18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #6 **Due: 22 Oct 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Prove this alternate form of the Nullstellensatz: Let k be a field of characteristic zero, and let B be a domain that is finitely generated k-algebra. If B is a field, then $[B : k] < \infty$.

2. Let A be a finite type domain over $\mathbb{C}$, $R = \mathbb{C}[t]$, $X = \mathrm{Spec}(A)$, and $Y = \mathrm{Spec} R$. Let $\varphi^* : A \to R$ be a homomorphism whose image is not a subset of $\mathbb{C}$, and let $\varphi : Y \to X$ be the corresponding morphism. Show that R is a finite A-module, and that the image φ is a closed subset of X.

3. Let α be an element of a domain A, and set $\beta = \alpha^{-1}$. Prove that if β is integral over A, the it is an element of A.

4. Let k be a field of characteristic zero. Let $A \subset B$ be finite type domains over k with fraction fields $K \subset L$ of characteristic zero, and let $Y \to X$ be the corresponding morphism of affine varieties. Prove the following:

   (a) There is a nonzero element s in A such A_s is integrally closed.

   (b) There is a nonzero element s in A such that B_s is a finite A-module over a polynomial ring $A_s[y_1, \ldots, y_d]$.

   (c) Supposed that L is a finite extension of K of degree d. There is a nonzero element $s \in A$ such that all fibres of the morphism $Y_s \to X_s$ consist of d points.

5. Identify the normalization of $A = \mathbb{C}[x, y, z]/(y^2 - xz^2)$.