18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #5 **Due: 15 Oct 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $X = \mathrm{Spec} A$ be an affine variety. The following conditions defined close subsets Y of $X \times \mathbb{P}^n$. Prove that they are equivalent.

   (a) Y is the locus of zeros of a finite number of homogeneous polynomials in $A[x_0, \ldots, x_n]$.

   (b) For the standard affine opens $U_i \subset \mathbb{P}^n$, the intersection $Y \cap (X \times U_i)$ is a closed subset of the affine variety $X \times U_i$.

2. Determine all the morphisms $\mathbb{P}^2 \to \mathbb{P}^1$.

3. Consider the Veronese embedding $\mathbb{P}^2_{x,y,z} \to \mathbb{P}^5_u$ by monomials of degree 2 defined by $(u_0, u_1, u_2, u_3, u_4, u_5) = (z^2, y^2, x^2, yz, xz, xy)$. If we drop the coordinate u_0, we obtain a map $\varphi : \mathbb{P}^2 \setminus \{q\} \longrightarrow \mathbb{P}^4$ that is defined at all points except $q = (0, 0, 1)$ by $(x, y, z) \mapsto (y^2, x^2, yz, xz, xy)$. Prove that the inverse map φ^{-1} is a morphism, and that the fibre over q is a projective line.

4. Let C be the projective plane curve $x^3 - y^2 z = 0$.

   (a) Show that the function field $K(C)$ of C is the field $C(t)$ of rational functions in $t = y/x$.

   (b) Show that the point $(t^2 - 1, t^3 - 1)$ of $\mathbb{P}^1(K(C))$ defines a morphism $C \to \mathbb{P}^1$.

5. Show that the conic C in $\mathbb{P}^2$ defined by the polynomial $x^2 + y^2 + z^2 = 0$ and the twisted cubic $V = V(v_0v_2 - v_1^2, v_0v_3 - v_1v_2, v_1v_3 - v_2^2)$ in $\mathbb{P}^3$ are isomorphic, by exhibiting a inverse morphisms between them.

6. Show that the morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $\varphi(t) = t^2$ is an integral morphism.