_2$ is a canonical divisor on Y, and $\Omega_Y \approx \mathcal{O}_Y(K)$.

Since K has zeros of order $g-1$ at infinity, the rational functions $1, x, x^2, ..., x^{g-1}$, when viewed as functions on Y, are among the global sections of $\mathcal{O}_Y(K)$. They are independent, and there are g of them. Since $\mathbf{h}^0\mathcal{O}_Y(K) = g$, they form a basis of $H^0(\mathcal{O}_Y(K))$. The map $Y \to \mathbb{P}^{g-1}$ defined by the global sections of $\mathcal{O}_Y(K)$ evaluates these powers of x, so it factors through X. $\square$

The map u is the one defined by the global sections of $\mathcal{O}_X((g-1)p)$, where p is the point at infinity. Since $X = \mathbb{P}^1$, all of its points are linearly equivalent. Therefore u is determined up to the choice of coordinates in $\mathbb{P}^{g-1}$, as is κ. It follows that π is unique.

8.8.22. Corollary. *A curve of genus $g \geq 2$ can be presented as a branched covering of $\mathbb{P}^1$ of degree 2 in at most one way.* $\square$ *onedblcover*

(8.8.23) canonical embedding canonemb

Let Y be a smooth projective curve of genus $g \geq 2$, and let K be a canonical divisor on Y. Since $\mathcal{O}(K)$ has no base point, its global sections define a morphism $Y \to \mathbb{P}^{g-1}$. This morphism is called the *canonical map*. Let's denote the canonical map by κ. The degree of κ is the degree $2g - 2$ of the canonical divisor.

Theorem 8.8.21 shows that, when Y is hyperelliptic, the image of the canonical map is isomorphixc to $X = \mathbb{P}^1$.

8.8.24. Theorem. *Let Y be a smooth projective curve of genus g at least two. If Y isn't hyperelliptic, the canonical map embeds Y as a closed subvariety of projective space $\mathbb{P}^{g-1}$.* *canonembtwo*

proof. We show first that Y is hypereliptic if the canonical map $Y \xrightarrow{\kappa} \mathbb{P}^{g-1}$ isn't injective.

Let p and q be distinct points of Y, and suppose that $\kappa(p) = \kappa(q)$. We choose an effective canonical divisor whose support doesn't contain p or q, and we inspect the global sections of $\mathcal{O}(K{-}p{-}q)$. Since $\kappa(p) = \kappa(q)$, any global section of $\mathcal{O}(K)$ that vanishes at p vanishes at q too. Therefore $\mathcal{O}(K-p)$ and $\mathcal{O}(K-p-q)$ have the same global sections, and q is a base point of $\mathcal{O}(K-p)$. We've computed the cohomology of $\mathcal{O}(K-p)$ before: $\mathbf{h}^0\mathcal{O}(K-p) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p) = 1$. Therefore $\mathbf{h}^0\mathcal{O}(K-p-q) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p-q) = 2$. The Serre dual of $\mathcal{O}(K-p-q)$ is $\mathcal{O}(p+q)$, so by Riemann-Roch, $\mathbf{h}^0\mathcal{O}(p+q) = 2$. Then $\mathcal{O}(p+q)$ has no base point, because $\mathbf{h}^0(\mathcal{O}(D)) \le 1$ for any divisor D of degree one on a curve of positive genus. So the global sections of $\mathcal{O}(p+q)$ define a morphism $Y \to \mathbb{P}^1$ of degree 2. Therefore Y is hyperelliptic.

If Y isn't hyperelliptic, the canonical map is injective, so Y is mapped bijectively to its image Y' in $\mathbb{P}^{g-1}$. This almost proves the theorem, but: can Y' have a cusp? We must show that the bijective map $Y \xrightarrow{\kappa} Y'$ is an isomorphism. We go over the computation made above for a pair of points p, q, this time taking $q = p$. The computation is the same. Since Y isn't hyperelliptic, p isn't a base point of $\mathcal{O}_Y(K-p)$. Therefore $\mathbf{h}^0\mathcal{O}_Y(K-2p) = \mathbf{h}^0\mathcal{O}_Y(K-p) - 1$. This tells us that there is a global section f of $\mathcal{O}_Y(K)$ that has a zero of order exactly 1 at p. When properly interpreted, this fact shows that κ doesn't collapse any tangent vectors to Y, and that κ is an isomorphism. Since we haven't discussed tangent vectors, we prove this directly.

