18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #7 **Due: 29 Oct 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let X be a variety and let $C_0 \supsetneq C_1 \supsetneq \ldots \supsetneq C_k$ be some chain of irreducible closed subvarieties of X. Prove that this chain can be extended to a chain of length $\dim(X)$, by putting new irreducible closed subvarieties between, before, or after the existing ones.

2. Let $f\colon X \to Y$ be a finite morphism. Prove that for every $y \in Y$ the fibre $f^{-1}(y)$ is finite.

3. For each of the following statement determine whether the statement is true. Give a proof or disproof.

   (a) Chevalley's finiteness theorem (corollary 4.6.8 in the 15 Oct version of the notes) also holds when Y is affine.

   (b) If $f\colon X \to Y$ is a surjective morphism of varieties and $\dim(X) = \dim(Y)$, then f is finite.

   (c) Let $f\colon X \to Y$ be a surjective morphism of curves. Suppose that X is smooth. Then Y is smooth.

   (d) Let $f\colon X \to Y$ be a surjective morphism of curves. Suppose that Y is smooth. Then X is smooth.

4. Prove that, if v is a (discrete) valuation on a field K that contains the complex numbers, every nonzero complex number c has value zero.

5. Prove that every nonconstant morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is finite. Do this by showing the fibres cannot have positive dimension.