18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #8 **Due: 5 Nov 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Prove that, when a variety X is covered by countably many constructible sets, a finite number of those sets will cover X.

2. (a) Let X be a proper affine variety. Show that X is a point.

   (b) Let $f\colon X \to Y$ be surjective morphism of varieties, and suppose X is a proper variety. Prove that Y is proper.

   (c) Let X and Y be varieties. Show that $X \times Y$ is proper if and only if both X and Y are proper.

3. Consider the space Z defined in Example 5.5.7 of the 22 October version of the lecture notes.

   (a) Consider the subspace Z_{irred} of irreducible polynomials. Prove that Z_{irred} is open in Z.

   (b) Consider the projection map $\pi\colon \Sigma \to Z$. Show that $\pi^{-1}(z)$ has a component of dimension greater than 0 if and only if the polynomial f corresponding to z is not square-free.

   (c) Show how the semicontinuity of the fibre dimension implies that the subspace $Z_{\text{sq-free}}$ of squarefree polynomials is open in Z.

4. Finish the proof of Theorem 5.6.2. *Hint: see also exercises 5.7.16 and 5.7.18 in the lecture notes (but you are not obliged to use these).*

5. Let $u\colon Y \to X$ be a surjective morphism of varieties and define $\Delta\colon X \to \mathbb{Z}$ by

$$\Delta(x) = \max_{y \in f^{-1}(x)} \delta(y).$$

   Suppose Y is proper, prove that Δ is upper semicontinuous.

6. Let $P = (0:0:1) \in \mathbb{P}^2$. Determine $\mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus \{P\})$.