Since κ is bijective, the function fields of Y and its image Y' are equal, and Y is the normalization of Y'. Moreover, κ is an isomorphism except on a finite set. We work locally at a point p' of Y', and we denote the unique point of Y that maps to Y' by p. When we restrict the global section f of $\mathcal{O}_Y(K)$ found above to the image Y', we obtain an element of the maximal ideal $\mathfrak{m}'_{p'}$ of $\mathcal{O}_{Y'}$ at p', that we denote by x. On Y, this element has a zero of order one at p, and therefore it is a local generator for the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_Y$. Let $\mathcal{O}'$ and $\mathcal{O}$ be the local rings at p. We apply the Local Nakayama Lemma 5.1.1, regarding $\mathcal{O}$ as a finite $\mathcal{O}'$-module. We substitute $V = \mathcal{O}$ and $M = \mathfrak{m}'_{p'}$ into the statement of that lemma. Since x is in $\mathfrak{m}'_{p'}$, $V/MV = \mathcal{O}/\mathfrak{m}'_p\mathcal{O}$ is the residue field $k(p)$ of $\mathcal{O}$, which is spanned, as $\mathcal{O}'$-module, by the element 1. The Local Nakayama Lemma tells us that $\mathcal{O}$ is spanned, as $\mathcal{O}'$-module, by 1, and this shows that $\mathcal{O} = \mathcal{O}'$. $\square$

lowgenus **(8.8.25) some curves of low genus**

Here Y will denote a smooth projective curve of genus g.

curves of genus 2.

When Y is a smooth projective curve of genus 2. The canonical map κ is a map from Y to $\mathbb{P}^1$, whose degree is $2g-2 = 2$. Every smooth projective curve of genus 2 is hyperelliptic.

curves of genus 3.

Let Y be a smooth projective curve of genus $g = 3$. The canonical map κ is a morphism of degree 4 from Y to $\mathbb{P}^2$. If Y isn't hyperelliptic, its image will be a plane curve of degree 4, that is isomorphic to Y. The genus of a smooth projective curve of degree 4 is $\binom{3}{2} = 3$, which checks (see (1.8.24)).

There is another way to arrive at the same result. We go through it because the same method can be used for curves of genus 4 or 5. Riemann-Roch determines the dimension of the space of global sections of $\mathcal{O}(dK)$. When $d > 1$,

$$\mathbf{h}^1\mathcal{O}(dK) = \mathbf{h}^0\mathcal{O}((1-d)K) = 0$$

Then

OdK (8.8.26)
$$\mathbf{h}^0\mathcal{O}(dK) = deg(dK) + 1 - g = d(2g-2) - (g-1) = (2d-1)(g-1)$$

In our case $g = 3$, so when $d > 1$, $\mathbf{h}^0\mathcal{O}(dK) = 4d-2$.

The number of monomials of degree d in $n+1$ variables $x_0, ..., x_n$ is $\binom{n+d}{d}$. Here $n = 2$, so that number is $\binom{d+2}{2}$.

We assemble this information into a table:

d	0	1	2	3	4	5
monos deg d	1	3	6	10	15	21
$\mathbf{h}^0\mathcal{O}(dK)$	1	3	6	10	14	18

Now, if $(\alpha_0, ..., \alpha_2)$ is a basis of $H^0\mathcal{O}(K)$, the products $\alpha_{i_1} \cdots \alpha_{i_d}$ of length d are global sections of $\mathcal{O}(dK)$. In fact, they generate the space $H^0\mathcal{O}(dK)$ of global sections. This isn't easy to prove, and it isn't very important here, so we omit the proof. What we see from the table is that there is at least one homogeneous polynomial $f(x_0, ..., x_2)$ of degree 4, such that $f(\alpha) = 0$. This means that the curve Y lies in the zero locus of that polynomial, which is a quartic curve. The table also shows that Y isn't in the zero locus of any curve of lower degree. So Y is a quartic curve. Then f is, up to scalar factor, the only homogeneous quartic that vanishes on Y. Therefore the monomials of degree 4 in α span a space of dimension 14, and therefore they span $H^0\mathcal{O}(4K)$. This is one case of the general fact that was stated above.

