18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #9 **Due: 13 Nov 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $X = \mathbb{P}^k$.

   (a) Prove that the multiplication morphism $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(m+n)$ is well-defined and an isomorphism. Deduce that the global sections of the tensor product does not have to be isomorphic to the tensor product of the global sections.

   (b) Prove that any non-zero morphism $\mathcal{O}_X(m) \to \mathcal{O}_X(n)$ is injective. Moreover, if $m > n$, prove that no such morphism exists.

2. Let $R = \mathbb{C}[x, y, z]$, and let $f = x^2 - yz$.

   (a) Determine generators and defining relations for the ring $R_{\{f\}}$ of homogeneous fractions of degree zero whose denominators are powers of f.

   (b) Prove that the twisting module $\mathcal{O}(1)$ is not a free module on the open subset of $\mathbb{P}^2$ at which $f \neq 0$.

3. Let $X = \mathbb{P}^1$. Let $\mathcal{M}$ be a locally free sheaf of modules of rank 1 on X, i.e. on for each U in some covering $\bigcup U$ of X by affine opens, there exists an isomorphism $\mathcal{M}|_U \to \mathcal{O}_U$ of $\mathcal{O}_U$ modules. Let U_0 and U_1 be the standard affines of X. You may assume that $\mathcal{M}(U_i)$ is a finitely generated $\mathcal{O}(U_i)$-module for $i = 0, 1$.

   (a) Prove that $\mathcal{M}(U_i)$ is isomorphic to $\mathcal{O}(U_i)$ for $i = 0, 1$. *Hint: use the structure theorem for finitely generated modules over a principal ideal domain.*

   (b) Prove that $\mathcal{M}$ is isomorphic to one of the twisting sheafs $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$.

4. Prove the following *coherence property* of an $\mathcal{O}$-module. Let Y be an open subset of a variety X, let s be a nonzero regular function on Y, and let Y_s be a localisation. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $\mathcal{M}(Y_s)$ is the localisation $\mathcal{M}(Y)_s$ of $\mathcal{M}(Y)$. In particular, $\mathcal{O}_X(U_s)$ is the localisation $\mathcal{O}_X(U)_s$. (This is a requirement for an $\mathcal{O}$-module, when Y is affine.)

5. Let $n \geq 0$ and consider the twisting module $\mathcal{O}(n)$ on $\mathbb{P}^1$ with homogeneous coordinates x_0, x_1.

(a) Show that the global sections x_0^n, x_1^n generate $\mathcal{O}(n)$, i.e., that the map

$$\begin{aligned} \mathcal{O} \oplus \mathcal{O} &\to \mathcal{O}(n) \\ (s_0, s_1) &\mapsto x_0^n s_0 + x_1^n s_1 \end{aligned}$$

is surjective.

(b) Show that the induced map on global sections is not surjective when $n > 1$.