18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #3 **Due: 1 Oct 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $\mathbb{C}_{\mathrm{std}}$ be $\mathbb{C}$ with the usual Euclidean topology, and let $\mathbb{C}_{\mathrm{Zar}}$ be $\mathbb{C}$ with the Zariski topology. Determine for each of the following statements whether they are true or false, and give a proof or disproof.

   (a) The identity map $\mathbb{C}_{\mathrm{std}} \to \mathbb{C}_{\mathrm{Zar}}$ is continuous.

   (b) The identity map $\mathbb{C}_{\mathrm{Zar}} \to \mathbb{C}_{\mathrm{std}}$ is continuous.

   (c) Let X be an irreducible topological space, and let $f\colon X \to Y$ be continuous. Then $f(X)$ is irreducible.

   (d) Let Y be an irreducible topological space, and let $f\colon X \to Y$ be continuous. Then X is irreducible.

2. Find generators for the ideal of $\mathbb{C}[x_1, x_2, x_3]$ that vanish on the four points

$$(0,0,0), (0,0,1), (0,1,0), (1,0,0).$$

3. (a) Suppose that $V(I)$ is irreducible, i.e. $V(I)$ is a variety. Prove that $\sqrt{I}$ is a prime ideal.

   (b) Suppose that X and Y are Zariski closed subsets of $\mathbb{A}^n$ and suppose that $X \subsetneq Y$. Prove that there exists a polynomial f such that f vanishes on every point of X, but f does not vanish on every point of Y.

   (c) Suppose that P is a prime ideal. Prove that $V(P)$ is irreducible.

4. Let X be the affine variety in $\mathbb{A}^3$ with coordinates x, y, z given by $x^7 - y^5 = x^{13} - z^5 = y^{13} - z^7 = 0$. Prove that X is irreducible.