The table also shows that there are (at least) three independent polynomials of degree 5 that vanish on Y. They don't give new relations because we can think of three such polynomials, namely x_0f, x_1f, x_2f.

curves of genus 4**.**

When Y is a smooth projective curve of genus 4 that isn't hyperelliptic, the canonical map embeds Y as a curve of degree $2g - 2 = 6$ in $\mathbb{P}^3$. Let's leave the analysis of this case as an exercise.

curves of genus 5**.**

With genus 5, things become more complicated.

Let Y be a smooth projective curves of genus 5 that isn't hyperelliptic. The canonical map embeds Y as a curve of degree 8 in $\mathbb{P}^4$. We make a computation analogous to what was done for genus 3. For $d > 1$, the dimension of the space of global sections of $\mathcal{O}(dK)$ is

$$\mathbf{h}^0\mathcal{O}(dK) = (2d-1)(g-1) = 8d - 4$$

and the number of monomials of degree d in 5 variables is $\binom{d+4}{4}$.

We form a table:

d	0	1	2	3
monos deg d	1	5	15	35
$\mathbf{h}^0\mathcal{O}(dK)$	1	5	12	20

This table predicts that there are (at least) three independent homogeneous quadratic polynomials q_1, q_2, q_3 that vanish on the curve Y. Let Q_i be the quadric $\{q_i = 0\}$. Then Y will be contained in the zero locus $Z = Q_1 \cap Q_2 \cap Q_3$.

Bézout's Theorem has a generalization that applies here. Let Q_1, Q_2, Q_3 be hypersurfaces in $\mathbb{P}^4$, of degrees r_1, r_2, r_3, respectively. Let $Z_1, ..., Z_k$ be the irreducible components of the zero locus $Z : \{q_1 = q_2 = q_3 = 0\}$. If Z has dimension 1, then the sum of the degrees $\deg Z_1 + \cdots + \deg Z_k$ is at most equal to the product $r_1r_2r_3$. We omit the proof of this, which is similar to the proof of the usual Bézout's Theorem.

When Q_i are the quadrics $\{q_i = 0\}$, $i = 1, 2, 3$, the intersection $Z = Q_1 \cap Q_2 \cap Q_3$ will contain Y, which has degree 8, and it follows from Bézout's Theorem that if $\dim Z = 1$, then $Y = Z$. In this case Y is called a *complete intersection* of the three quadrics.

However, it is possible that the intersection Z has dimension 2. This happens when Y is can be represented as a three-sheeted covering of $\mathbb{P}^1$. Such a curve is called a *trigonal curve* (another peculiar term).

8.8.27. Proposition. *A trigonal curve of genus* 5 *is not isomorphic to an intersection of three quadrics in* $\mathbb{P}^4$. *trigonal*

proof. A trigonal curve Y will have a degee three morphism to the projective line: $Y \to X = \mathbb{P}^1$. Let's suppose that the point at infinity of X isn't a branch point. Let the fibre over the point at infinity be $\{p_1, p_2, p_3\}$. With coordinates (x_0, x_1) on X, the rational function $u = x_1/x_0$ on X has poles $D = \sum p_i$ on Y, so $H^0(Y, \mathcal{O}(D))$ contains 1 and u, and therefore $\mathbf{h}^0\mathcal{O}(D) \geq 2$. By Riemann-Roch, $\chi\mathcal{O}(D) = 3 + 1 - g = -1$. Therefore $\mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(K - D) \geq 3$. There are (at least) three independent global sections of $\mathcal{O}(K)$ that vanish on D. Let them be $\alpha_0, \alpha_1, \alpha_2$. When Y is embedded into $\mathbb{P}^4$ by a basis $(\alpha_0, ..., \alpha_4)$, the three planes $\{x_i = 0\}$, $i = 0, 1, 2$ contain D. The intersection of these planes is a line L that contains the three points p_1, p_2, p_3.

