18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #2 **Due: 09/24/2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $f, g \in k[x]$, where $\deg f = \ell$ and $\deg g = m$. Show the following properties regarding resultants:
   (a) $\operatorname{Res}(f, g, x) = (-1)^{\ell m}\operatorname{Res}(g, f, x)$.
   (b) For $\lambda, \mu \neq 0$, we have $\operatorname{Res}(\lambda f, \mu g, x) = \lambda^m \mu^\ell \operatorname{Res}(f, g, x)$.
   (c) There are polynomials $A, B \in k[x]$ such that $Af + Bg = \operatorname{Res}(f, g, x)$.

2. Solve the following system of equations in $\mathbb{P}^3(x, y, z, t)$
$$\begin{aligned} x^2 + yt + zt &= t^2, \\ y^2 + xt + zt &= t^2, \\ z^2 + xt + yt &= t^2. \end{aligned}$$
   You may use a computer algebra system to carry intermediate calculations.

3. Let C be a cubic curve with a node.
   (a) Determine the degree of C^*.
   (b) Determine the number of flexes, bitangents, nodes, and cusps of C and of C^*.

4. Let C be a singular plane curve of degree d where all of its singularities are nodes or cusps. Let $\overline{C}$ be the desingularization of C. Compute the genus of $\overline{C}$ in the following settings:
   (a) C has a unique singular point, a cusp.
   (b) C has a unique singular point, a node.
   (c) C has a $k + n$ singular points, k cusps and n nodes.