18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #4 **Due: 8 Oct 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $X = \mathrm{Specmax}(A)$ be an affine algebraic variety, and let f, g be elements of its function field $K = \mathrm{Frac}(A)$.
   (a) Prove that there is a Zariski open subset $U \subset X$ such that both f and g are regular on U.
   (b) Suppose that for every $p \in U$, we have $f(p) = g(p)$. Prove that $f = g$.
2. Let $X = \mathrm{Specmax}(A)$ be an affine variety. Show that homomorphisms $X \to \mathbb{A}^1$ correspond exactly to elements of A.
3. Let n be an integer. Describe an affine variety X such that for any affine variety $Y = \mathrm{Specmax} B$, the set of homomorphisms $Y \to X$ can be naturally identified with the set of invertible $n \times n$ matrices with coefficients in B and having trace 0.
4. Consider the action of the Klein four group V_4 on $A = \mathbb{C}[x, y]$, where the first generator acts by swapping x and y, and the second generator sends x to $-x$ and y to $-y$. Determine the ring of invariants A^{V_4}.
5. Consider the action of the symmetric group S_3 on $A = \mathbb{C}[x, y, z]$. You may assume that $A^{S_3} = \mathbb{C}[s_1, s_2, s_3]$ with $s_1 = x + y + z$, $s_2 = xy + yz + zx$, and $s_3 = xyz$.
   (a) Prove that $\mathrm{Frac}(A^{S_3}) = \mathbb{C}(s_1, s_2, s_3)$ has transcendence degree 3 over $\mathbb{C}$.
   (b) Find a polynomial $f \in A^{S_3}[t]$ of degree 3 such that $B := A^{S_3}[x] \subset A$ is isomorphic to $A^{S_3}[t]/(f)$.
   (c) Find a polynomial $g \in B[t]$ of degree 2 such that $C := B[y] \subset A$ is isomorphic to $B[t]/(g)$.
   (d) Prove that $C = A$, and deduce that A is a free A^{S_3}-module of rank 6 with basis $1, x, x, y, xy, xy$.
   (e) Let $\sigma \in S_3$ be the cycle (123). Explain how you can reduce the task of finding $A^{\langle\sigma\rangle}$ to a linear algebra problem over A^{S_3}. *You are not asked to actually solve this linear algebra problem. If you like to do that, we recommend using a computer algebra package.*
6. Show that the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$ is not closed in the product topology.