We go back to the three quadrics Q_1, Q_2, Q_3 that contain Y. Since they contain Y, they contain D. A quadric Q intersects the line L in at most two points unless it contains L. Therefore each of the quadrics Q_i

contains L, and $Q_1 \cap Q_2 \cap Q_3$ contains L as well as Y. Suppose that $Z = Q_1 \cap Q_2 \cap Q_3$ has dimension 1. Then, according to Bézout, the sum $1 + 8$ of the degrees of L and Y, must be at most $2 \cdot 2 \cdot 2 = 8$. Nope: $Z = Q_1 \cap Q_2 \cap Q_3$ cannot have dimension 1. $\square$

It can be shown that this is the only exceptional case. A curve of genus 5 is either hyperelliptic, or trigonal, or else it is a complete intersection of three quadrics in $\mathbb{P}^4$.

Section 8.9 Exercises

8.9.1. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-modules. Prove that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ is a (quasicoherent) $\mathcal{O}$-module. chapeigh-tex xhomco-herent

8.9.2. ### Let D be a divisor of degree d on a smooth projective curve Y, and assume that $\mathbf{h}^0\mathcal{O}(D) = k > 0$. generic-notbp

(i) Prove that if p is a generic point of Y, then $\mathbf{h}^0\mathcal{O}(D - p) = k - 1$.

(ii) Show that $\mathbf{h}^0\mathcal{O}(D) \leq d + 1$, and that if $\mathbf{h}^0\mathcal{O}(D) = d + 1$, then X is isomorphic to $\mathbb{P}^1$.

8.9.3. Prove that an open subset of a smooth affine curve is affine. openaffine

8.9.4. Let Y be a smooth curve of genus 1. Use version 1 of Riemann-Roch to prove that, if $r \geq 1$, then $\dim H^0(Y, \mathcal{O}_Y(rp)) = r$ and $H^1(Y, \mathcal{O}_Y(rp)) = 0$. xgenone

8.9.5. Let D be a divisor of degree d on a smooth projective curve Y. Show that $\mathbf{h}^0(\mathcal{O}(D)) \leq d + 1$, and if $\mathbf{h}^0(\mathcal{O}(D)) = d + 1$, then Y is a smooth rational curve, isomorphic to $\mathbb{P}^1$. xsmrat

8.9.6. Prove that a projective curve Y such that $\mathbf{h}^1(\mathcal{O}_Y) = 0$, smooth or not, is isomorphic to the projective line $\mathbb{P}^1$. xHonezero

8.9.7. Let C be a plane projective curve of degree d, with δ nodes and κ cusps, and let C' be the normalization of C. Determine the Genus of C'. xnode-sandcusps

8.9.8. Let Y be a curve of genus two, and let p be a point p of Y. xgenust-wotwo

(i) Prove that there are two cases: Either $\mathbf{h}^0(\mathcal{O}_Y(2p)) = 1$ and $H^1(\mathcal{O}_Y(2p)) = 0$, or else $2p$ is a canonical divisor, in which case $\mathbf{h}^0(\mathcal{O}_Y(2p)) = 2$ and $\mathbf{h}^1(\mathcal{O}_Y(2p) = 1$.

(ii) Suppose we are in the first case. Show that then $\mathbf{h}^0(\mathcal{O}_Y(rp)) = r - 1$ and $H^1(\mathcal{O}_Y(rp)) = 0$, for all $r \geq 2$.

(iii) Show that there is a basis of global sections of $\mathcal{O}(4p)$ of the form $(1, x, y)$, where x and y have poles of orders 3 and 4 at p. This basis defines a morphism $Y \to \mathbb{P}^2$ whose image is a curve Y' of degree 4.

