18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #12 **Due: 3 Dec 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. (a) Let D be a divisor of degree d on a smooth projective curve Y, and assume that $h^0(D) > 0$. Prove that if p is a generic point of Y, then $h^0(\mathcal{O}(D-p)) = h^0(D) - 1$.

   (b) Let Y be a smooth projective curve of genus $g \geq 2$, and let $D = \sum_{i=1}^{g+1} q_i$ be a divisor that is the sum of $g+1$ generic points of Y. Prove that $h^0(\mathcal{O}(D)) = 2$.

2. Let Y be a smooth curve of genus 2 and suppose p is a point of Y such that $h^1(\mathcal{O}(2p)) = 0$. Show that there exists a basis of global sections of $\mathcal{O}(4p)$ of the form $(1, x, y)$, where x and y have poles of order 3 and 4 at p, respectively. Prove that this basis defines a morphism $Y \to \mathbb{P}^2$ whose image is a singular curve of degree 4.

3. Let Y be a smooth curve of genus g.

   (a) Show that if D is a divisor on Y with $\deg(D) < 0$, then $h^0(\mathcal{O}(D)) = 0$.

   (b) Show that if D is a divisor on Y and $\deg(D) > 2g-2$, then $h^0(\mathcal{O}(D)) = 1 - g + \deg(D)$.

   (c) Suppose that $g \geq 2$ and let p be a point of Y. Note that $h^0(\mathcal{O}(kp)) \leq h^0(\mathcal{O}((k+1)p))$ for all $k \in \mathbb{Z}$. Show that for k in the range $0 \leq k \leq 2g-2$, there are exactly g values of k where $h^0(kp) = h^0(\mathcal{O}((k+1)p))$. (These are called Weierstrass gaps.)