(iv) Prove that Y' is a singular curve.

8.9.9. Use version 1 of the Riemann-Roch Theorem to compute $\mathbf{h}^0(\mathcal{O}(rp))$ for a smooth projective curve of genus 1 xcohgenu-sone

8.9.10. The projective line $X = \mathbb{P}^1$ with coordinates x_0, x_1 is covered by the two standard affine open sets $U^0 = \operatorname{Spec} R_0$ and $U^1 = \operatorname{Spec} R_1$, $R_0 = \mathbb{C}[u]$ with $u = x_1/x_0$, and $R_1 = \mathbb{C}[v]$ with $v = x_0/x_1 = u^{-1}$. The intersection U^{01} is the spectrum of the Laurent polynomial ring $R_{01} = \mathbb{C}[u, v] = \mathbb{C}[u, u^{-1}]$. The units of R_{01} are the monomials cu^k, where k can be any integer. xproveBG

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible R_{01}-matrix. Prove that there is an invertible R_0-matrix Q and there is an invertible R_1-matrix P such that $Q^{-1}AP$ is diagonal.

(ii) Use part **(i)** to prove the Birkhoff-Grothendieck Theorem for torsion-free $\mathcal{O}_X$-modules of rank 2.

8.9.11. On $\mathbb{P}^1$, when is $\mathcal{O}(m) \oplus \mathcal{O}(n)$ isomorphic to $\mathcal{O}(r) \oplus \mathcal{O}(s)$? xOmplu-sOn

8.9.12. Let Y be an elliptic curve. xellgrplaw

(i) Prove that, with the law of composition $\oplus$ defined in (**8.8.13**), Y is an abelian group.

(ii) Let p be a point of Y. Describe the product $p^k = p \circ p \cdots \circ p$ of k copies of p.

(iii) Determine the number of points of order 2 on Y.

8.9.13. Let Y be an elliptic curve. Show that, if origin is a flex point, the other the flexes of C are the points of order 3. Determine the number of points of Y of order 3. xnumber-flex

8.9.14. How many real flex points can a real cubic curve have? xrealflex

8.9.15. Prove that a finite $\mathcal{O}$-module on a smooth curve is a direct sum of a torsion module and a locally free module. xmod-dirsum

8.9.16. Let A be a finite-type domain. xde-scomega

(i) Let $B = A[x]$ be the ring of polynomials in one variable with coefficients in A. Deascribe the module Ω_B in terms of Ω_A.

(ii) Let s be a nonzero element of A and let A' be the localization $A[x]/(sx - 1)$. Describe the module $\Omega_{A'}$.

xgenomega **8.9.17.** Let $Y = \operatorname{Spec} B$ a smooth affine curve, and let y be an element of B. At what points does dy generate Ω_Y locally?

xmorphcurvefin **8.9.18.** Prove a morphism of curves Y to $\mathbb{P}^1$ that doesn't map Y to a point is a finite morphism without appealing to Chevalley's Theorem.

xtracederiv **8.9.19.** Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth affine curves, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $B \xrightarrow{\delta} {}_A(B, \Omega_A)$ be the composition of the derivation $B \xrightarrow{d} \Omega_B$ with the trace map $\Omega_B \approx_B (B, \Omega_B) \xrightarrow{\tau\circ} {}_A(B, \Omega_A)$. Prove that δ is a derivation from B to the B-module ${}_A(B, \Omega_A)$.

xprovetrace **8.9.20.** Let $Y \to X$ be a branched covering, and let p be a point of X whose inverse image in Y consists of one point q. Use a basis to prove the main theorem on the trace map for differentials locally at p.

xgenustwo **8.9.21.** Let Y be a smooth projective curve of genus 2. Determine the possible dimensions of $H^q(Y, \mathcal{O}(D))$, when D is an effective divisor of degree n.

xdegfive **8.9.22.** **(i)** Let C be plane curve of degree 5 with a double point. Show that the projection of the plane to $X = \mathbb{P}^1$ with the double point as center of projection represents C as a trigonal curve of genus 5.

(ii) The canonical embedding of a trigonal genus 5 curve Y will have three colinear points $D = p_1 + p_2 + p_3$, Show that $\mathbf{h}^0\mathcal{O}(K - D) = 3$ and that $\mathcal{O}(K - D)$ has no base point. Show that a basis of $H^0\mathcal{O}(K - D)$ maps Y to a curve of degree 5 in $\mathbb{P}^2$ with a double point.

xmoduli **8.9.23.** When determining the number of moduli (parameters) of generic plane curves of degree d, there are several dimensions:

- m, the dimension $\binom{d+2}{2}$ of the space of homogeneous polynomials of degree d,
- ℓ, the dimension of the group of linear operators on $\mathbb{P}^2$, which is $\dim GL_3 - 1 = 8$, and
- a the dimension of the group of automorphisms of a generic curve C of degree d.

Putting these together, determine the number of moduli of curves of degrees $1, 2, 3,$ and 4.

bptfre **8.9.24.** *the basepoint-free trick.* Let D be an effective divisor on a smooth projective curve Y, ans suppose that $\mathcal{O}(D)$ has no base points. Choose global sections α, β of $\mathcal{O}(D)$ with no common zeros. Prove the following:

(i) The sections α, β generate $\mathcal{O}(D)$, and there is an exact sequence

$$0 \to \mathcal{O}(-D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}(D) \to 0$$

.

(ii) The tensor product of this sequence with $\mathcal{O}(E)$ is an exact sequence

$$0 \to \mathcal{O}(E - D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}(E)^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}(E + D) \to 0$$

(iii) If $H^1\mathcal{O}(E - D)$, the map $H^0\mathcal{O}(E)^2 \to H^0\mathcal{M}(E + D)$ is surjective. Then every global section of $\mathcal{O}(E + D)$ can be obtained as a product uv with $u \in H^0\mathcal{O}(E)$ and $v \in H^0\mathcal{O}(D)$. The map $H^0\mathcal{O}(E) \otimes H^0\mathcal{O}(E) \to H^0\mathcal{O}(E + D)$ is surjective.

xponcelet **8.9.25.** Let C and D be conics that meet in four distinct points in the projective plane $\mathbb{P}$, and let D^* be the dual conic of tangent lines to D. Let E be the locus of points (p, ℓ^*) in $\mathbb{P}\times\mathbb{P}^*$ such that $\ell^* \in D^*$ and $p \in \ell$.

(i) Prove that E is a smooth elliptic curve.

(ii) Show that, for most $p \in C$, there will be two tangent lines ℓ to D such that (p, ℓ^*) is in E, and that, for most $\ell^* \in D^*$, there will be two points p such that (p, ℓ^*) is in E. Identify the exceptional points.

(iii) If (p_1, ℓ_1) is given, let p_2 denote the second intersection of C with ℓ_1^*, and let ℓ_2^* denote the second tangent to D that contains p_2. Define a map, where possible, by sending $(p_1, \ell_1^*) \to (p_2, \ell_1^*) \to (p_2, \ell_2^*)$. Show that this map extends to a morphism $E \xrightarrow{\gamma} E$ on E, and that this morphism is a translation $p \to p \oplus a$, for some point a of E.

(iv) It might happen that for some point p of C and some n, $\gamma^n(p) = p$. Show that if this occurs, the same is true for every point of C. For example, if $\gamma^3(p) = p$, the lines ℓ_1, ℓ_2, ℓ_3 will form a triangle whos vertices are on C., and this will be true for all points p of C. This is *Poncelet's Theorem.*

8.9.26. Let $d(x,y)$ and $f(x,y)$ be polynomials such that d divides f_x and f_y, then f is constant on the locus $d=0$. bertiniforpencil

8.9.27. Let M be a module over a finite-type domain A, and let α be an element of A. Prove that for all but finitely many complex numbers c, scalar multiplication by $s=\alpha-c$ is an injective map $M\xrightarrow{s}M$. xmultbyfplusc

8.9.28. Let $Y\xrightarrow{u}X$ be a finite morphism of curves, and let K and L be the function fields of X and Y, respectively, and suppose $[L:K]=n$. Prove that all fibres of Y/X have order at most n, and all but finitely many fibres of Y over X have order equal to n. countfibres

8.9.29. Let x_0,x_1 and y_0,y_1 be the coordinates in the two factors of the product $X=\mathbb{P}^1\times\mathbb{P}^1$.A homogeneous fraction of bidegree m,n on X is a fraction g/h of bihomogeneous polynomials in x,y such that, if the bidegree of g is i,j an the bidegree of h is k,ℓ, then $m=i-k$ and $n=j-\ell$. Rational functions on X can be represented as bihomogeneous fractions of bidegree $0,0$. If a curve C in X has bidegree m,n if it is the zero locus of a bihomogeneous polynomial $f(x,y)$ of bidegree m,n. xOmn

Let $\mathcal{O}_X(m,n)$ denote the co-module whose sections on an open subset W of X are the rational functions f on X such that $fx_0^my_0^n$ is a regular function on W. We say that such a function f has poles of orders $\le m$ on V and $\le n$ on H, where H is the 'horizontal' line $y_0=0$, and V is the 'vertical' line $x_0=0$.

(i) Determine the cohomology of $\mathcal{O}_X(m,n)$.

(ii) Determine the genus of a smooth curve of bidegree m,n.

8.9.30. Prove that a finite $\mathcal{O}$-module on a smooth curve is a direct sum of a torsion module and a locally free module. dirsum

8.9.31. Let Y be a smooth projective curve Y of genus g, and let d be an integer. Prove that gminusone

(i) If $d<g-1$, then $\mathbf{h}^1\mathcal{O}(D)>0$ for every divisor D of degree d on Y.

(ii) If $d\le 2g-2$, there exist divisors D of degree d on Y such that $hh^1\mathcal{O}(D)>0$.

(iii) If $d\ge g-1$, there exist divisors D of degree d on Y such that $hh^1\mathcal{O}(D)=0$.

(iv) If $d>2g-2$, then $\mathbf{h}^1\mathcal{O}(D)=0$ for every divisor D of degree d on Y.

8.9.32. Let $Y\xrightarrow{\pi}X$ be a branched covering of smooth curves. Use the trace from$\mathcal{O}_Y$ to $\mathcal{O}_X$ to prove that the $\mathcal{O}_X$-module $\mathcal{O}_Y$ (meaning its direct image) is isomorphic to the direct sum $\mathcal{O}_X\oplus\mathcal{M}$ for some locally free $\mathcal{O}_X$-module $\mathcal{M}$. OYdirectsum

8.9.33. Let Y be a smooth projective curve of genus $g>1$, and let D be an effective divisor of degree $g+1$ on Y, such that $\mathbf{h}^1\mathcal{O}(D)=0$ and $\mathbf{h}^0\mathcal{O}(D)=2$ (see Exercise 8.9.31). Let $Y\xrightarrow{\pi}X$ be the morphism to the projective line X defined by a basis $(1,f)$ of $H^0\mathcal{O}(D)$. The $\mathcal{O}_X$-module $\mathcal{O}_Y$ is isomorphic to a direct sum $\mathcal{O}_X\oplus\mathcal{M}$, where $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of rank g (see Exercise 8.9.32). decomposeOY

(i) Let p be the point at infinity of X. Prove that $\mathcal{O}_Y(D)$ is isomorphic to $\mathcal{O}_Y\otimes_{\mathcal{O}_X}\mathcal{O}_X(p)$.

(ii) Determine the dimensions of cohomology of $\mathcal{M}$ and of $\mathcal{M}(p)$.

(iii) According to the Birkhoff-Grothendieck Theorem, $\mathcal{M}$ is isomorphic to a sum of twisting modules $\sum_{i=1}^g\mathcal{O}_X(r_i)$. Determine the twists r_i.

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